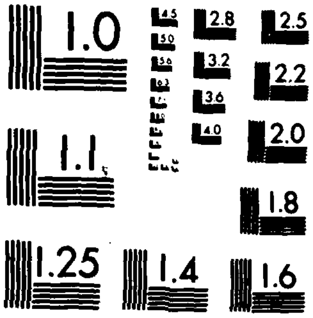
AD-A128 737 EFFICIENT ADI (ALTERNATING DIRECTION IMPLICIT) AND SPLINE ADI METHODS FOR..(U) DAYTON UNIV OH RESEARCH INST M NAPOLITANO FEB 83 AFWAL-TR-83-3001 1/1

UNCLASSIFIED F33615-79-C-3030 F/G 12/1 NL

END
DATE
FILMED
6-83
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

AFWAL-TR-83-3001

EFFICIENT ADI AND SPLINE ADI METHODS FOR THE NAVIER-STOKES EQUATIONS

MICHELE NAPOLITANO
Visiting Scientist

UNIVERSITY OF DAYTON RESEARCH INSTITUTE
300 COLLEGE PARK
DAYTON, OHIO 45469

FEBRUARY 1983

Final Report for Period July 1981 - September 1981

Approved for public release; distribution unlimited

FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This report has been reviewed by the Information Office (PA) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

Project Engineer

LOWELL C. KEEL, Lt Col, USAF
Chief, Aerodynamics and
Airframe Branch

JOHN R. CHEVALIER, Col, USAF
Chief, Aeromechanics Division

"If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization please notify AFWAL/FIMM, W-PAFB, OH 45433 to help us maintain a current mailing list."

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFWAL-TR-83-3001	2. GOVT ACCESSION NO. AD-A128737	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EFFICIENT ADI AND SPLINE ADI METHODS FOR THE NAVIER-STOKES EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Final Report July 1981 - September 1981 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Dr. Michele Napolitano Universita Degli Studi Di Bari Via Re David, 200 - 70125 Bari, ITALY		8. CONTRACT OR GRANT NUMBER(s) F33615-79-C-3030
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Dayton Research Institute 300 College Park Dayton, Ohio 45469		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Project 2307N603
11. CONTROLLING OFFICE NAME AND ADDRESS Flight Dynamics Laboratory (AFWAL/FIMM) Air Force Wright Aeronautical Labs (AFSC) Wright-Patterson AFB, Ohio 45433		12. REPORT DATE February 1983 13. NUMBER OF PAGES Thirty-two
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Numerical Solution
Incompressible, Laminar Viscous Flow
Alternating Direction Implicit (ADI) Technique
General (orthogonal) Curvilinear Coordinates
High Order Accurate Spline
Interpolating Polynomials

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This ~~The present~~ report provides an improved Alternating Direction Implicit (ADI) technique as well as a high-order-accurate spline ADI method, for the numerical solution of steady two-dimensional incompressible viscous flow problems. The vorticity-stream function Navier-Stokes equations are considered in a general curvilinear coordinate system, which maps an arbitrary two-dimensional flow domain in the physical plane into a rectangle in the computational plane. The stream function equation is parabolized in time by means of a relaxation-like time derivative and the steady state solution is obtained by a time marching two-sweep

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. (Cont'd)

ADI method, which requires to solve only linear 2 x 2 block-tridiagonal systems. The difference equations are written in incremental form; windward differences are used for the incremental variable for stability, whereas central differences approximate the nonincremental terms for accuracy. Thus, at convergence, the solution is free of numerical viscosity and second-order-accurate. The high-order accurate spline ADI technique proceeds exactly in the same manner. In addition, at the end of each two sweep ADI cycle, the solution is corrected by means of a fifth-order spline interpolating polynomial along each row and column of the computational grid, explicitly. Thus, at convergence, a formally fourth-order-accurate solution is obtained, for the case of orthogonal coordinate systems.

The validity and the efficiency of the present methods are demonstrated by means of their application to three test problems.

FOREWORD

This report is the result of the work performed by Dr. Michele Napolitano, visiting scientist in the AFWAL/FIMM, from July to September 1981 under Project Number 2307N603, whose AFWAL task engineer was Dr. W. L. Hankey. The report was completed by the Author in Italy in 1982 and the calculations of the flow in a channel of complex geometry were performed in cooperation with Vinicio Magi, whose care and thoroughness have been of invaluable help.

The Author is also very grateful to Dr. Hankey and Dr. Shang, for their continuous interest and encouragement, to all members of the Computational Aerodynamics group, for the cooperation and help provided with the computer on base and to Professor S. G. Rubin, of the University of Cincinnati, for many precious discussions and helpful suggestions.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF SYMBOLS

A,B	Arbitrary constants in the spline deferred corrector approach
c	Constant in Burgers equation
D	Incremental operator, applied to any variable
h	Step size in the space variable
J	Jacobian of the coordinate transformation
k	Time step
K	Spline coefficient, related to its second derivative
m	Spline first derivative
M	Spline second derivative
t	Nondimensional time
x	Nondimensional horizontal coordinate
y	Nondimensional vertical coordinate
u	Dependent variable in Burgers equation
Y_ℓ, Y_u	Lower and upper boundaries of the channel

Greek Symbols

$\alpha,\beta,\gamma,\sigma,\tau$	Scale factors of the coordinate transformation
Δ	Step size in anulus flow
∂	Partial derivative sign
η	Vertical transformed coordinate
ν	(Nondimensional) kinematic viscosity in Burgers equation
ξ	Longitudinal transformed coordinate
ψ	Stream function
ω	Vorticity

Subscripts

i, i+1, i-1	Longitudinal gridpoints
j, j+1, j-1	Vertical gridpoints
t,x,y,ξ,η	Partial derivative with respect to the indicated variable

Superscripts

n, $\sim$, n+1	(Time) levels of the ADI techniques

SECTION I

INTRODUCTION

The present Author has developed an Alternating Direction Implicit (ADI) technique for the calculation of viscous, incompressible, steady flows past an arbitrary, two-dimensional body[1]. Such an approach used the vorticity-stream function Navier-Stokes equations in a system of general body-fitted coordinates. The governing equations were parabolized by adding a relaxation-like time derivative to the stream function equation, linearized in time and solved by means of the ADI procedure of Douglas and Gunn[2]. The method used second-order-accurate finite differences and, at convergence, provided a second-order-accurate approximation to the steady flow of interest. The major limitation of the proposed ADI approach was due to its use of central differences, which limited its applicability to low Reynolds number flows, or to separation-free high Reynolds number flows. A first-order-accurate method, using windward differences for the convective terms in the equations, although feasible in principle, is not recommended, insofar as the effective Reynolds number of the numerical solution is lowered by the numerical viscosity introduced by the first-order-accurate windward differences. A more stable, viscosity-free numerical technique is obtainable by using windward differences for the convective terms, which are evaluated implicitly, and correcting them to second-order-accurate central differences, explicitly[3], or, more simply, by employing the incremental (delta) form of the equations[4] and using windward differences for the incremental variables and central differences for the nonincremental variables. Both approaches, which are shown here to coincide, have the desirable property of lowering the effective Reynolds number of the pseudo-transient problem by adding numerical viscosity, which is, however, completely removed from the solution as convergence is achieved. The second approach, using the incremental variables, is employed in this paper to provide an improved version of the ADI technique given in Ref. 1, which is more stable and capable of resolving high Reynolds number separated flows, while maintaining the second-order accuracy of the numerical-viscosity-free central differences, at convergence.

Furthermore, following the idea of Rubin and Khosla[5], it is possible and very straight-forward indeed to obtain a fourth-order-accurate spline ADI method by means of a spline deferred-corrector approach[5]. The present paper also provides a simplified spline ADI technique for the vorticity-stream function Navier-Stokes equations, which has all the features of the aforementioned improved ADI method. In particular, incremental variables are used and, in order to enhance the stability of the method, the convective terms are approximated by first-order-accurate windward differences. At the end of each two-sweep ADI cycle, the right hand side (RHS) of the difference equations, which is already second-order-accurate, is corrected by means of a spline interpolating procedure, explicitly, so that, at convergence, a fourth-order-accurate approximation to the steady flow of interest is obtained.

The present report develops as follows: In Section II the basic ideas of obtaining second- or fourth-order accuracy at convergence, while using first-order-accurate windward differences to stabilize the transient phenomenon, are provided. The approach of Rubin and Khosla[3,5] is briefly reviewed and the equivalent simpler and more elegant approach used in this study is presented. In Section III those ideas are applied to the ADI numerical technique previously developed by the Author[1], to provide an improved, more stable, second-order-accurate ADI method as well as a fourth-order-accurate spline ADI procedure.

Finally, the results obtained by applying the present techniques to two model problems (viscous flow between two concentric circles and the classical driven cavity flow) as well as to a problem of practical interest (viscous flow in a channel of complex geometry) are presented in Section IV.

SECTION II

THE DEFERRED CORRECTOR APPROACH OF RUBIN AND KHOSLA

Rubin and Khosla have presented their simplified spline technique[5], as applied to the numerical solution of the steady state Burgers equation, starting from the unsteady equation:

$$u_t + c\, u_x = \nu u_{xx}$$

The numerical procedure of Rubin and Khosla uses the following discrete form of Eqn. (1):

$$u_j^{n+1} - u_j^n + cA(u_j^{n+1} - u_{j-1}^{n+1})k/h - B\,\nu(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})k/h^2 =$$

$$c\,\{A(u_j^n - u_{j-1}^n) - hm_j^n\}\,k/h + \nu\,k\{M_j^n - B(u_{j+1}^n - 2\,u_j^n + u_{j-1}^n)/h^2\} \qquad (2)$$

where the subscript j, j-1 and j+1 indicate the spatial grid locations, the superscripts n and n+1 indicate the old and new time levels, k is the time step, h is the spatial meshwidth, m and M are the fourth-order-accurate spline approximations of u_x and u_{xx} and A and B are two arbitrary constants, necessary for the stability of the method, see Ref. 5 for details. Notice that the high-order spline correction terms only appear on the old time level, known RHS of Eqn. (2), so that at each time advancement only a tri-diagonal system has to be solved, exactly as for the case of a low-order-accurate finite difference scheme, but that, at convergence, a fourth-order-accurate solution is obtained. Also notice that, if m and M are replaced with simple second-order-accurate finite difference approximations, Eqn. (2) leads to the unconditionally stable KR scheme (A = B = 1) of Ref. 3.

In the present paper an ADI and a spline ADI technique will be developed for the Navier-Stokes equations, which are based on these two techniques.

However, an incremental (delta) formulation is preferred in this paper for its superior simplicity and elegance. For example, by writing Eqn. (2) in delta form, one obtains:

$$Du_j + cA\,(Du_j - Du_{j-1})k/h - B\,\nu(Du_{j+1} - 2Du_j + Du_{j-1})k/h^2 =$$

$$- c\,k\,m_j^n + \nu\,k\,M_j^n \qquad (3)$$

where $Du = u^{n+1} - u^n$. Eqn. (3) actually coincides with Eqn. (2) and will obviously produce the same numerical results, but it is simpler, more elegant and requires less computation effort. Notice, for example, that the RHS of Eqn. (3) is a fourth-order-accurate discrete approximation of the steady Burgers equation, which thus, at convergence ($Du = 0$ at all gridpoints), will be resolved with the desired level of accuracy. Moreover, the arbitrary constants A and B, which only multiply the incremental variables, are clearly seen not to influence the final steady state solution. By comparing the simplicity of Eqn. (3) with respect to Eqn. (2), it is easy to understand the advantage of using the delta approach in complex numerical techniques for the Navier-Stokes equations. Obviously, also in Eqn. (3), by replacing m and M with standard second-order-accurate finite differences, one obtains a scheme which has the stability of a windward difference scheme and the accuracy (at convergence) of a central difference scheme. In such a case, although unnecessary for stability, the use of values greater than one for A and B can further enhance the convergence of the numerical method. It remains to be said how m and M are evaluated. After all n+1 values have been obtained directly from Eqn. (2), or from Eqn. (3) and the definition of

Du, the new time level m and M values are explicitly evaluated as:

$$m_j = (u_{j+1} - u_{j-1})/2\,h + h(K_{j-1} - K_{j+1})/12 \qquad (4)$$

and

$$M_j = \{K_j + (u_{j+1} - 2u_j + u_{j-1})/h^2\}/2 \qquad (5)$$

where all K_j terms are easily obtained by solving the following tridiagonal system

$$K_{j+1} + 4K_j + K_{j-1} = 6(u_{j+1} - 2u_j + u_{j-1})/h^2 \qquad (6)$$

see Ref. 5 or Ref. 8 for details. It is noteworthy that, with respect to a standard second-order-accurate finite difference method, the simplified spline technique requires the solution of an additional tridiagonal system (Eqn. 6), at each time level, whereas a standard spline technique[8] would require the solution of a 2 x 2 block-tridiagonal system, at each time level. The convenience of the simplified approach is seen to increase when dealing with coupled systems of equations (e.g. with the Navier-Stokes equations).

SECTION III

THE PRESENT ADI AND SIMPLIFIED SPLINE ADI METHODS FOR THE NAVIER-STOKES EQUATIONS

The vorticity-stream function Navier-Stokes equations in a general system of curvilinear body-oriented coordinates (ξ,η) are given[7] as:

$$\omega_t + (\psi_\eta \omega_\xi - \psi_\xi \omega_\eta)/J - (\alpha\omega_{\xi\xi} - 2\beta\omega_{\xi\eta} + \gamma\omega_{\eta\eta} + \sigma\omega_\eta + \tau\omega_\xi)/J^2 Re = 0 \qquad (7)$$

and

$$(\alpha\psi_{\xi\xi} - 2\beta\psi_{\xi\eta} + \gamma\psi_{\eta\eta} + \sigma\psi_\eta + \tau\psi_\xi)/J^2 + \omega = \psi_t \qquad (8)$$

where α, β, γ, σ and τ are the metric coefficients and J is the jacobian of the coordinate transformation, and a relaxation-like time derivative has been added to the stream function equation, in order to deal with a parabolic system of equations[6,1]. For this reason, Eqns. (7) and (8) are not the time-*dependent Navier-Stokes equations and in the present time-marching numerical* techniques only the converged solutions will have physical meaning.

Equations (7, 8) are written in terms of the incremental variables, $D\psi = \psi^{n+1} - \psi^n$, $D\omega = \omega^{n+1} - \omega^n$, and linearized in time by a Taylor's series expansion, which neglects terms of order D^2, to give:

$$D\omega/k + (D\psi_\eta \omega_\xi^n + \psi_\eta^n D\omega_\xi - D\psi_\xi \omega_\eta^n - \psi_\xi^n D\omega_\eta)/J - (\alpha\, D\omega_{\xi\xi} + \gamma\, D\omega_{\eta\eta} +$$

$$\sigma\, D\omega_\eta + \tau\, D\omega_\xi)/J^2 Re = SSVE^n \qquad (9)$$

and

$$D\psi/k - D\omega - (\alpha\, D\psi_{\xi\xi} + \gamma\, D\psi_{\eta\eta} + \sigma\, D\psi_\eta + \tau\, D\psi_\xi)/J^2 = SSSFE^n \qquad (10)$$

where $SSVE^n$ and $SSSFE^n$ are shorthand notations for steady state vorticity (stream function) equation evaluated at the n time level. It is noteworthy that the linearized Eqns. (9) and (10) are fully implicit for the incremental variables except for the mixed derivatives, which are treated explicitly. Also, for more generality, two constants, A and B, can be introduced to multiply all convective and diffusive incremental terms, respectively. At this point, we are ready to solve Eqns. (9) and (10) by means of two-sweep ADI techniques, differing only in the level of accuracy used to approximate $SSVE^n$ and $SSSFE^n$. These ADI methods, derived from that of Douglas and Gunn[2], proceed as follows: at the first sweep, the η derivatives and the source-like terms in the LHS (left hand side) of Eqns. (9) and (10) are evaluated implicitly to give:

$$D\tilde{\omega}/k + (D\tilde{\psi}_\eta \, \omega_\xi - \psi_\xi D\tilde{\omega}_\eta)/J - (\gamma \, D\tilde{\omega}_{\eta\eta} + \sigma \, D\tilde{\omega}_\eta)/J^2 \, Re = SSVE^n \tag{11}$$

$$D\tilde{\psi}/k - D\tilde{\omega} - (\gamma D\tilde{\psi}_{\eta\eta} + \sigma \, D\tilde{\psi}_\eta)/J^2 = SSSFE^n \tag{12}$$

where the $\sim$ indicate that the solution is a *first sweep (predictor-type) one* and all the ξ derivatives, which are evaluated explicitly, give zero contribution in the incremental variables. At the second and final sweep, all the ξ derivatives and the source-like terms are evaluated implicitly, whereas the η derivatives are evaluated at the first sweep ($\sim$) level, explicitly. The resulting equations are not given here because, for computational convenience, they are replaced by the following ones, obtained by subtracting from them the first sweep Eqns. (11,12):

$$D\omega/k + (\psi^n_\eta \, D\omega_\xi - D\psi_\xi \, \omega^n_\eta)/J - (\alpha \, D\omega_{\xi\xi} + \tau \, D\omega_\xi)/J^2 Re = D\tilde{\omega}/k \tag{13}$$

$$D\psi/k - D\omega - (\alpha \, D\psi_{\xi\xi} + \sigma \, D\psi_\xi)/J^2 = D\tilde{\psi}/k - D\tilde{\omega} \tag{14}$$

In Eqns. (11) thru (14) all the incremental terms are approximated with central differences (the second derivatives) and windward differences (the first derivatives) whereas the nonincremental terms are approximated with standard central differences (for the case of the ADI method) or fourth-order-accurate spline approximations (for the case of the spline ADI method). Therefore, for both techniques, a series of 2 x 2 block-tridiagonal systems is to be solved at each sweep of the ADI procedure, exactly as in the former ADI technique due to the Author[1].

At the end of a complete ADI cycle the solution is updated as:

$$\psi^{n+1} = \psi^n + D\psi \tag{15}$$

$$\omega^{n+1} = \omega^n + D\omega \tag{16}$$

and, for the case of the ADI method, the process is repeated until a satisfactory convergence is achieved.

For the case of the spline ADI, however, in order to be able to evaluate the RHS in Eqns. (11) and (12) with fourth-order accuracy (at convergence), it is necessary to obtain a fifth-order interpolating polynomial approximating the new values of the stream function and of the vorticity along each row and column of the computational grid. This is done by solving two tridiagonal systems, formally identical to Eqn. (6), for each row and for each column of gridpoints, which allow to evaluate the four matrices $K\psi\xi_{i,j}$, $K\psi\eta_{i,j}$, $K\omega\xi_{i,j}$ and $K\omega\eta_{i,j}$; from these, all the corresponding first and second derivatives, $m\psi\xi_{i,j}$, $M\psi\xi_{i,j}$, etc., can then be evaluated *explicitly*, by means of expressions formally identical to Eqns. (4) and (5). As far as the boundary conditions are concerned, for the case of Eqn. (6), the first and last values of K are evaluated by linear or quadratic extrapolations from the neighboring points. It is noteworthy that in the present spline ADI technique, if a N x N mesh is used, at each ADI sweep it is necessary to solve 2N 2 x 2 block-tridiagonal systems (Eqns. 7-10) and 4N simple tridiagonal systems (equations of the type of Eqn. 6). A standard spline ADI would require the solution of 2N 4 x 4 block-tridiagonal systems, i.e., a lot more computational work. Also, the additional work of the present simplified spline ADI method with respect to the corresponding ADI approach is minimal, considering the accuracy improvement it provides.

The present approaches solve, at each sweep, the vorticity and the stream function equations as a coupled set on each row and column of the computational grid. For this reason it is possible, in the solution routine for each 2 x 2 block-tridiagonal system, to accommodate the double specification on the stream function at the boundary and to evaluate the vorticity at the wall, directly. For the case of the ADI method, the boundary conditions are imposed exactly as in Ref. 1, with the difference that, in the present case, the incremental approach is used also at the first sweep of the ADI method. For the case of the spline ADI method, the boundary conditions have to be imposed in such a way that, at convergence, they are to be fourth-order-accurate, consistently with the numerical scheme. Dirichlet boundary conditions are obvious, insofar as, if ω or ψ are prescribed at the boundary, it is required that

$$D\tilde{\psi} = D\tilde{\omega} = D\psi = D\omega = 0 \qquad (17)$$

at the appropriate boundary gridpoints. A Neumann boundary condition for the stream function is slightly more complicated to deal with. Here only one example will be given, namely how to impose the boundary condition $\psi_\eta = 0$

in the first sweep of the spline ADI method: The boundary condition itself is written in incremental form and replaced by its fourth-order-accurate spline approximation (Eqn. 4); the stream function equation at the boundary and the second boundary condition on the stream function are also considered. In this way, three linear equations are available for evaluating the vorticity and the stream function at the boundary point and the stream function at a mirror-image gridpoint(external to the flow field). The stream function equation basically allows to eliminate this extra unknown and the two boundary conditions on the stream function allow a direct evaluation of the wall stream function and vorticity. Obviously, all the K terms are known from the previous time level and the value corresponding to the aforementioned mirror-image gridpoint is obtained by linear or quadratic extrapolation. Several other boundary conditions are possible; for example, using alternate expressions of m and M, which do not introduce a mirror-image gridpoint, or two different expressions for m, which amounts to enforce that the spline interpolating polynomial has a continuous first derivative thru the boundary gridpoint. In the present study all the approaches described above have been used successfully.

In Ref. 1, an alternate, Crank Nicolson-type, linearization and discretization in time of the governing equations was also used. The present techniques also have this option: In the computer programs all coefficients of the block-tridiagonal systems, to be solved at every sweep of the ADI methods, contain a CR coefficient which can be either 1 or 0.5, to provide the implicit backward (see, e.g., Eqns. 9 and 10) or the Crank Nicolson time discretization, directly. The Crank Nicolson approach is obviously second-order-accurate in time, but, since the present techniques only provide the steady state solution, this is not necessarily an advantage. Furthermore, when using a Crank Nicolson averaging, in order to obtain a correct value of the vorticity at the wall, the boundary conditions have to be inconsistent with the difference equations; that is, it is necessary to use an implicit backward time discretization of the stream function equation at all boundary points in the ∿ sweep of the ADI methods, in order to obtain the correct vorticity at the wall at convergence. This result is consistent with that already observed by Davis et al[9] and Briley and McDonald[10]. For this reason, most of the results later presented in this report have been obtained with the fully implicit time discretization of Eqns. (9,10). However, in some calculations, the Crank Nicolson time linearization (which obviously produces identical results at convergence) has been found to provide faster convergence rates for the same values of Δt.

SECTION IV

RESULTS

A. Flow Between Two Rotating Circles

The present methods have been developed in order to compute viscous steady flows past two-dimensional airfoils, in connection with a method for generating a system or orthogonal curvilinear coordinates. The viscous flow between two concentric rotating circles has been considered as a model problem (somewhat simulating such a flow configuration), for which an exact solution is available for comparisons. The inner circle of radius equal to one is chosen to be stationary in order to test the no-slip, zero injection boundary conditions, usually given at the surface of a stationary airfoil, whereas the outer circle rotates at such a speed that the vorticity on its boundary is also equal to one. A given vorticity has been imposed at the outer circle (of radius equal two), because, for external flow configurations, the outer boundary is usually chosen at a sufficient distance from the surface of the airfoil, that a zero vorticity boundary condition is imposed. The physical flow field, divided into a system of equally spaced polar coordinates, has been transformed into a rectangle in the ξ,η plane. All metric coefficients and the jacobian have then been evaluated numerically with fourth-order-accurate spline interpolating polynomials, except for the mixed derivatives which are identically zero. It is noteworthy that, due to the coordinate transformation, the flow field is "opened up," so that periodic boundary conditions are needed in the ξ direction. For more details and for an algorithm capable of solving periodic 2 x 2 block-tridiagonal systems, the reader is referred to Ref. 1.

The results obtained with both the ADI and the spline ADI methods for such a test problem are given in Fig. 1, where the stream function at the center of the anulus and the vorticity at the wall of the inner circle are plotted versus the inverse of the number of gridpoints (Δ), square (for the ADI method), and to the fourth power (for the spline ADI method), for several values of Δ. Two sets of spline ADI results are given, corresponding to the use of a linear and a quadratic extrapolations for the K spline functions. All results are seen to tend to the exact solution as Δ tends to zero and with the correct second-order-accuracy and fourth-order-accuracy, respectively. Also, the higher order extrapolation is seen to produce more accurate results as expected. However, the most interesting point to make is that, whereas the computation cost is almost equivalent (the convergence rate is almost the same for both approaches, probably due to the one-dimensional nature of the problem), the more accurate spline ADI results, using a 10 x 10 mesh, have a discretization error up to five times smaller than that of the ADI results, using a 24 x 24 mesh.

B. The Driven Cavity Flow

The classical driven cavity problem (see, e.g., Rubin[11]) was also used as a test problem to verify the present numerical techniques. The Re = 100 case has been considered, using a uniform rather coarse 14 x 14 mesh. The values of the vorticity at the center of the moving wall of the cavity and the maximum value of the stream function are given in Table 1 for the ADI method as well as for the spline ADI method using either linear or quadratic extrapolation for the spline boundary conditions. The second-order-accurate results are seen to perfectly coincide with the results given by Rubin[11]. It is worth mentioning that the present approach does not use the divergence form of the equations.

The results in Table 1 have been obtained with both the backward and Crank Nicolson time discretization of the governing equations and with values of A and B equal to or greater than one. It is noteworthy that, if one uses CR = 0.5 and the consistent boundary conditions, the vorticity at the wall is completely wrong although a correct solution is obtained at all internal gridpoints. The spline ADI results given in Table 1 clearly indicate the superior accuracy obtained by means of the high order spline correction procedure (see Ref. 11 for very accurate results). It is noteworthy that in the present calculations it was unnecessary to use values of A and B greater than 1. The results in Table 1 have been obtained within 2 CPU minutes on an HP 1000/F minicomputer, thus verifying the efficiency of the proposed methods. The Re = 1000 rather difficult case has also been considered, using a uniform 20 x 20 mesh, and the results obtained with the ADI method are given in Table 1. Although insufficiently accurate, due to the use of a uniform mesh, which is completely incapable of capturing the thin boundary layers at the walls of the cavity, the results in Table 1 are the correct ones corresponding to a second-order-accurate central space discretization of the steady state equations. Furthermore, the most important point is that, whereas this Re = 1000 case was found to be an impossible task for the method of Ref. 1, a fully converged solution was obtained in less than 1000 ADI cycles (30 CPU minutes on the HP/1000F minicomputer) by using the present improved method.

C. Flow in a Channel of Complex Geometry

The present methods were finally used to compute viscous laminar flow inside a channel of complex geometry. The problem was first proposed by Roache[12] who numerically verified that, if the length of the channel is scaled proportionally to the Reynolds number of the flow, self similar flow conditions are obtained for very high Re values. Recently the same problem has been used as a numerical test-case for comparing the accuracy and efficiency of several numerical Navier-Stokes solvers by the IAHR working group on refined modelling of flows in its VI meeting held in Rome (June 24-25, 1982). The geometry of the channel is given in Figs. 2a for the Re = 10 case and 2b for the Re = 100 case. The lower wall of the channel is given analytically as

$$Y_{\ell} = \frac{1}{2}\,[\tanh\,(2 - 30\,x\,/Re) - \tanh\,2] \tag{18}$$

and its centerline as

$$Y_u = 1 \tag{19}$$

The inlet and outlet sections of the channel are finally given as

$$x = 0 \quad \text{and} \quad x = Re/3 \tag{20a,b}$$

respectively.

In the present study, a system of orthogonal, curvilinear coordinates has been used to map the physical (x,y) flow domain into a rectangle in the (ξ,η) computational domain. A simple algebraic transformation, as given by Blottner and Ellis[13] and described by Davis[14], has been used for simplicity as well as for taking full advantage of the shape of the channel, being prescribed analytically. The system of (η = constant) coordinate lines in the physical plane has been prescribed as follows: The line $\eta = 0$ coincides with the lower boundary (the wall) of the channel and the line $\eta = 1$ with its upper boundary (the symmetry line). All the other $\eta = (j - 1)\Delta\eta$ ($j = 2,3,\ldots N+1$, $\Delta\eta = \frac{1}{N}$) are given by the following expressions

$$Y_j = \frac{q^N - q^{j-1}}{q^N - 1}\,Y_{\ell}(x) + \frac{q^{j-1} - 1}{q^N - 1}\,Y_u(x) \tag{21}$$

A family of (ξ = constant) lines, orthogonal to the η = constant lines have then been obtained by integrating numerically with a fourth-order-accurate Runge Kutta procedure, the following equation

$$\frac{\partial x}{\partial \eta} = -\frac{(\frac{\partial y}{\partial \xi})_\eta (\frac{\partial y}{\partial \eta})_x}{1 + (\frac{\partial y}{\partial x})_\eta^2} \tag{22}$$

starting from prescribed points at the lower boundary of the channel. A few points are of interest: The distance between two successive η = constant lines in the physical plane has been chosen to increase at a constant rate ($q = 1.1$) starting from the lower boundary. In this way a finer resolution is obtained near the wall of the channel where viscous effects are more important. The distance between two successive ξ = constant lines along the x coordinate in the physical plane has also been chosen to increase at a constant rate ($r = 1.054$), starting from the point $x_i = Re/15$ (where the lower boundary of the channel has an inflection point) in *both* $x > x_i$ *and* $x < x_i$ directions. In this way a finer resolution is obtained in the region where a separation bubble is likely to develop. Finally, the $\xi = 0$ and $\xi = 1$ lines in the physical plane have been chosen to coincide with the entrance and the exit of the channel, i.e., with the $x = 0$ and $x = Re/3$ lines, for convenience, and are not perfectly orthogonal to the η = constant lines. Therefore, the metric coefficients multiplying the mixed derivatives in the governing Navier-Stokes equations are not identically zero at all gridpoints and the accuracy of the spline ADI method slightly deteriorates locally, insofar as the mixed derivatives are evaluated with standard second-order-accurate central differences. The curvilinear coordinates, generated as described above, are shown in Fig. 3 for the Re = 10 case, together with the boundary conditions used in the present calculations. After evaluating all the gridpoint locations in the physical x,y plane (corresponding to a uniform cartesian grid in the computational ξ,η plane), the metric coefficients α, β, γ, σ, τ, and the jacobian of the transformation are evaluated at all internal gridpoints by means of central differences or fourth-order-accurate spline approximations in the ADI and spline ADI methods, respectively. At the boundary points the metric coefficients necessary to evaluate ω from the stream function equation are obtained by linear or cubic extrapolation from the neighboring gridpoints. The numerical solutions obtained for the Re = 10 and the Re = 100 cases by means of both present ADI methods are given in Table 2 as the values of the vorticity at the wall versus the x_i locations of the ξ coordinate lines (the x_i values corresponding to Re = 100 have to be multiplied by 10). For convenience, these results are also plotted in Figs. 4 and 5 for the Re = 10 and Re = 100 cases, respectively; the results obtained by means of the spline ADI method, using a coarser 10 x 10 mesh, are also given. All solutions are seen to coincide, for all practical purposes, and favorably compare with those obtained by means of several other methods (see e.g., Ref. 15). The 20 x 20 mesh spline ADI solution is the most accurate and the other two solutions have comparable accuracy. For the two ADI calculations, the minimum value of the vorticity at the wall is plotted versus the normalized number of iterations, to provide an idea of the convergence properties of the technique. A full convergence (to machine accuracy) has been obtained within about 130 and 150 iterations ($\Delta t = 0.075$ and $\Delta t = 0.08$) for the Re = 10 and Re = 100 cases, corresponding to less than 5 CPU minutes on the HP 1000/F minicomputer. If one considers that a reasonable convergence is obtained in about 40% of the total number of iterations (see Figs. 6 and 7), the efficiency of the present ADI approach is self evident. The spline ADI procedure, using the same mesh, required from five to ten times more iterations to fully converge, because a stability limitation on Δt had to be satisfied for $A = B = 1$. An optimization of the convergence, obtained by using different values of A and B, or by correcting the ADI solution only after a given number of ADI cycles, has not been pursued. However, by comparing the ADI solutions with the 10 x 10 mesh spline ADI solutions (which have comparable accuracy), it turns out that these required about 20% less CPU time.

SECTION V

CONCLUSIONS

An improved ADI numerical technique for the solution of incompressible viscous steady flows has been developed, together with a fourth-order-accurate spline ADI method obtained by applying a spline deferred corrector approach to the present ADI technique. The validity and efficiency of the present approaches have been demonstrated by their application to the numerical solution of three viscous flow problems.

REFERENCES

1. M. Napolitano, "Simulation of Viscous Steady Flows Past Arbitrary Two-Dimensional Bodies," in Numerical Methods for Nonlinear Problems, Pineridge Press, 1980, pp. 721-733. Also AFWAL-TR-80-3038.

2. J. Douglas and J. E. Gunn, "A General Formulation of Alternating Direction Methods," Numerische Mathematik, Vol. 6, 1964, pp. 428-464.

3. P. K. Khosla and S. G. Rubin, "A Diagonally Dominant Second-Order-Accurate Implicit Scheme," Computers and Fluids, Vol. 2, 1974, pp. 207-209.

4. R. M. Beam and R. F. Warming, "An Implicit Factored Scheme for the Compressible Navier-Stokes Equations," AIAA J., Vol. 16, n. 4, 1978, pp. 393-402.

5. S. G. Rubin and P. K. Khosla, "A Simplified Spline Solution Procedure," Presented at the 6th ICNMFD, Tblisi, USSR, 1978, in Lecture Notes in Physics, n. 90, pp. 468-476, Springer Verlag.

6. R. T. Davis, "Numerical Solution of the Navier-Stokes Equations for Symmetric Laminar Incompressible Flow Past A Parabola," J. Fluid Mech., Vol. 51, Part 3, 1972, pp. 417-433.

7. F. C. Thames, J. F. Thompson, C. W. Mastin, and R. L. Walker, "Numerical Solutions for Viscous and Potential Flow About Arbitrary Two-Dimensional Bodies Using Body-Fitted Coordinate Systems," J. Computational Physics, Vol. 24, 1977, pp. 245-273.

8. S. G. Rubin and P. K. Khosla, "Polynomial Interpolation Methods for Viscous Flow Calculations," J. Comp. Phy., Vol. 24, 1977, pp. 217-244.

9. J. A. Hill, R. T. Davis, and G. L. Slater, "Development of a Factored ADI Scheme for Solving the Navier-Stokes Equations in Stream Function Vorticity Variables," University of Cincinnati, AFL Report No. 79-12-48, Dept. of Aerospace Engineering and Applied Mechanics, December 1979.

10. W. R. Briley and H. McDonald, "Solution of the Multidimensional Compressible Navier-Stokes Equations by a Generalized Implicit Method," J. Computational Physics, Vol. 24, 1977, pp. 372-397.

11. S. G. Rubin, "Incompressible Navier-Stokes and Parabolized Navier-Stokes Solution Procedures and Computational Techniques," VKI Lecture Series 1982-5, Rhode St. Genese, Belgium, March 29-April 2, 1982.

12. P. J. Roache, "Scaling of High Reynolds Number Weakly Separated Channel Flows," Symposium on Numerical and Physical Aspects of Aerodynamic Flows, California State University, Long Beach, CA January 19-21, 1981.

13. P. R. Blottner and M. A. Ellis, "Finite Difference Solution of the Incompressible Three-Dimensional Boundary Layer Equations for a Blunt Body," Computers and Fluids, Vol. 1, 1973, pp. 133-138.

14. R. T. Davis, "Notes on Coordinate Generation," VKI Lecture Series 1981-5, Rhode St., Genese, Belgium, March 30-April 3, 1981.

15. L. Quartapelle and M. Napolitano, "Finite Element Solution to the Time-Dependent Vorticity-Stream Function Equations in Split Form," International Symposium on Refined Modelling of Flows, Paris, September 7-10, 1982.

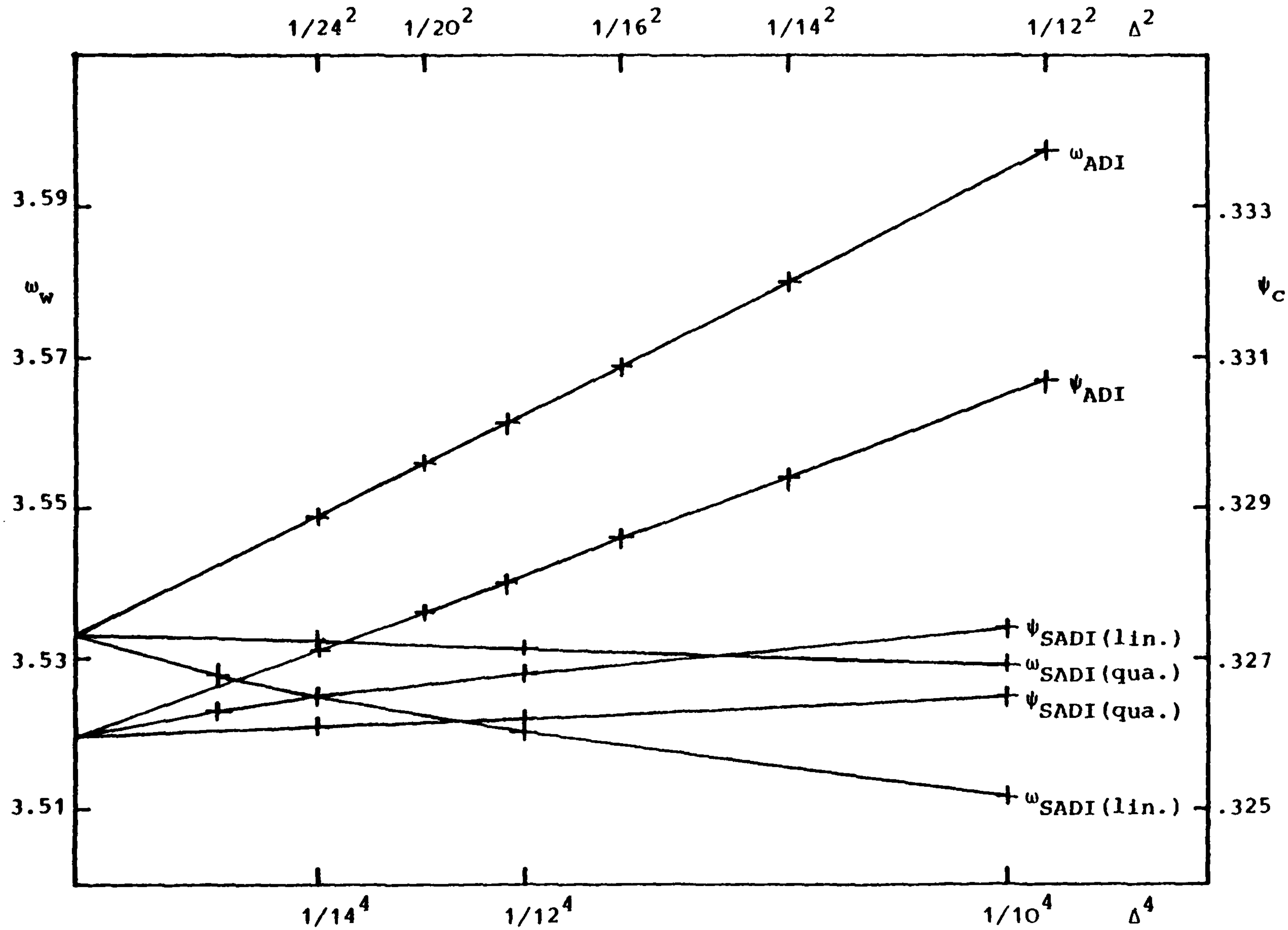


Figure 1. Numerical Results for Anulus Flow

TABLE 1

DRIVEN CAVITY RESULTS

Re = 100		14 x 14 Mesh
	ω_{MTP}	ψ_{MAX}
ADI	-8.196	-0.0874
Spline ADI (linear extrap.)	-8.557	-0.0941
Spline ADI (quad. extrap.)	-9.380	-0.0992

Re = 1000		20 x 20 Mesh
	ω_{MTP}	ψ_{MAX}
ADI	-28.471	-0.0344

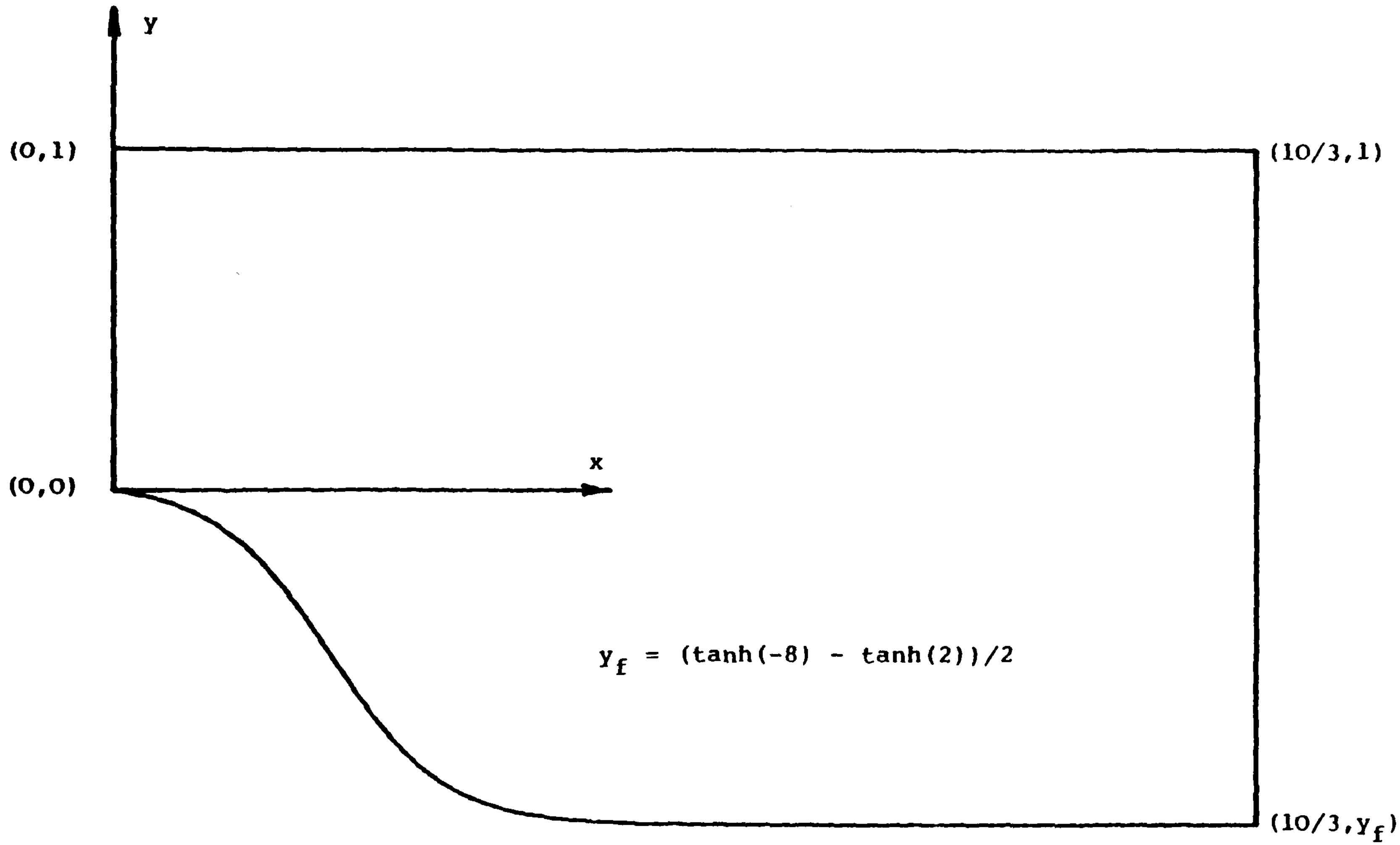
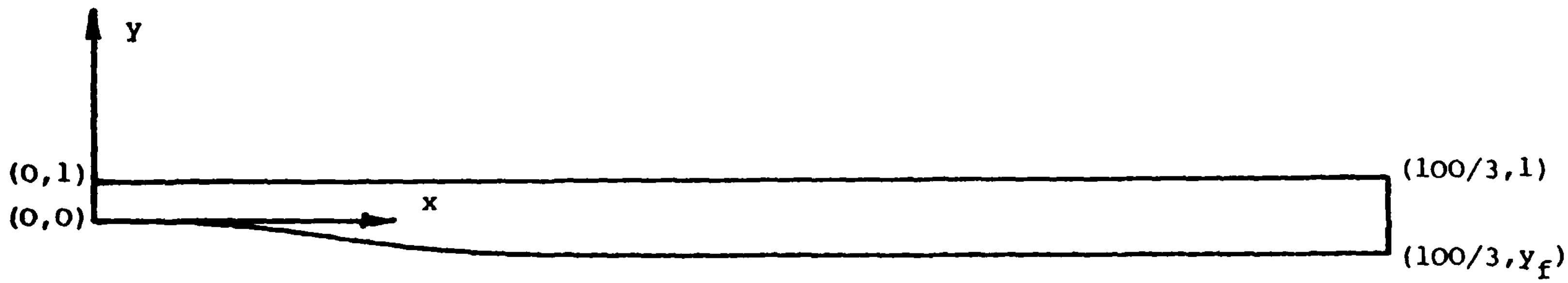


Figure 2a. Re = 10 Channel Geometry



$$y_f = (\tanh(-8) - \tanh(2))/2$$

Figure 2b. Re = 100 Channel Geometry

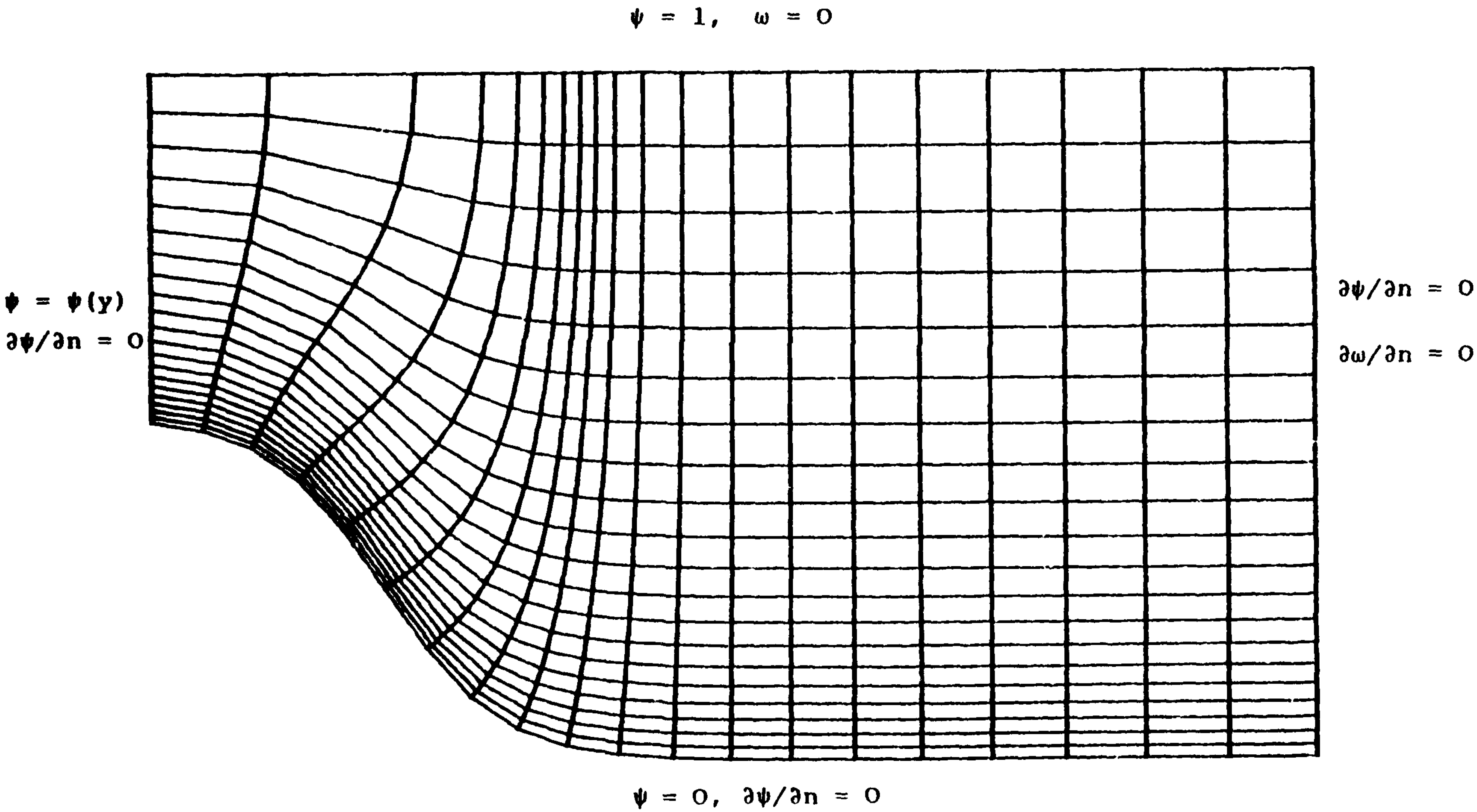


Figure 3. Re = 10 Computational Grid and Boundary Conditions

TABLE 2

CHANNEL FLOW ADI AND SPLINE ADI WALL VORTICITY RESULTS

i	x	SADI Re=10	SADI Re=100	ADI Re=10	ADI Re=100
1	0.0	3.0756	3.0769	3.0113	2.9970
2	0.1477	2.5877	2.5393	2.6612	2.5620
3	0.2879	2.1025	1.9546	2.0040	1.9564
4	0.4209	0.9908	1.1806	0.9675	1.1820
5	0.5470	0.1826	0.4915	0.2148	0.4999
6	0.6666	-0.0864	0.0992	-0.0837	0.1075
7	0.7862	-0.1314	-0.0679	-0.1365	-0.0621
8	0.9123	-0.1145	-0.1227	-0.1184	-0.1189
9	1.0453	-0.1024	-0.1261	-0.1037	-0.1245
10	1.1855	-0.1025	-0.0930	-0.1032	-0.0901
11	1.3333	-0.0986	-0.0349	-0.0993	-0.0327
12	1.4891	-0.0752	0.0431	-0.0761	0.0482
13	1.6534	-0.0292	0.1281	-0.0307	0.1307
14	1.8266	0.0337	0.2158	0.0313	0.2212
15	2.0092	0.1056	0.2966	0.1024	0.2985
16	2.2018	0.1798	0.3747	0.1762	0.3790
17	2.4048	0.2511	0.4399	0.2473	0.4396
18	2.6188	0.3147	0.5020	0.3114	0.5058
19	2.8455	0.3661	0.5507	0.3636	0.5470
20	3.0824	0.4001	0.5945	0.3986	0.5997
21	3.3333	0.4115	0.6415	0.4336	0.6525

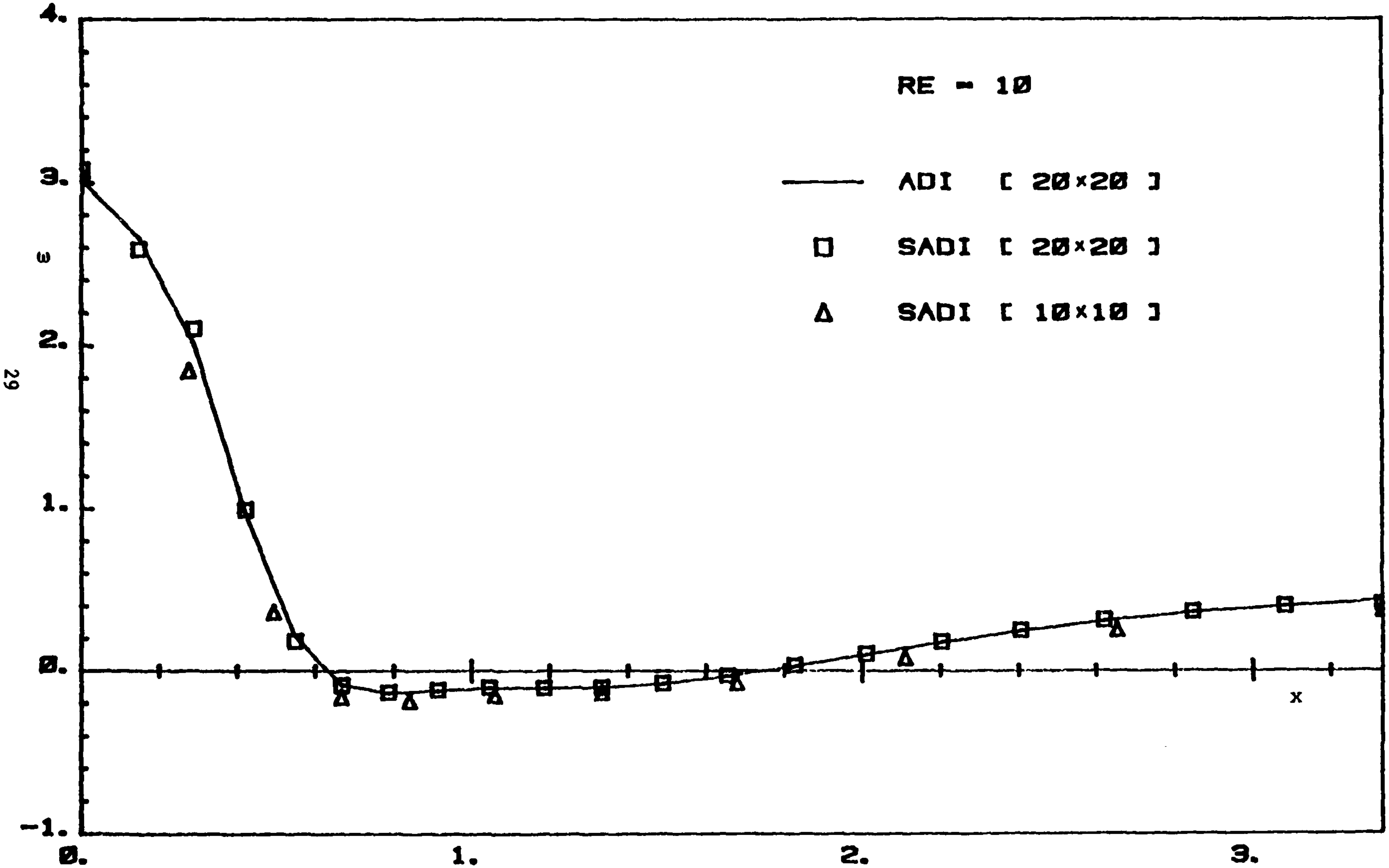


Figure 4. Re = 10 Wall Vorticity

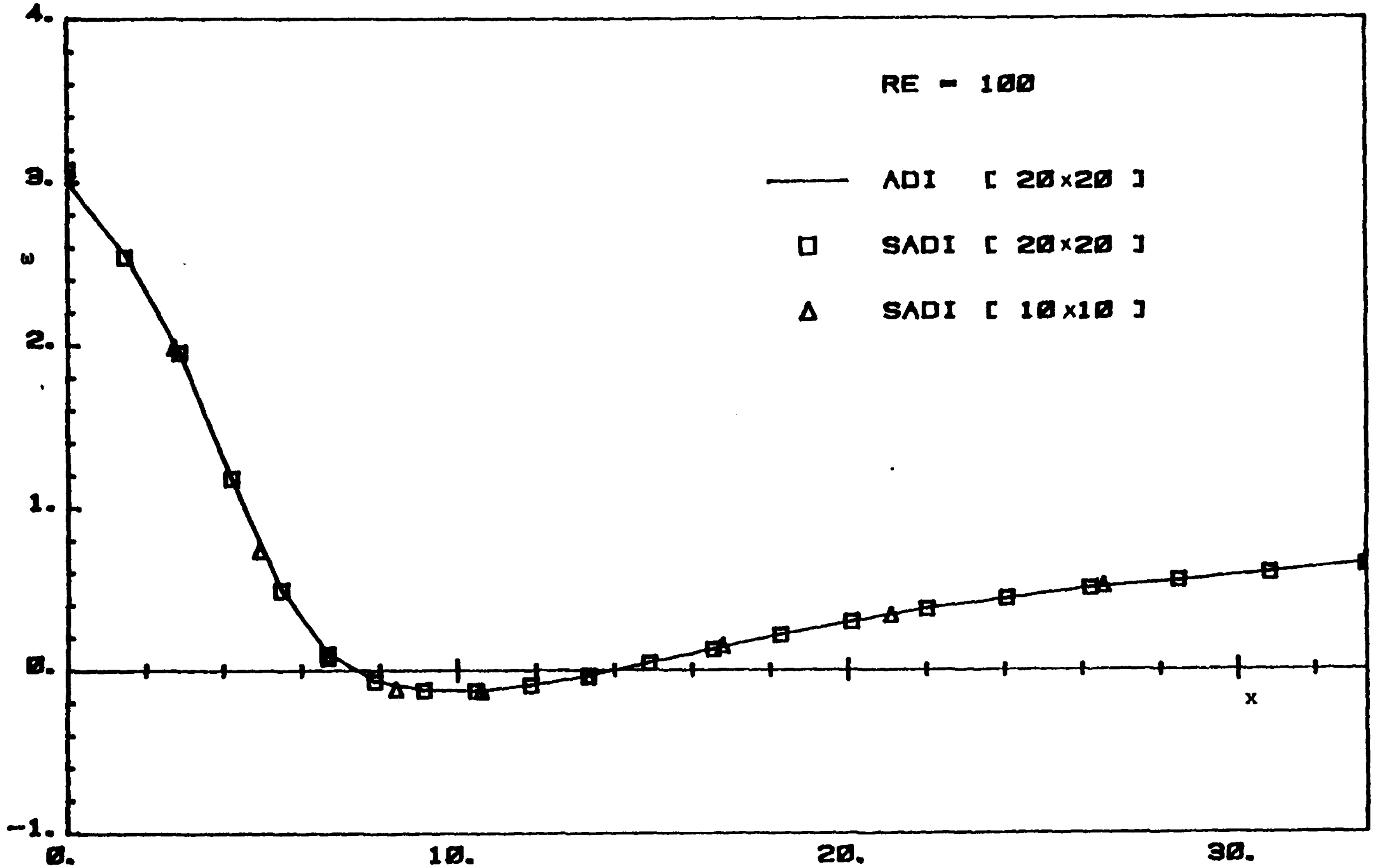


Figure 5. Re = 100 Wall Vorticity

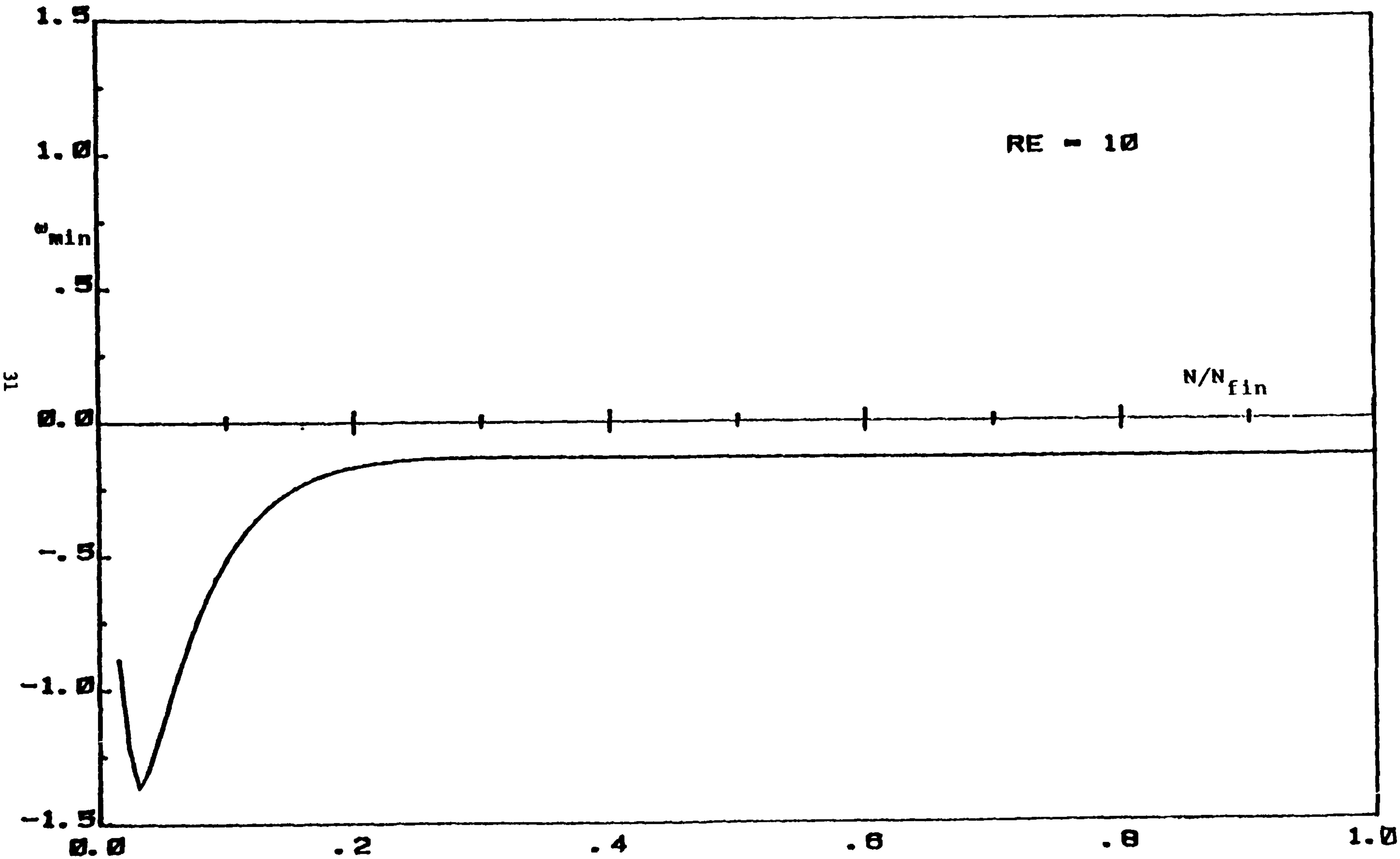


Figure 6. Minimum Wall Vorticity Versus Normalized Iteration Number (ADI, Re = 10)

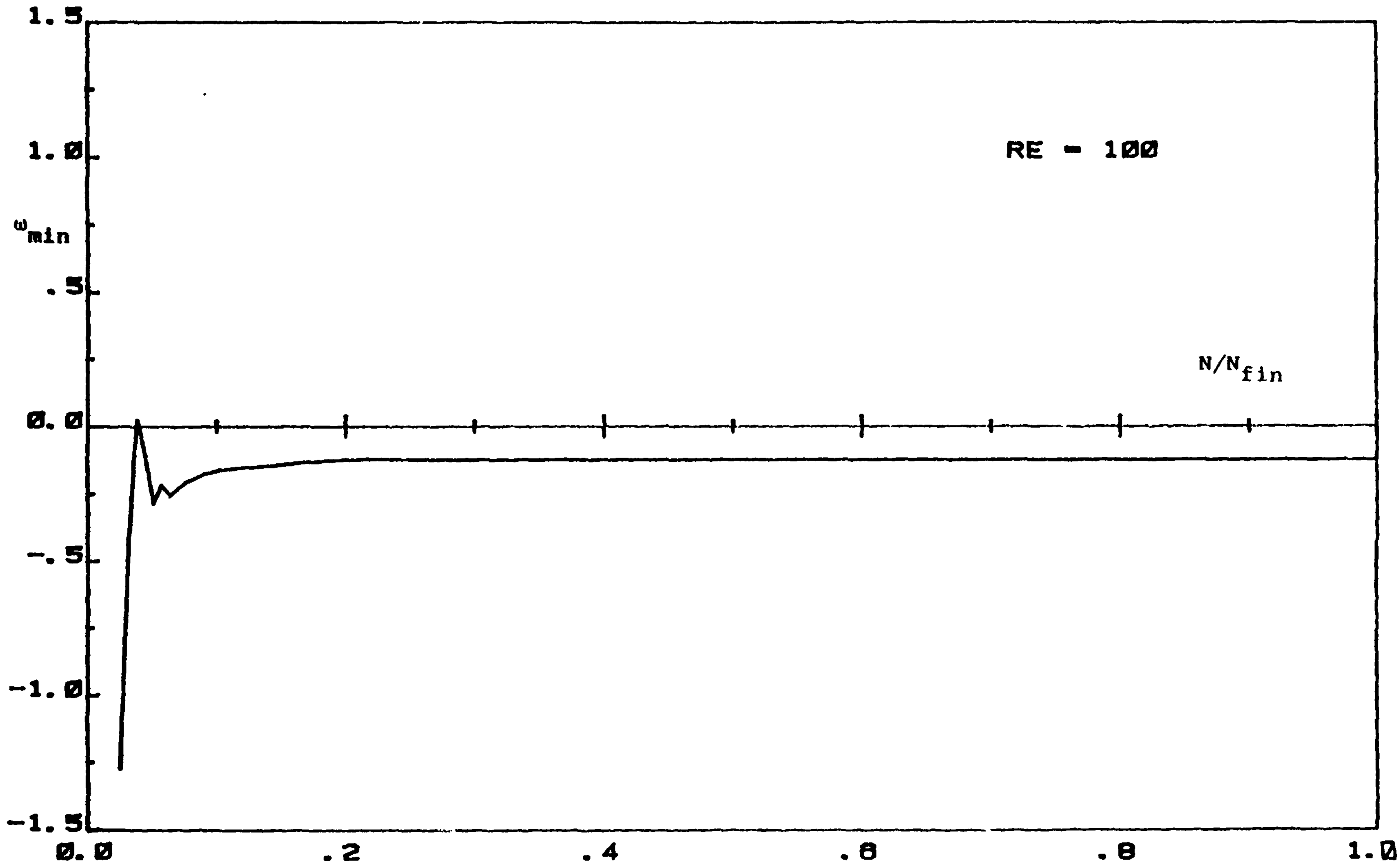


Figure 7. Minimum Wall Vorticity Versus Normalized Iteration Number (ADI, Re = 100)

END

DATE
FILMED

6-83

DTIC