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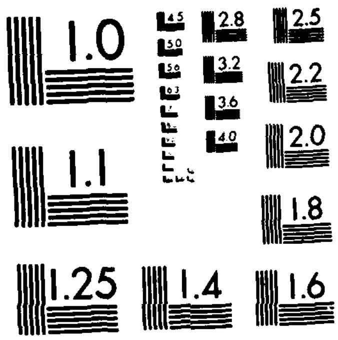

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18. SUPPLEMENTARY NOTES

This paper has been submitted to Aequationes Mathematicae for publication. Part of this paper was presented at the 19th International Symposium on Functional Equations in Brittany, France, in May 1981.

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Semigroup; the dominates relation; transitive relation; triangular norm.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper the dominates relation was characterized on a family of semigroups on the unit interval. The dominates relation was shown to be transitive on that family.

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Characterizing Dominates on a Family of Triangular Norms

By Howard Sherwood*

A t-norm is a function defined from I^2 to $I = [0,1]$ which is commutative, associative, nondecreasing in each place, having zero as a null element and one as an identity. A t-norm R dominates a t-norm T (and we write $R \gg T$) if, for each a,b,c,d in I,

$$R(T(a,b),T(c,d)) \geq T(R(a,c),R(b,d)).$$

While our interest in the dominates relation on t-norms arose in the study of products of fuzzy subgroups [4], R. Tardiff first introduced dominates to the literature [5] as a relation on triangle functions to study products of probabilistic metric spaces. In his paper he shows how the familiar Minkowski inequality merely states that addition dominates the mapping f from $R^+ \times R^+$ into R^+ given by $f(a,b) = (a^p + b^p)^{1/p}$ for $p \geq 1$.

To get another view of the connections between dominates and the Minkowski inequality we introduce notation suggested by a colleague, M. Taylor. If $a = (a_1, a_2)$ we write $T_i a_i$ for $T(a_1,a_2)$, if $a = (a_1,a_2,a_3)$ we again write $T_i a_i$ for $T(T(a_1,a_2),a_3)$, and the obvious extension defines $T_i a_i$ when $a = (a_1,a_2,\ldots,a_n)$. It is easy to prove that $R \gg T$ if and only if, for all a_{ij} in I with $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$,

$$R_j T_i a_{ij} \geq T_i R_j a_{ij} \quad .$$

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This form of the dominates relation bears a striking resemblance to the very elegant symmetrical form of Minkowski's inequality attributed to A. E. Ingham by G. H. Hardy, J. E. Littlewood and G. Polya [2;p31].

In [3] B. Schweizer and A. Sklar pose two problems we attack in this paper: (1) Is dominates transitive on the collection of all t-norms? (2) Given a specific t-norm such as Π, defined by $\Pi(a,b) = a\cdot b$, find all t-norms that dominate Π or are dominated by Π. While we do not address these problems in all their generality in this paper, we do completely resolve them as they pertain to a particular family $\{T_p\}$ of t-norms defined by Schweizer and Sklar [3].

For any real number $p \neq 0$, let g_p and f_p be defined by

$$g_p(x) = \frac{1-x^p}{p}$$

for each x in I, and

$$f_p(x) = \begin{cases} 0, & \text{if } 1-px \le 0, \\ (1-px)^{1/p}, & \text{if } 1-px > 0. \end{cases}$$

For any $p \neq 0$, the t-norm T_p defined by

$$T_p(a,b) = f_p(g_p(a) + g_p(b))$$

can also be given by

$$T_p(a,b) = \{\mathrm{Max}(a^p + b^p - 1,0)\}^{1/p}.$$

The remaining three members of the family $\{T_p\}$ are given by

$$T_0(a,b) = \lim_{p\to 0} T_p(a,b) = \Pi(a,b),$$

$$T_{-\infty}(a,b) = \lim_{p\to-\infty} T_p(a,b) = M(a,b), \text{ and}$$

$$T_{\infty}(a,b) = \lim_{p\to\infty} T_p(a,b) = Z(a,b).$$

To resolve the above-mentioned problems as they pertain to the family $\{T_p\}$, we shall show that

(1) $\qquad T_q >> T_p$ if and only if $q \leq p$

The terminology used throughout this paper is standard in probabilistic metric spaces. All the necessary definitions as well as the results listed in the following theorem are given in [3].

THEOREM 1. For any t-norms T and R, (i) $M>>T$, (ii) $T>>T$, (iii) $T>>Z$, and (iv) if $R>>T$, then $R \geq T$, i.e., $R(a,b) \geq T(a,b)$ for all $a,b \in I$.

Theorem 1 (iv) says there is no hope of proving $q \leq p$ implies $T_q >> T_p$ if we cannot prove $q \leq p$ implies $T_q \geq T_p$. Since the latter result is helpful in proving the former, we prove it first.

THEOREM 2. $T_q \geq T_p$ if and only if $q \leq p$.

PROOF. Suppose p and q are nonzero real numbers with $q < p$. Since (f_p, g_p) and (f_q, q_q) additively generate T_p and T_q, respectively, we can use Lemma 5.5.8 of [3] to conclude that $T_q \geq T_p$ if and only if $g_p \circ f_q$ is subadditive and that $T_p = T_q$ if and only if $g_p \circ f_q$ is linear. Now

$$g_p \circ f_q(x) = \begin{cases} 1/p, & \text{if } 1 - qx \leq 0, \\ [1-(1-qx)^{p/q}]/p, & \text{if } 1 - qx > 0. \end{cases}$$

Notice that $g_p \circ f_q$ is defined on $[0,\infty)$ and $g_p \circ f_q(0) = 0$.

Moreover,

$$(g_p\ f_q)''(x) = \begin{cases} 0, & \text{if } 1-qx \le 0, \\ (q-p)(1-qx)^{(p-2q)/q}, & \text{if } 1-qx > 0. \end{cases}$$

From this it follows that $g_p \circ f_q$ is concave. Lemma 2.2.6 of [3] now yields that $g_p \circ f_q$ is subadditive so that $T_q \geq T_p$. Furthermore, since $g_p \circ f_q$ is nonlinear, $T_q \neq T_p$.

Now, if $0 < p$, then for any q, q_1 with $0 < q < q_1 < p$, and for any x,y, in I,

$$T_q(x,y) \le T_{q_1}(x,y) \le T_p(x,y).$$

Consequently,

$$T_0(x,y) = \lim_{q\to 0} T_q(x,y) \le T_{q_1}(x,y).$$

But then $T_0 \le T_{q_1} \le T_p$ and, since $T_{q_1} \neq T_p$, we also have $T_0 \neq T_p$. A similar argument shows that whenever $q < 0$, $T_q \ge T_0$ with $T_q \neq T_0$.

Finally, for any real number q, parts (i), (iii) and (iv) of Theorem 1 yield $T_{-\infty} \ge T_q \ge T_\infty$. Also, it is easy to verify that $T_{-\infty} \neq T_q \neq T_\infty$ by showing that, for $a = (2/3)^{1/|q|}$,

$$T_{-\infty}(a,a) > T_q(a,a) > T_\infty(a,a).$$

Turning to the reverse implication, we suppose $T_q \geq T_p$. If p were less than q, then we would have $T_p \ge T_q$ whence $T_q = T_p$, a contradiction to what was proved above. Thus $T_q \geq T_p$ implies $q \leq p$, and the proof is complete.

The next theorem is an immediate consequence of Theorem 2 and part (iv) of Theorem 1.

THEOREM 3. If $T_q \gg T_p$ then $q \leq p$.

To prove the reverse implication we must be able to show that whenever $q \leq p$,

$$(2) \qquad T_q(T_p(a,b),T_p(c,d)) \geq T_p(T_q(a,c),T_q(b,d))$$

for all a,b,c,d in I. Because the definitions of T_p and T_q are essentially two-part rules, it is convenient to consider as special cases those values of a,b,c and d which make T_p and T_q zero. Motivated by this we prove the following.

LEMMA 1. Suppose R and T are t-norms. Let a,b,c,d be in I. If any of T(a,b), T(c,d), R(a,c), R(b,d) is zero, then

$$(3) \qquad R(T(a,b),T(c,d)) \geq T(R(a,c),R(b,d)).$$

PROOF. If either R(a,c) or R(b,d) is zero, then (3) is satisfied because its right side is zero. Now suppose T(a,b) is zero. Then

$$\begin{aligned} R(T(a,b),T(c,d)) &= R(0,T(c,d)) = 0 \\ &= T(a,b) \geq T(R(a,c),R(b,d)). \end{aligned}$$

Similarly (3) is satisfied when T(c,d) = 0.

When q is positive the left side of (2) could be zero even though neither $T_p(a,b)$ nor $T_p(c,d)$ is zero. In this event we must be able to show the right side is also zero. The following lemma, whose proof is in the appendix, will help us do that.

LEMMA 2. Suppose $r > 1$ and let

$$U = \{(x_1,x_2,x_3,x_4)\in I^4: \ x_1^r + x_2^r \geq 1,\ x_3^r + x_4^r \geq 1,$$
$$x_1 + x_3 \geq 1 \text{ and } x_2 + x_4 \geq 1\}.$$

Define F and G on U via

$$F(x_1,x_2,x_3,x_4) = (x_1 + x_3 - 1)^r + (x_2 + x_4 - 1)^r$$

and

$$G(x_1,x_2,x_3,x_4) = (x_1^r + x_2^r - 1)^{1/r} + (x_3^r + x_4^r - 1)^{1/r} - 1.$$

Then, $F(x_1,x_2,x_3,x_4) \leq 1$ at each (x_1,x_2,x_3,x_4) in U at which $G(x_1,x_2,x_3,x_4) = 0$.

LEMMA 3. Suppose $q < p$ with $p,q \neq 0$. Suppose $T_q(T_p(a,b), T_p(c,d))$ equals zero. Then $T_p(T_q(a,c), T_q(b,d))$ also equals zero and consequently (2) is satisfied.

PROOF. First consider the case $q < 0$. Since

$$0 \leq T_p(a,b)T_p(c,d) = T_0(T_p(a,b),T_p(c,d))$$

$$\leq T_q(T_p(a,b),T_p(c,d)) = 0$$

we must have $T_p(a,b) = 0$ or $T_p(c,d) = 0$. By Lemma 1 (2) is satisfied whence $T_p(T_q(a,c),T_q(b,d))$ must be zero.

Now suppose $q > 0$. If any of $T_p(a,b)$, $T_p(c,d)$, $T_q(a,c)$, $T_q(b,d)$ were zero, then the result would again follow from Lemma 1. Thus we assume $a^p + b^p > 1$, $c^p + d^p > 1$, $a^q + c^q > 1$ and $b^q + d^q > 1$. Since $T_q(T_p(a,b), T_p(c,d)) = 0$,

$$(a^p + b^p - 1)^{q/p} + (c^p + d^p - 1)^{q/p} - 1 \leq 0.$$

There exist a_1, b_1, c_1 and d_1 such that $a \leq a_1 \leq 1$, $b \leq b_1 \leq 1$, $c \leq c_1 \leq 1$, $d \leq d_1 \leq 1$, and

$$(a_1^p + b_1^p - 1)^{q/p} + (c_1^p + d_1^p - 1)^{q/p} - 1 = 0$$

Moreover, $a_1^p + b_1^p > 1$, $c_1^p + d_1^p > 1$, $a_1^q + c_1^q > 1$ and $b_1^q + d_1^q > 1$. Let $x_1 = a_1^q$, $x_2 = b_1^q$, $x_3 = c_1^q$, $x_4 = d_1^q$ and $r = p/q$. Notice that $r > 1$, $x_1^r + x_2^r > 1$, $x_3^r + x_4^r > 1$, $x_1 + x_3 > 1$, $x_2 + x_4 > 1$ and that

$$(x_1^r + x_2^r - 1)^{1/r} + (x_3^r + x_4^r - 1)^{1/r} - 1 = 0 \cdot$$

Thus, Lemma 2 implies

$$(x_1 + x_3 - 1)^r + (x_2 + x_4 - 1)^r \le 1,$$

i.e.,

$$(a_1^q + c_1^q - 1)^{p/q} + (b_1^q + d_1^q - 1)^{p/q} - 1 \le 0.$$

Since t-norms are nondecreasing in each place, the preceding inequality yields

$$T_p(T_q(a,c),\ T_q(b,d)) \le T_p(T_q(a_1,c_1),\ T_q(b_1,d_1)) = 0$$

and the proof is complete.

The next lemma shows that for nonzero p and q, $T_q \gg T_p$ provided $(T_q)^p$ satisfies a condition which resembles the 2-increasing condition of [3].

LEMMA 4. Let p and q be nonzero real numbers and let a,b,c,d in I be such that $T_p(a,b)$, $T_p(c,d)$, $T_q(a,c)$, $T_q(b,d)$ and $T_q(T_p(a,b),\ T_p(c,d))$ are all positive. If p is positive and

$$(4)\quad [T_q(T_p(a,b),T_p(c,d))]^p + [T_q(1,1)]^p \ge [T_q(a,c)]^p + [T_q(b,d)]^p$$

or if p is negative and

$$(5)\quad [T_q(T_p(a,b),T_p(c,d))]^p + [T_q(1,1)]^p \le [T_q(a,c)]^p + [T_q(b,d)]^p,$$

then

$$(6)\qquad T_q(T_p(a,b),\ T_p(c,d)) \ge T_p(T_q(a,c),\ T_q(b,d)) .$$

PROOF. Suppose $p > 0$ and (4) holds. Then

$$[T_q(T_p(a,b),\ T_p(c,d))]^p \geq [T_q(a,c)]^p + \ T_q(b,d)]^p - 1. \tag{7}$$

If the right side of (7) is less than or equal to zero, then (6) is obviously satisfied because its right side is zero. On the other hand, if the right side of (7) is positive, then

$$\begin{aligned} T_q(T_p(a,b),\ T_p(c,d)) &\geq \{[T_q(a,c)]^p + [T_q(b,d)]^p - 1\}^{1/p} \\ &= T_p(T_q(a,c),\ T_q(b,d)) \end{aligned}$$

so that (6) is again satisfied.

Now suppose $p < 0$ and (5) holds. Then

$$[T_q(T_p(a,b),\ T_p(c,d))]^p \leq [T_q(a,c)]^p + [T_q(b,d)]^p - 1$$

But $p < 0$ and the both sides of the preceding inequality are positive. Therefore, we can raise both sides to the power $1/p$ and, when we reverse the sense of the inequality, the result is (6).

The importance of Lemma 4 lies in the geometric view it gives the problem of verifying (6). Notice that the right sides of (4) and (5) are the sum of the values of $(T_q)^p$ at two points (a,c) and (b,d) of the unit square and that the left sides of (4) and (5) are the sum of the values of $(T_q)^p$ at two other points of the unit square. When (a,c) and (b,d) are given, it is an easy matter to locate the other two points. Of course $(1,1)$ is trivial to find. To locate the remaining point, $(T_p(a,b),\ T_p(c,d))$, we first locate (a,b) and (c,d) as indicated in Figure 1. Next we proceed along the level curves of T_p from (a,b) and (c,d) to $(T_p(a,b),\ 1)$ and $(1, T_p(c,d))$, respectively. The desired point is vertical to $(T_p(a,b),\ 1)$ and horizontal to $(1,\ T_p(c,d))$.

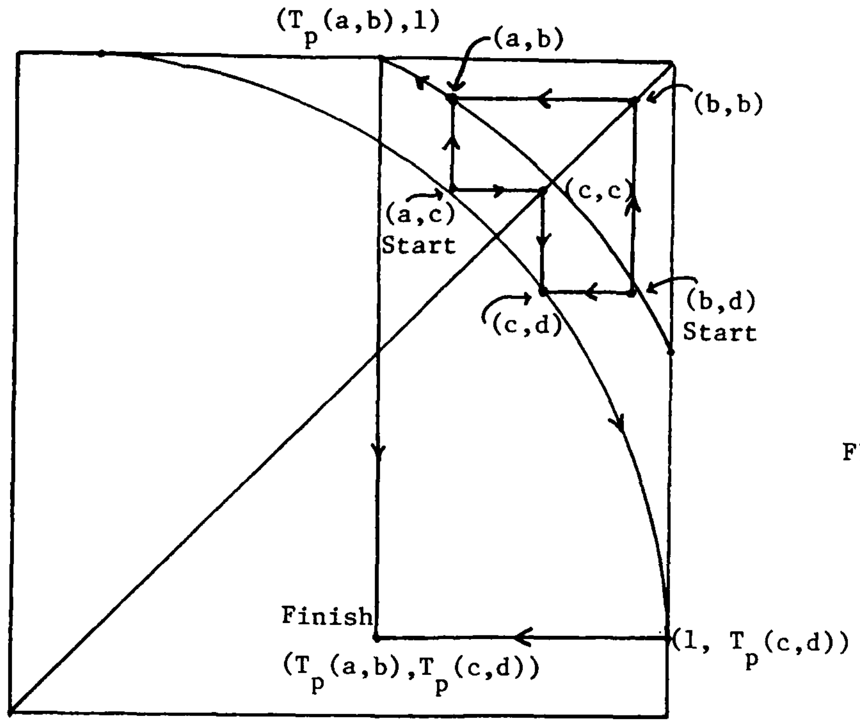

FIGURE 1

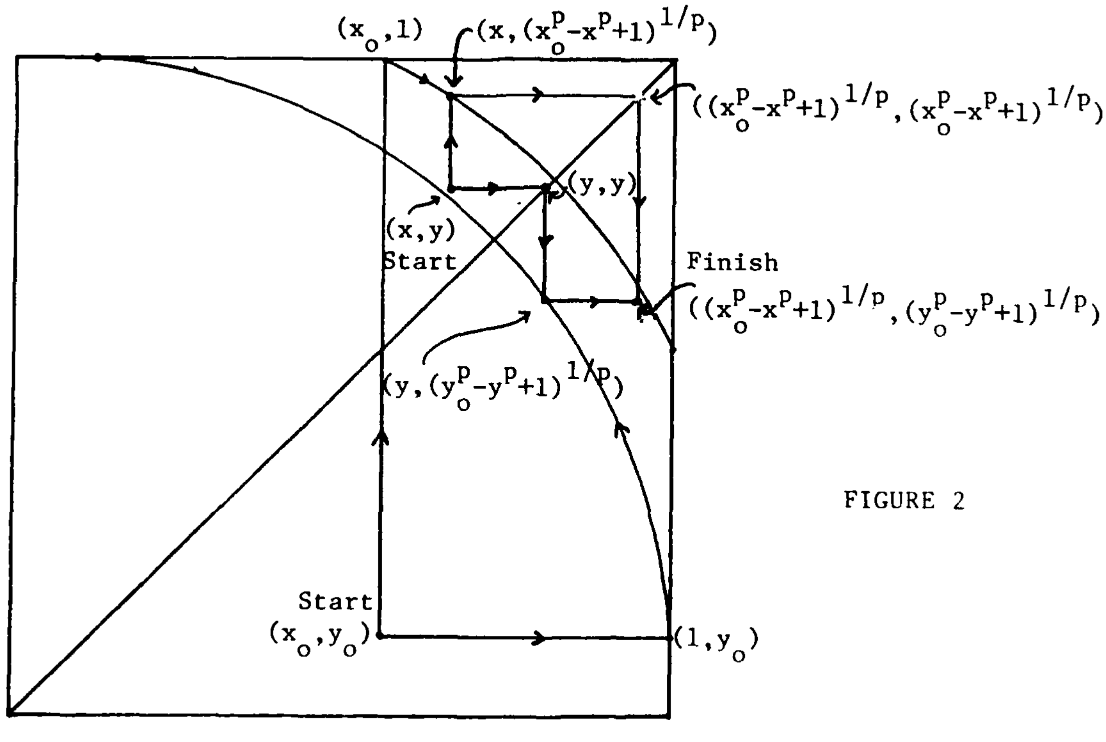

FIGURE 2

It is helpful to reverse the process as indicated in Figure 2. Start at any point $(x_o,y_o) \in I^2$ such that $T_q(x_o,y_o) > 0$, and at any point (x,y) in $[x_o,1] \times [y_o,1]$ then proceed to $((x_o^p - x^p + 1)^{1/p},(y_o^p - y^p + 1)^{1/p})$ as directed by the arrows. From our new point of view we now need to compare

$$[T_q(x_o,y_o)]^p + [T_q(1,1)]^p \tag{8}$$

with

$$[T_q(x,y)]^p + [T_q((x_o^p - x^p + 1)^{1/p},\ y_o^p - y^p + 1)^{1/p})]^p \quad . \tag{9}$$

Notice that (8) is merely the value of (9) when (x,y) is replaced by (x_o,y_o). These observations motivate the following lemma.

LEMMA 5. Suppose $p,q \neq 0$. For each $(x_o,y_o) \in I^2$ such that $T_q(x_o,y_o) > 0$ and for each (x,y) in $[x_o,1] \times [y_o,1]$, let

$$G_{(x_o,y_o)}(x,y) = [T_q(x,y)]^p + [T_q(x_o^p - x^p + 1)^{1/p},(y_o^p - y^p + 1)^{1/p}]^p.$$

Then,

(i) Each $G_{(x_o,y_o)}$ is a function defined on $[x_o,1] \times [y_o,1]$, and

(ii) If p is positive and each $G_{(x_o,y_o)}$ assumes its maximum value at the lower left corner of $[x_o,1] \times [y_o,1]$ or, if p is negative and each $G_{(x_o,y_o)}$ assumes its minimum value at the lower left corner of $[x_o,1] \times [y_o,1]$, then, for any a,b,c,d such that $T_p(a,b)$, $T_p(c,d)$, $T_q(a,c)$, $T_q(b,d)$ and $T_q(T_p(a,b),\ T_p(c,d))$ are all positive.

$$T_q(T_p(a,b),\ T_p(c,d)) \geq T_p(T_q(a,c),\ T_q(b,d)). \tag{10}$$

PROOF of (i). Let $(x,y) \in [x_o,1] \times [y_o,1]$. It is easy to verify that $((x_o^p - x^p + 1)^{1/p}, (y_o^p - y^p + 1)^{1/p})$ is also in $[x_o,1] \times [y_o,1]$. Since $T_q(x_o,y_o) > 0$, and T_q is nondecreasing in each place, it is immediate that $G_{(x_o,y_o)}$ is a function defined on $[x_o,1] \times [y_o,1]$.

PROOF of (ii). Suppose first that p is positive and each $G_{(x_o,y_o)}$ assumes its maximum value at the lower left corner of $[x_o,1] \times [y_o,1]$. Let a,b,c,d be in I such that $T_p(a,b)$, $T_p(c,d)$, $T_q(a,c)$, $T_q(b,d)$ and $T_q(T_p(a,b), T_p(c,d))$ are all positive. Choose $x_o = T_p(a,b)$ and $y_o = T_p(c,d)$. Since $T_p(a,b) \le a \le 1$ and $T_p(c,d) \le c \le 1$ we have $(a,c) \in [x_o,1] \times [y_o,1]$. Thus

$$G_{(x_o,y_o)}(x_o,y_o) \ge G_{(x_o,y_o)}(a,c).$$

But

$$G_{(x_o,y_o)}(x_o,y_o) = \left[T_q(T_p(a,b), T_p(c,d))\right]^p + \left[T_q(1,1)\right]^p$$

and, since $x_o^p = a^p + b^p - 1$ and $y_o^p = c^p + d^p - 1$,

$$G_{(x_o,y_o)}(a,c) = [T_q(a,c)]^p + [T_q(b,d)]^p$$

we conclude from Lemma 4 that (10) holds.

The proof for p negative is so much like the proof for p positive, we shall omit it.

The preceding lemma reduces the problem of showing $T_q >> T_p$ for nonzero p and q to infinitely many extreme value pro..ems on closed rectangles of the form $[x_o,1] \times [y_o,1]$. We shall show later that each $G_{(x_o,y_o)}$ has its only critical point interior to $[x_o,1] \times [y_o,1]$ at $([(x_o^p + 1)/2]^{1/p}, [y_o^p + 1/2]^{1/p})$ and we shall need to compare the value of $G_{(x_o,y_o)}$ at this point with $G_{(x_o,y_o)}(x_o,y_o)$. The following lemma will help make that comparison.

LEMMA 6. Suppose p and q are nonzero real numbers with $q < p$. For each x in I, let

$$\alpha(x) = \left(\frac{x^p + 1}{2}\right)^{1/p}$$

Then, for all x,y in I,

$$\alpha T_q(x,y) \geq T_q(\alpha(x), \alpha(y)) \tag{11}$$

PROOF. Using standard calculus techniques it is easily verified that α is an increasing function from I onto $[\alpha(0),1]$. Notice that if $p < 0$, then $\alpha(0) = 0$ while if $p > 0$, $\alpha(0) = 2^{-1/p}$. Define β from I into I via

$$\beta(x) = \begin{cases} 0, & \text{if } \quad 0 \leq x \leq \alpha(0), \\ (2x^p - 1)^{1/p}, & \text{if } \quad \alpha(0) < x \leq 1. \end{cases}$$

For each x in I,

$$\beta \circ \alpha(x) = x$$

while

$$\alpha \circ \beta(x) \geq x.$$

Therefore, (11) is satisfied if and only if

$$(12) \qquad T_q(x,y) \geq \beta T_q(\alpha(x),\alpha(y)).$$

The pair (f_q,g_q) of additive generators for T_q are quasi-inverses of each other and β is a quasi-inverse of α. Moreover, the domain of β is a subset of the range of f_q. Thus, Lemma 2.1.4 of [3] guarantees that $g_q \circ \alpha$ is a quasi-inverse of $\beta \circ f_q$. It is now easy to verify the hypotheses of Theorem 5.5.2 of [3]; therefore we conclude that the pair $(\beta \circ f_q, g_q \circ \alpha)$ additively generates a continuous Archimedean t-norm T. Moreover, (12) is satisfied if and only if

$$T_q(x,y) \geq \beta \circ f_q(g_q \circ \alpha\ (x) + g_q \circ \alpha\ (y)) = T(x,y).$$

According to Lemma 5.5.8 of [3], this last inequality will hold if and only if $g_q \circ \alpha \circ f_q$ is subadditive. This will follow from Lemma 2.2.6 of [3] if we can show that $g_q \circ \alpha \circ f_q(0) = 0$ and that $g_q \circ \alpha \circ f_q$ is concave. Now $g_q \circ \alpha \circ f_q(0) = g_q \circ \alpha\ (1) = g_q(1) = 0$ and it is easy to show that $(g_q \circ \alpha \circ f_q)''\ (x)$ is negative if $1 - q\,x > 0$ and zero if $1 - q\,x < 0$. From that it follows that $g_q \circ \alpha \circ f_q$ is concave, completing the proof.

As indicated immediately above the statement of Lemma 6, we need to compare the value of $G_{(x_o,y_o)}$ at its only critical point inside $[x_o,1] \times [y_o,1]$ with $G_{(x_o,y_o)}(x_o,y_o)$. The next lemma does that. In order to simplify the notation we shall drop the subscript from $G_{(x_o,y_o)}$.

LEMMA 7. Suppose p and q are nonzero real numbers with $q < p$. Let $(x_o,y_o) \in I^2$ such that $T_q(x_o,y_o) > 0$. For each (x,y) in $[x_o,1] \times [y_o,1]$, let

$$G(x,y) = [T_q(x,y)]^p + [T_q(x_o^p - x^p + 1)^{1/p},\ (y_o^p - y^p + 1)^{1/p})]^p \ . \tag{13}$$

If $p > 0$, then

$$G(x_o,y_o) \geq G\left(\left(\frac{x_o^p + 1}{2}\right)^{1/p}, \left(\frac{y_o^p + 1}{2}\right)^{1/p}\right)$$

while if $p < 0$ the inequality is reversed.

PROOF. According to Lemma 6,

$$\alpha T_q(x_o,y_o) \geq T_q(\alpha(x_o),\ \alpha(y_o))$$

from which it follows that

$$\left\{\frac{[T_q(x_o,y_o)]^p + 1}{2}\right\}^{1/p} \geq T_q\left(\left(\frac{x_o^p + 1}{2}\right)^{1/p},\ \left(\frac{y_o^p + 1}{2}\right)^{1/p}\right) .$$

If $p > 0$, this yields

$$[T_q(x_o,y_o)]^p + 1 \geq 2\left[T_q\left(\left(\frac{x_o^p + 1}{2}\right)^{1/p},\ \left(\frac{y_o^p + 1}{2}\right)^{1/p}\right)\right]^p \tag{14}$$

while if $p < 0$ the inequality is reversed. Now the left side of (14)is precisely $G(x_o,y_o)$ and the right side is $G\left(\left(\frac{x_o^p + 1}{2}\right)^{1/p}, \left(\frac{y_o^p + 1}{2}\right)^{1/p}\right)$. This proves the lemma.

LEMMA 8. Suppose p and q are nonzero real numbers with $q < p$. Let (x_o,y_o) be in I^2 such that $T_q(x_o,y_o) > 0$. For each (x,y) in $[x_o,1] \times [y_o 1]$ let $G(x,y)$ be given by (13). Then for all (x,y) in $[x_o,1] \times [y_o,1]$, if p is positive,

(15) $$G(x_o,y_o) \geq G(x,y)$$

while if $p > 0$ the inequality is reversed.

The proof of Lemma 8 is lengthy so it is also placed in the appendix.

THEOREM 4. Suppose p and q are nonzero real numbers with $q < p$ then $T_q >> T_p$.

PROOF. Let a, b, c and d be in I. We need to show that

(16) $$T_q(T_p(a,b),\ T_p(c,d)) \geq T_p(T_q(a,c),\ T_q(b,d)).$$

If any of $T_p(a,b)$, $T_p(c,d)$, $T_q(a,c)$, $T_q(b,d)$ or $T_q(T_p(a,b), T_p(c,d))$ is zero then Lemma 1 or Lemma 3 would yield (16). But if each of those quantities is positive, then Lemma 5 (ii) and Lemma 8 yield (16) and the proof is complete.

Parts (i), (ii) and (iii) of Theorem 1 yield

$$(17) \qquad q \le p \text{ implies } T_q >> T_p$$

when $q = -\infty$ or $q = p$ or $p = \infty$. Theorem 4 gives (17) when p and q are ordinary nonzero real numbers. Thus the only two cases remaining are $q < 0 = p$ and $q = 0 < p$. Because of the definition of T_0 and the continuity of each member of the family $\{T_p\}$, both of these cases follow immediately from Theorem 4 by taking limits. Thus we have the following theorem.

THEOREM 5. If $q \le p$ then $T_q >> T_p$.

Combining Theorems 3 and 5 we obtain

THEOREM 6. $T_q >> T_p$ if and only if $q \le p$.

As a corollary to Theorem 6 we have

COROLLARY. The dominates relation is transitive on the family $\{T_p\}$.

The author expresses his appreciation to M. D. Taylor for patiently forcing the author to justify each detail and also to B. Schweizer for improving the paper and shortening the proof of Lemma 2.

APPENDIX

PROOF OF LEMMA 2: We shall make use of the theorem on Lagrange multipliers given in [1; p. 381]. Functions F and G are of class C' on the open set int(U) in R^4. Let X_0 be that subset of int(U) on which G vanishes. Assume that $\underline{x}_0 = (a,b,c,d) \in X_0$ and assume there is a 4-ball $B(\underline{x}_0)$ such that $F(\underline{x}) \le F(\underline{x}_0)$ for all $\underline{x}$ in $X_0 \cap B(x_0)$. Notice also that $G_1(\underline{x}_0) \neq 0$. Thus there is a number λ such that

$$F_i(\underline{x}_0) + \lambda\, G_i(\underline{x}_0) = 0$$

for i = 1,2,3,4. Thus,

(18) $r(a+c-1)^{r-1} + \lambda a^{r-1}(a^r + b^r - 1)^{(1-r)/r} = 0$,

(19) $r(b+d-1)^{r-1} + \lambda b^{r-1}(a^r + b^r - 1)^{(1-r)/r} = 0$,

(20) $r(a+c-1)^{r-1} + \lambda c^{r-1}(c^r + d^r - 1)^{(1-r)/r} = 0$,

and

(21) $r(b+d-1)^{r-1} + \lambda d^{r-1}(c^r + d^r - 1)^{(1-r)/r} = 0.$

Moreover, since $\underline{x}_0$ is in X_0, $G(\underline{x}_0) = 0$. Thus,

(22) $(a^r + b^r - 1)^{1/r} + (c^r + d^r - 1)^{1/r} = 1.$

Combining (18) and (20), (19) and (21), and (18) and (19) we obtain, respectively

(23) $a^{r-1}\,(a^r + b^r - 1)^{(1-r)/r} = c^{r-1}\,(c^r + d^r - 1)^{(1-r)/r}$,

(24) $b^{r-1}\,(a^r + b^r - 1)^{(1-r)/r} = d^{r-1}\,(c^r + d^r - 1)^{(1-r)/r}$,

and

(25) $(a + c - 1)^{r-1}/\, a^{r-1} = (b + d - 1)^{r-1}/\, b^{r-1}$.

From (23) and (24) we get

$$\frac{c}{a} = \frac{d}{b} \tag{26}$$

while from (25) we get

$$1 + \frac{c}{a} - a^{-1} = 1 + \frac{d}{b} - b^{-1}$$

which, in light of (26), yields

$$a = b \quad \text{and} \quad c = d. \tag{27}$$

Using (27) in (23) we obtain

$$a^{r-1}(2a^r - 1)^{(1-r)/r} = c^{r-1}(2c^r - 1)^{(1-r)/r} \quad . \tag{28}$$

Raising both sides of (28) to the power $r/(1-r)$ and then performing the indicated multiplications, we get

$$2 - a^{-r} = 2 - c^{-r} \; ;$$

whence

$$a = c \; . \tag{29}$$

Finally, (22), (27) and (29) yield

$$a = b = c = d = ((1 + 2^r)/2^{r+1})^{1/r} \; .$$

Next we find the value of F at $\underline{x}_0 = (a,b,c,d)$ to be

$$F(\underline{x}_0) = \left[(1 + 2^r)^{1/r} - 2^{1/r}\right]^r .$$

To show that this number is less than 1, it suffices to define

$$f(x) = (1 + x^r)^{1/r}$$

for each $x \in [1,2]$ and observe by the mean value theorem that, for some ξ with $1 < \xi < 2$,

$$(1 + 2^r)^{1/r} - 2^{1/r} = f(2) - f(1) = f'(\xi)(2 - 1) = \left[\frac{\xi}{(1+\xi^r)^{1/r}}\right]^{r-1} < 1.$$

Thus, if F has a relative maximum at some point of X_0, the value there is less than 1.

Since U is a closed subset of R^4 and G is continuous on U, the set V of points of U at which G vanishes must also be closed in R^4. Moreover, since V is also bounded and F is continuous on V, F must attain a maximum value on V. Either it attains that maximum at an interior point of U or else on the boundary of U. We have already shown the maximum value is less than 1 if it is attained at an interior point of U. So now we shall see what happens on the boundary of U.

First we consider that boundary where $x_1^r + x_2^r = 1$. In that case, $T_r(x_1,x_2) = 0$ so that by Lemma 1,

$$T_1(T_r(x_1,x_2),T_r(x_3,x_4)) \geq T_r(T_1(x_1,x_3),T_1(x_2,x_4))$$

whence

$$\begin{aligned} F(x_1,x_2,x_3,x_4) &= (x_1 + x_3 - 1)^r + (x_2 + x_4 - 1)^r - 1 + 1 \\ &\leq [T_r(T_1(x_1,x_3),T_1(x_2,x_4))]^r + 1 = 1 \end{aligned}$$

Likewise on the boundary where $x_3^r + x_4^r = 1$, $F(x_1,x_2,x_3,x_4) \le 1$.

Next consider the boundary where $x_1 + x_3 = 1$ or $x_2 + x_4 = 1$. It is immediate in either of these cases that $F(x_1,x_2,x_3,x_4) \le 1$.

On the boundary where $x_1 = 0$, if (x_1,x_2,x_3,x_4) is in U then $x_2 = 1 = x_3$ so that

$$F(x_1,x_2,x_3,x_4) = x_4^r \le 1.$$

Similarly on the boundaries where any of x_2, x_3 or $x_4 = 0$ we must have $F(x_1,x_2,x_3,x_4) \le 1$.

Now consider the boundary where $x_1 = 1$. We desire to show that

$$F(x_1,x_2,x_3,x_4) = x_3^r + (x_2 + x_4 - 1)^r \le 1 \tag{30}$$

at all those points $(x_1,x_2,x_3,x_4) = (1,x_2,x_3,x_4)$ for which

$$G(x_1,x_2,x_3,x_4) = x_2 + (x_3^r + x_4^r - 1)^{1/r} - 1 = 0. \tag{31}$$

Solving (31) for x_3^r, substituting the result into (30) and rewriting the new equation, we conclude that we need only show

$$[x_4 - (1 - x_2)]^r \le x_4^r - (1 - x_2)^r .$$

Since $x_2 \le 1$ and $x_2 + x_4 - 1 \ge 0$ we have $0 \le 1 - x_2 \le x_4$. Thus, letting $1 - x_2 = t \cdot x_4$ we see it suffices to show

$$(1 - t)^r \le 1 - t^r$$

for $0 \le t \le 1$. It is easy to verify this. Therefore (30) is satisfied at all points on the boundary of U for which $x_1 = 1$ and $G(x_1,x_2,x_3,x_4) = 0$. Because of the symmetry in the variables, the same situation prevails at any boundary point of U at which any of x_2, x_3 or x_4 equals 1. This completes the proof.

PROOF OF LEMMA 8: Suppose $p > 0$. First we establish (15) on the boundary of the rectangle $[x_o,1] \times [y_o,1]$. Consider that portion of the boundary where $y = y_o$. For each x in $[x_o,1]$, let

$$k(x) = G(x,y_o) = (x^q + y_o^q - 1)^{p/q} + (x_o^p - x^p + 1).$$

Notice that

$$k(x_o) = G(x_o,y_o)$$

while an application of Theorem 2 yields

$$k(1) = y_o^p + x_o^p \le [T_p(x_o,y_o)]^p + 1$$

$$\le [T_q(x_o,y_o)]^p + 1 = G(x_o,y_o).$$

Now, for $x_o < x < 1$, $k'(x) = 0$ if and only if $y_o = 1$. But, if $y_o = 1$, then

$$k(x) = x_o^p + 1 = x_o^p + y_o^p \le G(x_o,y_o).$$

This shows that (15) is satisfied on the boundary where $y = y_o$.

Next consider the boundary where $y = 1$. For each x in $[x_o,1]$, let

$$m(x) = G(x,1) = x^p + [(x_o^p - x^p + 1)^{q/p} + y_o^q - 1]^{p/q} .$$

Notice that

$$m(x_o) = x_o^p + y_o^p \le G(x_o,y_o)$$

while

$$m(1) = 1 + [x_o^q + y_o^q - 1]^{p/q} = 1 + [T_q(x_o,y_o)]^p = G(x_o,y_o).$$

Also, for $x_o < x < 1$, $m'(x) = 0$ if and only if $y_o = 1$. If $y_o = 1$, then

$$m(x) = x_o^p + 1 = x_o^p + y_o^p \le G(x_o,y_o).$$

Thus (15) is satisfied on the boundary where $y = 1$.

Because of the symmetry in x and y we can assert that (15) is also satisfied on the other two boundaries.

Turning our attention to the interior of the rectangle we let (x,y) be an element of the open rectangle $(x_o,1) \times (y_o,1)$. Then,

$$G_1(x,y) = px^{q-1}(x^q+y^q-1)^{\frac{p-q}{q}} - px^{p-1}(x_o^p-x^p+1)^{\frac{q-p}{p}}[(x_o^p-x^p+1)^{\frac{q}{p}}+(y_o^p-y^p+1)^{\frac{q}{p}}-1]^{\frac{p-q}{q}}$$

and

$$G_2(x,y) = py^{q-1}(x^q+y^q-1)^{\frac{p-q}{q}} - py^{p-1}(y_o^p-y^p+1)^{\frac{q-p}{p}}[(x_o^p-x^p+1)^{\frac{q}{p}}+(y_o^p-y^p+1)^{\frac{q}{p}}-1]^{\frac{p-q}{q}} .$$

Since each of p, x, y, $x^q + y^q - 1$, $y_o^p - y^p + 1$, $x_o^p - x^p + 1$ is positive, we have, upon setting $G_1(x,y) = 0$ and $G_2(x,y) = 0$, that

$$(32) \qquad x^{p-q}(x_o^p-x^p+1)^{\frac{q-p}{p}}[(x_o^p-x^p+1)^{\frac{q}{p}} + (y_o^p-y^p+1)^{\frac{q}{p}} - 1]^{\frac{p-q}{q}}$$
$$= y^{p-q}(y_o^p-y^p+1)^{\frac{q-p}{p}}[(x_o^p-x^p+1)^{\frac{q}{p}} + (y_o^p-y^p+1)^{\frac{q}{p}} - 1]^{\frac{p-q}{q}} .$$

But, since $x_o \le (x_o^p - x^p + 1)^{1/p} \le 1$ and $y_o \le (y_o^p - y^p + 1)^{1/p} \le 1$,

$$[(x_o^p-x^p+1)^{q/p}+(y_o^p-y^p+1)^{q/p}-1]^{1/q} = T_q\left((x_o^p-x^p+1)^{1/p}, (y_o^p-y^p+1)^{1/p}\right)$$
$$\ge T_q(x_o,y_o) > 0.$$

Thus

$$x^{p-q}(x_o^p - x^p + 1)^{\frac{q-p}{p}} = y^{p-q}(y_o^p - y^p + 1)^{\frac{q-p}{p}}$$

Raising both sides to the power $\frac{p}{q-p}$, multiplying both sides of the result by $x^p y^p$ and solving for y we obtain

$$(33) \qquad y = \left(\frac{y_o^p + 1}{x_o^p + 1}\right)^{1/p} x .$$

Substituting this into $G_1(x,y) = 0$ we obtain

$$x^{q-1}\left(x^q + \left(\frac{y_o^p + 1}{x_o^p + 1}\right)^{\frac{q}{p}} x^q - 1\right)^{\frac{p-q}{q}}$$

$$= x^{p-1}(x_o^p - x^p + 1)^{\frac{q-p}{p}}\left[(x_o^p - x^p + 1)^{\frac{q}{p}} + \left(y_o^p - \frac{y_o^p+1}{x_o^p+1}\, x^p + 1\right)^{\frac{q}{p}} - 1\right]^{\frac{p-q}{q}} ,$$

i.e.,

$$\left[x^q + \left(\frac{y_o^p+1}{x_o^p+1}\right)^{\frac{q}{p}} x^q - 1\right]^{\frac{p-q}{q}} = x^{p-q}(x_o^p - x^p + 1)^{\frac{q-p}{p}}\left[(x_o^p - x^p + 1)^{\frac{q}{p}} + (y_o^p + 1)^{\frac{q}{p}}\left(1 - \frac{x^p}{x_o^p+1}\right)^{\frac{q}{p}} - 1\right]^{\frac{p-q}{q}} .$$

This implies

$$\left[x^q + \left(\frac{y_o^p+1}{x_o^p+1}\right)^{\frac{q}{p}} x^q - 1\right] = x^q\left(x_o^p - x^p + 1\right)^{-\frac{q}{p}}\left[(x_o^p - x^p + 1)^{\frac{q}{p}} + (x_o^p - x + 1)^{\frac{q}{p}}\left(\frac{y_o^p+1}{x_o^p+1}\right)^{\frac{q}{p}} - 1\right]$$

$$= x^q + \left(\frac{y_o^p + 1}{x_o^p + 1}\right)^{\frac{q}{p}} x^q - x^q\,(x_o^p - x^p + 1)^{-\frac{q}{p}} .$$

This reduces to

$$- 1 = -x^q(x_o^p - x^p + 1)^{-\frac{q}{p}}$$

which we can solve for x to obtain

$$x = \left(\frac{x_o^p + 1}{2}\right)^{1/p}$$

Substituting this into (33) gives

$$y = \left(\frac{y_o^p + 1}{2}\right)^{1/p} .$$

Therefore the only critical point inside the rectangle $[x_o,1] \times [y_o,1]$ is located at

$$\left(\left(\frac{x_o^p + 1}{2}\right)^{1/p}, \left(\frac{y_o^p + 1}{2}\right)^{1/p}\right).$$

An application of Lemma 7 yields the desired result.

The proof for negative p is so similar we shall omit it.

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