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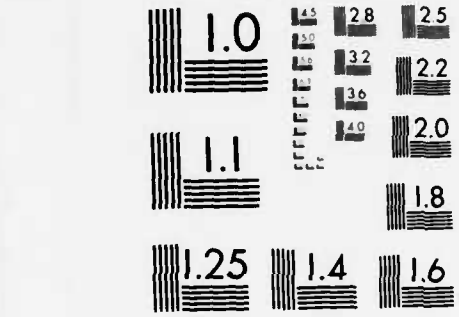

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

STATIONARY EXPONENTIAL TIME SERIES:

FURTHER MODEL DEVELOPMENT AND

A RESIDUAL ANALYSIS

by

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April 1983

Prepared for:
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correlation and exponential time series with mixed autoregressive-moving average structure. An application to the analysis of a set of wind speed data is included.

Stationary Exponential Time Series: Further model development and a residual analysis

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ABSTRACT

A second order autoregressive process in exponential variables, NEAR(2), is established: the distributional assumptions involved in this model highlight a very broad four parameter structure which combines five exponential random variables into a sixth exponential random variable. The dependency structure of the NEAR(2) process beyond and including autocorrelations is explored using some new ideas on residual analysis for non-normal processes with autoregressive correlation structure. Other applications of the exponential structure are considered briefly. These include exponential time series with negative correlation and exponential time series with mixed autoregressive-moving average structure. An application to the analysis of a set of wind speed data is included.

January 1983

1. Introduction

There are several aspects of many observed univariate time series which are not satisfactorily accounted for in standard time series analysis: they include non-Gaussian marginal distributions, dependence beyond second-order moments (autocorrelations) and directionality in the time series. Quite often a Gaussian distribution will be inappropriate because the variable being modelled has a positive and highly skewed distribution, i.e., the service times in a queue or daily flows in a river. Many particular such distributions can be envisaged and time series models have been constructed for them. Examples are Gamma distributions (Gaver and Lewis, 1980; Lewis, 1981; McKenzie, 1982; Lawrance, 1982) and mixed exponential distributions (Gaver and Lewis, 1980; Lawrance, 1980a; Lawrance and Lewis, 1982).

However the simplest, most widely used and most tractable analytically of these distribution models is the exponential distribution. Like Gaussian random variables, exponentially distributed random variables enjoy many special properties; also they can be mildly transformed quite easily into distributions which are more skewed, or less skewed, than the exponential. The Weibull distribution is an example, being just a power transformation of an exponentially distributed random variable. Thus the approach here, following earlier work (Gaver and Lewis, 1980; Lawrance and Lewis, 1980, 1981) is to regard the exponential variables as canonical and to develop their use in time series modelling. It should also be noted that time series of uniformly distributed random variables are obtained by exponential transformations of the exponential time series; this has many possibilities and the process of uniformly distributed random variables will be considered elsewhere.

The work cited above has concentrated for the most part on first-order, non-Gaussian autoregressive models, both of the standard type (constant coefficient additive linear combinations) and a random coefficient type

introduced by the authors. The extension of the models to higher order autoregression is clearly necessary to attain flexibility in modelling correlation and dependency structure of the processes, but these extensions are in no way as immediate as in the standard linear Gaussian case. A simple mixing device can be used (Jacobs and Lewis, 1983) but the range of correlations attained is much narrower than the range attained in the standard linear, second-order autoregressive structure. A broader extension, called the EAR(2) model, was obtained in the exponential case by Lawrance and Lewis (1980), but its innovation variable has a zero component and this will often be hard to justify.

A major part of the present work consists of obtaining a very broad and rich extension of the NEAR(1) model (Lawrance and Lewis, 1981) to a second-order autoregressive process; this NEAR(2) model was first proposed in Lawrance (1980b) and later reviewed in Raftery (1981), although necessary analysis of its innovation structure was not given. The innovation random variable for the NEAR(2) process is proved here to exist without unnatural boundaries on its (four) parameter region and does not have a zero component. Taken out of the context of second-order autoregressive exponential time series this result gives a very broad structure for combination of exponential random variables. In fact it gives a way to combine three (possibly dependent) exponential random variables with an independent pair of (possibly dependent) exponential random variables to give another exponential random variable. The result has a number of uses beyond the second-order time series model giving, for instance, a first-order model which is broader than the NEAR(1) model, and several second-order mixed autoregressive-moving average models. Schemes for obtaining negative correlation can also be accommodated. These models, as in the first-order case, include aspects of dependence beyond autocorrelation and also directionality in the time series.

The richness of the four-parameter NEAR(2) model and the fact that an infinite number of cases of the model with identical correlation structure are available forces consideration of higher-order aspects of dependence. The analysis of the higher order aspects of the exponential time series is at a fairly early stage and is as follows. First it will be shown that the autocorrelations $\rho(\ell)$, $\ell = 0, \pm 1, \pm 2, \ldots$ for the NEAR(2) process satisfy the Yule-Walker equations with constants a_1 and a_2 which are functions of the four parameters of the model. This follows immediately from the fact that X_n is a random-coefficient, linear additive combination of X_{n-1}, X_{n-2} and the innovation r.v. E_n. Secondly, it can be shown (Lewis and Lawrance, 1983a) that the residuals $X_n - a_1X_{n-1} - a_2X_{n-2}$, which are the usual residuals for second-order constant coefficient, linear additive autoregressive processes are uncorrelated.

Thus although the standard analysis of time series stops with uncorrelated residuals, i.e. a flat spectrum for the residuals, such residuals can also be used to good effect to investigate higher order aspects of dependence in the NEAR(2) model. In fact, if the autoregression is not of the standard type (constant coefficient, linear additive) the (uncorrelated) residuals will have non-random scatters and higher order dependence properties. In particular, the squared residuals will have non-zero autocorrelations and the cross-correlations of residuals and squared residuals except in the zero-lag case will be non-zero; both sets of correlations are effectively zero when a standard second-order autoregressive model is appropriate.

An extension of this residual analysis is suggested by the fact that autocorrelations are time reversible, while the process itself is not time reversible, i.e. is directional. Thus one constructs reversed residuals $X_n - a_1X_{n+1} - a_2X_{n+2}$ which are to be used to assess directionality. Analysis

of both types of residuals can be fruitful; because of space limitations we present here only an investigation of cross-correlations of residuals and squared-residuals. This particular case of the possible residual analyses is illustrated by some theoretical calculations, some simulations and a brief application to a transformed series of wind speed data. The complete residual analysis including directional aspects and their investigation using the additional tool of reversed residuals will be given elsewhere.

2. Exponential Time Series Models

Our aim in this section is to give in outline the ideas leading to the time series model of main concern in this paper, and called NEAR(2), following the earlier terminology NEAR(1). The NEAR(2) model has four-parameters, and incorporates and broadens the earlier two-parameter EAR(2) model (Lawrance and Lewis, 1980). The NEAR(2) model will be exponential in marginal distribution, have second-order autoregressive Markov dependence, and have autocorrelations satisfying second-order difference equations of the familiar Yule-Walker type. In addition it will have dependence beyond autocorrelation, and will have adjustable directionality. It is not linear in the standard sense, having random-coefficient, linear additive autoregressive structure, but neither is it non-linear in the standard sense of incorporating powers or products of lagged variables.

Writing $\{X_n\}$ for the time series variables, and $\{E_n\}$ for an i.i.d. exponential innovation sequence of unit mean, the two-parameter NEAR(1) model, as previously defined, is given by

$$X_n = \begin{cases} \beta X_{n-1} & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha \end{cases} + \begin{cases} E_n & \text{w.p. } p\,, \\ bE_n & \text{w.p. } 1-p\,, \end{cases} \tag{2.1}$$

with $b = (1-\alpha)\beta$ and $p = (1-\beta)/\{1-(1-\alpha)\beta\}$. The parameter region is in general $0 \leq \alpha, \beta \leq 1$, $\alpha = \beta \neq 1$. The case $\beta = 1$, $0 \leq \alpha < 1$ is rather special, and has been called the TEAR(1) model, and when $\alpha = 1$, $0 \leq \beta < 1$, the earlier EAR(1) model is recovered. Except for this EAR(1) case, the NEAR(1) model does not allow zero innovations (Gaver and Lewis, 1980) and so is more statistically acceptable. The zero innovation implies that $X_n = \beta X_{n-1}$ and thus β can be determined exactly from runs down in the sample path of the process.

In general the i.i.d. innovation in the NEAR(1) process are formed as the probabilistic mixture of two exponentials, and are thus easily simulated.

The NEAR(2) model is a direct generalization of (2.1) and takes the form

$$X_n = \begin{cases} \beta_1 X_{n-1} & \text{w.p. } \alpha_1 \\ \beta_2 X_{n-2} & \text{w.p. } \alpha_2 \\ 0 & \text{w.p. } 1-\alpha_1 - \alpha_2 \end{cases} + E_n \qquad (2.2)$$

with parameter region $0 \le \beta_1, \beta_2 \le 1$, $\alpha_1 \ge 0$, $\alpha_2 \ge 0$, $\alpha_1 + \alpha_2 \le 1$; $\{E_n\}$ is an appropriately chosen innovation sequence. Many special cases can arise when the above restrictions include some of the equalities and, for the purposes of a general development, it is best to regard the inequalities as strict. Given that $\{X_n\}$ is required to have an exponential marginal distribution, the main question concerns whether there is a valid probability distribution for E_n . The Theorem proved in Section 2.3 will show that this is the case, and that the distribution, when the inequalities on $\alpha_1 + \alpha_2$ and β_1, β_2 in the parameter region are strict, takes the form

$$E_n = \begin{cases} E_n & \text{w.p. } 1-p_2 - p_3 , \\ b_2 E_n & \text{w.p. } p_2 , \\ b_3 E_n & \text{w.p. } p_3 , \end{cases} \qquad (2.3)$$

a probabilistic mixture of three exponentials. To establish this result a fairly detailed analysis of a derived moment generating function is required. This is necessary since a direct moment generating function solution of (2.2) for E_n does not establish that E_n has a proper distribution; all that is shown is that the solution is a possibly improper mixture of three exponentials.

3. Validity of the NEAR(2) Model

In this section we prove the following

THEOREM. Let $\{E_n\}$ be an i.i.d. sequence of unit mean exponential random variables. Then if the four parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy $\alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 < 1, 0 < \beta_1, \beta_2 < 1$, the relationship

$$X_n = \left\{ \begin{array}{ll} \beta_1 X_{n-1} & \text{w.p. } \alpha_1 , \\ \beta_2 X_{n-2} & \text{w.p. } \alpha_2 , \\ 0 & \text{w.p. } 1-\alpha_1 - \alpha_2 , \end{array} \right\} + E_n , \quad n = 0, \pm 1, + 2, \ldots , \tag{3.1}$$

where

$$E_n = \left\{ \begin{array}{ll} E_n & \text{w.p. } 1-p_2 - p_3 , \\ b_2 E_n & \text{w.p. } p_2 , \\ b_3 E_n & \text{w.p. } p_3 , \end{array} \right. \tag{3.2}$$

defines a stationary sequence of (marginally) exponentially distributed random variables with mean one. Here

$$p_2 = \{(\alpha_1\beta_1 + \alpha_2\beta_2)b_2 - (\alpha_1 + \alpha_2)\beta_1\beta_2\}/\{(b_2 - b_3)(1 - b_2)\} , \tag{3.3}$$

$$p_3 = \{(\alpha_1 + \alpha_2)\beta_1\beta_2 - (\alpha_1\beta_1 + \alpha_2\beta_2)b_3\}/\{(b_2 - b_3)(1 - b_3)\} , \tag{3.4}$$

and

$$0 < b_3 = \frac{1}{2}\left\{s - (s^2 - 4r)^{1/2}\right\} < b_2 = \frac{1}{2}\left\{s + (s^2 - 4r)\right\} < 1 , \tag{3.5}$$

where

$$s = (1 - \alpha_1)\beta_1 + (1 - \alpha_2)\beta_2 , \tag{3.6}$$

and

$$r = (1 - \alpha_1 - \alpha_2)\beta_1\beta_2 . \tag{3.7}$$

PROOF. For the NEAR(2) model specified by (3.1) - (3.7), let $\phi_X(t)$ and $\phi_E(t)$ be the moment generating functions of the $\{X_n\}$ and $\{E_n\}$ sequences; then quite simply if stationarity of the $\{X_n\}$ series is assumed,

$$\phi_X(t) = \phi_E(t)\{\alpha_1\phi_X(\beta_1 t) + \alpha_2\phi_X(\beta_2 t) + (1 - \alpha_1 - \alpha_2)\} . \tag{3.8}$$

Assuming an exponential marginal distribution of unit mean for $\{X_n\}$, then the independent distribution of $\{E_n\}$ has moment generating function, possibly not proper, given by

$$\phi_E(t) = \frac{(1+\beta_1 t)(1+\beta_2 t)}{(1+t)[(1-\alpha_1 - \alpha_2)\beta_1\beta_2 t^2 + \{(1-\alpha_1)\beta_1 + (1-\alpha_2)\beta_2\}t + 1]} . \tag{3.9}$$

First we remark that (3.9) simplifies in special cases, such as when one of β_1, $\beta_2 = 1$, $\beta_1 = \beta_2$, or $\alpha_1 + \alpha_2 = 1$. For simplicity in the development, the parameter region considered in the theorem is that defined by $0 < \beta_1$, $\beta_2 < 1$, $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 < 1$. Interesting special cases are more easily treated separately; these include NEAR(1) models and particularly tractable types of NEAR(2) models.

It is convenient to establish right away that the quadratic term in the denominator of (3.9) has real distinct and positive roots, b_1 and b_2; this eliminates any subsequent need to invert such a term as a whole. The required condition for real distinct roots is that

$$[(1-\alpha_1)\beta_1 + (1-\alpha_2)\beta_2]^2 - 4(1-\alpha_1 - \alpha_2)\beta_1\beta_2$$

be positive: this is so from its equality to the expression

$$[(1-\alpha_1)\beta_1 - (1-\alpha_2)\beta_2]^2 + 4\alpha_1\alpha_2\beta_1\beta_2$$

which is clearly positive; the positivity of the roots b_1 and b_2 is obvious from (3.9) since their product and sum given in (3.11) and (3.12) below are both positive.

Subject to the above qualifications concerning b_1 and b_2, a partial fraction expansion of (3.2) can be written in the suggestive form

$$\phi_E(t) = (1-p_2-p_3)\frac{1}{1+t} + p_2\frac{1}{1+b_2t} + p_3\frac{1}{1+b_3t} . \tag{3.10}$$

Comparisons between (3.9) and (3.10) then show that b_2, b_3 and p_2, p_3 may be obtained in terms of β_1, β_2 and α_1, α_2 by solving the equations

$$b_2 + b_3 = (1-\alpha_1)\beta_1 + (1-\alpha_2)\beta_2 ; \tag{3.11}$$

$$b_2b_3 = (1-\alpha_1 - \alpha_2)\beta_1\beta_2 ; \tag{3.12}$$

$$(1-b_2)p_2 + (1-b_3)p_3 = \alpha_1\beta_1 + \alpha_2\beta_2 ; \tag{3.13}$$

$$b_3(1-b_2)p_2 + b_2(1-b_3)p_3 = (\alpha_1 + \alpha_2)\beta_1\beta_2 . \tag{3.14}$$

A difficulty with this apparently straightforward solution is that the inversion of (3.9) or (3.10) could lead to a function which is not a probability density, or it could yield a probability density but not one which is a probabilistic mixture of three exponentials. In fact, neither of these possibilities is the case, as will be shown by establishing that p_2 and p_3 are positive and subject to the condition $p_2 + p_3 < 1$, and hence can represent probabilities.

Explicit expressions for p_2 and p_3 can be obtained from (3.13) and (3.14) and are given at (3.3) and (3.4). From now on it will be assumed, in accordance with the theorem, that b_2 is the larger of b_2 and b_3, these being obtained by solving the quadratic pair (3.11) and (3.12). To establish that $p_2 + p_3 < 1$, we have, by adding (3.3) and (3.4),

$$p_2 + p_3 = \frac{(\alpha_1\beta_1 + \alpha_2\beta_2) - (\alpha_1 + \alpha_2)\beta_1\beta_2}{(1-b_2)(1-b_3)} . \tag{3.15}$$

Multiplying out $(1-b_2)(1-b_3)$ in the denominator and using (3.11) and (3.12) gives, after some rearrangement,

$$p_2 + p_3 = 1 - \frac{(1-\beta_1)(1-\beta_2)}{(1-\beta_1)(1-\beta_2) + \alpha_1\beta_1(1-\beta_2) + \alpha_2\beta_2(1-\beta_1)} . \tag{3.16}$$

The algebraic expression here is clearly positive and less than one, from which it follows that $p_1 + p_2 < 1$.

The positivity of p_2 and p_3 will now be proved by showing that the numerators and denominators of (3.3) and (3.4) are positive. For the denominators, this requires that $0 < b_2, b_3 < 1$ which will be verified by showing that $0 < b_2b_3 < 1$ and $0 < (1-b_2)(1-b_3) < 1$. The first of these latter two inequalities is obvious from (3.12); for the second consider the expressions

$$\begin{aligned}(1-b_2)(1-b_3) &= 1 - (b_2 + b_3) + b_2b_3 \\ &= (\alpha_1\beta_1 + 1-\beta_1)(\alpha_2\beta_2 + 1 - \beta_2) - (\alpha_1\beta_1)(\alpha_2\beta_2)\end{aligned} \tag{3.17}$$

after using (3.11) and (3.12), and then

$$\begin{aligned}1 - (1-b_2)(1-b_3) &= b_2 + b_3 - b_2b_3 \\ &= (1-\alpha_1)\beta_1(1-\beta_2) + (1-\alpha_2)\beta_2(1-\beta_1) + \beta_1\beta_2 .\end{aligned} \tag{3.18}$$

Both the right hand sides of (3.17) and (3.18) are obviously positive. This concludes the proof that $0 < b_1, b_2 < 1$ and hence that the denominators of p_2 and p_3 are positive.

For the numerators of p_2 and p_3 to be positive (3.3) and (3.4) indicate that $b = (\alpha_1 + \alpha_2)\beta_1\beta_2/(\alpha_1\beta_1 + \alpha_2\beta_2)$ must satisfy the inequalities

$$b_3 < b < b_2 . \tag{3.19}$$

At this last stage, explicity expressions for b_2 and b_3 must be used, and from (3.11) and (3.12) are given, after writing

$$s = (1-\alpha_1)\beta_1 + (1-\alpha_2)\beta_2 \quad \text{and} \quad r = (1-\alpha_1-\alpha_2)\beta_1\beta_2 ,$$

by

$$b_2 = \tfrac{1}{2}\{s + (s^2-4r)^{1/2}\} \text{ and } b_3 = \tfrac{1}{2}\{s - (s^2-4r)^{1/2}\}. \tag{3.20}$$

Then (3.19) is equivalent to

$$-(s^2-4r)^{1/2} < s - 2b < (s^2-4r)^{1/2}$$

or

$$s^2-4r > (s-2b)^2$$

or

$$sb - b^2 - r > 0 . \tag{3.21}$$

After some algebraic rearrangement the left hand side of (3.21) becomes

$$\alpha_1\alpha_2\beta_1\beta_2(\beta_1-\beta_2)^2/(\alpha_1\beta_1+\alpha_2\beta_2)^2 \tag{3.22}$$

which is again clearly strictly positive, excluding special cases, as was to be proved.

This concludes the proof that p_2 and p_3 are both positive subject to $p_2 + p_3 < 1$; hence $1-p_2 - p_3$, p_2 , and p_3 can all be regarded as probabilities. Thus E_n has a proper probability distribution which can be generated as the $(1-p_2 - p_3, p_2, p_3)$ mixture of three exponentials of means 1, b_2 and b_3 respectively; further, both b_2 and b_3 are less than unity and $b_2 \neq b_3$.

In the special cases mentioned earlier, there are valid and simpler results for the distribution of E_n . For instance, when $\beta_1 = \beta_2 = 1$, E_n has a simple exponential distribution of mean $(1-\alpha_1 - \alpha_2)$. When $\beta_1 = \beta_2 \neq 1$ the innovation has a mixed exponential distribution of the NEAR(1) form given in (2.1) with $\alpha = \alpha_1 + \alpha_2$. Delineation of all these special cases is needed for successful simulation of the NEAR(2) process for the full parameter range. This will be considered elsewhere.

4. Other Uses of the NEAR(2) Exponential Construction

The NEAR(2) process was established by showing that (3.2) was a valid innovation distribution for the relation (3.1). The distributional assumptions implied by this result can also be taken out of the time series context in which they were derived and viewed generally as a way to combine a pair of (possibly dependent) unit exponential variables (L_1, L_2) with an independent triple of (possibly dependent) unit exponential variables (M_1, M_2, M_3) so as to yield a further unit exponential variable. Specifically, with $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and (b_2, b_3, p_2, p_3) as previously related by (3.3) - (3.7), the Theorem has established that

$$\left.\begin{array}{lll} \beta_1 L_1 & \text{w.p.} & \alpha_1 \\ \beta_2 L_2 & \text{w.p.} & \alpha_2 \\ 0 & \text{w.p.} & 1-\alpha_1-\alpha_2 \end{array}\right\} + \left\{\begin{array}{lll} M_1 & \text{w.p.} & 1-p_1-p_2 \\ b_2 M_2 & \text{w.p.} & p_2 \\ b_3 M_3 & \text{w.p.} & p_3 \end{array}\right. \tag{4.1}$$

has a unit exponential distribution.

First of all, the idea of "switching" will be illustrated; in the NEAR(2) context, this suggests taking (M_1, M_2, M_3) as $(X_{n-1}, X_{n-2}, X_{n-3})$ and (L_1, L_2) as (E_n, E_n). Then (4.1) gives the time series model

$$X_n = \left\{\begin{array}{lll} X_{n-1} & \text{w.p.} & 1-p_2-p_3 \\ b_2 X_{n-2} & \text{w.p.} & p_2 \\ b_3 X_{n-3} & \text{w.p.} & p_3 \end{array}\right. + \left\{\begin{array}{lll} \beta_1 E_n & \text{w.p.} & \alpha_1 \\ \beta_2 E_n & \text{w.p.} & \alpha_2 \\ 0 & \text{w.p.} & 1-\alpha_1-\alpha_2 \end{array}\right. . \tag{4.2}$$

This is a third-order autoregression, actually a case of the EAR(3) model cited in Lawrance and Lewis (1979); note, however, that this third-order autoregressive exponential process allows zero innovations. Another, better

behaved higher-order exponential model -in fact a p-th order model- is obtained by the following application of the result (4.1) in its original form (3.1). Let the indices $1,2,\ldots,p$ be partitioned into two non-empty sets I_1 and I_2 of size t_1 and t_2 respectively. Then in the model

$$X_n = \left\{ \begin{array}{lll} \beta_1' X_{n-1} & \text{w.p.} & \alpha_1' \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \beta_p' X_{n-p} & \text{w.p.} & \alpha_p' \\ 0 & \text{w.p.} & 1-\alpha_1'-\ldots\alpha_p' \end{array} \right\} + E_n , \quad n = 0, \pm 1, \pm 2, \ldots, \qquad (4.3)$$

let $\beta_i' = \beta_1$, $i \in I_1$; $\beta_i' = \beta_2$, $i \in I_2$; $\sum_{i \in I_1} \alpha_i' = \alpha_1$ and $\sum_{i \in I_2} \alpha_i' = \alpha_2$. Then if $\alpha_1 + \alpha_2 < 1$, $0 < \beta_1, \beta_2 < 1$ the distribution of E_n is given by the Theorem. Thus we have a pth-order exponential autoregressive process with four-parameters. However, while this may seem satisfying it is not clear that four parameters would be sufficient to characterize the sample path behaviour of an exponential process with very high order dependence.

Another use of (4.1) is to allow L_1 and L_2 to both be X_{n-1}, and so obtain a four parameter first-order model of the form

$$X_n = \left\{ \begin{array}{lll} \beta_1 X_{n-1} & \text{w.p.} & \alpha_1 \\ \beta_2 X_{n-1} & \text{w.p.} & \alpha_2 \\ 0 & \text{w.p.} & 1-\alpha_1-\alpha_2 \end{array} \right. + \left\{ \begin{array}{lll} E_n & \text{w.p.} & 1-p_2-p_3 , \\ b_2 E_n & \text{w.p.} & p_2 , \\ b_3 E_n & \text{w.p.} & p_3 . \end{array} \right. \qquad (4.4)$$

Four parameters seems excessive for a first-order autoregressive process but simulations show a wide range of behaviour in sample paths with different choices of parameters. Equation (4.4) in turn suggests a first-order model allowing negative dependence. This is obtained by letting the variable in

(4.4) which is multiplied by β_2 be the antithetic transformation of X_{n-1}, that is $-\log(1-e^{-X_{n-1}})$. A two parameter version of the model could be obtained, for example, taking $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.

A third type of use of the construction is to give mixed autoregressive moving average models; for this, (L_1, L_2) is (X_{n-1}, X_{n-2}) as previously, but (M_1, M_2, M_3) are chosen to be (E_n, E_{n+1}, E_{n+1}) for a second-order moving average compoent, or as (E_n, E_{n+1}, E_{n+2}) for a third-order moving average component.

Out of the time series context, the construction suggests ways to obtain multivariate exponential distributions, rather as in Lawrance and Lewis (1983b).

Further possibilities are numerous, but it is not the intention here to exhaustively list them, or to derive the details of those cited at this time. Analysis in the following sections will deal with the basic NEAR(2) model.

5. Autocorrelation Structure of the NEAR(2) process

In this section we show that the autocorrelations $\rho(\ell) = \text{corr}(X_n, X_{n-\ell})$, $\ell = 0, \pm 1, \pm 2, \ldots$ of the NEAR(2) process satisfy Yule-Walker type difference equations; thus the second moment dependency aspects of the process are indistinguishable from those of a standard autoregressive model, AR(2). To show this it is convenient to write the equation (3.1) as an additive autoregressive combination of X_{n-1}, X_{n-2} and E_n. Thus we have a random coefficient, linear additive autoregressive process

$$X_n = \beta_1 K_n' X_{n-1} + \beta_2 K_n'' X_{n-2} + L_n E_n \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{5.1}$$

where
$$L_n = \begin{cases} 1 & \text{w.p. } 1-p_2-p_3, \\ b_2 & \text{w.p. } p_2, \\ b_3 & \text{w.p. } p_3; \end{cases} \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{5.2}$$

$$\{K_n', K_n''\} = \begin{cases} (1,0) & \text{w.p. } \alpha_1, \\ (0,1) & \text{w.p. } \alpha_2, \\ (0,0) & \text{w.p. } 1-\alpha_1-\alpha_2; \end{cases} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{5.3}$$

and the i.i.d. sequences $\{L_n\}$ and $\{K_n', K_n''\}$ are mutually independent and independent of the independent exponential sequence $\{E_n\}$. The E_n's are assumed to have mean and variance 1, as do the X_n's by construction.

Now $E(K_n') = \alpha_1$ and $E(K_n') = \alpha_2$, so that $E(L_n) = 1-\alpha_1\beta_1 - \alpha_2\beta_2$. Then multiplying X_n in (5.1) by $X_{n-\ell}$ we have, for $\ell > 1$,

$$\begin{aligned} E(X_n X_{n-\ell}) &= \alpha_1\beta_1 E(X_{n-1}X_{n-\ell}) + \alpha_2\beta_2 E(X_{n-2}X_{n-\ell}) + E(L_n)E(E_n)E(X_{n-\ell}) \\ &= \alpha_1\beta_1 E(X_{n-1}X_{n-\ell}) + \alpha_2\beta_2 E(X_{n-2}X_{n-\ell}) + 1 - \alpha_1\beta_1 - \alpha_2\beta_2, \end{aligned}$$

so that

$$E(X_n X_{n-\ell}) - 1 = \alpha_1\beta_1\{E(X_{n-1}X_{n-\ell}) - 1\} + \alpha_2\beta_2\{E(X_{n-2}X_{n-\ell}) - 1\}$$

and $$\rho(-\ell) = a_1\rho(-(\ell-1)) + a_2\rho(-(\ell-2)) ,$$

where $a_1 = \alpha_1\beta_1$ and $a_2 = \alpha_2\beta_2$. Using $\rho(-\ell) = \rho(\ell)$, $\ell > 1$, we have finally

$$\rho(\ell) = a_1\rho(\ell-1) + a_2\rho(\ell-2) , \qquad \ell > 1 , \tag{5.4}$$

which are the Yule-Walker equations for the AR(2) process. The conditions for a solution to exist (Box and Jenkins, 1970, p.58), $a_1 + a_2 < 1$, $a_1 - a_2 > -1$, $a_2 > -1$ are clearly satisfied when the conditions on α_1, α_2, β_1, β_2 given in the Theorem of Section 3 hold. We then have

$$\rho(1) = a_1/(1-a_2) \quad \text{and} \quad \rho(2) = a_1\rho(1) + a_2 . \tag{5.5}$$

Note, however, the restriction to positive correlations since a_1 and a_2 are positive. The possible region of $(\rho(1),\rho(2))$ values is bounded below by $\rho(2) = \rho^2(1)$ and otherwise bounded by $\rho(1) > 0$ and $\rho(2) \leq 1$. Broadening of the model to negative dependency may be achieved using antithetic ideas, or the bivariate scheme given in Gaver and Lewis (1980), but is not pursued here.

Note too that the parameters in (5.4) enter only as products $a_1 = \alpha_1\beta_1$ and $a_2 = \alpha_2\beta_2$. Thus for small enough α_1 and α_2, values of β_1 and β_2 greater than one could be allowed, and (5.4) would still have a stable solution. However the sequence E_n in the defining equation (3.2) may not exist; it is not known if $\beta_1 \leq 1$ and $\beta_2 \leq 1$ is a necessary condition for this existence.

Solutions to (5.4) are given in detail in Box and Jenkins (1970, pp. 58-60).

Specifying allowable values of $\rho(1)$ and $\rho(2)$, as may be done by an initial second order analysis of data, leaves two parameters to be specified in the model, say α_1 and α_2 which could produce very different sample path behaviour in the time series. It is important to notice that this specification of $\rho(1)$ and $\rho(2)$ further constrains the range of possible α_1 and α_2 values. Recalling that $\rho(1)$ and $\rho(2)$ fix $a_1 = \alpha_1\beta_1$ and $a_2 = \alpha_2\beta_2$, as well as that $\alpha_1 + \alpha_2 \leq 1$, it is easily shown that we must have

$$a_1 \leq \alpha_1 \quad \text{and} \quad a_2 \leq \alpha_2 \tag{5.6}$$

which implies that $a_1 + a_2 \leq \alpha_1 + \alpha_2 \leq 1$. Thus α_1 and α_2 are forced to lie in a triangular subregion of the triangular (α_1,α_2) region which is bounded below by a_2, bounded on the left by a_1, and bounded above by the line $\alpha_1 + \alpha_2 = 1$.

6. An analysis of wind velocity data

6.1. Discussion of the data

Lewis and Hugus (1983) have given an analysis of a set of 3-hourly wind velocity readings taken by ship PAPA in the Gulf of Alaska over a 15 year period. After suitable detrending to remove 1 year, 6 month, 12 hour and 6 hour cyclic trend components a first-order autoregressive Gamma model (Lewis, 1981) was fitted to the data, the use of the model being suggested by the shape of the (marginal) histogram of the data (Figure 6.1) and the shape of the normalized log periodogram of the data (Figure 6.2). Note that the sample size is N = 43,800; also there is a residual 6-hour effect (P = 21900) because this cycle varies in intensity over the 15 years. In what follows this will be ignored and the data will be treated as stationary.

It is not the object here to discuss the above analysis in detail but to discuss a different analysis of the data using an assumption of a Weibull marginal distribution and a transformation to exponential variables. This is suggested firstly by the fact that a Weibull distribution is commonly used for this type of data by meteorologists and secondly by the fact that Weibull and Gamma distributions fit the data equally well (Lewis and Hugus, 1983).

The histogram of the transformed data, $X_n' = X_n^{2.205}$, is shown in Figure 6,3, where the power transformation to exponentiality has been determined by fitting the empirical coefficient of variation, 0.479, to the theoretical Weibull coefficient of variation, $C(X) = \{\Gamma(2/c + 1)/[\Gamma(1/c + 1)]^2\} - 1$ to give $c = 2.205$. This transformation does affect the correlation structure of the data, as shown in Table 6.1.

Column 2 in Table 6.1 gives the estimated auto correlations, $\hat{\rho}(\ell)$, of the detrended data; rough 95% confidence bands for these estimates are given by adding and subtracting $2/(N)^{1/2} = 2/(43,800)^{1/2} \simeq .01$. The first

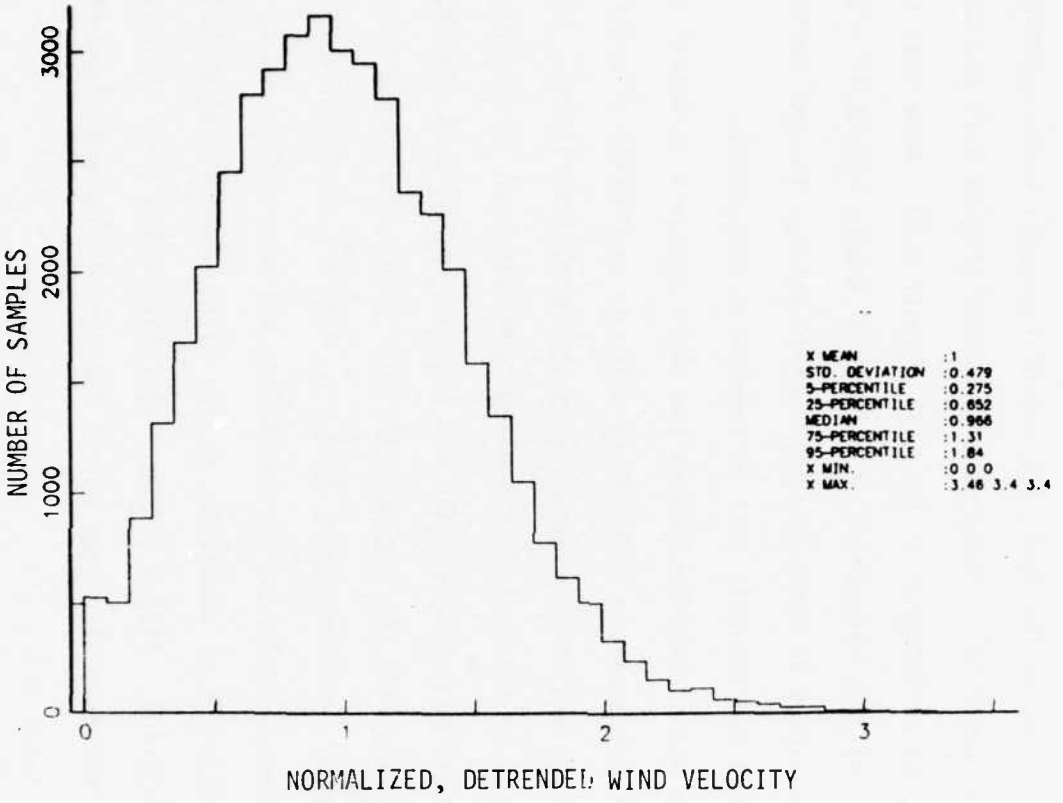

Figure 6.1. Histogram of detrended wind velocity data. Positive skewness is evident, suggesting either a Gamma or Weibull model for the marginal distribution. Sample size N = 43,800.

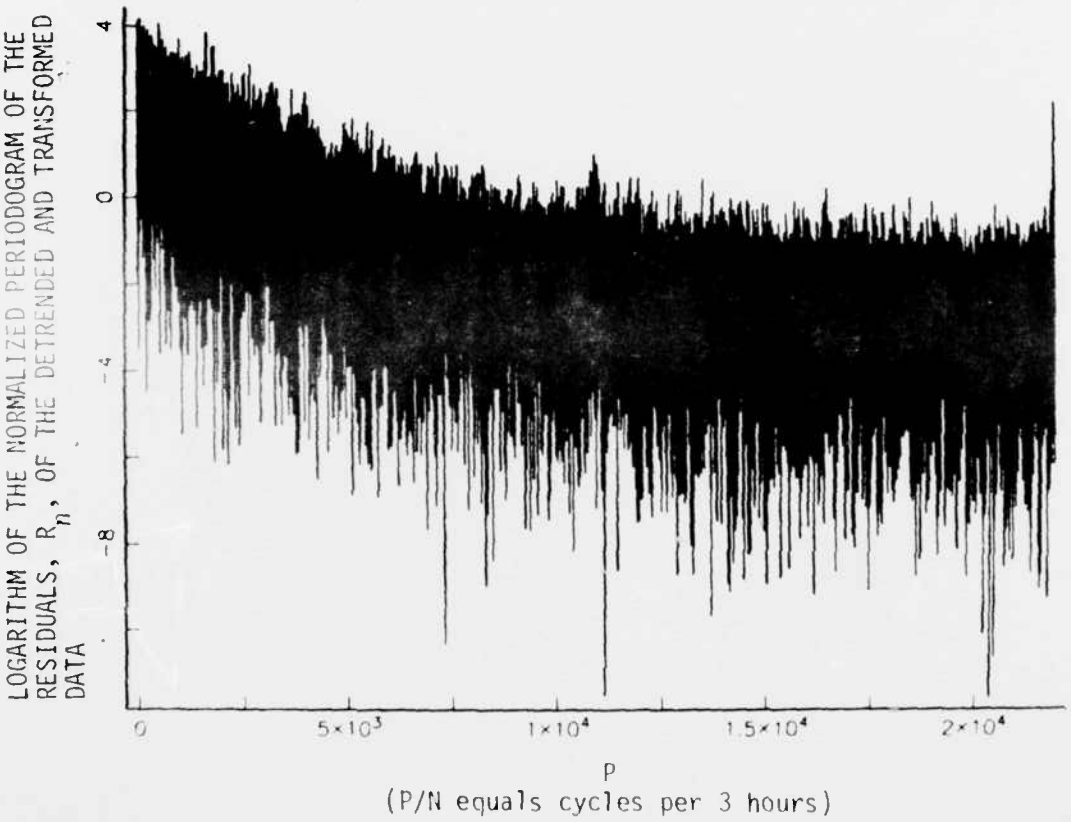

Figure 6.2. Log periodogram of 3-hourly detrended wind velocity data. Note the slight presence of the 6-hours cycle (P = 21900). The periodogram clearly shows that the data is serially correlated. Sample size N = 43,800.

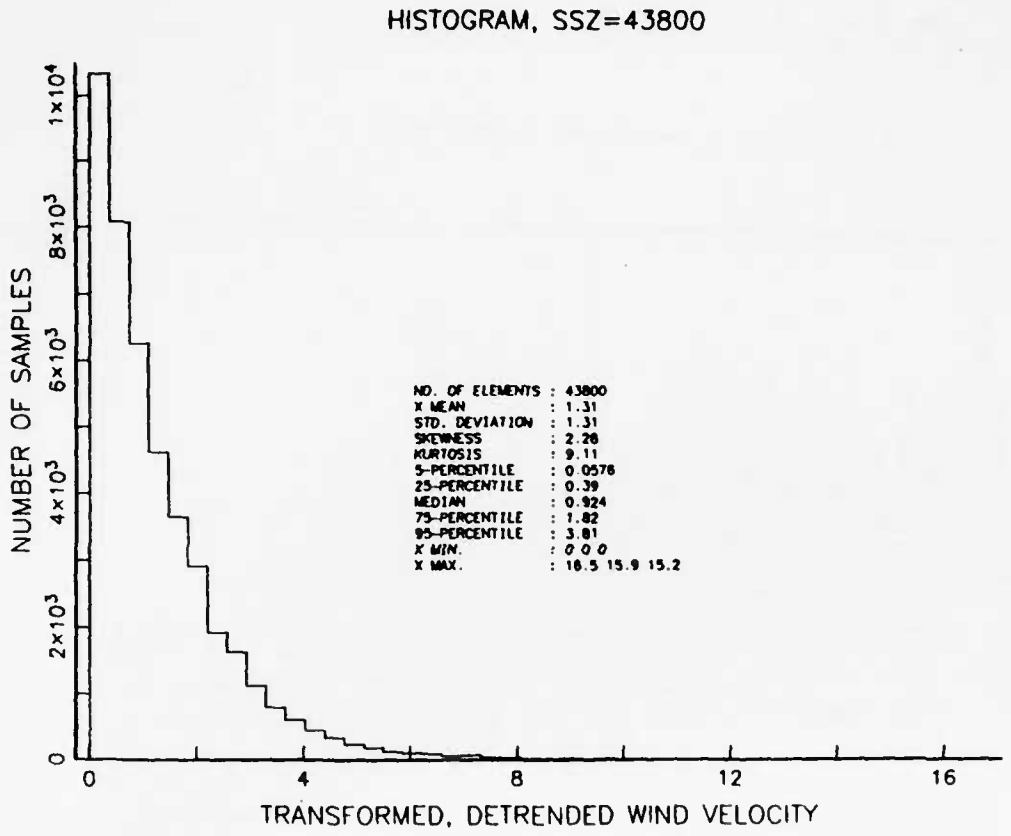

Figure 6.3. Histogram of transformed, detrended wind velocity data. The transformation is $X'_n = X_n^{2.205}$, where the X_n, $n = 1,\ldots,43{,}800$ are the original detrended data. The shape is clearly exponential, but the estimated skewness and kurtosis have values greater than the theoretical values of 2.0 and 6.0.

FIT OF WIND SPEED DATA AUTO-CORRELATIONS

ORIGINAL SERIES $\{X_n\}$			TRANSFORMED SERIES $\{X_n^{2.205}\}$				
AR(1)	RAW	$\lvert\Delta\rvert$	$\lvert\Delta\rvert$	AR(1)	RAW	AR(2)	$\lvert\Delta\rvert$
	$\hat{\rho}_1 = .8214$	.0000	0.0000	0.7985	$\tilde{\rho}_1 = .7985$	.7983	.0002
$\hat{\rho}_1^2 = .6747$	$\hat{\rho}_2 = .6856$	.0109	.0198	.6376	$\tilde{\rho}_2 = .6574$	.6572	.0002
$\hat{\rho}_1^3 = .5543$	$\hat{\rho}_3 = .5606$	.0063	.0255	.5091	$\tilde{\rho}_3 = .5346$	.5397	.0051
$\hat{\rho}_1^4 = .4569$	$\hat{\rho}_4 = .4608$	.0039	.0310	.4065	$\tilde{\rho}_4 = .4375$	.4432	.0057
$\hat{\rho}_1^5 = .3753$	$\hat{\rho}_3 = .3683$	.0070	.0252	.3246	$\tilde{\rho}_5 = .3498$	.3640	.0142

Table 6.1

Table 6.1. The second column shows the estimated auto-correlations for the original series; these are close in value to the powers of $\hat{\rho}_1 = 0.6747$ indicating a good fit to a model with AR(1) auto-correlation structure. After transformation this AR(1) fit is no longer valid (Columns 4,5 and 6) since the absolute values of the differences are large compared to the approximate standard deviations of the estimated auto-correlations, $1/\sqrt{N} = 1/\sqrt{43.800} \simeq .005$. An adequate fit is obtained (Columns 6, 7 and 8) to a model with AR(2) auto-correlation structure.

column gives the fitted auto correlations for a model with AR(1)-type auto correlations, just $\tilde{\rho}(\ell) = (\hat{\rho}(1))^{\ell} = (.8214)^{\ell}$, for lags $\ell = 1,\ldots,5$. The maximum difference (3rd column) is 0.0109 and the fit is clearly good.

Column 6 in Table 6.1 gives the estimated auto corelations, $\tilde{\rho}(\ell)$, for the transformed data; the transformation changes the serial correlations slightly. However, columns 5 and 7, which give the fitted AR(1) and AR(2) correlation models respectively, show that a model with AR(1) correlation structure is not adequate, but that a model with AR(2) correlation fits well. This fit is borne out by a periodogram analysis, which is not given here.

Thus, a NEAR(2) model is a candidate for representing the transformed data, and if $\rho(1)$ and $\rho(2)$ are fixed at the estimated values of 0.7985 and 0.6574, then the corresponding $a_1 = (\alpha_1\beta_1)$ and $a_2 = (\alpha_2\beta_2)$ from (5.5) are, respectively, 0.75488 and 0.05463. There are still two degrees of freedom left in fitting the model, represented by choice of parameters α_1 and α_2 greater than an equal to 0.75488 and 0.05463 respectively, with $\alpha_1 + \alpha_2 \leq 1$.

Figure 6.4 shows the logarithm of the periodogram of the usual AR(2) model linear residuals, $R_n = X'_n - a_1X'_{n-1} - a_2X'_{n-2}$, of the transformed data. This is flat (ignoring the slight effect at period 6 hours). At this point it might be thought that the usual second-order autoregressive model is adequate. We shall however now develop an extended residual analysis for higher order dependence which justifies fitting the NEAR(2) model to the transformed data.

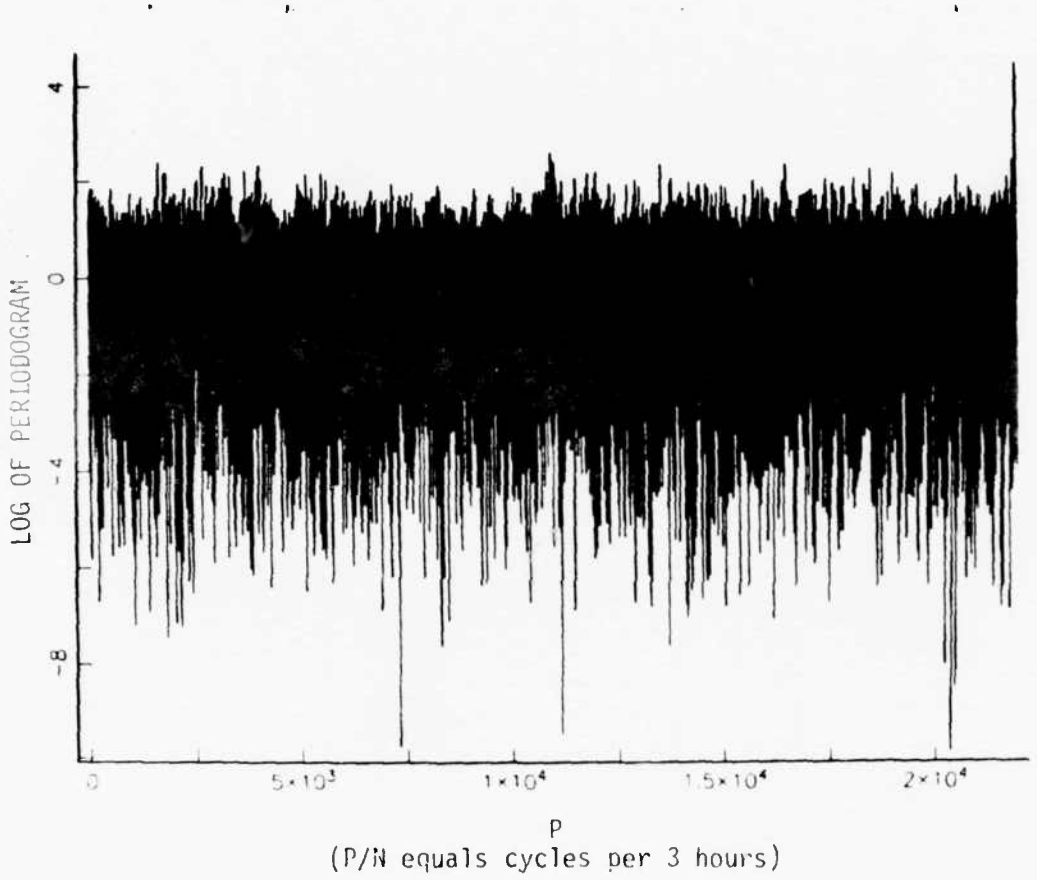

Figure 6.4. Log periodogram of linear residuals $R_n = X'_n - 0.75488\ X'_{n-1} - 0.05463\ X'_{n-2}$ for the detrended and transformed wind velocity data.

7. A Residual Analysis for the NEAR(2) Model

7.1. General results

It has already been remarked that the autocorrelations $\rho(\ell)$ are insufficient to describe the dependency structure of NEAR(2) models. A natural next step might be to examine higher joint moments and consider their associated spectra (see e.g., Priestley, 1982). The functions so obtained, e.g. the bispectrum, are often found to be difficult to calculate and hard to interpret. Rather than follow this course, it is proposed to adapt some ideas from a residual analysis for autoregressive models which has recently been suggested by the authors (Lewis and Lawrance, 1983). The thrust of this analysis is that the standard process of fitting and validating a linear autoregressive model is carried out beyond the customary final stage at which uncorrelated residuals are obtained (as in the previous section). The usual presumption is that the residuals are not only uncorrelated but also independent. This need not be the case, as will be exemplified for the wind velocity data. Moreover, dependent but uncorrelated residuals are obtained (Lewis and Lawrance, 1983) even for the NEAR(2) process. Thus the residuals should be subjected to further analysis in respect of their remaining dependency. Any found is then evidence that a standard linear, constant coefficient second-order autoregressive model is deficient. With normally distributed time series data this might suggest that non-linear modelling should be explored. With data marginally distributed in some other identifiable manner, then the exploration of a particular type of model with specified marginal distribution and autocorrelation function is suggested. This latter course is envisaged here.

Higher order dependency properties of the uncorrelated residuals are obtained for the proposed model and compared with their data counterparts; this stage can be informative from both exploratory and estimation considerations, and can be thought of as part of the model-refinement process common to much statistical methodology.

It might be thought that the specific class of NEAR(2) models could be incorporated in a residual analysis in the standard manner. However, a moments reflection indicates that even after estimating parameters of (2.2), it will not be possible, because of the mixture involved, to write down an expression or recursion for the residual innovations, namely E_n. However, the corresponding autoregressive (or linearized) residual is available and given, as in the previous section, by

$$R_n = X_n - a_1 X_{n-1} - a_2 X_{n-2} . \tag{7.1}$$

We now show that these are uncorrelated for the NEAR(2) process.

7.2. The residual theorem.

The autocovariances of the residuals (7.1) are

$$\begin{aligned} \mathrm{Cov}(R_n, R_{n+\ell}) &= \mathrm{Cov}(X_n, R_{n+\ell}) - a_1 \mathrm{Cov}(X_{n-1}, R_{n+\ell}) - a_2 \mathrm{Cov}(X_{n-2}, R_{n+\ell}) \\ &= \mathrm{Cov}(X_n, R_{n+\ell}) - a_1 \mathrm{Cov}(X_n, R_{n+\ell+1}) - a_2 \mathrm{Cov}(X_n, R_{n+\ell+2}) , \end{aligned} \tag{7.2}$$

since the $\{X_n\}$ process and consequently the $\{R_n\}$ process is stationary. The covariances on the right hand side are all of the same type and given by

$$\begin{aligned} \mathrm{Cov}(X_n, R_{n+\ell}) &= \mathrm{Cov}\{X_n, (X_{n+\ell} - a_1 X_{n+\ell-1} - a_2 X_{n+\ell-2})\} \\ &= \{\mathrm{Var}(X)\}(\rho(\ell) - a_1 \rho(\ell-1) - a_2 \rho(\ell-2)), \quad \ell = 1,2,... \end{aligned} \tag{7.3}$$

By the Yule-Walker equations (5.4) the expression in brackets is zero, and hence also

$$\text{Corr}(R_n, R_{n+\ell}) = 0 , \qquad \ell = \pm 1, \pm 2,... \tag{7.4}$$

as was to be proved. That these residuals are uncorrelated is an immediate consequence of the autocorrelations following Yule-Walker equations; this emphasizes that this type of residuals will be uncorrelated for <u>any</u> model whose autocorrelations satisfy Yule-Walker equations.

The analysis of the uncorrelated residuals $R_n, n = 3,4,...$ should begin with scatter plots of the low lag adjacent values; any patterns or concentrations will be evidence of dependency in the residuals. Many possibilities present themselves but only one is pursued in the following Section 8, and then applied in Section 9 to a continued analysis of the wind velocity data.

8. Cross-correlation analysis of $\{R_n\}$ and $\{R_n^2\}$

After the satisfactory fit to data of an ordinary linear model, the residual, R_n, should not only be uncorrelated but also independent; the latter is customarily investigated by seeking a flat spectrum, while for the independence, a flat spectrum of the squared residuals can be sought. As a method for probing model validity, the examination of squared residuals has been employed by McLeod and Li (1983), following Cranger and Andersen (1978); these authors suggested bilinear modelling for dependent but uncorrelated residuals of ARMA models. It is suggested here that autocorrelations of squared residuals and cross correlations of residuals of squared residuals be used in the analysis of higher order dependence of the detrended transformed wind speed data and its modelling by the NEAR(2) process. The residuals for this data have already been shown to have a flat spectrum (Figure 6.4) while the curved plot in Figure 8.1 for the cumulative periodogram shows that the spectrum of the squared residuals is far from flat.

Theoretical investigation of the squared residuals of the NEAR(2) model is pursued here. Whilst the autocorrelations of the squared residuals have just been mentioned, for the NEAR(2) model this involves computation of 36 terms, mostly distinct types of $4^{\underline{th}}$ order moments. A simpler suggestion which involves only third-order moments, and is thus the next step up after autocorrelations, is to use the cross correlations of the R_n and R_n^2 sequences; apart from lag 0, zero values will be found for linear models. This suggestion is more tractable than the squared residual analysis and will now be described. Sampling properties of third-order moments are also likely to be less extreme than those of fourth-order moments.

The starting point of the calculation is to note that from the definition of R_n at (7.1) that

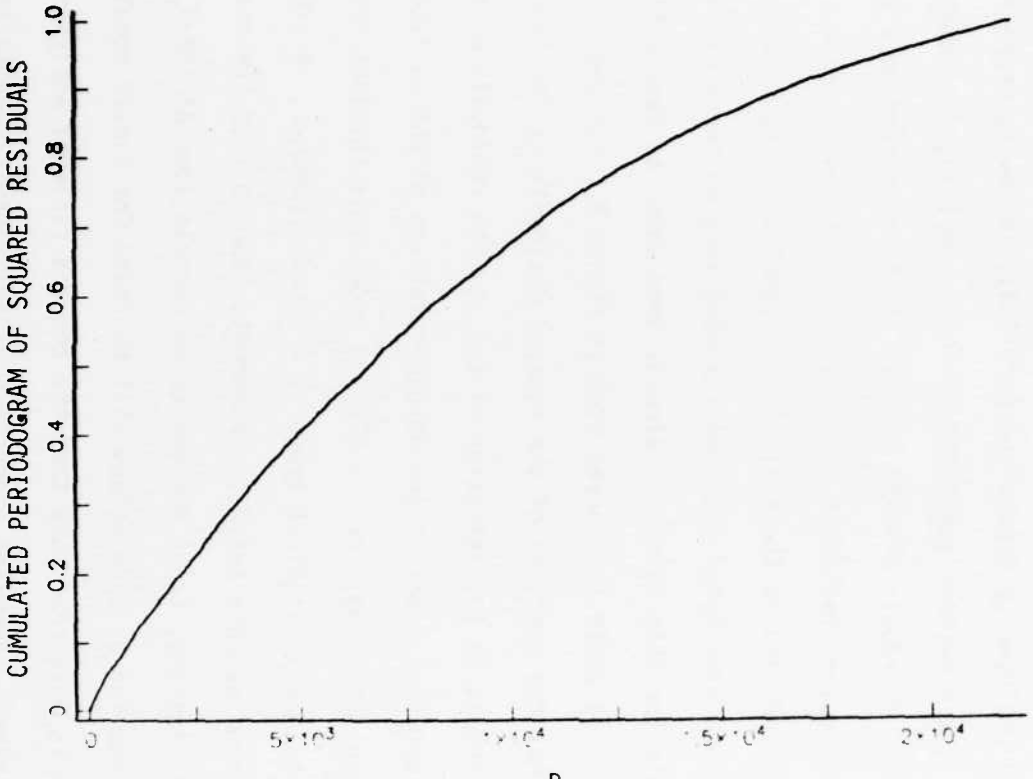

Figure 8.1. Cumulated periodogram values for the squared-residuals, R_n^2, of the transformed, detrended wind-speed data. The underlying spectrum is clearly not flat; value of the Kolmogorov-Smirnov test statistic is 3.08 which is very high compared to the upper .99 quantile of the statistic.

$$E(R_n^2 R_{n-\ell}) = E(R_n^2 X_{n-\ell}) - a_1 E(R_n^2 X_{n-\ell-1}) - a_2 E(R_n^2 X_{n-\ell-1}) \ , \qquad (8.5)$$

whence there is the structural form,

$$\mathrm{Cov}(R_n^2, R_{n-\ell}) = \mathrm{Cov}(R_n^2, X_{n-\ell}) - a_1 \mathrm{Cov}(R_n^2, X_{n-\ell-1}) - a_2 \mathrm{Cov}(R_n^2, X_{n-\ell-2}) \ . \qquad (8.6)$$

Calculation of the covariance terms in (8.6) requires the expanding out of R_n^2, taking expectations and expression in covariance terms. Thus

$$\begin{aligned} R_n^2 X_{n-\ell} = {} & X_n^2 X_{n-\ell} + a_1^2 X_{n-1}^2 X_{n-\ell} + a_2^2 X_{n-2}^2 X_{n-\ell} \\ & - 2a_1 X_n X_{n-1} X_{n-\ell} - 2a_2 X_n X_{n-2} X_{n-\ell} \\ & + 2a_1 a_2 X_{n-1} X_{n-2} X_{n-\ell} \ . \end{aligned} \qquad (8.7)$$

The conversion to covariances yields

$$\begin{aligned} \mathrm{Cov}(R_n^2, X_{n-\ell}) = {} & J_1(\ell) + a_1^2 J_1(\ell-1) + a_2^2 J_1(\ell-2) \\ & - 2a_1 J_2(\ell) - 2a_2 J_3(\ell) + 2a_1 a_2 J_2(\ell-1) \ , \end{aligned} \qquad (8.8)$$

where

$$J_1(\ell) = \mathrm{Cov}(X_n^2, X_{n-\ell}) \ ; \ J_2(\ell) = \mathrm{Cov}(X_n X_{n-1}, X_{n-\ell}) \ ,$$

$$J_3(\ell) = \mathrm{Cov}(X_n X_{n-2}, X_{n-\ell}) \ . \qquad (8.9)$$

We thus see the types of third-order joint moments which are involved in the $\mathrm{Cov}(R_n^2, R_{n-\ell})$ calculation: each of these has to be obtained using two difference equations, one for $\ell > 0$ and one for $\ell < 0$.

Taking $J_1(\ell)$ for illustration, square each side of the NEAR(2) defining equation (2.2) and multiply by $X_{n-\ell}$. After converting to the required covariances, the recursion is found to be

$$J_1(\ell) = a_1 \beta_1 J_1(\ell-1) + a_2 \beta_2 J_1(\ell-2) + 2(1-a_1 - a_2)\rho(\ell) \ . \qquad (8.10)$$

For negative ℓ there is the corresponding equation

$$J_1(\ell) = a_1 J_1(\ell+1) + a_2 J_1(\ell+2) \ , \qquad 8.11)$$

obtained by multiplying each side of (2.2) by $X^2_{n-\ell}$ and converting to covariances. These equations are given for illustration: there are similar equations for $J_2(\ell)$ and $J_3(\ell)$, and various special cases. Also required is the variance of R^2_n which involves many terms.

The complete algorithm for computing the cross-covariances is as follows.

Algorithm for computing cross-covariances for $\ell = 0, \pm 1, \ldots \pm L$.

0. Input a_1, a_2, β_1, β_2 .
   Set $\rho(0) = 1$; $\rho(1) = a_1/(1-a_2)$; $\rho(2) = a_1\rho_1 + a_2$ and, for $\ell = 3,\ldots,K$, $\rho(\ell) = a_1\rho(\ell-1) + a_2\rho(\ell-2)$. Note that $\rho(-\ell) = \rho(\ell)$ but these are not needed in computations.

1. Set $J_1(0) = 4$; $J(1) = [4a_1(\beta_1+a_2\beta_2) + 2(1-a_1-a_2)]/(1-a_2^2\beta_2)$.

2a. $J_1(\ell) = a_1 J_1(\ell+1) + a_2 J_1(\ell+2)$ for $\ell = -1, -2,\ldots, -(L+4)$.

2b. $J_1(\ell) = a_1\beta_1 J_1(\ell-1) + a_2\beta_2 J_1(\ell-2) + 2(1-a_1-a_2)\rho(\ell)$ for $\ell = 2,3,\ldots L$.

3. Set $J_2(0) = J_1(1) - \rho(1) + 1$; $J_2(1) = J_1(-1) - \rho(1) + 1$.

4a. $J_2(\ell) = a_1 J_2(\ell+1) + a_2 J_2(\ell+2)$ for $\ell = -1,-2,\ldots, -(L-3)$.

4b. $J_2(\ell) = a_1 J_1(\ell-1) + a_2 J_2(\ell-1) + 2a_1 + (1+\rho(1))a_2$
$-[1+\rho(1)] + [1 + \rho(\ell-1)](1-a_1-a_2)$ for $\ell = 2,3,\ldots, L$.

5. Set $J_3(0) = J_1(2) - \rho(2) + 1$; $J_3(1) = J_2(2) + \rho(1) - \rho(2)$; $J_3(2) = J_1(-2) - \rho(2) + 1$.

6a. $J_3(\ell) = a_1 J_3(\ell+1) + a_2 J_3(\ell+2)$ for $\ell = -1, -2,\ldots, -(L+2)$.

6b. $J_3(\ell) = a_1 J_2(\ell-1) + a_2 J_1(\ell-2) + [1+\rho(1)]a_1 + 2a_2$
$+ [1+\rho(\ell-2)](1-a_1-a_2) - (1+\rho(2))$ for $\ell = 3,4,5,\ldots, L$.

7. $J(\ell) = J_1(\ell) + a_1^2 J_1(\ell-1) + a_2^2 J(\ell-2) - 2a_1 J_2(\ell)$
$- 2a_2 J_3(\ell) + 2a_1 a_2 J_2(\ell-1)$ for $\ell = -(L+2), \ldots, (L)$.

8. Cross-covariance $(R_n^2, R_{n-\ell}) = J(\ell) - a_1 J(\ell+1) - a_2 J(\ell+2)$
for $\ell = -L, \ldots, L$.

9. Further Analysis and Modelling of the Wind Velocity Data

Dependence in the uncorrelated linear residuals for the NEAR(2) model has already been demonstrated (Figure 8.1). Further evidence of this is provided by the cross-correlations of R_n and R_n^2 given in Figure 9.1. The corresponding theoretical cross-correlations for the NEAR(2) model will next be presented, having been computed using the algorithm given in Section 8. At this point, it will be recalled, the NEAR(2) model has not been fitted in terms of all 4 parameter; the residuals involve the model parameter only through $a_1 = \alpha_1\beta_1$ and $a_2 = \alpha_2\beta_2$ and $\alpha_1,\beta_1,\alpha_2,\beta_2$ have not been separately estimated. In the present rather exploratory analysis the estimation problem will be circumvented; the cross-correlations of (R_n,R_n^2) will be given for four representative sets of parameter values in the reduced allowable region, as constrained by $(\alpha_1 \geq a_1,\ \alpha_2 \geq a_2,\ \alpha_1 + \alpha_2 \leq 1)$, according to (5.6). For the detrended transformed wind speed data, $a_1 = 0.75488$ at $a_2 = 0.05463$, and the four chosen sets are given by

(A) $\alpha_1 = 0.760,\ \alpha_2 = 0.06;\ \beta_1 = 0.99326,\ \beta_2 = 0.91050$

(B) $\alpha_1 = 0.925,\ \alpha_2 = 0.06;\ \beta_1 = 0.81608,\ \beta_2 = 0.91050$

(C) $\alpha_1 = 0.760,\ \alpha_2 = 0.20;\ \beta_1 = 0.99326,\ \beta_2 = 0.27315$

(D) $\alpha_1 = 0.850,\ \alpha_2 = 0.10;\ \beta_1 = 0.88809,\ \beta_2 = 0.54636.$

In Figure 9.2 the cross-correlations of (R_n,R_n^2) for each of these cases are presented; there is considerable differentiation amongst the figures, although the zero values for all but one or two negative lags are evident in all four cases. This effect will be due to the directional nature of the process. For matching with the wind speed data, Case D appear the most promising, although the suggestion of a negative correlation at lag minus one lends a little support to Case A . Cases B and C are

definitely unsympathetic to the data. Hence, a choice of parameters intermediate between Cases A and D is suggested by this exploratory analysis. A fuller analysis would require estimates of all four parameters and comparison of the resulting residual-squared residual cross-correlations with the estimated cross-correlations obtained from the data, as shown in Figure 9.1.

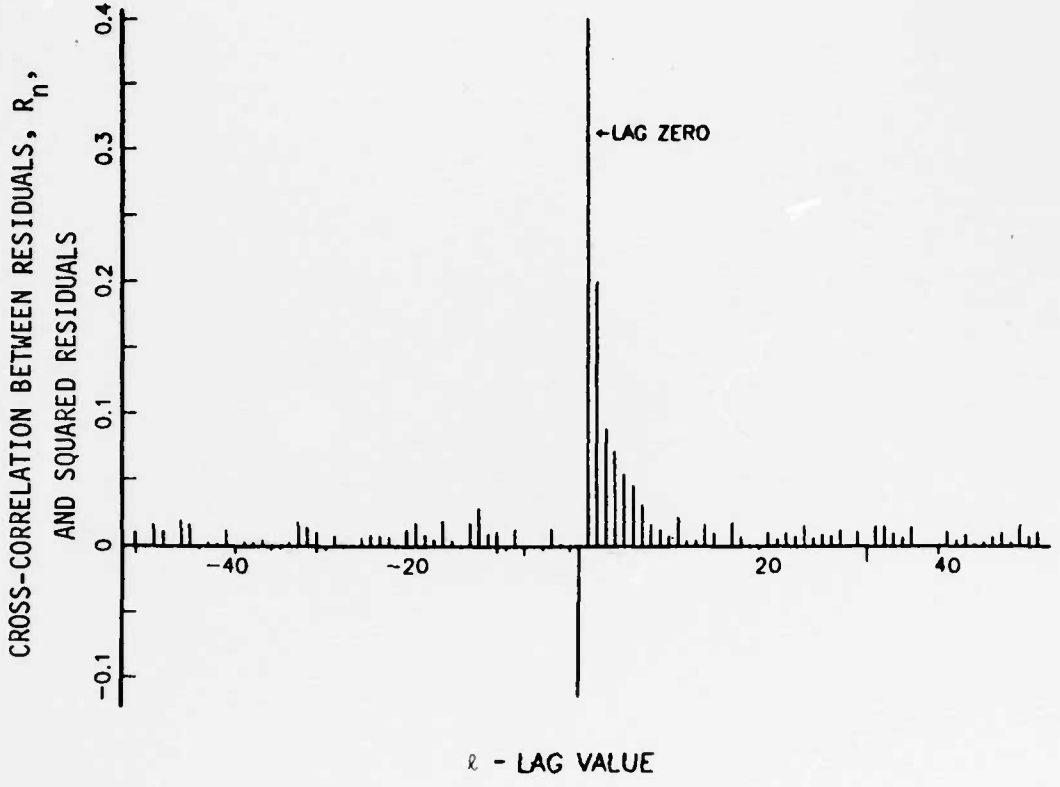

Figure 9.1. Cross-correlation between the residuals, R_n, and the squared residuals, R_n^2, for the transformed detrended wind-speed data. Sample size is N = 43,800. There is a significant negative value for lag one, and significant positive values from the first positive negative lags.

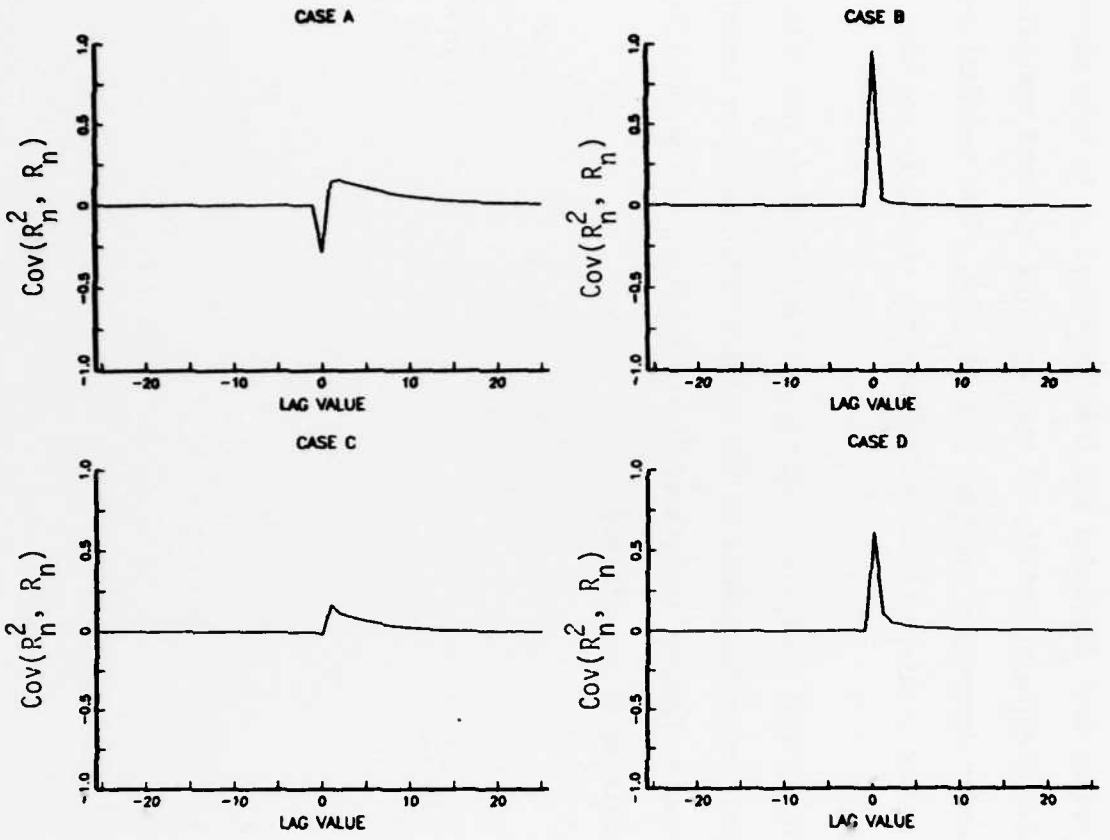

Figure 9.2. Theoretical crosscovariances for NEAR(2) residual sequences for $a_1 = 0.75488$ and $a_2 = 0.05463$ for values of the pair (α_1,α_2). Since the parameter a_2 is small, the process departs only slightly from a first-order autoregression and in Case B it is almost the (linear)EAR(1) process; hence the spike almost only at lag zero, which measures skewness of the marginal distribution of the residuals in a linear process.

10. Conclusions and Further Analysis

The very broad four parameter NEAR(2) time series model having exponential marginals and the correlation structure of an AR(2) model has been established; further developments will be reported elsewhere. A preliminary fit of the NEAR(2) model has been made to a very long series of wind speed data, the data having been detrended and transformed so as to have exponentially distributed marginals; utility of the suggested residual analysis in probing higher order dependence has been demonstrated. This residual analysis has been based on the cross-correlations between the residuals and the squared-residuals.

An extension to this analysis using reversed residuals is possible; more of the higher order dependency of the NEAR(2) model would be revealed and this would enable further aspects of its suitability of the model for the wind speed data to be ascertained.

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