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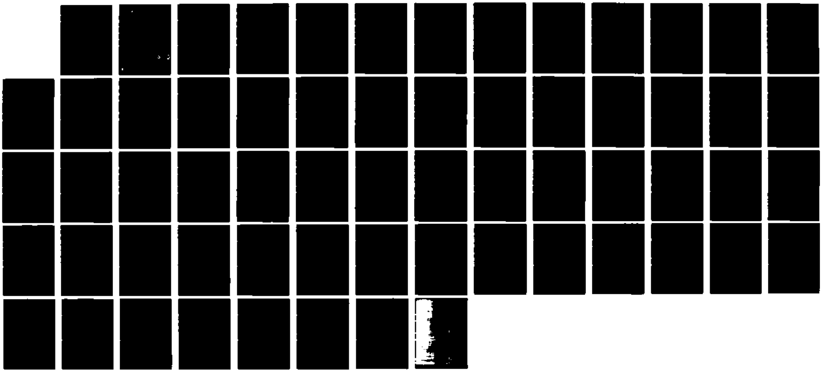

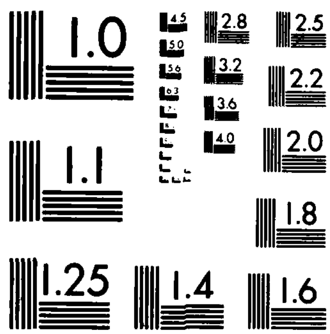
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AN ANALYSIS OF A FINITE ELEMENT METHOD FOR CONVECTION-DIFFUSION PROBLEMS

PART I: QUASI-OPTIMALITY

by

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and

I. Babuška

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An Analysis of a Finite Element Method for Convection-Diffusion Problems.

Part I: Quasi-Optimality.[*]

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Laboratory for Numerical Analysis

Technical Note BN-1001

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Abstract

A detailed analysis is performed for a finite element method applied to the general one-dimensional convection diffusion problem. Piecewise polynomials are used for the trial space. The test space is formed by locally projecting L-spline basis functions onto "upwinded" polynomials. The error is measured in the L_p mesh dependent norm. The method is proven to be quasi-optimal (yielding nearly the best approximation from the trial space), provided that the input data is piecewise smooth. This assumption is usually observed in practice. These results are used to establish a posteriori error estimates and an adaptive mesh refinement strategy in Part II of this series (35).

CHAPTER 1

INTRODUCTION

Reliable numerical solutions to singularly perturbed boundary value problems are of great importance in engineering. Because of the degenerative nature of these problems, conventional numerical methods produce approximations whose optimality degenerates as well.

In this paper we consider the model problem

$$
\begin{aligned}
(1.1) \qquad & -\varepsilon u'' + a(x)u' + b(x)u = f(x), \quad \text{in} \quad (0,1), \\
& u(0) = \alpha, \\
& \beta_1 u'(1) + \beta_2 u(1) = \beta,
\end{aligned}
$$

where

$u = u(x)$ is the solution and may measure, for example, temperature or concentration, $\varepsilon > 0$ is the diffusivity of $u(x)$, $a(x) \geq \underline{a} > 0$ is the velocity of the medium carrying $u(x)$, $b(x)$ is the coefficient for $u(x)$ used to represent a "loss" if $b(x) > 0$, or a "source" if $b(x) < 0$, and $f(x)$ is the external source term. The order of this problem degenerates from two to one as $\varepsilon \to 0$, and particular interest will be focused on the case when ε is small.

If $a(x) \geq \underline{a} > 0$ and the ratio $\underline{a}/\varepsilon$ is large-in practice this ratio may be as large as 10^8, the solution to (1.1) will often exhibit boundary layer behavior. A boundary layer can be loosely defined as a small region near the boundary where the solution changes rapidly. If the function f is "rough", the solution may also have interior layers where the solution changes rapidly near some points $\{x_i\} \in (0,1)$. In cases when a, b and f are smooth, the solution will generally behave like $e^{a(1)(x-1)/\varepsilon}$ near $x = 1$ (see e.g. [10], [15] or [36]).

Much attention has recently been focused on the application of finite element methods to problems of this type. Because conventional finite element methods employ piecewise polynomial trial and test spaces, one problem is the ability of these polynomials to approximate the exact solution well, particularly in the boundary layer region. However, the most serious problem by far is the loss of stability or quasi-optimality with these conventional methods.

A finite element solution u_h is called quasi-optimal in the norm $\|\cdot\|$ if there exists a C, independent of ε and the mesh, such that

$$\|u-u_h\| \leq C \inf_{w\in S_h} \|u-w\|, \tag{1.2}$$

where S_h is the trial space from which the approximation u_h to the exact solution u is taken. Whenever (1.2) holds we are guaranteed to have nearly the best approximation from the trial space S_h. With conventional methods, however, the constant C in (1.2) becomes unbounded as $\varepsilon \to 0$.

This loss of quasi-optimality can be seen numerically by considering the following problem:

$$\begin{aligned} -\varepsilon u'' + u' &= f \quad \text{in } (0,1) \\ u(0) = u(1) &= 0, \end{aligned} \tag{1.3a}$$

with

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1/3, \\ 0 & \text{if } 1/3 < x < 1,\ x \neq 2/3, \\ \delta(x-2/3) & \text{if } x = 2/3, \end{cases} \tag{1.3b}$$

where $\delta(x-2/3)$ is the Dirac delta function representing a point source at x = 2/3. Besides having a boundary layer at x = 1, this problem has an interior layer at x = 2/3. The exact and conventional piecewise linear finite element solutions are shown in Figure 1.1 with ε = .0001, and N = 24 elements. The loss of quasi-optimality expresses itself in the form of spurious oscillations of the finite element approximation.

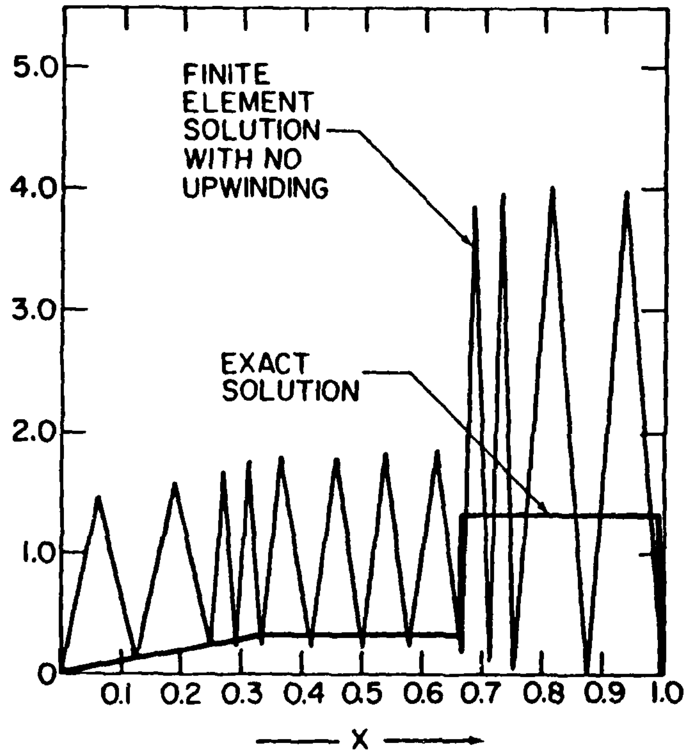

FIGURE 1.1: The exact and conventional finite element solutions to problem (1.3).

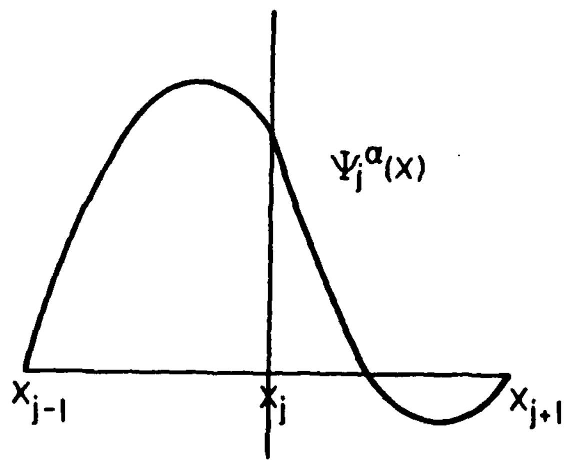

FIGURE 1.2: An α-quadratic upwinded basis function.

The most common way to alleviate the problem of oscillations is to "upwind" the test space. In the case of linear elements this is done by adding a quadratic term, multiplied by some parameter α, to each linear basis function of the test space (see [8], [9], [17]-[21], [30], or , [37]). This procedure will be referred to as α-quadratic upwinding throughout this paper. A typical basis function ψ_j^α upwinded in this way is displayed in Figure 1.2.

In all of the papers mentioned in the preceding paragraph, the criteria used for the selection of α was either to eliminate oscillations, or to produce exact nodal solutions for the model problem (1.3a). For example, Christie, et. al. [8], Heinrich, et. al. [19]-[21], Mitchell, et. al. [30] and Zienkiewicz, et. al. [37] have displayed the "optimal" α which produces the exact nodal values for problem (1.3a), when $f(x) \equiv 1$.

It is pointed out by Gresho and Lee [16] that "ad hoc" upwinding can be deceptive to the analyst by smoothing out the results, and any solution obtained by upwinding does not represent a solution to the original problem. Instead of upwinding, they advocate the use of conventional finite element methods, and propose to use the information given by the oscillations to refine and/or relocate the mesh points in the areas where any "wiggles" occur.

Although we agree that the upwinding criteria of damping the oscillations is incorrect, the oscillations themselves may be misleading in determining how the mesh refinement should proceed. The approximation to problem (1.3a,b) displayed in Figure 1.1 would mislead one into refining the mesh everywhere.

The criteria for α-quadratic upwinding should not be to eliminate oscillations but should be to obtain quasi-optimality as in (1.2) with the constant C independent of ε, and the mesh spacing h. In [18], Griffiths and Lorenz attempted to select α in a way to minimize this constant C. Unfortunately, even with α chosen in this way, if $\varepsilon \ll h$ their value of C increased with rate

h^{-1} as $h \to 0$.

In [3], we have proven that quasi-optimality is attainable for problem (1.3a) (in a norm different from the one studied by Griffiths) with σ-quadratic upwinding if and only if the input data f is piecewise smooth - a reasonable assumption in practice.

Upwinding can also be done with the use of L-spline basis functions. Methods using these spaces have been studied by Hemker and De Groen ([11], [12], [22]) who prove a-priori estimates of the error at the nodes and in the norm

$$(1.4) \qquad ||u||^2_{1,\varepsilon} = \varepsilon||u'||^2_{L_2} + ||u||^2_{L_2} \ .$$

In this norm, however, the error arising from any piecewise linear approximation cannot be made small unless $h < \varepsilon$. They also propose to upwind the trial space in order to get a better fit to the exact solution in the boundary layer region. However, this upwinding can introduce spurious internal layers in the approximation.

Also, whenever the norm (1.4) is used, the assumption $a(x) \geq \underline{a} > 0$ and the additional assumptions

$$(1.5) \qquad b(x) \geq 0, \text{ and } b(x) - \frac{1}{2} a'(x) \geq \gamma > 0,$$

are needed to prove coercivity of the bilinear form used to pose (1.1) variationally. When $a(x) \geq \underline{a} > 0$, a much weaker additional assumption, namely -

$$(1.6) \qquad \text{if} \quad b(x) > \underline{b} \quad \text{then } \underline{a}^2 + 4\varepsilon\underline{b} \geq \gamma > 0,$$

is sufficient to guarantee that zero is not an eigenvalue for (1.1) and hence the solution will be unique. This assumption together with some smoothness of a, b and f, are sufficient for the results in this paper. Note that in (1.6) $\underline{b}$ and hence b(x) may be negative. Furthermore, no condition on the expression $b(x) - \frac{1}{2} a'(x)$ is required.

Since the L-spline basis functions are exponential and have boundary layers themselves, a special quadrature rule must be devised in order to perform the integrations needed to assemble the matrix equations with sufficient accuracy. Diaz-Munio and Wellford [13] use exponentially upwinded basis functions which are local asymptotic expansions of the solution, and describe a special numerical quadrature rule which is exact for integrands of the form $t^n e^{at}$.

Kellogg and Han [27] present a scheme in which they add one singular function of boundary layer type to both the test and trial spaces. Using this method they were able to prove the error estimate

$$\|u - u_h\|_{1,\varepsilon} \leq Ch, \tag{1.7}$$

where C is bounded independently of ε and h, and $\|\cdot\|_{1,\varepsilon}$ is defined in (1.4). This method presumes a-priori knowledge of the location of the boundary layer. For example, in order to solve problem (1.3) which also has an interior layer at $x = 2/3$, some modifications must be made in their algorithm.

In this paper we develop a finite element method which produces a quasi-optimal approximation to (1.1) for all values of $\varepsilon \in (0,1]$. The norm used to measure the error is closely related to the L_p norm. An L_p (and in particular an L_1) type norm is appropriate especially when the location of the boundary or interior layers are of importance. In the second paper of this series: Part II - A-Posteriori Error Estimates and Adaptivity, the numerical results presented are based on the L_1 type norm.

The norms, spaces, and bilinear form used to pose (1.1) variationally are presented in Chapter 2. Chapters 3 and 4 show that a basis for an exponentially upwinded test space can be found, which produces a quasi-optimal approximation. In Chapter 5 these exponential basis functions are projected onto a polynomially upwinded test space. This type of upwinding is a generalization of α-quadratic upwinding. Finally, in Chapter 6 some remarks are made on the extension of these results to more general problems.

CHAPTER 2

MATHEMATICAL FRAMEWORK

This chapter sets up the mathematical framework in which the convection-diffusion problem (1.1) will be studied. First, two theorems are presented which are used to prove existence for variationally posed problems, and quasi-optimality for finite dimensional approximations. Next some results concerning the Green's function to (1.1) are proven. After the norms and spaces needed to pose (1.1) variationally are provided, the Green's function results are used to prove some important embedding theorems.

SOME ABSTRACT RESULTS

Two crucial results concerning variationally formulated boundary value problems and finite element approximations are given in this section.

<u>Theorem 2.1.</u> Let $K_{1,\Delta}$ and $K_{2,\Delta}$ be two reflexive Banach spaces, indexed by a parameter Δ with Δ varying over some index set, with norms $||\cdot||_{1,\Delta}$ and $||\cdot||_{2,\Delta}$ respectively, and let B_Δ be a bilinear form on $K_{1,\Delta} \times K_{2,\Delta}$. We suppose the following are satisfied:

(2.1) $$|B_\Delta(u,v)| \leq C_1 ||u||_{1,\Delta} ||v||_{2,\Delta} \text{ for all } u \in K_{1,\Delta},\ v \in K_{2,\Delta},$$

(2.2) $$\inf_{\substack{u\in K_{1,\Delta} \\ ||u||_{1,\Delta}=1}} \ \sup_{\substack{v\in K_{2,\Delta} \\ ||v||_{2,\Delta}=1}} |B_\Delta(u,v)| \geq C_2 > 0,$$

and

(2.3) $$\sup_{u\in K_{1,\Delta}} |B_\Delta(u,v)| > 0, \quad \text{for each } 0 \neq v \in K_{2,\Delta},$$

where C_1 and C_2 are positive constants, possibly depending on Δ. Then if $f \in (K_{2,\Delta})'$, there exists a unique solution $u \in K_{1,\Delta}$ to the problem

$$B_\Delta(u,v) = f(v) \qquad \text{for each} \qquad v \in K_{2,\Delta} .$$

Moreover, u satisfies

$$||u||_{1,\Delta} \leq C_2^{-1} ||f||_{K'_{2,\Delta}} .$$

If the bilinear form $B_\Delta(.,.)$ satisfies the assumptions (2.1), (2.2) and (2.3), B_Δ is said to be a (C_1, C_2)-proper bilinear form over the space $K_{1,\Delta} \times K_{2,\Delta}$. It should be noted that (2.2) and (2.3) can be shown to be equivalent to

$$(2.2)^* \qquad \inf_{\substack{v \in K_{2,\Delta} \\ ||v||_{2,\Delta} = 1}} \quad \sup_{\substack{u \in K_{1,\Delta} \\ ||u||_{1,\Delta} = 1}} |B_\Delta(u,v)| \geq C_2^* > 0,$$

and

$$(2.3)^* \qquad \sup_{v \in K_{2,\Delta}} |B_\Delta(u,v)| > 0, \qquad 0 \neq u \in K_{1,\Delta} .$$

This observation will be specifically used in this paper.

Since we will be studying finite element approximations to u, we let $S_{1,\Delta}$ and $S_{2,\Delta}$ be finite dimensional subspaces of $K_{1,\Delta}$ and $K_{2,\Delta}$ respectively. Clearly, condition (2.1) holds on $S_{1,\Delta} \times S_{2,\Delta}$ with the same constant C_1. We will be invoking the following theorem concerning the finite element solution u_h.

<u>Theorem 2.2.</u> Suppose B_Δ is (C_1, C_2') - proper over $S_{1,\Delta} \times S_{2,\Delta}$ furnished with norms $||\cdot||_{1,\Delta}$ and $||\cdot||_{2,\Delta}$, respectively. Let $u \in K_{1,\Delta}$, and let $u_h \in S_{1,\Delta}$ be the unique solution to $B_\Delta(u_h,v) = B_\Delta(u,v)$ for all $v \in S_{2,\Delta}$.

Then

$$||u - u_h||_{1,\Delta} \leq \left(1 + \frac{C_1}{C_2'}\right) \inf_{w \in S_{1,\Delta}} ||u - w||_{1,\Delta} .$$

For the proof of Theorems 2.1 and 2.2, see e.g. [1].

We remark that the reflexivity of the space $K_{1,\Delta}$ is not necessary for Theorem 2.2 to be valid - the reflexivity is needed only on the spaces $S_{1,\Delta}$ and $S_{2,\Delta}$, which is guaranteed since they are finite dimensional. However, Theorem 2.1 will not imply the existence of a solution $u \in K_{1,\Delta}$, if $K_{1,\Delta}$ and $K_{2,\Delta}$ are not reflexive. Therefore, if $K_{1,\Delta}$ and $K_{2,\Delta}$ are not reflexive, the solution u must be assumed or shown to exist in $K_{1,\Delta}$ by some other method, in order to apply Theorem 2.2.

THE GREEN'S FUNCTION

Consider the operator L defined by $Lu \equiv -\varepsilon u'' + a(x)u' + b(x)\,u$, for $u \in C^2[0,1]$, where $C^k[0,1]$ denotes the space of functions with k continuous derivatives on [0,1]. Let $G(x|y)$ denote the classical Green's function for the operator L with boundary conditions:

$$\text{(2.4)} \qquad \begin{aligned} &G(0|y) = 0 \\ &\Gamma(G(x|y)) \equiv \beta_1 G_x(1|y) + \beta_2\, G(1|y) = 0 \quad . \end{aligned}$$

The following lemma will be used to establish the existence of this function.

<u>Lemma 2.3.</u> Assume $a(x) \in C^1[0,1]$ and $b(x) \in C^0[0,1]$. If there exists a positive function $w \in C^2[0,1]$ which satisfies

$$\text{(2.5)} \qquad \begin{aligned} Lw &\geq 0, \\ w(0) &> 0, \\ w &\geq 0, \end{aligned}$$

then the Green's function $G(x|y)$ to L exists, is unique, and is non-negative.

<u>Proof:</u> This lemma follows from the maximum principle (see e. g. [32], Chapter 1, Theorem 1.1), and standard results concerning the Green's function (see e.g. [33], Sections 1.3 and 1.5). A detailed proof of this lemma can be found in [34], (Lemma 2.5).

In order to find a function w satisfying (2.5) we make the following assumption for the operator L.

A1:
$$a(x) \in C^1[0,1], \quad a(x) \geq \underline{a} > 0$$
$$b(x) \in C^0[0,1], \quad b(x) \geq \underline{b} \quad \text{and}$$
$$\underline{b} \text{ is such that } \underline{a}^2 + 4\varepsilon\underline{b} = \gamma > 0.$$

For the boundary operator Γ we assume

A2:
$$\beta_1, \beta_2 \geq 0, \quad \beta_1 + \beta_2 > 0.$$

The following is a corollary to Lemma 2.3.

Corollary 2.4. Suppose assumptions A1 and A2 hold. Then the Green's function to L, satisfying (2.4), exists, is unique, and is non-negative.

Proof: Let $\sigma = \frac{1}{2\varepsilon}\left(\underline{a} + (\underline{a}^2 + 4\varepsilon\underline{b})^{\frac{1}{2}}\right) > 0$ and $w(x) = e^{\sigma x}$. Then w satisfies (2.5).

An important fact used in this paper is that the Green's function is bounded independently of ε. The proof of the following theorem which establishes this fact is similar to one found in Lorenz [29].

Theorem 2.5. Suppose assumptions A1 and A2 hold. Then the Green's function $G(x|y)$ for L, which satisfies (2.4), is bounded by a constant which is independent of x, y and ε.

Proof: Let $\hat{G}(x|y)$ be the Green's function for $\hat{L}$ where

$$\hat{L}w \equiv -\varepsilon w'' + a(x)w',$$

with boundary conditions

$$w(0) = 0$$
$$\beta_1 w'(1) + \beta_2 w(1) = 0.$$

Then $\hat{G}(x|y)$ is given by

$$(2.6)\qquad \hat{G}(x|y) = \begin{cases} \left[\dfrac{\beta_1 p(1) + \beta_2 \int_y^1 p(t)dt}{\varepsilon p(y)\chi}\right] \int_0^x p(t)dt, & x \le y, \\[2ex] \dfrac{\beta_1 p(1) \int_0^y p(t)dt}{\varepsilon p(y)\chi} + \dfrac{\beta_2 \int_0^y p(t)dt}{\varepsilon p(y)\chi} \int_x^1 p(t)dt, & x \ge y, \end{cases}$$

where $p(t) = \exp(\frac{1}{\varepsilon}\int_0^t a(s)ds)$, and $\chi = \beta_1 p(1) + \beta_2\int_0^1 p(t)dt$. By assumption A2, β_1, $\beta_2 \ge 0$, and $\beta_1 + \beta_2 > 0$, and hence $\chi > 0$. Therefore, from (2.6) we have

$$\hat{G}(x|y) \le \hat{G}(y|y) \le \frac{\int_0^y p(t)dt}{\varepsilon p(y)} = \frac{1}{\varepsilon}\int_0^y \exp(-\frac{1}{\varepsilon}\int_t^y a(s)ds)dt$$

$$\le \frac{1}{\varepsilon}\int_0^y e^{(t-y)\underline{a}/\varepsilon}dt \le (1 - e^{-\underline{a}/\varepsilon})/\underline{a} < 1/\underline{a}.$$

If $\underline{b} = 0$ then $b(x) \ge 0$. Let $w_y(x) = G(x|y) - \hat{G}(x|y)$.

Then

$$\hat{L}w_y(x) = -b(x)G(x|y) \quad \text{for each } x \in (0,1).$$

Hence, $w_y(x) = -\int_0^1 \hat{G}(x|\xi)b(\xi)\, G(\xi|y)d\xi$. Since both G and $\hat{G}$ are non-negative by Corollary 2.4, $w_y(x) \le 0$. Thus, $G(x|y) \le \hat{G}(x|y) \le 1/\underline{a}$.

If $\underline{b} < 0$, let $\sigma = \frac{1}{2\varepsilon}(\underline{a} - (\underline{a}^2 + 4\varepsilon\underline{b})^{1/2}) > 0$, and $L_\sigma w = e^{-\sigma x}L(e^{\sigma x}w) = -\varepsilon w'' + (a(x) - 2\varepsilon\sigma)w' + (-\varepsilon\sigma^2 + a(x)\sigma + b(x))w$. Since $-\varepsilon\sigma^2 + a(x)\sigma + b(x) \ge -\varepsilon\sigma^2 + \underline{a}\sigma + \underline{b} = 0$, and $a(x) - 2\varepsilon\sigma \ge \underline{a} - 2\varepsilon\sigma = (\underline{a}^2 + 4\varepsilon\underline{b})^{1/2} = \gamma^{1/2} > 0$, we are in the case when $\underline{b} = 0$ for the operator L_σ. Let $G_\sigma(x|y)$ be the Green's function for L_σ satisfying the boundary conditions $G_\sigma(0|y) = 0$,

$$\Gamma_\sigma G_\sigma \equiv \beta_1\frac{\partial G_\sigma}{\partial x}(1|y) + (\beta_2 + \sigma\beta_1)\, G_\sigma(1|y) = 0.$$

Since $\sigma > 0$, the boundary operator Γ_σ satisfies the analogous assumption as A2 was to the boundary operator Γ, and therefore, $G_\sigma(x|y) \le 1/\gamma^{1/2}$.

$G_\sigma(x|y)$ is related to $G(x\ y)$ through the identity $G(x|y) = e^{\sigma(x-y)}G_\sigma(x|y)$. Therefore, $G(x|y) \le e^{\sigma}/\gamma^{1/2}$. Now, $\sigma = \frac{\underline{a}}{2\varepsilon}(1 - (1 + 4\varepsilon\underline{b}/\underline{a}^2)^{1/2}) \le \frac{\underline{a}}{2\varepsilon}(1 - 1 + 4\varepsilon\underline{b}/\underline{a}^2)) = -2\underline{b}/\underline{a}$. Thus, $G(x|y) \le e^{-2\underline{b}/\underline{a}}/\gamma^{1/2}$.

NOTATIONS, BILINEAR FORMS, SPACES, AND NORMS

We now define the various norms, spaces, and bilinear forms used throughout this paper. The norms introduced here are analogous to those defined in [2].

The space $H_p^k(I)$, $k = 0,1,\ldots$, $1 \le p \le \infty$ is the usual Sobolev space on the interval $I = [0,1]$ consisting of functions with k derivatives in $L_p(I)$. On this space we have the usual norms given by

$$||u||_{H_p^k(I)} = \begin{cases} \left[\sum_{j=0}^{k} \int_I |u^{(j)}(x)|^p dx \right]^{1/p}, & 1 \le p \le \infty \\ \sum_{j=0}^{k} \text{ess. sup. } |u^{(j)}|, & p = \infty \end{cases}$$

The space $\overset{o}{H}{}_p^1(I)$ denotes the subspace of $H_p^1(I)$ of functions which vanish at the endpoints of I. This has sense because $H_p^1 \subsetneq C^0(\bar{I})$. Note that $H_p^0 = L_p$.

Let $\Delta = \{0 = x_0 < x_1 < \ldots < x_N = 1\}$, where $N = N(\Delta)$, be an arbitrary mesh on the interval $I = [0,1]$. Let $h_j = x_j - x_{j-1}$, $I_j = (x_{j-1}, x_j)$, $j = 1,\ldots,N$, $\rho_j = (h_j + h_{j+1})/2$, $j = 1,\ldots, N-1$, $\rho_N = h_N$, and $h = \max_j h_j$.

We seek a variational setting for the problem

$$\text{(2.7)} \qquad \begin{aligned} Lu &\equiv -\varepsilon u'' + a(x)u' + b(x)u = f \quad \text{in} \quad (0,1), \\ u(0) &= \alpha, \\ \Gamma u &\equiv \beta_1 u'(1) + \beta_2 u(1) = \beta, \end{aligned}$$

where the functions a(x), b(x) satisfy assumption A1, and β_1, β_2 satisfy assumption A2. These assumptions A1 and A2 will be assumed to hold throughout the entire paper, and will not be repeated before each theorem.

Let L* denote the formal adjoint operator to L, i.e.,

$$L^* \equiv -\varepsilon \frac{d^2}{dx^2} - a(x)\frac{d}{dx} + (b-a')(x). \tag{2.8}$$

The boundary operator adjoint to Γ is Γ^*, where for u sufficiently smooth

$$\Gamma^* u = \begin{cases} \left(\varepsilon \dfrac{\beta_2}{\beta_1} + a(1)\right) u(1) + \varepsilon u'(1), & \text{if } \beta_1 \neq 0, \\ u(1), & \text{if } \beta_1 = 0. \end{cases} \tag{2.9}$$

We must associate a bilinear form to L, and describe the spaces over which this form is defined.

First, we define the space $H^o_{p,\Delta}$, $1 \leq p \leq \infty$, to be the completion of

$$H_1 = \{u \in H^1_p(I): \ u(0) = 0, \quad u(1) = 0 \ \text{ if } \beta_1 = 0\},$$

with respect to the norm

$$\|u\|_{H^o_{p,\Delta}} = \begin{cases} \left[\displaystyle\int_0^1 |u|^p dx + \sum_{j=1}^{N_1} \rho_j |u(x_j)|^p\right]^{1/p}, & 1 \leq p < \infty, \\ \|u\|_{L_\infty(I)}, & p = \infty, \end{cases} \tag{2.10}$$

where $N_1 = \begin{cases} N-1, & \text{if } \beta_1 = 0 \\ N, & \text{if } \beta_1 \neq 0. \end{cases}$

The space $H^o_{p,\Delta}$ can be easily identified with $L_p \oplus R^{N_1}$, that is,

$u = (\tilde{u}, d_1, \ldots, d_{N_1}) \in H^o_{p,\Delta} = L_p \oplus R^{N_1}$, and

$$(2.11)\qquad \|u\|_{H^o_{p,\Delta}} = \begin{cases} [\|\tilde{u}\|^p_{L_p(I)} + \sum_{j=1}^{N_1} \rho_j |d_j|^p]^{1/p}, & 1 \le p < \infty, \\ \max_j [\|\tilde{u}\|_{L_\infty(I_j)}, |d_j|], & p = \infty. \end{cases}$$

In consistency with our definition, we say $u \in H^o_{p,\Delta} \cap H^1_p(I)$ if $\tilde{u} \in H^1_p(I)$ and $d_j = \tilde{u}(x_j)$ for $j = 1,\dots,N_1$.

Note that the norm $\|\cdot\|_{H^o_{p,\Delta}}$ is very close to the L_p norm. The term $\sum_{j=1}^{N_1} \rho_j |d_j|^p$ is the trapezoid quadrature rule for the function $|u(x)|^p$, when $u(x)$ is continuous and $d_j = u(x_j)$. Therefore, for any continuous function u $\lim_{h\to o}\|u\|_{H^o_{p,\Delta}} = 2^{1/p}\|u\|_{L_p}$. Because of the boundary layer behavior of the solutions, an L_p (particularly L_1) type norm is appropriate for measuring the errors. The quality of our approximations is measured in the $H^o_{p,\Delta}$ norm. In particular, the computational results and adaptivity presented in [35] are based on the $H^o_{1,\Delta}$ norm.

Let us also define $H^2_{q,\Delta} = \{v \in H^1_q(I)\colon v(0) = 0,\ v(1) = 0$ if $\beta_1 = 0$, and $v|_{I_j} \in H^2_q(I_j)$, $j = 1,\dots,N\}$, for $1 \le q \le \infty$. We will equip this space with a norm to be defined later.

On $H^o_{p,\Delta} \times H^2_{q,\Delta}$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p \le \infty$, we define a bilinear form $B_\Delta(\cdot,\cdot)$ by

$$(2.12)\qquad B_\Delta(u,v) = \sum_{j=1}^{N} \int_{I_j} \tilde{u}L^*v\,dx - \sum_{j=1}^{N-1} \varepsilon d_j \mathcal{J}(v'(x_j)) + d_N\Gamma^*(v),$$

where $\mathcal{J}(v'(x_j)) = v'(x_j+0) - v'(x_j-0)$ for $1 \le j \le N-1$, and $d_N = 0$ if $\beta_1 = 0$.

The limits $v'(x_j \pm 0)$ are well defined because $v|I_j \in H_q^2(I_j)$ for each j. Now we will furnish the space $H_{q,\Delta}^2$ with the norm $|||\cdot|||$, defined by

$$(2.13) \qquad |||v||| = \sup_{u \in H_{p,\Delta}^o} \frac{|B_\Delta(u,v)|}{\|u\|_{H_{p,\Delta}^o}} .$$

In order to verify that $|||\cdot|||$ is indeed a norm, we must show positive definiteness - linearity and the triangle inequality are evident. To prove positive definiteness we will use the identity

$$(2.14) \qquad v(y) = \sum_{j=1}^{N} \int_{I_j} G(x|y)(L^*v)(x)dx - \sum_{j=1}^{N-1} \varepsilon J(v'(x_j))G(x_j|y) + G(1|y)\Gamma^*v.$$

That (2.14) holds for $v \in C^\infty[0,1]$ follows from the properties of the Green's function. By a density argument, and using the fact that H_q^2 is continuously embedded in L_∞ and H_∞^1 together with the fact that L^* is a continuous mapping from H_q^2 into L_q, it follows that (2.14) also holds for each $v \in H_{q,\Delta}^2$. Again, see [34] for a more detailed proof.

For $v \in H_{q,\Delta}^2$, $1 \le q \le \infty$, select $u_o \in H_{p,\Delta}^o$, $1/p + 1/q = 1$, where $u_o = (\tilde{u}_o, d_1, \ldots, d_{N_1})$ and

$$\tilde{u}_o|_{I_j} = \operatorname{sgn}(L^*v)|_{I_j}, \quad \text{for } j = 1,\ldots, N,$$

$$d_j = -\operatorname{sgn}(J(v'(x_j))), \quad \text{for } j = 1,\ldots, N-1,$$

and $\qquad d_N = \operatorname{sgn} \Gamma^*(v) \quad \text{if } \beta_1 \neq 0.$

Then

$$|B_\Delta(u_o,v)| = \sum_{j=1}^{N} \int_{I_j} |L^*v|dx + \sum_{j=1}^{N-1} \varepsilon |J(v'(x_j))| + |\Gamma^*(v)|.$$

If $B_\Delta(u_o,v) = 0$, then $L^*v|_{I_j} = 0$ a.e., $J(v'(x_j)) = 0$, $j = 1,\ldots,N-1$ and $\Gamma^*(v) = 0$. By (2.14), $v = 0$. Thus, $|||\cdot|||$ is a norm on $H^2_{q,\Delta}$, for $1 \le q \le \infty$.

Let us conclude this section by introducing another norm. For any $v \in H^2_{q,\Delta}$, define $||\cdot||_{H^2_{q,\varepsilon,\Delta}}$ by

$$(2.15)\qquad ||v||_{H^2_{q,\varepsilon,\Delta}} = \begin{cases} \left[\sum_{j=1}^{N} \int_{I_j} |L^*v|^q dx + \sum_{j=1}^{N-1} \varepsilon^q |J(v'(x_j))|^q \rho_j^{1-q} + h_N^{1-q}|\Gamma^*(v)|^q\right]^{1/q}, & 1 \le q < \infty, \\ \max\left[\max_{1\le j\le N} ||L^*v||_{L_\infty(I_j)}, \max_{1\le j\le N-1} \varepsilon|J(v'(x_j))|\rho_j^{-1}, |\Gamma^*(v)h_N^{-1}\right], & q = \infty. \end{cases}$$

EQUALITY OF NORMS

We shall now prove that the norms $||\cdot||_{H^2_{q,\varepsilon,\Delta}}$ and $|||\cdot|||$ are equal.

Lemma 2.6. Let $v \in H^2_{q,\Delta}$ then

$$(2.16)\qquad ||v||_{H^2_{q,\varepsilon,\Delta}} = |||v|||, \qquad 1 \le q \le \infty.$$

Proof: That $||v||_{H^2_{q,\varepsilon,\Delta}} \le |||v|||$ follows from Holder's inequality. For the inequality in the other direction, for a given v, $u = u_v$ is selected such that

$$|||v||| = \sup_{u \in H^o_{p,\Delta}} \frac{|B_\Delta(u,v)|}{||u||_{H^o_{p,\Delta}}} \ge \frac{|B_\Delta(u_v,v)|}{||u_v||_{H^o_{p,\Delta}}} = ||v||_{H^2_{q,\varepsilon,\Delta}}$$

If $1 \leq q < \infty$ this is done by selecting $u_v = (\tilde{u}, d_1, \ldots, d_{N_1})$ such that

$$\tilde{u}|_{I_j} = |L^*v|^{q-1}\operatorname{sgn}(L^*v)|_{I_j}, \qquad 1 \leq j \leq N$$

$$d_j = -\varepsilon^{q-1}|J(v'(x_j))|^{q-1}\rho_j^{\,q-1}\operatorname{sgn}(J(v'(x_j))), \quad 1 \leq j \leq N-1$$

and

$$d_N = |\Gamma^*(v)|^{q-1}h_N^{1-q}\operatorname{sgn}(\Gamma^*(v)) \quad \text{if } \varepsilon_1 \neq 0.$$

If $q = \infty$, first assume $||v||_{H^2_{\infty,\varepsilon,\Delta}} = ||L^*v||_{L_\infty(I_J)}$. Let $\eta > 0$ be given and define

$$E_\eta = \{x \in I_J : |(L^*v)(x)| \geq ||L^*v||_{L_\infty(I_J)} - \eta\}$$

Then $m(E_\eta) > 0$, where $m(A)$ is the Lebesgue measure of A. Select $u_v = (\tilde{u}, 0, \ldots, 0)$ such that $\tilde{u} = -\chi_{E_\eta}(m(E_\eta))^{-1}\operatorname{sgn}(L^*v)$, where χ_A denotes the characteristic function of the set A. Then $|||v||| \geq ||v||_{H^2_{\infty,\varepsilon,\Delta}} - \eta$, and since η was arbitrary,

$$|||v||| \geq ||v||_{H^2_{\infty,\varepsilon,\Delta}} .$$

If $||v||_{H^2_{\infty,\varepsilon,\Delta}} = \varepsilon|J(v'(x_J))|\rho_J^{-1}$ or $||v||_{H^2_{\infty,\varepsilon,\Delta}} = |\Gamma^*(v)|h_N^{-1}$ the selection of u_v is obvious.

Note that Lemma 2.6 implies that B_Δ is 1-1 proper over $H^0_{p,\Delta} \times H^2_{q,\varepsilon,\Delta}$ for $1 \leq p \leq \infty$, when $1/p + 1/q = 1$.

EMBEDDING RESULTS

The following lemma which is a slight modification of a result from [28] is used to prove the embedding result. All constants, C, C_1, $C_2, \ldots$ appearing in Lemma 2.7 and Theorem 2.8 are independent of p, q, v, ε and Δ.

Lemma 2.7. Let $\hat{G}(x/y)$ be as defined in (2.6). Then

$$\frac{\partial \hat{G}}{\partial y}(x/y) \leq \frac{1}{\varepsilon} e^{-ay/\varepsilon} + C, \quad \text{for } 0 \leq y < x,$$

and

$$\frac{\partial \hat{G}}{\partial y}(x/y) \leq \frac{C_1}{\varepsilon} e^{-a(x-y)/\varepsilon} + C_2, \quad \text{for } x < y \leq 1.$$

Proof: See [28] or [34].

Theorem 2.8. (Embedding result). If $v \in H^2_{q,\varepsilon,\Delta}$, then $v \in L_\infty(I) \cap H^1_q(I)$ with

(2.17) $$||v||_{L_\infty(I)} \leq C_1 ||v||_{H^2_{1,\varepsilon,\Delta}} \leq C_1 ||v||_{H^2_{q,\varepsilon,\Delta}}, \quad 1 \leq q \leq \infty,$$

and

(2.18) $$||v'||_{L_q(I)} \leq C_2 \varepsilon^{1/q-1} ||v||_{H^2_{q,\varepsilon,\Delta}}, \quad 1 \leq q \leq \infty,$$

where C_1 and C_2 are independent of v, q, ε and Δ.

Proof. Inequality (2.17) follows directly from (2.14), (2.15) and Theorem 2.5.

In order to prove (2.18), let $\hat{G}(x|y)$ be as defined in (2.6). Then, for $v \in H^2_{q,\varepsilon,\Delta}$ we have

$$v(y) = \int_0^1 \hat{G}(x|y)\, K_v(x)\, dx - \int_0^1 \hat{G}(x|y) c(x) v(x) dx$$

$$- \sum_{j=1}^{N-1} \varepsilon J(v'(x_j)) \hat{G}(x_j|y) + \hat{G}(1,y)\Gamma^*(v),$$

where $K_v(x) \in L_q(I)$, $K_v|_{I_j} = L^*v|_{I_j}$, $1 \leq j \leq N$, and $c(x) = (b-a')(x)$. So,

(2.19) $$v'(y) = \int_0^1 \frac{\hat{G}(x|y)}{\partial y} K_v(x) dx - \int_0^1 \frac{\partial \hat{G}}{\partial y}(x|y) c(x) v(x) dx$$

$$- \sum_{j=1}^{N-1} \varepsilon J(v'(x_j)) \frac{\partial \hat{G}}{\partial y}(x_j|y) + \frac{\partial \hat{G}}{\partial y}(1|y)\Gamma^*(v)$$

$$= w_1(y) + w_2(y) + w_3(y) + w_4(y).$$

First consider $w_1(y)$, where

$$(2.20)\quad w_1(y) = \int_0^1 \frac{\partial \hat{G}(x|y)}{\partial y} K_v(x)dx = \int_0^y \frac{\partial \hat{G}(x|y)}{\partial y} K_v(x)dx + \int_y^1 \frac{\partial \hat{G}(x|y)}{\partial y} K_v(x)dx$$

$$= z_1(y) + z_2(y).$$

Using Lemma 2.7, we have

$$|z_1(y)| \le \int_0^y \frac{C_1}{\varepsilon} e^{-a(y-x)/\varepsilon} |K_v(x)|dx + C_2||K_v||_{L_1}$$

$$= C_1 \int_0^1 \frac{1}{\varepsilon} \phi(\frac{x-y}{\varepsilon}) |K_v(x)| dx + C_2 ||K_v||_{L_1},$$

where

$$\phi(x) = \begin{cases} e^{-ax}, & \text{if } x \le 0, \\ 0, & \text{if } x > 0. \end{cases}$$

Extend $|K_v|$ by 0 to all of R^1. Then through Young's inequality,[+] we have

$$(2.21)\quad ||z_1||_{L_q(I)} \le C_1||\tfrac{1}{\varepsilon}\phi(\tfrac{x}{\varepsilon})||_{L_1(R)}||K_v||_{L_q(I)} + C_2||K_v||_{L_1(I)}$$

$$\le C||K_v||_{L_q(I)}, \qquad \text{for } 1 \le q \le \infty.$$

Again, using Lemma 2.7, we have

$$(2.22)\quad ||z_2||_{L_q(I)} \le C\varepsilon^{1/q-1}||K_v||_{L_1(I)}.$$

Also, from (2.19) we have $w_2(y) = -\int_0^1 \frac{\partial \hat{G}}{\partial y}(x|y)c(x)v(x)dx$. Using a similar agrument to the one used to bound $w_1(y)$, together with inequality (2.17), we have

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[+] Young's inequality states that for $1 \le q \le \infty$, if $s \in L_q(R^n)$, and $g \in L_1(R^n)$, then $h = s*g$ exists a.e., belongs to $L_q(R^n)$, and $||h||_{L_q} \le ||s||_{L_q}||g||_{L_1}$.

(2.23) $$||w_2||_{L_q(I)} \leq C\varepsilon^{1/q-1}||v||_{L_q} \leq C\varepsilon^{1/q-1}||v||_{H^2_{q,\varepsilon,\Delta}} ,$$

for $1 \leq q \leq \infty$.

Next,

$$w_3(y) = -\sum_{j=1}^{N-1} \varepsilon J(v'(x_j)) \frac{\partial \hat{G}}{\partial y}(x_j|y) = \sum_{j=1}^{N-1} z_j(y),$$

and

$$\int_0^1 |z_j(y)|^q dy \leq \varepsilon^q |J(v'(x_j))|^q \left\{ \int_0^{x_j} |\frac{\partial \hat{G}}{\partial y}(x_j|y)|^q + \int_{x_j}^1 |\frac{\partial \hat{G}}{\partial y}(x_j|y)|^q \right\}, \quad \text{for } 1 \leq q < \infty$$

By Lemma 2.7,

$$||z_j||_{L_q(I)} \leq C\varepsilon^{1/q-1}[\varepsilon|J(v'(x_j))|], \quad \text{for } 1 \leq q \leq \infty.$$

Therefore,

(2.24) $$||w_3||_{L_q} \leq \sum_{j=1}^{N-1} ||z_j||_{L_q} \leq C\varepsilon^{1/q-1} \sum_{j=1}^{N-1} \varepsilon|J(v'(x_j))| ,$$

for $1 \leq q \leq \infty$.

Finally,

$$w_4(v) = \frac{\partial \hat{G}}{\partial y}(1,v)\Gamma_1^*(v),$$

and Lemma 2.7 implies that

(2.25) $$||w_4||_{L_q(I)} \leq C\varepsilon^{1/q-1}|\Gamma^*v|.$$

Expressions (2.19), (2.20)-(2.25), and the triangle inequality imply that

$$||v'||_{L_q} \leq C\varepsilon^{1/q-1}||v||_{H^2_{q,\varepsilon,\Delta}} , \quad \text{for } 1 \leq q \leq \infty.$$

EXISTENCE OF SOLUTIONS

In addition to the assumptions A1 and A2, we make the following assumption on the input data.

A3: i) f is of the form $f = f_o + f_1$, with $f_o \in L_1$ (I) and $f_1 = \sum_{i=1}^{N-1} C_i\, \delta(x-x_i)$ where $\delta(x-x_i)$ is the Dirac delta function at the mesh point x_i. Furthermore, $\sum_{i=1}^{N-1} C_i = K$ is independent of N and f is independent of ε.

ii) α is bounded independently of ε. If $\beta_1 \neq 0$, then $\frac{\varepsilon\beta}{\beta_1}$ is bounded independently of ε, and if $\beta_1 = 0$, then $\frac{\beta}{\beta_2}$ is bounded independently of ε.

Under this assumption, we have the following representation for the solution $u(x)$, (see e.g., [33]);

$$u(x) = \sum_{i=1}^{N-1} C_i G(x|x_i) + \int_o^1 f_o(y)\, G(x|y)dy$$

$$+ \frac{\beta u_1(x)}{B(u_1)} + \frac{\alpha u_2(x)}{u_2(0)} ,$$

where u_1 and u_2 are non-trivial solutions to $Lu_1 = 0$, $u_1(0) = 0$, and $Lu_2 = 0$, $Bu_2 = 0$, respectively. From the maximum principle and assumption A3ii), it can be shown that $\frac{\beta u_1}{B(u_1)}$ and $\frac{\alpha u_2}{u_2(0)}$ are bounded independently of ε. This fact, together with Theorem 2.5 and assumption A3i), implies that $u(x)$ is bounded independently of ε.

For the variational formulation, first assume that the essential boundary conditions are homogeneous, that is $\alpha = 0$ and $\beta = 0$ if $\beta_1 = 0$. Let

$$F(v) = (f,v) - \begin{cases} 0 & \text{if } \beta_1 = 0 \\ \frac{\varepsilon\beta v(1)}{\beta_1} & \text{if } \beta_1 \neq 0. \end{cases}$$

Using assumption A3 and Theorem 2.8, it follows that F is a bounded linear functional on $H^2_{q,\varepsilon,\Delta}$ for $1 \le q \le \infty$. Equation 2.13, Lemma 2.6 and a simple verification of (2.3) imply that $B_\varepsilon(u,v)$ is (1-1) proper on $H^0_{p,\Delta} \times H^2_{q,\varepsilon,\Delta}$ for $1 \le p \le \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. For $1 < p < \infty$ the spaces $H^0_{p,\Delta}$ and $H^2_{q,\varepsilon,\Delta}$ are reflexive. In this case, we may apply Theorem 2.1 which leads to the existence of a unique $w \in H^0_{p,\Delta}$ solving

(2.26) $\quad B_\varepsilon(w,v) = F(v)$ for each $v \in H^2_{q,\varepsilon,\Delta}$.

If $\alpha \neq 0$ or $\beta \neq 0$ if $\beta_1 = 0$ we proceed in the standard way writing $u=w+u_o$ where u_o is a piecewise linear function on Δ which satisfies the essential boundary conditions, and $w \in H^0_{p,\Delta}$. We remark that since u is bounded independently of ε, $w \in H^0_{\infty,\Delta}$ and $||w||_{H^0_{\infty,\Delta}}$ is bounded independently of ε. By treating the boundary conditions in this way we can restrict the theory to the case of homogeneous essential boundary conditions without any loss of generality. Therefore, this restriction is made for the remainder of the theorems in this paper. These restrictions are not imposed in the numerical examples (see [35]).

Another assumption which will be used later is

A4: $a(x)/_{I_j} \in C^{k+1}(I_j)$,

$b(x)/_{I_j} \in C^k(I_j)$,

$f_o(x)/_{I_j} \in C^k(I_j)$, and

a(x) and b(x) are independent of ε. The specific value of k will depend on the finite element trial space.

CHAPTER 3

L*-SPLINE TEST FUNCTIONS AND THE INF-SUP CONDITION

In order to obtain a finite dimensional approximation to the solution of (2.26), the finite dimensional spaces $S_{1,\Delta}$ and $S_{2,\Delta}$ must be specified. To obtain quasi-optimality, these spaces must have the property that the inf-sup constant, C_2, of Theorem 2.2 is bounded away from zero independently of ε and Δ. When both $S_{1,\Delta}$ and $S_{2,\Delta}$ are the conventional piecewise polynomial spaces, this condition is violated.

For the trial space, $S_{1,\Delta}$, we take the space $S_r = \{u \in C^0 \cap H^0_{p,\Delta} : u|_{I_j}$ is a polynomial of degree $\leq r\}$, that is, the usual space of piecewise polynomials of degree r. For the test space, $S_{2,\Delta}$, first consider the space of L* splines:

$$(3.1) \qquad S_L = \{v \in H^2_{q,\varepsilon,\Delta} : L^*v|_{I_j} \text{ is a polynomial of degree } r-2 \text{ if } r > 1,$$

$$L^*v|_{I_j} = 0, \text{ if } r = 1\} .$$

It would be ideal if we could use the test space S_L - not only would quasi-optimality result (Theorem 3.3), but also the nodal errors would be zero (Theorem 3.4). Unfortunately, since the functions a(x) and b(x) are not constant, it is in general impossible to determine the basis functions for S_L exactly. However, it will be shown (Theorem 3.3) that quasi-optimality is preserved if basis functions can be found which are sufficiently accurate approximations to the "ideal" basis functions of S_L.

Suppose we can find basis functions $\{\psi^{(\eta)}_{\ell,j}\}$ $j = 1,\ldots,N$; $\ell = -1,\ldots r - 2$, satisfying

$$(3.2a)\qquad L^*\psi^{(\eta)}_{-1,j} = \begin{cases} \eta_{-1,j}(x), & \text{on } I_j \text{ and } I_{j+1}, \\ 0, & \text{elsewhere}, \end{cases}$$

$$\psi^{(\eta)}_{-1,j}(x_i) = \delta_{i,j}, \qquad \text{for } i,j = 1,\ldots,N.$$

$$(3.2b)\qquad L^*\psi^{(\eta)}_{\ell,j} = \begin{cases} \left(\dfrac{x-x_{j-1}}{h_j}\right)^{\ell} + \eta_{\ell,j}(x), & \text{on } I_j, \\ 0, & \text{elsewhere}, \end{cases}$$

$$\psi^{(\eta)}_{\ell,j}(x_i) = 0, \quad \text{for } \ell = 0,\ldots,r-2,\ i,j = 1,\ldots,N.$$

Let $\eta_j = \max_{\ell=-1,\ldots r-2} ||\eta_{\ell,j}(x)||_{L_\infty}$, and $\eta = \max_j(\eta_j)$. Denote by $S_L^{(\eta)}$ the space spanned by these basis functions. Note that $S_L^{(0)} = S_L$.

THE INF-SUP CONDITION OVER $S_r \times S_L^{(\eta)}$

We will now show that the inf-sup constant C_2, is bounded away from zero independently of ε and Δ when $S_{1,\Delta} = S_r$, and $S_{2,\Delta} = S_L^{(\eta)}$ and η is sufficiently small (independently of ε).

Before proving this result, we need to define some additional norms over the space of polynomials of degree r. These norms, as well as the basic idea of the proof of the inf-sup condition, are taken from [2].

On the space of polynomials of degree r over the interval $[\bar{x},\bar{x}+h]$, $f(x) = \sum_{i=0}^{r} b_i(x-\bar{x})^i$, we define the following norms:

$$\|f\|'_{L_p[\bar{x},\bar{x}+h]} = \begin{cases} \left[\sum_{i=0}^{r} |b_i|^p h^{ip+1}\right]^{1/p}, & 1 \le p < \infty, \\ \max_{0 \le i \le r} |b_i h^i|, & p = \infty, \end{cases}$$

$$\|f\|''_{L_p}[\bar{x},\bar{x}+h] = \begin{cases} \left[\sum_{i=0}^{r} \frac{\left|\int_{\bar{x}}^{\bar{x}+h} f(x)\,(x-\bar{x})^i dx\right|^p}{h^{pi+p-i}}\right]^{1/p}, & 1 \le p < \infty, \\ \max_{0 \le i \le r} \frac{\left|\int_{\bar{x}}^{\bar{x}+h} f(x)(x-\bar{x})^i dx\right|}{h^{i+1}}, & p = \infty, \end{cases}$$

and

$$\|f\|'''_{L_p}[\bar{x},\bar{x}+h] = \begin{cases} \left[(|f(\bar{x})|^p + |f(\bar{x}+h)|^p)\,h + \sum_{i=0}^{r-2} \frac{\left|\int_{\bar{x}}^{\bar{x}+h} f(x)(x-\bar{x})^i dx\right|^p}{h^{pi+p-i}}\right]^{1/p}, \\ \qquad 1 \le p < \infty, \\ \max\left[|f(\bar{x})|, |f(\bar{x}+h)|, \frac{\left|\int_{\bar{x}}^{\bar{x}+h} f(x)(x-\bar{x})^i dx\right|}{h^{i+1}}, \quad i = 0,\ldots r-2\right], \\ \qquad p = \infty . \end{cases}$$

<u>Lemma 3.1.</u> $\|\cdot\|'$, $\|\cdot\|''$, and $\|\cdot\|'''$ are norms over S_r, and there exists a constant $C = C(r)$, independent of h, f, and $\bar{x}$, such that

$$C^{-1}\|f\|'_{L_p[\bar{x},\bar{x}+h]} \le \|f\|_{L_p[\bar{x},\bar{x}+h]} \le C\|f\|'_{L_p[\bar{x},\bar{x}+h]}, \tag{3.3a}$$

$$C^{-1}\|f\|''_{L_p[\bar{x},\bar{x}+h]} \le \|f\|_{L_p[\bar{x},\bar{x}+h]} \le C\|f\|''_{L_p[\bar{x},\bar{x}+h]}, \tag{3.3b}$$

and

(3.3c) $$C^{-1}\|f\|'''_{L_p}[\bar{x},\bar{x}+h] \le \|f\|_{L_p}[\bar{x},\bar{x}+h] \le C\|f\|'''_{L_p}[\bar{x},\bar{x}+h],$$

The proof of this result can be found in [2].

We will now prove the main result of this section. All constants C, C_1, C_2, D_o, D_1 appearing in the theorem or the proof are independent of ε, Δ, p, v, and η.

Theorem 3.2. Let η be sufficiently small and $\frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\inf_{\substack{v\in S_L^{(\eta)} \\ \|v\|_{H^2_{q,\varepsilon,\Delta}} = 1}} \sup_{\substack{u\in S_r \\ \|u\|_{H^o_{p,\Delta}} = 1}} |B_\Delta(u,v)| \ge D_o(1-D_1\eta) \qquad \text{for } 1 \le p \le \infty.$$

Proof. Let $v\in S_L^{(\eta)}$ be given. In terms of the basis functions

$$v(x) = \sum_{j=1}^{N} v(x_j)\psi_{-1,j}(x) + \sum_{j=1}^{N}\sum_{\ell=0}^{r-2} b_{\ell,j}h_j^\ell\psi_{\ell,j}(x).$$

Then, by (3.2a, b)

(3.4) $$L^*v|_{I_j} = \sum_{i=0}^{r-2} b_{i,j}(x-x_{j-1})^i + \sum_{i=0}^{r-2} b_{i,j}h_j^i\eta_{i,j}$$

$$+ (v(x_{j-1})\eta_{-1,j-1} + v(x_j)\eta_{-1,j})|_{I_j}\,.$$

Hence,

(3.5) $$\|L^*v\|_{L_q(I_j)}$$

$$\le \Big\|\sum_{i=0}^{r-2} b_{i,j}(x-x_{j-1})^i\Big\|_{L_q(I_j)} + \Big\|\sum_{i=0}^{r-2} b_{i,j}h_j^i\eta_{i,j}\Big\|_{L_q(I_j)}$$

$$+ \|v(x_{j-1})\eta_{-1,j-1} + v(x_j)\eta_{-1,j}\|_{L_q(I_j)}, \quad \text{for } 1 \le q \le \infty.$$

Since,

$$\left\|\sum_{i=0}^{r-2} b_{i,j}(x-x_{j-1})^i\right\|_{L_q(I_j)} \le \begin{cases} \left[\sum_{i=0}^{r-2} |b_{i,j}|^q h_j^{qi+1}\right]^{1/q}, & 1 \le q < \infty, \\ (r-1)\max_i |b_{i,j}|h_j^i, & q = \infty, \end{cases}$$

$$\left\|\sum_{i=0}^{r-2} b_{i,j}h_j^i \eta_{i,j}\right\|_{L_q(I_j)} \le \begin{cases} \eta_j \left[\sum_{i=0}^{r-2} |b_{i,j}|^q h_j^{qi+1}\right]^{1/q}, & 1 \le q < \infty, \\ \eta_j (r-1)\max_i |b_{i,j}|h_j^i, & q = \infty, \end{cases}$$

and

$$||v(x_{j-1})\eta_{-1,j-1} + v(x_j)\eta_{-1,j}||_{L_q(I_j)} \le 2\eta_j ||v||_{L_\infty(I_j)} h_j^{1/q},$$

$$1 \le q \le \infty,$$

it follows from (3.5) that

$$\text{(3.6a)} \qquad \sum_{j=1}^{N} ||I^*v||^q_{L_q(I_j)} \le 2^q(1+\eta)^q \sum_{j=1}^{N} \sum_{i=0}^{r-2} |b_{i,j}|^q h_j^{qi+1}$$

$$+ 4^q \eta^q ||v||^q_{L_\infty}, \qquad 1 \le q < \infty,$$

and

$$\text{(3.6b)} \qquad \max_j ||L^*v||_{L_\infty(I_j)} \le (1+\eta)\ (r-1) \max_{1\le j\le N} \max_{0\le i\le r-2} |b_{i,j}|h_j^i$$

$$+ 2\eta||v||_{L_\infty}, \qquad q = \infty.$$

From (3.6a,b) and the embedding result (2.17), it follows that if η is sufficiently small

$$(3.7)\qquad \|v\|_{H^2_{q,\varepsilon,\Delta}} \le \begin{cases} C\left[\sum_{j=1}^{N}\sum_{i=0}^{r-2}|b_{i,j}|^q h_j^{qi+1} + \sum_{j=1}^{N-1}\varepsilon^q|J(v'(x_j))|^q\rho_j^{1-q} + |\Gamma^*(v)|^q h_N^{1-q}\right]^{1/q}, & 1\le q\le\infty, \\ C\max[|b_{L,J}|h_J^L, \varepsilon|J(v'(x_K))|\rho_K^{-1}, |\Gamma^*(v)|h_N^{-1}], & q=\infty, \end{cases}$$

where L and J are the indices such that

$$(3.8)\qquad |b_{L,J}|h_J^L = \max_{1\le j\le N}\ \max_{0\le i\le r-2} |b_{i,j}|h_j^i,$$

and K is the index such that

$$(3.9)\qquad \varepsilon|J(v'(x_K))|\rho_K^{-1} = \max_{1\le j\le N-1} \varepsilon|J(v'(x_j))|\rho_j^{-1}.$$

With $v \in S_L^{(\eta)}$ given, select $u_v = \phi_1 + \phi_2$ in the following way. Select $\phi_1 \in S_r$ such that

$$(3.10)\qquad \begin{gathered} \phi_1(x_j) = 0, \quad j = 0,\dots,N, \quad \text{and if } r > 1, \\ \int_{I_j}(x-x_{j-1})^i\phi_1\,dx = |b_{i,j}|^{q-1}h_j^{qi+1}\operatorname{sgn} b_{i,j}, \\ \text{for } i = 0,\dots,r-2, \quad j = 1,\dots,N, \quad 1\le q\le\infty. \end{gathered}$$

Select $\phi_2 \in S_r$ such that

$$(3.11)\qquad \phi_2(x_j) = \varepsilon^{q-1}|J(v'(x_j))|^{q-1}\rho_j^{1-q}\operatorname{sgn}(J(v'(x_j))), \quad 1\le j<N,$$

$$\phi_2(x_N) = |\Gamma^*(v)|^{q-1}\mathrm{sgn}(\Gamma^* v), \qquad 1 \le q < \infty,$$

and if $r > 1$

$$\int_{I_j} (x-x_{j-1})^i \phi_2 = 0, \quad \text{for } i = 0,\ldots,r-2 \text{ and } j = 1,\ldots,N.$$

That ϕ_1, and ϕ_2 are uniquely determined follows from Lemma 3.1.

Recall,

$$B_\Delta(u,v) = -\sum_{j=1}^{N} \int_{I_j} \tilde{u}(L^* v)dx - \sum_{j=1}^{N-1} \varepsilon J(v'(x_j))d_j + d_N \Gamma^*(v). \tag{3.12}$$

By (3.4), (3.10) and (3.11), we have

$$\begin{aligned} &B_\Delta(\phi_1 + \phi_2, v) \\ &= \sum_{j=1}^{N} \sum_{i=0}^{r-2} |b_{i,j}|^q h_j^{qi+1} + \sum_{j=1}^{N-1} \varepsilon^q |J(v'(x_j))|^q \rho_j^{1-q} \\ &+ |\Gamma^*(v)|^q h_N^{1-q} + \sum_{j=1}^{N} \int_{I_j} \left(\sum_{i=0}^{r-2} b_{i,j} h_j^i \eta_{i,j} \right) (\phi_1 + \phi_2) dx \\ &+ \sum_{j=1}^{N} \int_{I_j} (v(x_{j-1})\eta_{-1,j} + v(x_j)\eta_{-1,j})(\phi_1 + \phi_2)dx, \quad 1 \le q < \infty . \end{aligned}$$

Therefore,

$$\begin{aligned} &B_\Delta(\phi_1 + \phi_2, v) \\ &\ge \sum_{j=1}^{N} \sum_{i=0}^{r-2} |b_{i,j}|^q h_j^{qi+1} + \sum_{j=1}^{N-1} \varepsilon^q |J(v'(x_j))|^q \rho_j^{1-q} \\ &\qquad + |\Gamma_1^*(v)|^q h_N^{1-q} \\ &- \eta \left\{ \left[\sum_{j=1}^{N} \sum_{i=0}^{r-2} |b_{i,j}|^q h_j^{qi+1} \right]^{1/q} + 2||v||_{L_\infty} \right\} ||\phi_1 + \phi_2||_{H^o_{p,\Delta}} , \\ &\qquad 1 \le q < \infty . \end{aligned} \tag{3.13}$$

From the definitions of ϕ_1 and ϕ_2 ((3.10), (3.11)), (3.3c), and (2.11) we have

$$
(3.14)\qquad \begin{aligned}
||\phi_1+\phi_2||^p_{H^o_{p,\Delta}} \le\ & C^p \sum_{j=1}^{N}\sum_{i=0}^{r-2} |b_{i,j}|^q\, h_j^{qi+1} \\
&+ \sum_{j=1}^{N-1} \varepsilon^q |J(v'(x_j))|^q \rho_j^{1-q} + |\Gamma^*(v)|^q\, h_N^{1-q}\ , \\
&\qquad\qquad 1 < p < \infty .
\end{aligned}
$$

Inequalities (3.13), (3.14) and (3.7) yield

$$
\begin{aligned}
\frac{B_\Delta(\phi_1+\phi_2,v)}{||\phi_1+\phi_2||_{H^o_{p,\Delta}}} &\ge C_1||v||_{H^2_{p,\varepsilon,\Delta}} - C_2\eta||v||_{L_\infty} \\
&\ge D_1(1-D_2\eta)||v||_{H^2_{q,\varepsilon,\Delta}}\ ,\quad 1 < q < \infty,
\end{aligned}
$$

with the last inequality following from the embedding result (2.17).

Next, consider the case when $p = 1$ and $q = \infty$. In this case, we modify the definition of ϕ_1 in (3.10) such that

$$
\int_{I_j} (x-x_{j-1})^i \phi_1\,dx = \begin{cases} 0, & \text{if } j \ne J, \\ \delta_{i,L} h_J^L \operatorname{sgn}(b_{L,J}), & \text{if } j = J, \end{cases}
$$

where L and J are defined in (3.8) and $\delta_{i,j}$ is the Kronocker delta. The definition of ϕ_2 in (3.11) is also modified such that $\phi_2(x_j) = \delta_{j,K}\,\rho_K^{-1}\operatorname{sgn} J(v'(x_K))$, for $1 \le j \le N-1$, and $\phi_2(x_N) = h_N^{-1} \operatorname{sgn} \Gamma^*(v)$, where K is defined in (3.9). By (3.3c) $||\phi_1+\phi_2||_{H^o_{1,\Delta}} \le C$. From the modified definitions of ϕ_1 and ϕ_2, (3.12) and (3.7) it follows that

$$
\frac{B_\Delta(\phi_1+\phi_2,v)}{||\phi_1+\phi_2||_{H^0_{1,\Delta}}} \le D_1(1-D_2\eta)||v||_{H^2_{\infty,\varepsilon,\Delta}}\ ,
$$

as desired.

When $p = \infty$ and $q = 1$ the formulas for ϕ_1 and ϕ_2 given in (3.10) and (3.11) are still valid, and from (3.3c) we have $||\phi_1 + \phi_2||_{H^0_{\infty,\Delta}} = ||\phi_1 + \phi_2||_{L_\infty} \leq C$. The desired result then follows from this fact, and inequalities (3.13), (3.7) and (2.17)

Theorems 3.2 and 2.2 yield the following result.

Theorem 3.3. Let $u \in H^0_{p,\Delta}$ be the solution to $B_\Delta(u,v) = F(v), \forall\, v \in H^2_{q,\varepsilon,\Delta}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, assume $u \in H^0_{\infty,\Delta}$. Then, for η sufficiently small, there exists a unique solution $u_L \in S_r$ to the problem $B_\Delta(u_L,v) = F(v), \forall\, v \in S_L^{(\eta)}$, and $||u - u_L||_{H^0_{p,\Delta}} \leq C \inf_{w \in S_r} ||u - w||_{H^0_{p,\Delta}}$, $1 \leq p \leq \infty$, with C independent of u, p, ε, and Δ.

The next theorem shows that if $\eta = 0$, then $(u - u_L)(x_j) \doteq 0$, for $j = 1,\ldots,N$.

Theorem 3.4. Let $u \in H^0_{p,\Delta}$, $1 \leq p \leq \infty$, and $u_L \in S_r \subset H^0_{p,\Delta}$, with

(3.15) $$B_\Delta(u_L,v) = B_\Delta(u,v), \quad \forall\, v \in S_L^{(0)} .$$

Then $u_L(x_i) - d_i = 0$, $i = 1,\ldots,N_1$, where $u \equiv (\tilde{u},d_1,\ldots,d_{N_1})$.

Proof. Let $1 < q < \infty$. By Theorem 2.1 there exists $v_i \in H^2_{q,\varepsilon,\Delta}$ such that

(3.16) $$B_\Delta(u,v_i) = d_i \qquad \forall\, u \in H^0_{p,\Delta}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

From (3.16) and the definition of B_Δ (3.12), it follows that $L^*(v) = 0$ on every I_j,

$$J(v_i'(x_j)) = -\frac{1}{\varepsilon}\,\delta_{i,j} \qquad \text{for } i,j = 1,\ldots,N-1,$$

and

$$\Gamma^*(v_i) = \delta_{i,N} \qquad\qquad i = 1,\ldots,N.$$

Because $v_i(x)$ is continuous, and $L^*(v) = 0$, on $(0,x_i)$ and $(x_i,1)$, v_i is the Green's function at $x = x_i$. Thus, $v_i \in H^2_{\infty,\varepsilon,\Delta}$ and also $v_i \in S_L^{(0)}$. This implies that (3.16) holds for $1 \leq q \leq \infty$, and

$$B_\Delta(u_L - u, v_i) = 0 = u_L(x_i) - d_i,$$

which finishes the proof.

Theorem 3.4 is a restatement of the well known fact that when the Green's function at $x = x_i$, $x_i \in \Delta$, belong to the test space, then the error at the nodal points is zero.

As pointed out earlier, the space $S_L^{(0)}$ is in general unobtainable. However, in light of Theorem 3.3 it will be satisfactory if we can generate the basis functions for the space $S_L^{(\eta)}$, (3.2a,b), provided η is sufficiently small. This will be done in the following chapter.

CHAPTER 4

EXPLICIT REPRESENTATION OF THE TEST SPACE-$S_L^{(\eta)}$

In this chapter, we will determine the basis functions (3.2a,b) for $S_L^{(\eta)}$ explicitly, with the condition that $\eta_j = \max_{i=-1,\ldots,r-2} \|\eta_{i,j}\|_{L_\infty(I_j)} \leq Ch_j^k$. The constant C will depend on $1/\min_{x \in I_j} a(x)$, and the local smoothness of a(x) and b(x) on I_j, but will be independent of h_j and ε.

In order to determine the basis functions for $S_L^{(\eta)}$ in I_j, we first rescale the interval I_j to I = [0,1], and then drop the index j for simplicity. A "tilda" will be used to denote this rescaling, for example, if g(x) is defined for $x \in I_j$, $\tilde{g}(y)$ is the function defined for $y \in I$ such that $\tilde{g}(y) = g(yh_j + x_{j-1})$. Recall that c(x) = b(x) - a'(x).

After rescaling, our goal is to seek approximations to the solutions of

$$\text{(4.1a)} \qquad \tilde{L}^*v \quad -\frac{\varepsilon}{h^2} v'' - \frac{\tilde{a}(y)}{h} v' + \tilde{c}(y)v = 0, \quad \text{in I},$$

$$v(0) = 1, \quad v(1) = 0,$$

$$\text{(4.1b)} \qquad \tilde{L}^*v = 0, \quad \text{in I},$$

$$v(0) = 0, \quad v(1) = 1,$$

and

$$\text{(4.1c)} \qquad \tilde{L}^*v = y^{\ell}, \quad \text{in I}, \quad \ell = 0,\ldots,r-2$$

$$v(0) = v(1) = 0.$$

Because of our requirement on the space $S_L^{(\eta)}$ in (3.2a,b), we must estimate $||L^*\psi^{(o)}_{i,j} - L^*\psi^{(n)}_{i,j}||_{L_\infty(I_j)}$.

Notice that the rescaled operator has the coefficient $-\frac{\varepsilon}{h^2}$ in front of the highest order derivative. If a standard asymptotic expansion in powers of ε is used (see e.g. [10], [15], [36]), then the errors can be shown to be no better than $O((\frac{\varepsilon}{h})^k)$. This is undesirable because one of our goals is to use adaptive mesh refinements which will quickly lead to intervals having size h, with $h < \varepsilon$.

Although the asymptotic expansion in ε will fail as h becomes small, it will be shown that an asymptotic expansion in h will produce errors of size $O(h^k)$ independently of ε. In order to validate our asymptotic expansion the following lemmas are needed.

<u>Lemma 4.1</u>. Let v solve

$$Hv \equiv \frac{\varepsilon}{h^2} v'' + \frac{\tilde{a}(y)}{h} v' = g \quad \text{in } I,$$

$$v(0) = v(1) = 0,$$

with $\tilde{a}(y) \geq \alpha > 0$ and g bounded.

Then

$$||v||_{L_\infty} \leq \min (h/\alpha,\ 2h^2/\varepsilon)||g||_{L_\infty} .$$

<u>Proof</u>. From the maximum principle ([32] Chapter, 1, Theorem 11), it follows that $|v(y)| \leq |w_1(y)|$ where $w_1(y) = ||g||_{L_\infty} h(1-y)/\alpha$, and $|v(y)| \leq |w_2(y)|$ where $w_2(y) = h^2(e-e^y)||g||_{L_\infty}/\varepsilon$.

<u>Lemma 4.2</u>. Let w solve

$$-\frac{\varepsilon}{h^2} w' - \frac{a_o}{h} w = g \quad \text{in } I$$

$$w(0) = 0,$$

where $a_o > 0$ is constant and g is bounded. Then $||w||_{L_\infty} \le h(1-e^{-a_o h/\varepsilon})||g||_{L_\infty}/a_o$.

Proof. This result follows immediately from the identity

$$w(y) = -\frac{h^2}{\varepsilon}\int_0^y e^{-a_o h(s-y)/\varepsilon} g(s)ds.$$

We seek an approximation $v^{(n)}$ to the function v, where v is the solution to (4.1a), (4.1b) or (4.1c). Assuming sufficient smoothness of the coefficients $\tilde{a}(y)$ and $\tilde{c}(y)$, we may expand by the Taylor series around $y = 0$ to obtain

$$\begin{aligned}
\tilde{L}^*v \equiv & -\frac{\varepsilon}{h^2}v'' - \frac{\tilde{a}(y)}{h}v' + \tilde{c}(y)v \\
= & -\frac{\varepsilon}{h^2}v'' - \frac{\tilde{a}(0)}{h}v' \\
& + h[-\frac{y\tilde{a}'(0)}{h}v' + \frac{\tilde{c}(0)}{h}v] \\
& + h^2[-\frac{y^2}{2!}\frac{\widetilde{a''}(0)}{h}v' + \frac{y\widetilde{c'}(0)}{h}v] \\
& + \cdots \\
& + h^k[-\frac{y^k}{k!}\frac{\widetilde{a^{(k)}}(\xi_a)}{h}v' + \frac{y^{k-1}}{(k-1)!}\frac{\widetilde{c^{(k-1)}}(\xi_c)}{h}v],
\end{aligned}$$

where $\xi_a, \xi_c \in (0,1)$. Therefore, we can write

(4.2) $$L^* = L_o + hL_1 + \cdots + h^{\ell-1}L_{\ell-1} + h^{\ell}R_{\ell},$$

where

(4.3a) $$L_o = -\frac{\varepsilon}{h^2}\frac{d^2}{dy^2} - \frac{\tilde{a}(0)}{h}\frac{d}{dy},$$

(4.3b) $$L_j = \frac{1}{h}[-\frac{y^j}{j!}\widetilde{a^{(j)}}(0)\frac{d}{dy} + \frac{y^{j-1}}{(j-1)!}\widetilde{c^{(j-1)}}(0)],$$

$$j = 1,\ldots,\ell-1,$$

and

$$R_\ell = \frac{1}{h}\left[-\frac{y^\ell}{\ell!}\,\widetilde{a^{(\ell)}(\xi_a)}\,\frac{d}{dy} + \frac{y^{\ell-1}}{(\ell-1)!}\,\widetilde{c^{(\ell-1)}(\xi_c)}\right], \tag{4.3c}$$

$$\ell = 1,\ldots,k.$$

We seek $v^{(\eta)}$ in the form

$$v^{(\eta)} = v_o + hv_1 + h^2 v_2 + \cdots + h^{k-1} v_{k-1} \quad . \tag{4.4}$$

From (4.2) and (4.4) it follows that

$$\begin{aligned} \tilde{L}^* v^{(\eta)} = {} & L_o v_o \\ & + h[L_o v_1 + L_1 v_o] \\ & + h^2[L_o v_2 + L_1 v_1 + L_2 v_o] \\ & + \cdots \\ & + h^{k-1}[L_o v_{k-1} + L_1 v_{k-2} + \cdots L_{k-1} v_o] \\ & + h^k[R_1 v_{k-1} + R_2 v_{k-2} + \cdots + R_k v_o] \ . \end{aligned} \tag{4.5}$$

Based on this formula, the functions v_o, $v_1, \ldots$ should be defined recursively as follows:

(4.6a) $\quad L_o v_o = 0,\quad v_o(0) = 1,\quad v_o(1) = 0,\quad$ if approximating the solution to (4.1a),

(4.6b) $\quad L_o v_o = 0,\quad v_o(0) = 0,\quad v_o(1) = 1,\quad$ if approximating the solution to (4.1b), (4.6c) $$L_o v_o = v^\ell, \qquad \ell = 0,\ldots,r-2,$$

$$v_o(0) = v_o(1) = 0,$$ if approximating the solution to (4.1c),

$$L_o v_i = -\sum_{j=1}^{i} L_j v_{i-j},$$

(4.7)

$$v_i(0) = v_i(1) = 0, \qquad i = 1,\ldots,k-1.$$

From (4.7) and (4.5) it follows that

(4.8) $$L^* v^{(\eta)} = L_o v_o + h^k \Big[\sum_{i=1}^{k} R_i v_{k-i}\Big] \quad .$$

This leads us to the following theorem which is the main result in this chapter.

Theorem 4.3. Assume that $\tilde{a} \in C^k(I)$, and $\tilde{b} \in C^{k-1}(I)$. Let v_o be defined by either (4.6a), (4.6b) or (4.6c), and v_i, $i = 1,\ldots k-1$, defined recursively by (4.7). Let $v^{(\eta)} = v_o + hv_1 + \ldots + h^{k-1}v_{k-1}$. Then

$$||L^* v^{(\eta)}||_{L_\infty} \leq Ch^{k-1}$$

if v_o is defined by (4.6a) or (4.6b), and

$$||L^* v^{(\eta)} - L_o v_o||_{L_\infty} \leq Ch^k$$

if v_o is defined by (4.6c), and in each case C is independent of ε and h.

Proof. First assume v_o is defined by (4.6a). Because of equation (4.8) it must be shown that $h||\sum_{i=1}^{k} R_i v_{k-i}||_{L_\infty} \leq C$. We prove this by induction on k.

First, take k=1. From (4.3c) (with ℓ=1), we have $hR_1v_o = -y\tilde{a}'(\xi_a)v_o' + \tilde{c}(\xi_c)v_o$. From (4.6a) and the maximum principle, it follows that

(4.9) $$0 \le v_o(y) \le 1.$$

Let $w(y) = yv_o' - (v_o - 1)$. Then

$$-\frac{\varepsilon}{h^2} w' - \frac{a(0)}{h} w = \frac{a(0)}{h} (v_o - 1),$$

and $w(0) = 0$. By Lemma 4.2 and (4.9) we obtain $||w||_{L_\infty} \le 1$. Hence, $||yv_o'||_{L_\infty} = ||w + (v_o-1)||_{L_\infty} \le 2$, which implies that

$$||hR_1v_o|| \le 2||\tilde{a}'||_{L_\infty} + ||\tilde{c}||_{L_\infty} \le C$$

with C independent of ϵ and h. Thus, our assertion is true for k = 1.

Next, assume that

(4.10)
$$||v_i||_{L_\infty} \le C, \qquad i = 0,1,\dots,k-1 \quad \text{and}$$
$$||yv_i'||_{L_\infty} \le C, \qquad i = 0,1,\dots,k-1.$$

This is actually our induction assumption because from (4.10) it follows that $||hR_{k-i}v_i|| \le C$, for $i = 1,\dots,k-1$. We must show

$$||v_k||_{L_\infty} \le C,$$

and

$$||yv_k'||_{L_\infty} \le C.$$

From the induction assumption, (4.10), the definition of v_k, (4.7) and (4.3b) we have

(4.11) $$||L_ov_k||_{L_\infty} = \left|\left|\sum_{i=1}^{k} L_iv_{k-i}\right|\right|_{L_\infty} \le \frac{C}{h} ,$$

and so by Lemma 4.1

(4.12) $$||v_k||_{L_\infty} \leq C.$$

As before, let $w = yv_k' - v_k$. Then

$$-\frac{\varepsilon}{h^2}\, w' - \frac{\tilde{a}(0)}{h}\, w = yL_o v_k + \frac{\tilde{a}(0)}{h}\, v_k \ ,$$

and

$$w(0) = 0.$$

Inequalities (4.11), (4.12) and Lemma 4.2 imply that $||w||_{L_\infty} \leq C$. Therefore, $||yv_k'||_{L_\infty} \leq C$ which, with (4.12) implies that $||hR_1 v_k||_{L_\infty} \leq C$. This, together with the induction assumption, proves the result when v_o is defined by (4.6a). When v_o is defined by (4.6b) the proof is almost identical.

When v_o is defined by (4.6c) we use Lemma 4.1 and deduce that $||v_o||_{L_\infty} \leq Ch$. If we set $w = yv_o' - v_o$ and use Lemma 4.2 it will follow that $||yv_o'||_{L_\infty} \leq Ch$. Induction on the assertions

$$||v_i||_{L_\infty} \leq Ch,$$

and

$$||yv_i'||_{L_\infty} \leq Ch$$

yields the desired result.

This theorem proves that it is possible to choose the basis function $\Psi_{i,j}^{(\eta)}$ in such a way that

$$||L^*\Psi_{i,j}^{(o)} - L^*\Psi_{i,j}^{(\eta)}||_{L_\infty(I_j)} = ||\eta_{i,j}||_{L_\infty(I_j)} \leq Ch_j^{k-1} \ ,$$

where $C = C(a|_{I_j}, b|_{I_j}, k)$ is independent of ε, and h_j. Recall that by Theorem 3.3 a small value of η will guarantee a quasi-optimal finite element solution. Since we are approximating the exact solution with a piecewise polynomial of degree r, the accuracy of the method will not increase in order as we increase the order of η.[⊥] Therefore, it is sufficient to take $k = 2$, in which case $\eta = O(h)$.

These basis functions $\Psi_{i,j}^{(\eta)}$ are easily derivable and their explicit formulas are given in [34]. In general, they have the form $P_1(y) + P_2(y)e^{-\lambda y}$, where P_1 and P_2 are polynomials, and $\lambda = a_0 h/\varepsilon$ is often referred to as the local or cell Peclet number. For large values of λ these basis functions themselves exhibit boundary layer behavior. When the value of λ is small these functions are close to polynomials.

[⊥] However, the nodal errors will decrease as η decreases.

CHAPTER 5

PROJECTION ONTO UPWINDED POLYNOMIALS

When the space $S_L^{(\eta)}$ is used as the test space for the finite element method Theorems 3.3 and 4.3 yield quasi-optimality for the approximate solution in S_r. However, since the basis functions of $S_L^{(\eta)}$ are of the form $P_1(y) + P_2(y)\, e^{-\lambda y}$, with $\lambda = \frac{a_0 h}{\varepsilon}$, the bilinear form requires the integration of functions with boundary layers and smooth functions as well. Unless a special quadrature rule is used, which integrate terms of the form $y^m\, e^{-\lambda y}$ exactly, large quadrature errors will result whenever λ is large. Standard quadrature is also needed for the smooth terms.

In order to avoid this inconvenience we propose to project these "exponentially upwinded" basis functions of $S_L^{(\eta)}$ onto a space of polynomially upwinded functions. These projected basis functions will have the form

$$\psi_j(x) = \phi_j(x) + \sum_{i=1}^{M} \alpha_i g_i(x) \qquad \text{for } x \in I_j \tag{5.1}$$

where $\phi_j(x)$ is the standard piecewise linear "hat" function and $\alpha_i = \alpha_i(\varepsilon, h)$ are the upwind parameters. This is a direct generalization of the commonly used α-quadratic upwinding in which case M=1 and $g_1(x)$ is quadratic on I_j. In particular, it was shown in [3] that when $b(x) \equiv 0$, and $a(x)$ is constant the value α, computed from projecting the space S_L is identical to the so called "optimal" value of α presented in ([8], [9], [17]-[21], [30], and [37]), which yields the exact nodal solutions when $f(x)$ is constant.

Let $\psi_{\ell,j}^{(\eta)}$ be as defined in (3.2a,b) and as constructed in (4.6a,b,c). Define $S_\alpha^{(k)}$ by

(5.2a) $$S_\alpha^{(k)} = \mathrm{Span}\left\{\chi_{\ell,j}\right\} \quad j=1, \ldots, N; \; \ell=-1, \ldots, r-2,$$

where

(5.2b) $$\chi'_{\ell,j} = P_k((\psi_{\ell,j}^{(n)})') = \sum_{i=0}^{k} \alpha_i^{(\ell,j)} \phi_i,$$

with P_k denoting the L_2 projection operator onto the first k Legendre polynomials $\phi_0, \ldots, \phi_k$ on each interval I_j. Also, we take the convention

(5.2c) $$\chi_{\ell,j}(x) = \int_{x_{j-1}}^{x} \chi'_{\ell,j}(t) + \psi_{\ell,j}^{(n)}(x_{j-1}),$$

and hence

$$\chi_{\ell,j}(x_{j-1}) = \psi_{\ell,j}^{(n)}(x_{j-1}),$$

and

$$\chi_{\ell,j}(x_j) = \psi_{\ell,j}^{(n)}(x_j).$$

From the results of Chapter 4 these upwind coefficients $\alpha_i = (\psi', \phi_i)_{I_j}$ can be computed exactly, since only integrations of the form $\int_0^1 y^m e^{\lambda y} dy$, or integrations with polynomial integrands are required. Once these coefficients α_i, $i = 1, \ldots, k$ are computed on each interval I_j, all integrations remaining will be of the form $\int_0^1 P(y)g(y)dy$, where $g(y)$ is smooth independently of ε.

Let $v_L \in S_L^{(n)}$. Then $P_k(v_L') = v_\alpha'$ is a piecewise polynomial of degree k with the property that

(5.3) $$(g, v_L' - v_\alpha') = 0, \quad \text{for each } g(x) \in S_k,$$

a piecewise polynomial of degree $\leq k$. Note, that since $v_\alpha(0) = v_L(0) = 0$, it follows from (5.3) that

(5.4) $$v_L(x_j) = v_\alpha(x_j), \qquad j = 0, \ldots, N.$$

Before proving the main result of this section, we first prove two lemmas.

Lemma 5.1. Let $v_L \in S_L^{(n)}$. Let $P_k(v_L') = v_\alpha'$ where P_k is the local L_2 projection operator onto piecewise polynomials of degree k. Then

$$||v_L' - v_\alpha'||_{L_1(I)} \leq (1+C_k)||v_L'||_{L_1(I)},$$

where $C_k = \sum_{i=0}^{k} \sqrt{2i+1}$.

Proof. Let ϕ_i, $i = 0,\ldots,k$ denote the Legendre polynomials of degree i on the interval I_j. Then

$$v_\alpha'\Big|_{I_j} = \sum_{i=0}^{k} \frac{(v_L',\phi_i)_{I_j}}{(\phi_i,\phi_i)} \phi_i,$$

where $(\cdot,\cdot)_{I_j}$ denotes the L_2 inner product on I_j.

Therefore,

$$||v_\alpha'||_{L_1(I_j)} \leq ||v_L'||_{L_1(I_j)} \sum_{i=0}^{k} \frac{||\phi_i||_{L_\infty(I_j)}||\phi_i||_{L_1(I_j)}}{||\phi_i||^2_{L_2(I_j)}} .$$

If we normalize the Legendre polynomials in such a way that $\phi_i(x_j) = 1$, then

$$||\phi_i||_{L_\infty(I_j)} = 1, \quad \text{and} \quad ||\phi_i||^2_{L_2(I_j)} = \frac{h_j}{2i+1} ,$$

and hence, $||\phi_i||_{L_1(I_j)} \leq h_j^{1/2}||\phi_i||_{L_2(I_j)}$. Thus,

$$\sum_{i=0}^{k} \frac{||\phi_i||_{L_\infty(I_j)}||\phi_i||_{L_1(I_j)}}{||\phi_i||^2_{L_2(I_j)}} \leq \sum_{i=0}^{k} \sqrt{2i+1} = C_k .$$

So

$$||v_\alpha'||_{L_1(I_j)} \leq C_k||v_L'||_{L_1(I_j)}$$

and consequently,

$$\|v_\alpha'\|_{L_1(I)} \le C_k \|v_L'\|_{L_1(I)}.$$

The lemma now follows from the triangle inequality.

Lemma 5.2. Let $w \in S_r$. For each $v_L \in S_L^{(\eta)}$, let $v_\alpha \in S_\alpha^{(k)}$ be such that $v_\alpha' = P_k(v_L')$, with $k \ge r$. Assume $a|_{I_j} \in C^{k+1}(I_j)$, and $b|_{I_j} \in C^k(I_j)$ for $j = 1,\dots,N$. Then

$$\sup_{v_L \in S_L^{(\eta)}} \frac{|B_\Delta(w, v_L - v_\alpha)|}{\|v_L\|_{H^2_{q,\varepsilon,\Delta}}} \le C \max_j h_j^{k+1-r} \{\|a^{(k+1-r)}\|_{H^r_\infty(I_j)} + \|b^{(k-r)}\|_{H^r_\infty(I_j)}\} \|w\|_{L_\infty(I_j)},$$

for $1 \le q \le \infty$.

Proof. Since $v_\alpha(x_j) = v_L(x_j)$ for $j = 0,\dots,N$, we can rewrite the bilinear form as

$$B_\Delta(w, v_L - v_\alpha) = \sum_{j=1}^N \int_{I_j} \left[\varepsilon w' - aw - \int_{x_{j-1}}^x (b-a')w\,dt\right] (v_L' - v_\alpha')\,dx .$$

Because of (5.3) and the fact that $r \le k$, we have

$$B_\Delta(w, v_L - v_\alpha) = \sum_{j=1}^N \int_{I_j} \left[-aw - \int_{x_{j-1}}^x (b-a')w\,dt - p(x)\right] (v_L' - v_\alpha')\,dx,$$

where $p(x)$ is any piecewise polynomial of degree $\le k$. Consequently,

$$(5.5) \qquad |B_\Delta(w, v_L - v_\alpha)| \le \max_j [\|aw - p_1(x)\|_{L_\infty(I_j)} + \|\int_{x_{j-1}}^x (b-a')w\,dt - p_2(x)\|_{L_\infty(I_j)}] \cdot \|v_L' - v_\alpha'\|_{L_1(I)} .$$

Let $p_1(x)$ be the k^{th} interpolant of aw on each I_j.

That is,

$$p_1(x_{j-1}+ih_j/k) = (aw)(x_{j-1}+ih_j/k)$$

$$\text{for } i=0,\dots,k, \quad \text{and} \quad j=1,\dots,N.$$

Then,

$$(5.6) \qquad ||aw - p_1\ (x)||_{L_\infty(I_j)} \leq \frac{h_j^{k+1}}{4(k+1)!}\, ||(aw)^{(k+1)}||_{L_\infty(I_j)} \quad .$$

From Leibnitz' rule, we have

$$(aw)^{(k+1)} = \sum_{i=0}^{k+1} \binom{k+1}{i}\, a^{(k+1-i)} w^{(i)}$$

$$= \sum_{i=0}^{r} \binom{k+1}{i}\, a^{(k+1-i)} w^{(i)}, \quad \text{since } k \geq r \ .$$

Hence,

$$||(aw)^{(k+1)}||_{L_\infty(I_j)} \leq (r+1)(k+1)!\,||a^{(k+1-r)}||_{H^r_\infty(I_j)}||w||_{H^r_\infty(I_j)},$$

$$\leq C(k+1)!\,||a^{(k+1-r)}||_{H^r_\infty(I_j)} h_j^{-r}||w||_{L_\infty(I_j)},$$

the last inequality following from the inverse theorem. From this and (5.6) it follows that

$$(5.7) \qquad ||aw - p_1(x)||_{L_\infty(I_j)} \leq Ch_j^{k+1-r}||a^{(k+1-r)}||_{H^r_\infty(I_j)}||w||_{L_\infty(I_j)}.$$

Similarly, if $p_2(x)$ is the k^{th} interpolant of $\int_{x_{j-1}}^{x}$ (b-a')w on I_j, then

$$(5.8) \qquad \left|\left| \int_{x_{j-1}}^{x}(b-a')w\,dt - p_2(x)\right|\right|_{L_\infty(I_j)}$$

$$\leq Ch_j^{k+1-r}||(b-a')^{(k-r)}||_{H^r_\infty(I_j)}||w||_{L_\infty(I_j)}.$$

From (5.5), (5.7), (5.8), Lemma 5.1, and the embedding result (2.18), it follows that

$$\sup_{v_L \in S_L^{(n)}} \frac{|B_\Delta(w, v_L - v_\alpha)|}{||v_L||_{H^2_{\eta,\varepsilon,\Delta}}} \leq C \max_j h_j^{k+1-r}\{||a^{(k+1-r)}||_{H^r(I_j)}$$

$$+ ||b^{(k-r)}||_{H^r_\infty(I_j)}\}||w||_{L_\infty(I_j)}.$$

We are now ready to prove the main result of this chapter.

<u>Theorem 5.3.</u> Suppose that assumptions A1 - A4 hold, and that S_α is defined by (5.2a, b, c) with $k \geq r$.

Then there exist an h_o independent of ε, such that for all $h \leq h_o$, there exists a unique solution $u_\alpha \in S_r$ to

(5.9a) $\qquad B_\Delta(u_\alpha, v_\alpha) = F(v_\alpha)$ for each $v_\alpha \in S_\alpha^{(k)}$.

Also, let $u_L \in S_r$ be the unique solution to

(5.9b) $\qquad B_\Delta(u_L, v_L) = F(v_L)$ for each $v_L \in S_L^{(n)}$.

Then, for $1 \leq p \leq \infty$,

$$||u_L - u_\alpha||_{H^o_{p,\Delta}} \leq C_1 \max_j h_j^{k+1}||f_o^{(k)}||_{L_\infty(I_j)}$$

$$+ C_2 \max_j h_j^{k+1-r}\{||a^{(k+1-r)}||_{H^r_\infty(I_j)} + ||b^{(k-r)}||_{H^r_\infty(I_j)}\}$$

with C_1 and C_2 independent of ε and Δ.

Proof. The existence and uniqueness of u will be established if the homogeneous problem,

$$(5.10) \qquad B_\Delta(w,v_\alpha) = 0 \quad \text{for each} \quad v_\alpha \in S_\alpha^{(k)} \quad ,$$

has only the zero solution in S_r.

Suppose (5.10) holds. Let $v_L \in S_L^{(\eta)}$ be s.t. $v_L(x_j) = v_\alpha(x_j)$, $j = 0,\dots,N$. Then $v_\alpha' = P_k(v_L')$. From (5.10) it follows that

$$B_\Delta(w,v_L) = B_\Delta(w,v_L-v_\alpha) \text{ for each } v_L \in S_L^{(\eta)}, \text{ and } v_\alpha \in S_\alpha^{(k)},$$

and hence

$$(5.11) \qquad \frac{|B_\Delta(w,v_L)|}{||v_L||_{H^2_{1,\varepsilon,\Delta}}} = \frac{|B_\Delta(w,v_L-v_\alpha)|}{||v_L||_{H^2_{1,\varepsilon,\Delta}}}$$

for each $v_L \in S_L^{(\eta)}$ and $v_\alpha \in S_\alpha^{(k)}$.

Because the inf-sup condition holds for $B_\Delta(\cdot,\cdot)$ over the spaces $S_r \times S_L^{(\eta)}$ for h sufficiently small (Theorem 3.2 & Theorem 4.3), there exists a $v_L \in S_L^{(\eta)}$ such that the left hand side of (5.11) is larger than $C||w||_{H^o_{\infty,\Delta}}$. The right hand side of (5.11) can be bounded by Lemma 5.2, and hence it follows that

$$(5.12) \qquad C_1||w||_{H^o_{\infty,\Delta}} \leq C_2 h^{k+1-r}||w||_{H^o_{\infty,\Delta}} \quad ,$$

where C_1 and C_2 are independent of h and ε. Since $k \geq r$, there exists an h_o such that for $h \leq h_o$, the only way that (5.12) can be satisfied is if $||w||_{H^o_{\infty,\Delta}} = 0$, which implies $w = 0$.

Since

$$B_\Delta(u_\alpha,v_L) = B_\Delta(u_\alpha,v_\alpha) + B_\Delta(u_\alpha,v_L-v_\alpha),$$

it follows from (5.9a,b) that

$$B_\Delta(u_L-u_\alpha,v_L) = F(v_L-v_\alpha) - B_\Delta(u_\alpha,v_L-v_\alpha).$$

From Theorem 3.2 it follows that

$$(5.13) \qquad ||u_L - u_\alpha||_{H^o_{p,\Delta}} \leq C_1 \sup_{v_L\in S_L^{(\eta)}} \frac{|F(v_L-v_\alpha)|}{||v_L||_{H^2_{q,\varepsilon,\Delta}}}$$

$$+ C_2 \sup_{v_L\in S_L^{(\eta)}} \frac{|B_\Delta(u_\alpha,v_L-v_\alpha)|}{||v_L||_{H^2_{q,\varepsilon,\Delta}}}$$

for each $v_\alpha \in S_\alpha^{(k)}$, and $1 \leq p \leq \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$.

First, consider the term $\sup_{v_L\in S_L^{(\eta)}} \frac{|F(v_L-v_\alpha)|}{||v_L||_{H^2_{p,\varepsilon,\Delta}}}$. Let $F_j(x) = \int_{x_{j-1}}^{x} f_o(s)dx$

for each $x \in I_j$. Let $\tilde{F}_j$ be the polynomial of degree k on I_j such that

$\tilde{F}_j(x_j+ih_j/k) = F_j(x_j+ih_j/k)$ for $i = 0,1,\ldots,k$. Then

$$(5.14) \qquad ||F_j - \tilde{F}_j||_{L_\infty(I_j)} \leq \frac{||f_o^{(k)}||_{L_\infty(I_j)} h_j^{k+1}}{4(k+1)!} .$$

As before, let $v_\alpha \in S_\alpha^{(k)}$ be such that $v_\alpha(x_j) = v_L(x_j)$, $j = 0,\ldots,N$. Then,

$$F(v_L-v_\alpha) = \sum_{j=1}^{N} \int_{I_j} f_o(x)(v_L-v_\alpha)(x)dx$$

$$= - \sum_{j=1}^{N} \int_{I_j} F_j(x)(v_L'-v_\alpha')(x)dx$$

$$= \sum_{j=1}^{N} \int_{I_j} (F_j - \tilde{F}_j)(v_L' - v_\alpha')dx,$$

the last equality following from (5.3). Thus,

(5.15) $$|F(v_L - v_\alpha)| \leq \sum_{j=1}^{N} ||F_j - \tilde{F}_j||_{L_\infty(I_j)} ||v_L' - v_\alpha'||_{L_1(I_j)} .$$

From (5.15), (5.14), Lemma 5.1, inequality (2.18), and the fact that $||v||_{H^2_{1,\varepsilon,\Delta}} \leq ||v||_{H^2_{q,\varepsilon,\Delta}}$, $1 \leq q \leq \infty$, it follows that

(5.16) $$\sup_{v_L \in S_L^{(\eta)}} \frac{|F(v_L - v_\alpha)|}{||v_L||_{H^2_{q,\varepsilon,\Delta}}} \leq C \max_j h_j^{k+1} ||f_o^{(k)}||_{L_\infty(I_j)}.$$

Using Lemma 5.2, and inequality (5.16) it follows from (5.13) that

(5.17) $$||u_L - u_\alpha||_{H^o_{p,\Delta}} \leq C_1 \max_j h_j^{k+1} ||f_o^{(k)}||_{L_\infty(I_j)}$$

$$+ C_2 \max_j h_j^{k+1-r} \{ ||a^{(k+1-r)}||_{H^r_\infty(I_j)}$$

$$+ ||b^{(k-r)}||_{H^r_\infty(I_j)} \} ||u_\alpha||_{L_\infty(I_j)} .$$

By hypothesis, $||u||_{L_\infty}$ is bounded indenendently of ε. Since u_L is a quasi-optimal approximation to u, it follows that $||u_L||_{L_\infty}$ is bounded independently of ε. From (5.17) it follows that if h is sifficiently small, then $||u_\alpha||_{L_\infty}$ is also bounded independently of ε. That $||u_\alpha||_{L_\infty}$ is bounded independently of ε, and Δ (provided h is sufficiently small), combined with (5.17), proves the theorem.

This theorem shows that if we project the space $S_L^{(\eta)}$ onto k upwinded polynomials in $S_\alpha^{(k)}$, then the finite element solution with $S_\alpha^{(k)}$ as the test space will have an error composed of two parts:

$$||u - u_\alpha||_{H^o_{p,\Delta}} \leq ||u - u_L||_{H^o_{p,\Delta}} + ||u_L - u_\alpha||_{H^o_{p,\Delta}} \quad .$$

The first part $||u - u_L||_{H^o_{p,\Delta}}$ is quasi-optimal and hence the best order of this term that can be expected is $O(h^{r+1})$. If $k = 2r + 1$, the theorem says that the second term $||u_L - u_\alpha||_{H^o_{p,\Delta}} = O(h^{r+2})$, which is one higher order than the optimal error.

Corollary 5.4. Suppose that all the assumptions of Theorem 5.3 hold with k=2r+1. Then there exists an h_o independent of ε such that for all $h \leq h_o$

$$||u - u_\alpha||_{H^o_{p,\Delta}} \leq C_1 \inf_{w \in S_r} ||u - w||_{H^o_{p,\Delta}} + C_2 h^{r+2} \quad .$$

We remark that when using a polynomially upwinded test space the term $C_2 h^{r+2}$ is unadvoidable. It was proven in [3] and [34] that quasi-optimality is unobtainable when a test space containing basis functions of the form (5.1), with $g_i(x)$ independent of ε and h, is used. However, if local smoothness on a, b and f is assumed-a condition always satisfied is practice, we can obtain an additional error of $O(h^{r+2})$. This error can in general be neglected because the best approximation from a function $w \in S_r$ has order $O(h^{r+1})$.

Because of the second part of this work - the a-posteriori error estimates, it is important to keep the projection error of one higher order than the optimal error. This should also be true of the numerical quadrature errors. A quadrature rule, which is exact when a(x) is a piecewise polynomial of degree r+2, and b and f are piecewise polynomials of degree r+1, is derived in [34], and shown to produce an error of order $O(h^{r+2})$ as well.

CHAPTER 6

CONCLUSION

In this paper it has been shown that quasi-optimality is obtainable for a finite element solution when the test space is composed of functions which are "nearly" L* splines. The norm used to measure the errors is very close to an Lp norm which is important particularly if the location and shape of the boundary layer are important. Furthermore, although it was shown in [3] and [34] that quasi-optimality is unobtainable when using a polynomially upwinded test space, we have shown that a "nearly" quasi-optimal result is possible if the input functions a(x), b(x) and f(x) are piecewise smooth. This "nearly" quasi-optimal result is sufficient for finding a-posteriori error estimates and proving that the error estimate converges to the true error as $h = h(\Delta) \to 0$. This result is proven and an adaptive mesh refinement procedure and numerical results are presented in the second part of this paper [35].

Many of the results of this paper (particularly the embedding result) used bounds on the Green's function. For turning point problems, in which there is a point $x_0 \in I$ such that $a(x_0) = 0$, the Green's function is not bounded independently of ε. However, in [6] sharp bounds are given for the Green's functions of turning point problems. The numerical results for a turning point problem are given in Part II, [35] and suggest that analogous results hold for this case as well.

Upwinding can easily be implemented in two dimensional problems. On a rectangular mesh, upwinding can be done by simply upwinding in each direction separately. This is done in [19] - [21] for ϵ- quadratic upwinding, in [23] for upwinding the integration point, and in [21] for upwinded elements which are local asymptotic expansions of the solution.

In [7], Brooks and Hughes describe a method in which upwinding is done only in the direction of the flow (streamline diffusion method). A mathematical analysis of this method was performed by Navert [31]. Interior error estimates were proven to decrease with rate $h^{k+\frac{1}{2}}$ in L_2 which is $\frac{1}{2}$ of a power lower than the optimal rate. However, the optimal rate was observed in the numerical results.

The polynomial upwinding presented in this paper can also be implemented in two dimensions. Nevertheless, major theoretical questions still remain unsolved.

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