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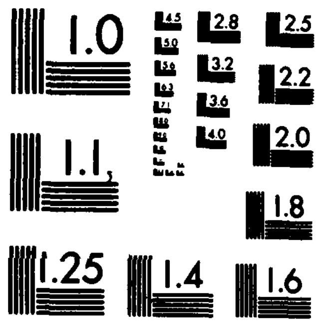

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The Theory of Optimal Confidence Limits for Systems Reliability with Counterexamples for Results on Optimal Confidence Limits for Series Systems

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Abstract

The paper treats the general theory of optimal confidence limits for systems reliability introduced by Buehler. ~~(1957)~~. These general statements are specialized to the case of series systems. It is noted that many results previously given are false. In particular, counterexamples for results of Sudakov, ~~(1974)~~, Winterbottom (1974) and Harris and Soms (1980,1981) are given. Numerical examples are provided, which suggest that despite the deficiencies of these results, they are nevertheless valid for those significance levels likely to be used in practice.

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1. Introduction and Summary

A problem of substantial importance to practitioners in reliability is the statistical estimation of the reliability of a system of stochastically independent components using experimental data collected on the individual components. In the situations discussed in this paper, the component data consist of a sequence of Bernoulli trials. Thus, for component i, $i=1,2,\ldots,k$, the data is the pair (n_i,Y_i), where n_i is the number of trials and Y_i is the number of observations for which the component functions. $Y_1,Y_2,\ldots,Y_k$ are assumed to be mutually independent random variables.

This problem was treated in Sudakov (1974), Winterbottom (1974), and Harris and Soms (1980,1981); one purpose of the present paper is to exhibit counterexamples to theorems in the above papers.

In Section 2 we discuss the general theory of optimal confidence limits for system reliability so that the notation and definitions to be employed in the balance of the paper have been prescribed. Some general results on optimal confidence limits are established.

In Section 3 the counterexamples previously mentioned are exhibited and the specific errors in the proofs of the theorems are indicated.

Section 4 presents the proof of a special case of the key test theorem (Winterbottom (1974)), the general form of which was invalidated by a counterexample in Section 3.

The consequences for reliability applications are discussed in Section 5.

2. Buehler's Method for Optimal Lower Confidence Bounds for System Reliability

We now introduce the notation, definitions, and assumptions that will be used throughout the balance of this paper.

1. Let p_i, $i=1,2,\ldots,k$ denote the probability that the i^{th} component functions. The components will be assumed to be stochastically independent. The reliability of the system will be denoted by $h(\tilde{p})$, where $\tilde{p} = (p_1,p_2,\ldots,p_k)$, $0 \le p_i \le 1$. It is assumed that $h(0,0,\ldots,0) = 0$, $h(1,1,\ldots,1) = 1$, and that $h(\tilde{p})$ is non-decreasing in each p_i, $i=1,2,\ldots,k$. Further, $h(\tilde{p})$ is continuous on $\{\tilde{p}\,|\,0 \le p_i \le 1\}$, which follows readily from the assumption of independence. These properties hold for coherent systems (see Barlow and Proschan (1975)).

2. Let $S = \{\tilde{x}\,|\,x_i=0, 1,\ldots; n_i, i=1,2,\ldots,k\}$. $g(\tilde{x})$ is said to be an ordering function if for $x_1 \le z_1$, $x_2 \le z_2,\ldots, x_k \le z_k$, $\tilde{x}$, $\tilde{z} \in S$, $g(\tilde{x}) \ge g(\tilde{z})$. (It is often convenient to normalize $g(\tilde{x})$ by letting $g(\tilde{0}) = 1$ and $g(n_1,n_2,\ldots,n_k) = 0$. With such a normalization, $g(\tilde{x})$ is often selected to be a point estimator of $h(\tilde{p})$.)

3. Let $R = \{r_1,r_2,\ldots,r_s,\ s \ge 2\}$ be the range set of $g(\tilde{x})$. With no loss of generality we order R so that $r_1 > r_2 > \ldots > r_s$.

4. Let $A_i = \{\tilde{x}\,|\,g(\tilde{x}) = r_i,\ \tilde{x} \in S,\ i=1,2,\ldots,s\}$. The sets A_i constitute a partition of S induced by $g(\tilde{x})$.

5. We assume throughout that the data is distributed by

$$f(\tilde{x};\tilde{p}) = P_{\tilde{p}}(\tilde{X}=\tilde{x}) = \prod_{i=1}^{k}\binom{n_i}{x_i}p_i^{n_i-x_i}\,q_i^{x_i} = \prod_{i=1}^{k}\binom{n_i}{y_i}p_i^{y_i}\,q_i^{n_i-y_i}, \quad (2.1)$$

where $q_i = 1-p_i$, $x_i = n_i-y_i$, $i=1,2,\ldots,k$. With no loss of generality, we assume $n_1 \le n_2 \le \dots \le n_k$.

From these definitions, it follows that

$$P_{\tilde{p}}\left\{X \in \bigcup_{i=1}^{j} A_i\right\} = P_{\tilde{p}}\left\{g(\tilde{X}) \ge r_j\right\} . \qquad (2.2)$$

From (2.1) and (2.2), we have

$$P_{\tilde{p}}\left\{g(\tilde{X}) \ge r_j\right\} = \sum_{i_1=0}^{u_1} \sum_{i_2=0}^{u_2} \cdots \sum_{i_k=0}^{u_k} f(\tilde{i};\tilde{p}) , \qquad (2.3)$$

where $\tilde{i} = (i_1, i_2, \dots, i_k)$ and $u_2 = u_2(i_1), \dots, u_k = u_k(i_1, i_2, \dots, i_{k-1})$ are integers determined by r_j.

6. Subsequently we will need to extend the definitions of S and $g(\tilde{x})$ to real values. We denote this as follows. Let

$$S^* = \left\{\tilde{x} \mid 0 \le x_i \le n_i,\ i=1,2,\dots,k\right\}$$

and let $\bar{g}(\tilde{x})$ be defined on S^* with $\bar{g}(\tilde{x}) \ge \bar{g}(\tilde{z})$, $\tilde{x}$, $\tilde{z} \in S^*$, whenever $\tilde{x} \le \tilde{z}$, and $\bar{g}(\tilde{x}) = g(\tilde{x})$ for $\tilde{x} \in S$.

Then

$$P_{\tilde{p}}\left\{g(\tilde{X}) \ge r_j\right\} = \sum_{i_1=0}^{[t_1]} \sum_{i_2=0}^{[t_2]} \cdots \sum_{i_k=0}^{[t_k]} f(\tilde{i};\tilde{p}) , \qquad (2.4)$$

where $t_2 = t_2(i_1), \dots, t_k = t_k(i_1, i_2, \dots, i_{k-1})$, with $t_1 = \sup\left\{t \mid t \in S^* \text{ and } g(t,0,0,\dots,0) \ge r_j\right\}$ and $t_\ell(i_1, i_2, \dots, i_{\ell-1}) = \sup\left\{t \mid t \in S^* \text{ and } g(i_1, i_2, \dots, i_{\ell-1}, t, 0, \dots 0) \ge r_j\right\}$, $\ell = 2,3,\dots,k$.

We now introduce the notion of Buehler optimal confidence bounds. Let $g(x) = r_j$. Then define

$$a_{g(\tilde{x})} = \inf\left\{h(\tilde{p}) \mid P_{\tilde{p}}\left\{\tilde{i} \mid g(\tilde{i}) \ge g(\tilde{x})\right\} \ge \alpha\right\} . \qquad (2.5)$$

Equivalently, by (2.2), we can also write

$$a_{g(\tilde{x})} = \inf\left\{h(\tilde{p}) \,\middle|\, P_{\tilde{p}}\left\{X \in \bigcup_{i=1}^{j} A_i\right\} \geq \alpha\right\} . \tag{2.6}$$

We now establish the following theorem.

<u>Theorem 2.1</u>. Let assumptions 1-5 be satisfied. Then, for $\tilde{x} \in S$, $a_{g(\tilde{x})}$ is a 1-α lower confidence bound for $h(\tilde{p})$. If $b_{g(\tilde{X})}$ is any other 1-α lower confidence bound for $h(\tilde{p})$ with $b_{r_1} \geq b_{r_2} \geq \ldots \geq b_{r_j}$, then $b_{g(\tilde{x})} \leq a_{g(\tilde{x})}$ for all $\tilde{x} \in S$.

<u>Proof</u>. Fix $\tilde{p}$ and let $m(\tilde{p})$ be the smallest integer such that

$$P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{m(\tilde{p})} A_i\right\} \geq \alpha .$$

Then

$$P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=m(\tilde{p})}^{s} A_i\right\} \geq 1-\alpha .$$

Let

$$D_{r_m} = \left\{\tilde{p} \,\middle|\, P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{m} A_i\right\} \geq \alpha\right\} .$$

Then $D_{g(\tilde{X})}$ is a 1-α confidence set for $\tilde{p}$, since

$$P_{\tilde{p}}\left\{\tilde{p} \in D_{g(\tilde{X})}\right\} = P_{\tilde{p}}\left\{g(\tilde{X}) \leq r_{m(\tilde{p})}\right\} \geq 1-\alpha .$$

By assumption 1, $h(\tilde{p})$ is continuous and the set of parameter points satisfying (2.5) is compact; therefore the infimum in (2.5) and (2.6) is attained.

Assume that there is an integer j, $1 \leq j \leq s-1$, such that $b_{r_j} > a_{r_j}$. Then there exists a $\tilde{p}_o$ such that

$$b_{r_j} > a_{r_j} = \inf\left\{h(\tilde{p}) \,\middle|\, P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{j} A_i\right\} \geq \alpha\right\} = h(\tilde{p}_o) . \tag{2.7}$$

In addition, there exists a $\tilde{p}_1$ such that

$$P_{\tilde{p}_1}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{j} A_i\right\} > \alpha \ , \quad h(\tilde{p}_1) < b_{r_j} \ . \tag{2.8}$$

Since $b_{r_1} \geq b_{r_2} \geq \cdots \geq b_{r_s}$, from (2.7) we have

$$h(\tilde{p}_1) < b_{r_\ell} \ , \quad \ell = 1,2,\ldots,j \ . \tag{2.9}$$

Therefore

$$\alpha < P_{\tilde{p}_1}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{j} A_i\right\} \leq P_{\tilde{p}_1}\left\{h(\tilde{p}_1) < b_{g(\tilde{X})}\right\}, \tag{2.10}$$

which is a contradiction. Consequently, there is no integer j, $1 \leq j \leq s-1$, for which $b_{r_j} > a_{r_j}$.

Remarks. From (2.6), it follows that $a_{r_s} = 0$ and b_{r_s} is also necessarily zero. Note further that in (2.7) it is possible that the infimum is attained at a point for which $P_{\tilde{p}}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{j} A_i\right\} > \alpha$. To see this consider the following example.

Let $k = 2$, $n_1 = 5$, $n_2 = 10{,}000$, $x_1 = 0$, $x_2 = 5$, $g(\tilde{x}) = n_1+n_2-x_1-x_2$, $h(\tilde{p}) = p_1p_2$. It is easily seen that the hypotheses of Theorem 2.1 are satisfied. Thus, for the data given, $g(\tilde{x}) = 10{,}000 = r_6$. The set $\bigcup_{i=1}^{6} A_i$ consists of all points (x_1,x_2) for which $x_1+x_2 \leq 5$, that is, $A_1 = \left\{(0,0)\right\}$, $A_2 = \left\{(1,0),(0,1)\right\}$, and so on. Consequently,

$$D_{r_6} = \left\{\tilde{p} \,|\, P_{\tilde{p}}\left\{\tilde{X} \varepsilon \bigcup_{i=1}^{6} A_i\right\} \geq \alpha\right\}$$

includes the parameter points $(0, p_{2\alpha})$ where $p_{2\alpha}$ satisfies $P_{p_{2\alpha}}\left\{X_2=0\right\} \geq \alpha$, since $P_{\tilde{p}_1}\left\{X_1 \leq 5\right\} = 1$ when $p_1 = 0$. Thus inf $h(\tilde{p}) = 0$ for all $0 < \alpha < 1$.

The reader should also note that the monotonicity of $h(\tilde{p})$ is not utilized in the proof, which is valid whenever $h(\tilde{p})$ is continuous.

It is easy to see that $a_{g(\tilde{x})}$ is monotone, i.e., $a_{r_1} \geq a_{r_2} \geq \ldots \geq a_{r_s}$. This follows from (2.7) upon noting that as j increases, the set of $\tilde{p}$ satisfying (2.7) increases and the infimum is taken over a larger set.

Corollary. For a series system $h(\tilde{p}) = \prod_{i=1}^{k} p_i$. Then if $g(\tilde{x}) = \prod_{i=1}^{k} (n_i-x_i)/n_i = \prod_{i=1}^{k} y_i/n_i$, the hypotheses of Theorem 2.1 are satisfied and the conclusion follows.

Remark. Note that $g(\tilde{x}) = \prod_{i=1}^{k} (n_i-x_i)/n_i$ is the maximum likelihood estimator as well as the minimum variance unbiased estimator of $\prod_{i=1}^{k} p_i$ and is therefore a natural choice of an ordering function for this case.

We now establish the following theorem.

Theorem 2.2. Let $g(\tilde{x}) = r_j$ and let

$$f^*(x;a) = \sup_{h(\tilde{p})=a} P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\}, \quad 0 < a < 1 . \qquad (2.10)$$

Then

$$\sup_{0<a<1} f^*(\tilde{x};a) = 1$$

and $f^*(\tilde{x};a)$ is non-decreasing in a.

Proof. Since $h(\tilde{p})$ is continuous and $h(\tilde{1}) = 1$,

$$\lim_{a\to 1} \sup_{h(\tilde{p})=a} P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} = 1 .$$

Now choose a and b such that $0 < a < b < 1$,

$$P_{\tilde{p}_a}\left\{g(\tilde{X}) \geq r_j\right\} = f^*(\tilde{x};a)$$

and

$$P_{\tilde{p}_b}\left\{g(\tilde{X}) \geq r_j\right\} = f^*(\tilde{x};b) .$$

Let I_a be the set of indices i such that $p_{ia} < 1$. Then it is possible to replace p_{ia} by p'_{ib}, $i \in I_a$, where $p_{ia} < p'_{ib} < 1$, so that $h(\tilde{p}'_b) = b$, where $p'_{ib} = p_{ia}$, $i \in I_a^c$. This follows since $h(\tilde{1}) = 1 > a$ and $h(\tilde{p})$ is continuous. The conclusion follows from the monotone likelihood ratio property of the binomial distribution.

<u>Remark</u>. Only the continuity of $h(\tilde{p})$ was used in the proof of Theorem 2.2.

For the case of series systems, it is possible to strengthen Theorem 2.2 and to exhibit the above construction. This is done below.

<u>Corollary</u>. Let $g(\tilde{x}) = r_j$. If $h(\tilde{p}) = \prod_{i=1}^{k} p_i$, then $\inf_{0<a<1} f^*(\tilde{x};a) = 0$ and $f^*(\tilde{x};a)$ is strictly increasing in a whenever all $u_j < n_j$ (see (2.3) for the definition of u_j), j=1,2,...k.

<u>Proof</u>. From the hypotheses,

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} \leq 1-q_i^{\,n_i} \quad , \quad i=1,2,\ldots,k ,$$

and since $\prod_{i=1}^{k} p_i \to 0$ implies at least one $p_i \to 0$, this gives

$$\inf_{0<a<1} f^*(\tilde{x};a) = 0 .$$

To show that $f^*(\tilde{x};a)$ is strictly increasing in a, consider 0<a<b<1 and let $\tilde{p}_a = (p_{a1},\ldots,p_{ak})$ satisfy $f^*(\tilde{x};a) = P_{\tilde{p}_a}\left\{g(\tilde{X}) \geq r_j\right\}$. Similarly, let $\tilde{p}_b$ satisfy $f^*(\tilde{x};b) = P_{\tilde{p}_b}\left\{g(\tilde{X}) \geq r_j\right\}$. Let $I_a = \left\{i_1,i_2,\ldots,i_r\right\}$ be any non-empty set of indices such that $p_{ai_j}(\frac{b}{a})^{1/r} < 1$ (non-empty because otherwise multiplying the components would give $b > 1$, a contradiction) and let I_a^c be the remaining indices. Then

$$\Big(\prod_{j\varepsilon I_a} p_{ai_j}\big(\tfrac{b}{a}\big)^{1/r}\Big)\prod_{j\varepsilon I_a^c} p_{ai_j} = b \ . \tag{2.11}$$

From the monotone likelihood ratio property of the binomial distribution,

$$P_{\tilde{p}_a}\Big\{g(\tilde{X}) \geq r_j\Big\} < P_{\tilde{p}^*}\Big\{g(\tilde{X}) \geq r_j\Big\} \ ,$$

where the components of p^* are given by (2.11). This gives

$$f^*(\tilde{x};a) < f^*(\tilde{x};b) \ ,$$

which is the desired conclusion.

Remarks. Note that if at least one $u_j = n_j$, it follows immediately from (2.5) that $h(\tilde{p}) = 0$. For $g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i)/n_i$ the condition $u_j < n_j$ is equivalent to $x_j < n_j$, $j=1,2,\ldots,k$.

We now establish the following result, which may be interpreted as a duality theorem. This will prove useful in some of the subsequent material.

Theorem 2.3. If $f^*(\tilde{x};a) = \alpha$, $0<\alpha<1$, has at least one solution in a, then

$$a_{g(\tilde{x})} = \inf\Big\{a \,|\, f^*(\tilde{x};a) = \alpha\Big\}. \tag{2.12}$$

If $f^*(\tilde{x};a) > \alpha$ for all a, then $a_{g(\tilde{x})} = 0$.

Proof. Let

$$c = \inf\Big\{a \,|\, f^*(\tilde{x};a) \geq \alpha\Big\} \ . \tag{2.13}$$

The infimum in (2.13) is attained. Thus, there exists a $\tilde{p}_o$ such that $c = h(\tilde{p}_o)$. If $f^*(\tilde{x};a) > \alpha$ for all a, let $p_i \to 0$, $i=1,2,\ldots,k$. Then $h(\tilde{p}) \to 0$, since $h(\tilde{0}) = 0$ and $h(\tilde{p})$ is continuous, and $a_{g(\tilde{x})} = 0$.

Now assume there is at least one a with $f^*(\tilde{x};a) = \alpha$. Then $f^*(\tilde{x};a_{g(\tilde{x})}) \geq \alpha$ and therefore $c \leq a_{g(\tilde{x})}$. If $c < a_{g(\tilde{x})}$, then $c = h(\tilde{p}_o)$ and $f^*(\tilde{x};c) = \alpha$, which is a contradiction.

Remarks. Again, only the continuity of $h(\tilde{p})$ was used in the proof of Theorem 2.3. Under the hypotheses of the Corollary to Theorem 2.2, for a series system, $a_{g(\tilde{x})}$ is the solution in a of

$$f^*(\tilde{x};a) = \alpha \ . \tag{2.14}$$

The general theory described in this section applies as well to what is known as systems with repeated components (see, e.g., Harris and Soms (1973)). For such systems, there are $1 \leq m \leq k$ unknown parameters $p_1,p_2,\ldots,p_m$, since the "repeated components" are assumed to have identical failure probabilities. This assumption permits the experimenter to regard the data as (n_i,Y_i), $i=1,2,\ldots,m$, and employ the previous results.

For example, if a series system of k components has α_1 of one type, α_2 of a second, ..., α_m of an m^{th} type, then

$$h(\tilde{p}) = p_1^{\alpha_1}\, p_2^{\alpha_2} \cdots p_m^{\alpha_m} \ , \qquad \sum_{i=1}^{k} \alpha_i = k_i \ .$$

3. Counterexamples

In this section we restrict attention to series systems and employ the ordering function

$$g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i)/n_i \ ,$$

introduced following Theorem 2.1. As noted previously, in this case the reliability function $h(\tilde{p}) = \prod_{i=1}^{k} p_i$. With this specialization we have for (2.4)

$$t_1 = n_1(1-r_m) \tag{3.1}$$

and for each fixed $0 \le i_1 \le t_1$, $0 \le i_2 \le t_2$, ..., $0 \le i_{j-1} \le t_{j-1}$,

$$t_j = n_j(1-r_m/[\prod_{\ell=1}^{j-1} (n_\ell - i_\ell)/n_\ell]), \qquad 2 \le j \le k , \tag{3.2}$$

whenever $g(\tilde{x}) = r_m$, $1 \le m < s$. If $m = s$, then $r_s =$ and $a_o = 0$.

For $\kappa > 0$, $\lambda > 0$, let

$$I_p(\kappa,\lambda) = \frac{1}{\beta(\kappa,\lambda)} \int_0^p t^{\kappa-1}(1-t)^{\lambda-1}dt , \qquad 0 \le p \quad , \tag{3.3}$$

the incomplete beta function.

It is well-known that if t is an integer, $t < n$, we have

$$\sum_{i=0}^{t} \binom{n}{i} p^{n-i}q^i = I_p(n-t,t+1) . \tag{3.4}$$

In Sudakov (1974) the following inequality was published.

$$P_{\tilde{p}}\left\{g(\tilde{X}) \ge r_j\right\} \le I_{\prod_{i=1}^{k} p_i}(n_1-t_1,t_1+1) . \tag{3.5}$$

This inequality and generalizations of it were further studied in Harris and Soms (1980,1981). (3.5) implies

$$f^*(\tilde{x};a) \le I_a(n_1-t_1,t_1+1) ,$$

hence its usefulness. However, as we now establish, (3.5) is not universally valid, as was claimed in Sudakov (1974).

Let $(x_1,x_2) = (x_1,0)$ and let $(n_1,n_2) = (n_1,2n_1)$. Then $g(\tilde{x}) = (n_1-x_1)/n_1$ and $t_1 = x_1$. Consider $P_{\tilde{p}}\left\{g(\tilde{X}) \ge r_m\right\}$. If

$\tilde{p} = (1,a)$, $0 < a < 1$, we have

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} = P_a\left\{(n_2 - X_2)/n_2 \geq r_m\right\} ,$$

since $P\left\{X_1 = 0\right\} = 1$, by (2.1). Consequently,

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} = P_a\left\{X_2 \leq n_2(1-r_m)\right\}$$

$$= P_a\left\{X_2 \leq 2n_1(1-r_n)\right\} .$$

Since $r_m = (n_1 - x_1)/n_1$,

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} = P_a\left\{X_2 \leq 2x_1\right\} .$$

Thus from (3.4),

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_m\right\} = I_a(2(n_1 - x_1),\ 2x_1 + 1) .$$

The Sudakov inequality implies that

$$I_a(2(n_1 - x_1),\ 2x_1 + 1) \leq I_a(n_1 - x_1,\ x_1 + 1)$$

or

$$I_a(2n_1 r_m,\ 2n_1(1-r_m)+1) \leq I_a(n_1 r_m,\ n_1(1-r_m)+1) \quad . \tag{3.6}$$

Let $h_2(t;n_2,r_m)$ and $h_1(t;n_1,r_m)$ denote the beta density functions corresponding to the left and right hand side of (3.6), respectively. Then, provided $n_1 r_m > 1$ there is an $\varepsilon > 0$ such that

$$h_2(t;n_2,r_m) < h_1(t;n_1,r_m) \quad 0 < t < \varepsilon,\ 1-\varepsilon < t < 1 .$$

This implies that $h_1(t;n_2,r_m)$ and $h_2(t;n_2,r_m)$ intersect in at least two points. If t^* is such an intersection, setting $h_1(t;n_1,r_m)/h_2(t;n_2,r_m) = 1$ gives

$$t^{n_1 r_m}(1-t)^{n_1(1-r_m)} = c(n_1,r_m) > 0 \ .$$

Thus, for $1 \leq m < s$, there are exactly two such intersections. Therefore there is a z_o such that

$$I_{z_o}(n_1 r_m,\ n_1(1-r_m)+1) = I_{z_o}(n_2 r_m,\ n_2(1-r_m)+1) \quad ,$$

for $z > z_o$,

$$I_z(n_1 r_m,\ n_1(1-r_m)+1) < I_z(n_2 r_m,\ n_2(1-r_m)+1)$$

and for $z < z_o$,

$$I_z(n_1 r_m, n_1(1-r_m)+1) > I_z(n_2 r_m,\ n_2(1-r_m)+1) \ .$$

Thus for $z > z_o$, (3.6) is violated. (3.6) was used as a lemma by Sudakov (1974) to prove the inequality (3.5). This lemma was also employed in Harris and Soms (1980, 1981). It is the falsity of this lemma which invalidates (3.5).

Table 1 provides some illustrations of the violation of (3.5) for $k = 2$ and selected values of (n_1,n_2), (x_1,x_2). The smallest value of $p_1 p_2$ for which this violation occurs is also given in the table, where it is denoted by a^*. In addition, $f^*(\tilde{x};a^*)$ is tabulated. Thus for $\alpha < f^*(\tilde{x};a^*)$, (3.5) is valid.

The calculations were made by means of a FORTRAN program.

Note that for $(n_1,n_2) = (5,5)$ and $(x_1,x_2) = (1,1)$, the inequality was not violated.

Table 1. The Smallest a, a^*, and $f^*(\tilde{x};a^*)$

(n_1,n_2)	(x_1,x_2)	a^*	$f^*(\tilde{x};a^*)$
(5,5)	(1,1)	1.0000	1.0000
(5,5)	(3,3)	.7454	.9998
(5,10)	(1,0)	.8798	.8909
(5,15)	(0,3)	.8698	.8791
(5,30)	(1,0)	.8498	.8467

4. The Theory of Key Test Results

If for $n_1 \le n_2 \le \cdots \le n_k$, $(x_1,x_2,\ldots,x_k) = (x_1,0,\ldots,0)$, $k \ge 2$, then $\tilde{x}$ is called a key test result. Winterbottom (1974) asserted that subject to $x_1 < f(k,n_1)$, where $f(k,n_1)$ is the solution in f of

$$n_1k-f-1 = k[(n_1-f)n_1^{k-1}]^{1/k} , \qquad (4.1)$$

we have $a_{g(\tilde{x})}$ is the solution in a of

$$I_a(n_1-x_1,\ x_1+1) = \alpha \quad , \quad 0 < \alpha < 1 . \qquad (4.2)$$

This would imply the inequality (3.5), which we have disproved in Section 3.

As we subsequently establish, the error in Winterbottom's (1974) result is a consequence of falsely concluding that $f(k,n_1)$ depends only on n_1. It is easy to be led to this conclusion on intuitive grounds, since $(n_1-x_1,\ n_1,\ldots,\ n_1)$ would seem to be a less favorable experimental result than $(n_1-x_1,n_2,\ldots,n_k)$, whenever $n_i > n_1$ for at least one index i, $2 \le i \le k$. However, we will now establish that a "modified" key test result holds for $x_1 < f(k,n_1,n_2,\ldots,n_k)$, where $f(k,n_1,n_2,\ldots,n_k)$ is the solution in f of

$$n_1 - f - 1 + \sum_{i=2}^{k} n_i = k[(n_1-f) \prod_{i=2}^{k} n_i]^{1/k} . \qquad (4.3)$$

Theorem 4.1. If $n_1 \leq n_2 \leq \ldots \leq n_k$ and $\tilde{x} = (x_1,0,\ldots,0)$, with $x_1 < f(k,n_1,n_2,\ldots,n_k) < n_1$, then

$$P_{\tilde{p}}\left\{g(\tilde{X}) \geq r_j\right\} \leq I_{\prod_{i=1}^{k} p_i}(n_1-x_1,x_1+1) , \qquad (4.4)$$

where $g(x_1,0,\ldots,0) = r_j$.

Proof. For $\sum_{i=1}^{k} x_i < n_1$, from Marshall and Olkin (1979, p. 78), $\prod_{i=1}^{k} (n_i-x_i)$ is a strictly Schur-concave function of $(n_1-x_1,n_2-x_2,\ldots,n_k-x_k)$. Thus, if $\sum_{i=1}^{k} (n_i-x_i)$ is fixed, $\prod_{i=1}^{k} (n_i-x_i)$ is minimized at $(n_1 - \sum_{i=1}^{k} x_i,n_2,\ldots,n_k)$. Equivalently, $\prod_{i=1}^{k} (n_i-x_i)$ is minimized for vectors of the type $\tilde{x} = (\sum_{i=1}^{k} x_i,0,\ldots,0)$ when $\sum_{i=1}^{k} x_i$ is fixed.

Let $\tilde{z} = (z_1,z_2,\ldots,z_k)$ with $\sum_{i=1}^{k} z_i \leq x_1 < n_1$. Then

$$\sum_{i=1}^{k} (n_i-z_i) \geq n_1-x_1 + \sum_{i=2}^{k} n_i . \qquad (4.5)$$

For each fixed value of $\sum_{i=1}^{k} (n_i-z_i)$, we have

$$\prod_{i=1}^{k} (n_i-z_i) \geq (n_1 - \sum_{i=1}^{k} z_i) \prod_{i=2}^{k} n_i \geq (n_1-x_1) \prod_{i=2}^{k} n_i . \qquad (4.6)$$

In order that

$$\left\{\tilde{z}\,\middle|\,\prod_{i=1}^{k}(n_i-z_i) \geq \prod_{i=1}^{k}(n_i-x_i)\right\} = \left\{\tilde{z}\,\middle|\,\sum_{i=1}^{k}(n_i-z_i) \geq \sum_{i=1}^{k}(n_i-x_i)\right\}, \quad (4.7)$$

we must have $x_2 = x_3 = \ldots = x_k = 0$. Note that if $\sum_{i=1}^{k} x_i \geq n_1$, $x_1 < n_1, x_2 < n_2, \ldots, x_k < n_k$, the two sets cannot coincide, because $(0, n_2, \ldots, n_k)$ is in the right hand set but not the left. From (4.6) it follows that

$$\left\{\tilde{z}\,\middle|\,\sum_{i=1}^{k}(n_i-z_i) \geq n_1-x_1+\sum_{i=2}^{k}n_i\right\} \subset \left\{\tilde{z}\,\middle|\,\prod_{i=1}^{k}(n_i-z_i) \geq (n_1-x_1)\prod_{i=2}^{k}n_i\right\}. \quad (4.8)$$

Equality holds if $\max \prod_{i=1}^{k}(n_i-z_i) < (n_1-x_1)\prod_{i=2}^{k}n_i$ when $\sum(n_i-z_i) = (n_1-x_1) + \sum_{i=2}^{k}n_i-1$. From the arithmetic-geometric mean inequality this is true whenever

$$\left[\frac{n_1-x_1+\sum_{i=2}^{k}n_i-1}{k}\right]^k < (n_1-x_1)\prod_{i=2}^{k}n_i \ . \quad (4.9)$$

Note that equality in (4.7) may still hold if (4.9) is violated since $(n_1-x_1+\sum_{i=2}^{k}n_i-1)/k$ may not be an integer or may be bigger than some of the n_i, $i=1,2,\ldots,k$. Thus if x_1 is the smallest x_1 value for which equality holds in (4.9), then $f(k,n_1,n_2,\ldots,n_k) = x_1$.

If $x_1 < f(k,n_1,n_2,\ldots,n_k)$, then, from the above,

$$f^*(\tilde{x};a) = \sup_{\prod_{i=1}^{k}p_i=a} P\left\{\sum_{i=1}^{k}Y_i \geq n_1-x_1+\sum_{i=2}^{k}n_i\right\} . \quad (4.10)$$

Writing (4.10) as an iterated sum and noting that $I_t(n-x,x+1)$ is a decreasing function of n for fixed x, we have

$$\sup_{\prod_{i=1}^{k} p_i = a} P\left\{\sum_{i=1}^{k} Y_i \geq n_1 - x_1 + \sum_{i=2}^{k} n_i\right\} \leq \sup_{\prod_{i=1}^{k} p_i = a} P\left\{Y_1 + \sum_{i=2}^{k} U_i \geq n_1 - x_1 + (k-1)n_1\right\},$$

where the U_i are independent binomial random variables with parameters (n_1, p_i), $i=2,\ldots,k$. Writing

$$Y_1 + \sum_{i=2}^{k} U_i = \sum_{i=1}^{k} \sum_{j=1}^{n_1} Y_{ij} ,$$

where the Y_{ij} are independent Bernoulli random variables with parameter p_i, a result of Pledger and Proschan (1971) may be employed to show that the upper tail of $\sum_{i=1}^{k} \sum_{j=1}^{n} Y_{ij}$ is a Schur-convex function of $(-\ln p_1, -\ln p_1, \ldots, -\ln p_1, -\ln p_2, \ldots, -\ln p_2, \ldots, -\ln p_k, \ldots, -\ln p_k)$ and therefore $f^*(\tilde{x};a) = I_a(n_1 - x_1, x_1 + 1)$, as required.

As is discussed below, (4.3) may have no solutions. In such cases, and in general, it is possible to strengthen (4.3).

<u>Corollary</u>. For each f, form the vector $\tilde{z} = (z_1, z_2, \ldots, z_k)$ from $\tilde{n} = (n_1, n_2, \ldots, n_k)$ by continually reducing the maximum (s) until the subtractions total $f+1$, $f \geq 0$. Denote by $f'(k, n_1, n_2, \ldots, n_k)$ the first f for which

$$\prod_{i=1}^{k} z_i \geq (n_1 - f) \prod_{i=2}^{k} n_i .$$

Then (4.4) holds for $x_1 < f'(k, n_1, n_2, \ldots, n_k)$.

<u>Proof</u>. The proof proceeds exactly as for Theorem 4.1 by noting that $\tilde{z}$ maximizes $\prod_{i=1}^{k} r_i$ subject to $0 < r_i \leq n_i$ and $\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} n_i - f - 1$. This follows since $\tilde{z}$ is majorized by $\tilde{r}$ and the product is strictly Schur-concave.

<u>Remarks</u>. For $0 \leq f \leq n_1$, the right hand side of (4.3) is concave decreasing. The left hand side exceeds the right hand side when $f = n_1$. If the left hand side is less than the right at $f = 0$, there is exactly one solution f, $0 < f < n_1$. If not, there are no solutions. There is always a solution if $n_1 = n_2 = \ldots = n_k$. From the *Corollary following* Theorem 4.1, $x_1 = 0$ satisfies (4.4). If $n_1 = n_2 = \ldots = n_k$, (4.3) reduces to (4.1) which is Winterbottom's (1974) condition. However, s should be replaced by s+1 in his formula, which also has a sign error. As an example, for $k = 2$, $n_1 = n_2 = 50$, from Winterbottom (1974), (4.4) is stated to hold for $x_1 \leq 17$ or $n_1 - x_1 \geq 33$. However, $33 \cdot 50 < 41 \cdot 41$, and therefore (4.4) only holds for $x_1 \leq 13$ or $n_1 - x_1 \geq 37$, as the Corollary to Theorem 4.1 shows, or the solution of (4.3), which gives $f(2,50,50) = 13.14$.

The dependence of f on $\tilde{n}$ may be seen by considering an example. Let $k = 2$, $n_1 = 5$, $n_2 = 10$. Then from the Corollary *following* Theorem 4.1, (4.4) only holds for $x_1 = 0$, whereas for $n_1 = n_2 = 5$, it holds for $x_1 = 0,1,2$, and 3. Thus the case of equal n_i, $i=1,2,\ldots,k$, does not give the minimal f. In fact, it may be seen that if $n_k \geq 2n_1$, then (4.4) holds only for $x_1 = 0$.

5. Concluding Remarks

From Table 1, it seems reasonable to conjecture that (3.5) is valid for those values of $\alpha, k, \tilde{n}$ likely to arise in practice. The authors are continuing to investigate the problem and hope to report more precise conditions for the validity of (3.5) in subsequent work.

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The paper treats the general theory of optimal confidence limits for systems reliability introduced by Buehler (1957). These general statements are specialized to the case of series systems. It is noted that many results previously given are false. In particular, counterexamples for results of Sudak (1974), Winterbottom (1974) and Harris and Soms (1980,1981) are given. Numeri examples are provided, which suggest that despite the deficiencies of these results, they are nevertheless valid for those significance levels likely to used in practice.

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