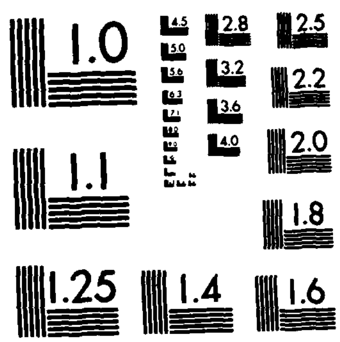
AD-A128 093   BLOW-UP OF RADIAL SOLUTIONS OF LINEAR WAVE EQUATION U =   1/1
PARTIAL DERIVATIV..(U) WISCONSIN UNIV-MADISON
MATHEMATICS RESEARCH CENTER F JOHN MAR 83 MRC-TSR-2493

UNCLASSIFIED   DAAG29-80-C-0041   F/G 12/1   NL


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

MRC Technical Summary Report # 2493

BLOW-UP OF RADIAL SOLUTIONS OF $\Box u = \frac{\partial F(u_t)}{\partial t}$ IN THREE SPACE DIMENSIONS

Fritz John

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

March 1983

(Received December 13, 1982)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

BLOW-UP OF RADIAL SOLUTIONS OF $\Box u = \frac{\partial F(u_t)}{\partial t}$ IN THREE SPACE DIMENSIONS

FRITZ JOHN

Technical Summary Report #2493

March 1983

SUMMARY

Under consideration are strict solutions $u = u(x_1,x_2x_3,t) = u(x,t)$ of the differential equation

$$\Box u = u_{tt} - \Delta u = \frac{\partial F(u_t)}{\partial t} \tag{*}$$

which are "radial" (or have "spherical symmetry") in the sense that u only depends on t and on $r = |x|$. We prescribe initial conditions

$$u(x,0) = \varepsilon f(r), \ u_t(x,0) = \varepsilon g(r) \tag{**}$$

for u, where f and g are even functions in r of class C^∞ (for simplicity) and of compact support, and $\varepsilon > 0$ is a parameter that measures the "amplitude" of the initial data. We assume that equation (*) reduces to the linear wave equation $\Box u = 0$ for "infinitesimal" u, that is we assume that

$$F'(0) = 0 \quad .$$

In addition we postulate that (*) is "genuinely nonlinear" in the sense that

$$F''(0) \neq 0 \quad .$$

Without restriction of generality we can always assume that

$$F''(0) > 0$$

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

(if necessary changing u into $-u$) and that f and g have their support in the unit ball:

$$f(r) = g(r) = 0 \quad \text{for} \quad |r| > 1 \quad .$$

We show here that every non-trivial solution u blows up after a finite time T if ε is sufficiently small. More precisely for given f, g, F there exists a constant ε_0 and a function $A(\varepsilon)$ such that

$$T < \exp(\frac{A(\varepsilon)}{\varepsilon}) \tag{***}$$

for all $\varepsilon < \varepsilon_0$. Here $A(\varepsilon)$ is bounded independently of ε:

$$C = \limsup_{\varepsilon \to 0} A(\varepsilon) < \infty \quad .$$

This result has to be compared with the known lower and upper bounds for T. In [4] the author showed that $T=T(\varepsilon)$ increases faster than any reciprocal power of ε, as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \varepsilon^N T(\varepsilon) = \infty \quad \text{for any} \quad N \quad .$$

This lower bound for T has been improved dramatically by S. Klainerman, [1], who showed that for radial solutions

$$T > \exp(\frac{B}{\varepsilon})$$

with a positive constant B. In view of (***), Klainerman's lower bound for T is optimal in the general case. Upper bounds for T had been given previously (see [2], [3]) without requiring the initial data to be radial or ε to be small. But then certain inequalities for f and g had to be postulated, and in addition assumptions had to be made on the behavior of F for large arguments. (Results of this latter type have also been derived for other types of differential equations with spherical symmetry by Th. C. Sideris [5].)

The argument used in the present paper is based on the use of differential equations for the second derivatives of u along characteristic curves (as was done in [6] in the case of one space dimension). This emphasizes blow-up as a <u>local</u> phenomenon. We show that for small radial initial data singularities are formed, even if the differential equation (*) is imposed on u just for small values of u_t. For the singularities in question u and its first derivatives stay small, while certain second derivatives become infinite. (This does however not exclude the possibility that other types of singularities with different behavior form earlier in other parts of the domain of u.) Blow-up takes place only after an exceedingly long time, and only after the solution has passed through a phase where the second derivatives are exceedingly small. Qualitatively the behavior of the second derivatives resembles that of the function

$$\phi(t) = \frac{\varepsilon}{t(1-\varepsilon \log t)} .$$

Setting $T = e^{1/\varepsilon}$ we have for that function

$$\phi(1) = \varepsilon$$

$$\phi(T/e) = \varepsilon e^{-1/\varepsilon}$$

$$\phi(T-1) \sim 1$$

$$\phi(T) = \infty .$$

AMS (MOS) Subject Classifications: 35L15, 35L67, 35L70, 35B40

Key Words: Nonlinear hyperbolic equations, wave equations, radial solutions, blow-up in three space dimensions.

Work Unit Number 1 - Applied Analysis

BLOW-UP OF RADIAL SOLUTIONS OF $\Box u = \frac{\partial F(u_t)}{\partial t}$ IN THREE SPACE DIMENSIONS

Fritz John

The differential equation for $u(x,t) = u(x_1,x_2,x_3,t)$ is

$$\Box u = u_{tt} - \Delta u = F'(u_t)u_{tt} \tag{1a}$$

where it is assumed only

$$F''(0) = a > 0 \quad ; \quad F'(0) = 0 \quad . \tag{1b}$$

We can always continue $F(\lambda)$ outside of a neighborhood of the origin in such a way that also

$$F'(\lambda) < \frac{1}{2} \quad \text{for all} \quad \lambda \tag{1c}$$

so that equation (1a) is hyperbolic for all arguments; but actually we will only need $F'(\lambda)$ for λ near 0. The aim is to show that all nontrivial solution of (1a) with spherical symmetry and with initial data of compact support and sufficiently small, blow up in finite time. We thus take u of the form

$$u = u(r,t) \quad ; \quad r = |x| \tag{1d}$$

with initial data

$$u = \varepsilon f(r) \quad ; \quad u_t = \varepsilon g(r) \quad \text{for} \quad t = 0; \ \varepsilon > 0 \tag{1e}$$

where:

$$f(r) = f(-r) \quad ; \quad g(r) = g(-r) \quad \text{for all} \quad r \tag{1f}$$

$$f(r) = g(r) = 0 \quad \text{for} \quad |r| > 1 \quad . \tag{1g}$$

THEOREM

If ε is sufficiently small the solution u either vanishes identically or has a finite life span T. Here T satisfies an estimate of the form

$$T < \exp\left(\frac{A(\varepsilon)}{\varepsilon}\right) \tag{1h}$$

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

where $A(\varepsilon)$ is a certain function which is bounded for sufficiently small $\varepsilon > 0$; more precisely

$$\limsup_{\varepsilon \to 0} A(\varepsilon) = \frac{2}{a \sup k(\lambda)} \tag{1i}$$

where k is the function defined by

$$k(\lambda) = \frac{1}{2}\,(\lambda f''(\lambda) + 2f'(\lambda) - \lambda g'(\lambda) - g(\lambda)) \quad . \tag{1j}$$

PROOF.

In the spherically symmetric case we can write (1a) as

$$(ru)_{tt} - (1+p)^2(ru_{rr}) = 0 \tag{2a}$$

where p is defined by

$$1+p = \frac{1}{\sqrt{1-F'(u_t)}} \quad . \tag{2b}$$

We introduce the vector

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (ru)_r \\ (ru)_t \end{pmatrix} \quad . \tag{2c}$$

Then (2a) yields the system

$$V_t + AV_r = 0 \quad \text{with} \quad A = \begin{pmatrix} 0 & -1 \\ -(1+p)^2 & 0 \end{pmatrix} \quad . \tag{2d}$$

Here A has the eigenvalues $\pm(1+p)$ and eigenvectors

$$\xi^1 = \binom{1}{-(1+p)} \quad ; \quad \xi^2 = \binom{1}{1+p} \quad . \tag{2e}$$

Set

$$V_r = w_1\xi^1 + w_2\xi^2 \tag{3a}$$

where then

$$\begin{aligned} w_1 &= \frac{1}{2}\,v_{1r} - \frac{1}{2(1+p)}\,v_{2r} \\ &= \frac{r((1+p)u_{rr} - u_{tr}) + 2(1+p)u_r - u_t}{2(1+p)} \end{aligned} \tag{3b}$$

$$w_2 = \frac{1}{2} v_{1r} + \frac{1}{2(1+p)} v_{2r}$$

$$= \frac{r((1+p)u_{rr}+u_{tr})+2(1+p)u_r+f_t}{2(1+p)} , \tag{3c}$$

By (2d), (3b,c)

$$w_{1t}+(1+p)w_{1r} = \frac{1}{2(1+p)} p_t(-w_1+w_2) + \frac{1}{2} p_r(-3w_1-w_2) \tag{3d}$$

$$w_{2t}-(1+p)w_{2r} = \frac{1}{2(1+p)} p_t(w_1-w_2) + \frac{1}{2} p_r(w_1+3w_2) \quad . \tag{3e}$$

By (2b)

$$p_r = \frac{(1+p)^3 F''(u_t)}{2r} ((1+p)(w_2-w_1)-u_t) \tag{3f}$$

$$p_t = \frac{(1+p)^5 F''(u_t)}{2r} (w_1+w_2) \quad . \tag{3g}$$

By (1b), (2b) for small u_t

$$p = \frac{a}{2} u_t + 0(u_t^2) \quad . \tag{3h}$$

We can express then u_t in terms of p from (2b), writing for small u_t

$$\frac{1}{2} (1+p)^4 F''(u_t) = P(p) = \frac{1}{2} a + 0(p) \tag{4a}$$

$$\frac{1}{2} (1+p)^3 F''(u_t)u_t = Q(p) = \frac{1}{a} p + 0(p^2) \quad . \tag{4b}$$

Then by (3f,g)

$$p_r = \frac{-P(p)(w_1-w_2) - Q(p)}{r} ; \; p_t = \frac{(1+p)P(p)(w_1+w_2)}{r} , \tag{4c}$$

and by (3d,e)

$$p_t+(1+p)p_r = \frac{(1+p)(2P(p)w_2-Q(p))}{r} \tag{5a}$$

$$p_t-(1+p)p_r = \frac{(1+p)(2P(p)w_1+Q(p))}{r} \tag{5b}$$

$$w_{1t}+(1+p)w_{1r} = \frac{2P(p)(w_1^2 -w_1w_2)+Q(p)(3w_1+w_2)}{2r} \tag{5c}$$

$$w_{2t}-(1+p)w_{2r} = \frac{2P(p)(w_2^2-w_1w_2)-Q(p)(w_1+3w_2)}{2r} \quad . \tag{5d}$$

In what follows we shall just deal with equations (5a,b,c,d). Blow-up will be based on the occurence of the term with w_1^2 in (5c). The aim will be to show that the other terms in (5c) can be neglected near the wave front for large positive r. Equations (5a,b,c,d) represent ordinary differential equations along the characteristic curves defined by

$$\frac{dr}{dt} = \pm(1+p) \quad . \tag{5e}$$

Here

$$0 < 1+p < \sqrt{2} \tag{5f}$$

by (2b), (1c). If $u(r,t)$ is a strict solution for $r \in \mathbb{R}$, $0 < t < T$, each of the two families of characteristic curves will cover that region in a "schlicht" manner. Since $u(r,t)$ by (1g) vanishes for $|r| > t+1$, we also have

$$w_1(r,t) = w_2(r,t) = p(r,t) = 0 \quad \text{for} \quad |r| > t+1,\ 0 < t < T \quad . \tag{5g}$$

Hence the characteristics reduce to the line $r \pm t = \text{const.}$ for $|r| > t+1$. We introduce the characteristics as coordinate lines in the region of interest to us. For that purpose we denote by C_s the characteristic $\frac{dr}{dt} = -(1+p)$ passing through the point $(s+1,s)$ of the wave front. Choosing a positive fixed s_0 (actually $s_0 = 1/\sqrt{\varepsilon}$) we denote by Γ_τ the characteristic $\frac{dr}{dt} = 1+p$ passing through the point of C_{s_0} for which $t = s_0 + \tau$. (See Figure 1.)

We always take

$$s > s_0 \quad ; \quad 0 < \tau < 1 \quad . \tag{5h}$$

If C_s and Γ_τ intersect in the domain of existence $0 < t < T$, we denote the intersection by

$$(s+1-R,s+L) \tag{5i}$$

where R and L are functions of s and τ:

$$R = R(s,\tau),\ L = L(s,\tau) \quad . \tag{5j}$$

In particular

$$L(s_0,\tau) = \tau \quad ; \quad R(s,0) = 0 \quad . \tag{5k}$$

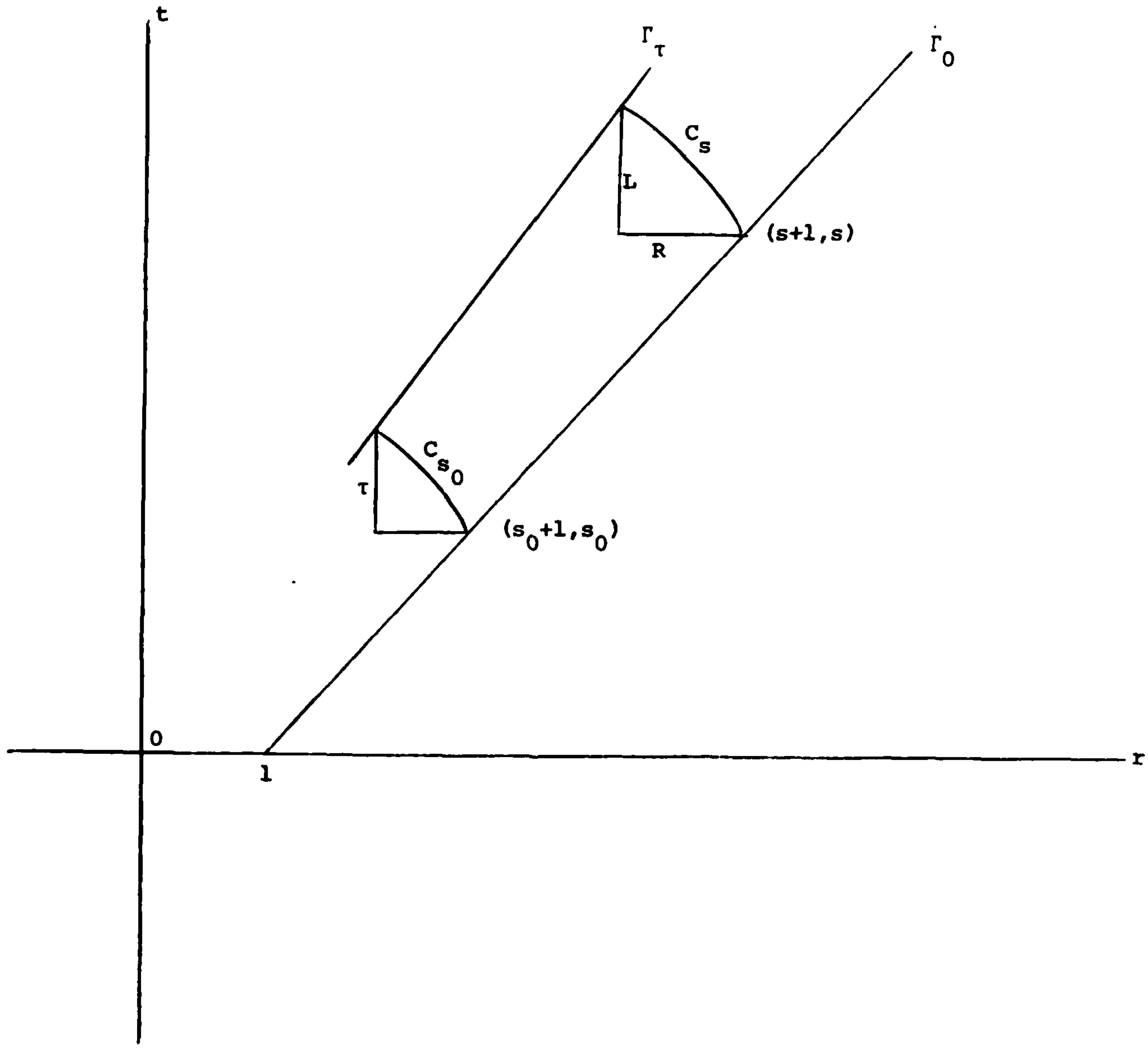


Figure 1

We can approximate u by the solution of the linear homogeneous wave equation with the same initial data. Let u^1 be the solution of (see (2a))

$$(ru^1)_{tt} - (ru^1)_{rr} = 0 \tag{6a}$$

with

$$u^1(r,0) = f(r) \quad ; \quad u_t^1(r,0) = g(r) \quad . \tag{6b}$$

Then

$$u(r,t) - \varepsilon u^1(r,t)$$

and its first and second derivatives are of order $O(\varepsilon^2)$ for $t < 1/\varepsilon^2$ for small ε (see John [7], [2]). Here

$$u^1(r,t) = \frac{(r+t)f(r+t)+(r-t)f(r-t)}{2r} + \frac{1}{2r}\int_{r-t}^{r+t} \rho\, g(\rho)d\rho \quad . \tag{6c}$$

We shall restrict ourselves to the region

$$r+t > 1 \tag{6d}$$

where by (1f,g)

$$u^1(r,t) = \frac{(r-t)f(r-t)}{2r} + \frac{1}{2r}\int_{r-t}^{1} \rho\, g(\rho)d\rho \quad . \tag{6e}$$

Using the expressions (3b,c) for w_1, w_2 we find that for $r+t > 1$, $t < 1/\varepsilon^2$

$$\begin{aligned} w_1 &= \frac{\varepsilon}{2}\left(r(u^1_{rr}-u^1_{tr})+2u^1_r-u^1_t\right) + O(r\varepsilon^2) \\ &= \varepsilon k(r-t) + O(r\varepsilon^2) \end{aligned} \tag{6f}$$

with k defined by (1j). Similarly

$$w_2 = \frac{\varepsilon}{2}\left(r(u^1_{rr}+u^1_{tr})+2u^1_r+u^1_t\right) + O(r\varepsilon^2) = O(r\varepsilon^2) \quad . \tag{6g}$$

It is plausible from (6f,g) that the leading term in w_1, w_2 is given by $\varepsilon k(r-t)$. We have to make sure of the behavior of the function $k(\lambda)$. By (1g,k)

$$k(\lambda) = 0 \quad \text{for} \quad |\lambda| > 1 \quad . \tag{7a}$$

Moreover

$$k(\lambda) = \frac{1}{2}\frac{d}{d\lambda}\left(\lambda f'(\lambda)+f(\lambda)-\lambda g(\lambda)\right) \quad . \tag{7b}$$

It follows that

$$\int_{-1}^{1} k(\lambda)d\lambda = 0 \quad . \tag{7c}$$

Then

$$K = \max_{-1<\lambda<1} k(\lambda) > 0 \tag{7d}$$

unless u is the trivial solution of (1a). For $K \leq 0$ implies by (7a,c) that $k(\lambda)$ vanishes identically. But then also by (1f,k)

$$0 = k(\lambda)-k(-\lambda) = \lambda f''(\lambda)+2f'(\lambda)$$

$$0 = k(\lambda)+k(-\lambda) = -\lambda g'(\lambda)-g(\lambda) \quad .$$

Since f and g have compact support, it would follow that f, g and hence also u vanish identically. We exclude the trivial solution so that (7d) holds.

We can find constants K_1, K_2, K_3, λ_1, λ_2 such that

$$-1 < \lambda_1 < \lambda_2 < 1 \quad ; \quad K_1 > K_2 > K_3 > 0 \tag{8a}$$

$$k(\lambda_1) > K_1 \tag{8b}$$

$$k(\lambda) > K_3 \quad \text{for} \quad \lambda_1 \leq \lambda \leq \lambda_2 \tag{8c}$$

$$k(\lambda) < K_2 \quad \text{for} \quad \lambda_2 \leq \lambda \leq 1 \tag{8d}$$

(see Figure 2). We need only to choose for λ_1 the "last" total maximum point of k, i.e. such that

$$k(\lambda_1) = K \quad ; \quad k(\lambda) < K \quad \text{for} \quad \lambda_1 < \lambda < 1 \quad . \tag{8e}$$

Since $K > 0$ we can find a λ_2 such that

$$\lambda_1 < \lambda_2 < 1 \quad ; \quad k(\lambda) > 0 \quad \text{for} \quad \lambda_1 < \lambda < \lambda_2 \quad . \tag{8f}$$

We then take for K_1, K_2, K_3 any numbers with

$$\sup_{\lambda_2 \leq \lambda \leq 1} k(\lambda) < K_2 < K_1 < K \tag{8g}$$

$$0 < K_3 < \inf_{\lambda_1 < \lambda < \lambda_2} k(\lambda) \quad ; \quad K_3 < K_2 \quad . \tag{8h}$$

(Notice that K_1 can be chosen arbitrarily close to K.)

We set

$$\delta = \sqrt{\varepsilon} \quad ; \quad s_0 = \frac{1}{\delta} \quad . \tag{9a}$$

In what follows we always assume that δ is "sufficiently" small, that ih δ is small compared to any positive constant that turns up in the computation.

We first discuss the behavior of w_1, w_2 on C_{s_0} for $s_0+1 > t > s_0$. Since here $dr/dt = -(1+p)$ we have by (5f)

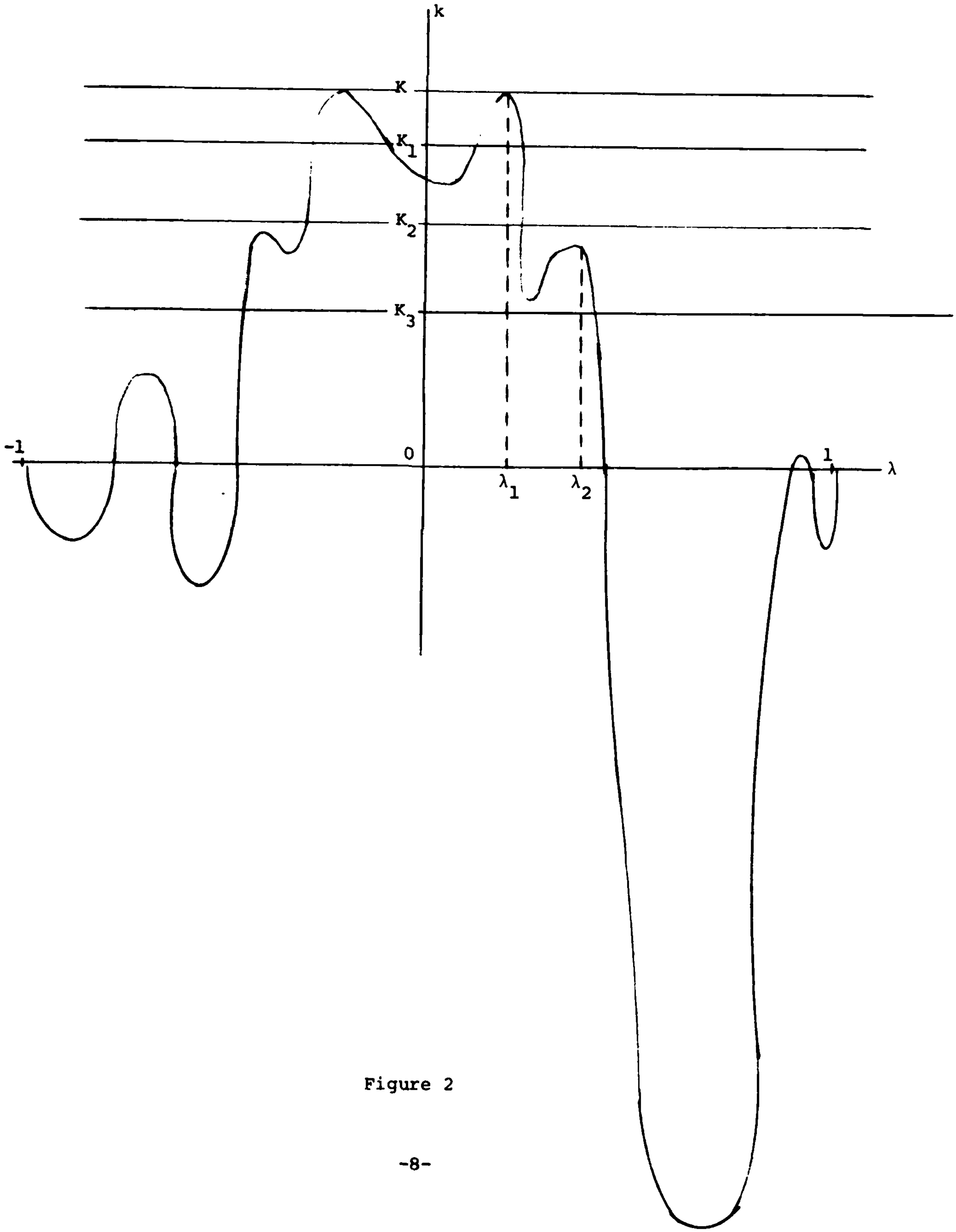


Figure 2

$$r < s_0+1 = s_0(1+\delta) = \delta^{-1}(1+\delta) \tag{9b}$$

$$r > s_0 - \sqrt{2}\ (t-s_0) > s_0 - \sqrt{2} > s_0(1-0(\delta)) > 0 \quad . \tag{9c}$$

Thus on C_{s_0} for $s_0 < t < s_0+1$

$$r+t > t > s_0 > 1 \quad ; \quad t < s_0+1 < \varepsilon^{-2} \tag{9d}$$

so that (6f,g) hold. Moreover by (6e)

$$u_t = \varepsilon u_t^1 + 0(\varepsilon^2) = 0(\frac{\varepsilon}{r}) + 0(\varepsilon^2) = 0(\delta^3) \tag{9e}$$

and hence by (3h)

$$p = 0(\delta^3) \quad . \tag{9f}$$

Then, because of $dr/dt = -(1+p)$

$$r = s_0+1 - \int_{s_0}^{t} (1+p)dt = 2s_0+1-t - \int_{s_0}^{t} p\ dt = 2s_0+1-t+0(\delta^3) \quad . \tag{9g}$$

It follows from (6f,g) that on C_{s_0} for $s_0 < t < s_0+1$

$$w_1 = \delta^2[k(1-2(t-s_0)) + 0(\delta)] = \delta^2[k(1-2\tau) + 0(\delta)] \tag{9h}$$

$$w_2 = 0(\delta^3) \quad . \tag{9i}$$

Considering w_1 on C_{s_0} as a function of $\tau = t-s_0$ for $0 < \tau < 1$, and setting

$$\tau_i = \frac{1}{2}(1-\lambda_i) \quad \text{for} \quad i = 1,2 \tag{9j}$$

we have from (8a,b,c,d), (9h)

$$w_1 = 0 \quad \text{for} \quad \tau = 0$$

$$w_1 = \delta^2(k(\lambda_1) + 0(\delta)) > \delta^2 K_1 \quad \text{for} \quad \tau = \tau_1 \tag{9k}$$

$$w_1 > \delta^2 K_3 \qquad \text{for} \quad \tau_2 < \tau < \tau_1 \tag{9l}$$

$$w_1 < \delta^2 K_2 \qquad \text{for} \quad 0 < \tau < \tau_2 \tag{9m}$$

provided ε and hence δ are sufficiently small. (See Figure 3.)

Introducing the characteristics as coordinate lines we have by (5i,j) for the coordinates (r,t) of a point in terms of the characteristic parameters s, τ

$$r = s+1-R(s,\tau) \quad ; \quad t = s + L(s,\tau) \quad . \tag{10a}$$

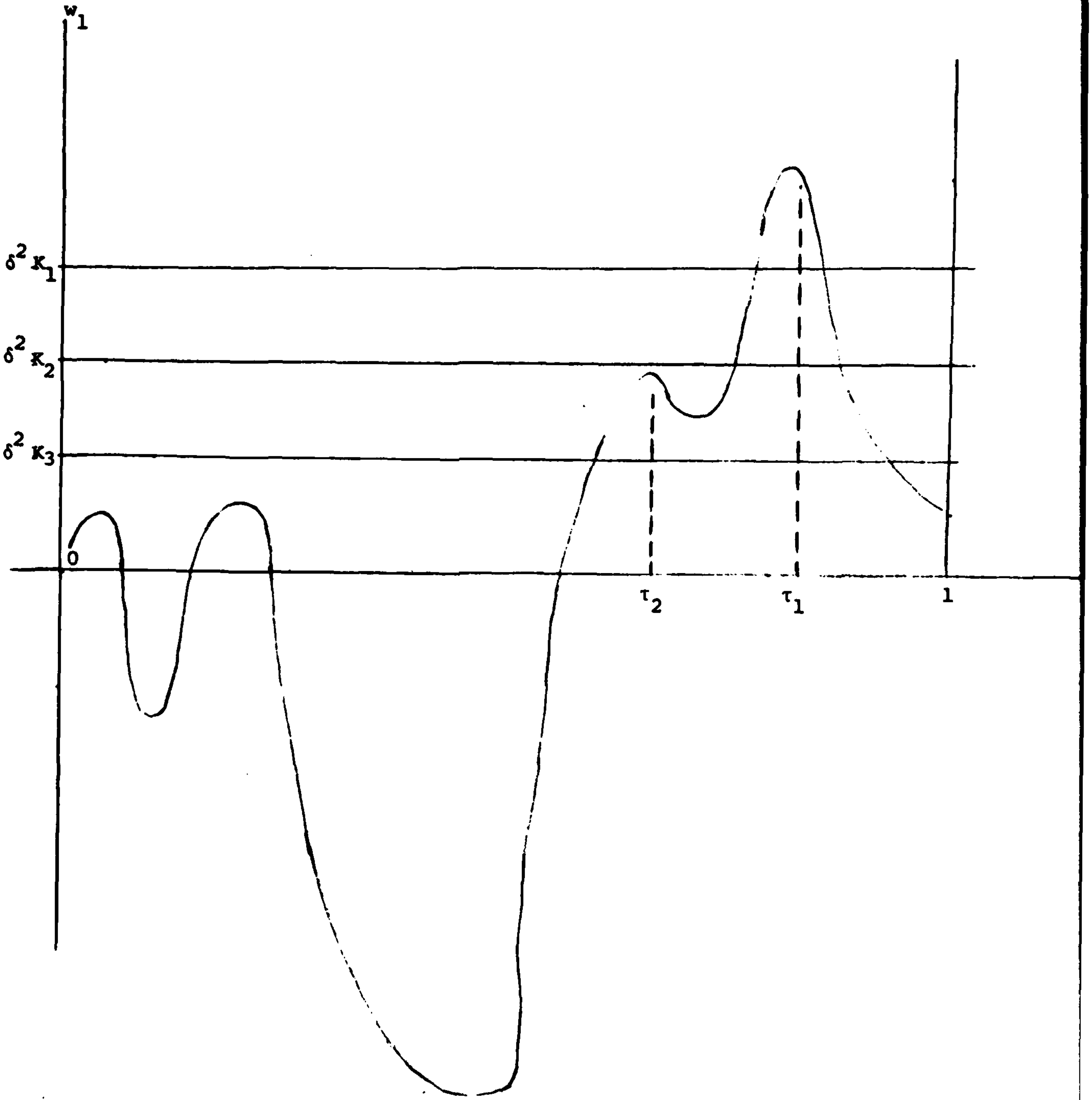


Figure 3

We introduce the extrema of the quantities w_1, w_2, p on the portion of C_s lying between Γ_0 and Γ_τ:

$$M_i(s,\tau) = \sup_{0<\sigma<\tau} w_i(s+1-R(s,\sigma),\ s+L(s,\sigma)) \quad \text{for} \quad i = 1,2 \tag{10b}$$

$$m_i(s,\tau) = \sup_{0<\sigma<\tau} -w_i(s+1-R(s,\sigma),\ s+L(s,\sigma)) \quad \text{for} \quad i = 1,2 \tag{10c}$$

$$N(s,\tau) = \sup_{0<\sigma<\tau} p(s+1-R(s,\sigma),\ s+L(s,\sigma)) \tag{10d}$$

$$n(s,\tau) = \sup_{0<\sigma<\tau} -p(s+1-R(s,\sigma),\ s+1+L(s,\sigma)) \ . \tag{10e}$$

Here

$$M_i, m_i, N, n > 0 \tag{10f}$$

since w_1, w_2, p vanish for $\tau = 0$.

We introduce the abbreviations

$$\begin{aligned} &M_i^0(\tau) = M_i(s_0,\tau) \ ; \ m_i^0(\tau) = m_i(s_0,\tau) \ , \\ &\qquad\qquad N^0(\tau) = N(s_0,\tau) \ ; \ n^0(\tau) = n(s_0,\tau< \quad W \end{aligned} \tag{10g}$$

By (9f), (9h,i)

$$m_1^0, M_1^0 = 0(\delta^2) \ ; \ N^0, n^0, m_2^0, M_2^0 = 0(\delta^3) \quad \text{for} \quad 0 < \tau < 1 \ . \tag{10h}$$

More specifically we have from (9k,ℓ,m)

$$M_1^0(\tau_1) > \delta^2 K_1 \tag{10i}$$

$$M_1^0(\tau) > \delta^2 K_3 > 0; \ m_1^0(\tau) = m_1^0(\tau_2) \quad \text{for} \quad \tau_2 < \tau < \tau_1 \tag{10j}$$

$$M_1^0(\tau) < \delta^2 K_2 \quad \text{for} \quad 0 < \tau < \tau_2 \ . \tag{10k}$$

We show first that we can estimate L, n, N, M_2 in terms of m_1 and M_1^0 without involving m_2 and M_1. For that purpose assume that we have a bound μ on m_1

$$m_1(s,\tau) < \mu(\tau) \quad \text{for} \quad 0 < \tau < \tau_3,\ s_0 < s < S \tag{11a}$$

where

$$\mu = O(\delta^2);\ 0 < \tau_3 < 1;\ s_0 < S \quad . \tag{11b}$$

We shall show that then (for δ sufficiently small)

$$L(s,\tau) < 2\tau \left(\frac{s}{s_0}\right)^{2a\mu(\tau)} \tag{11c}$$

$$n(s,\tau) < \frac{3a\mu(\tau)L(s,\tau)}{2s} \tag{11d}$$

$$N(s,\tau) < \frac{3aM_1(s,\tau)L(s,\tau)}{2s} \tag{11e}$$

$$M_2(s,\tau) < \frac{5\mu(\tau)L(s,\tau)}{2as} \tag{11f}$$

$$N(s,\tau) < \frac{3\tau}{s_0}\left(\frac{a}{2} M_1^0(\tau) + 3\mu(\tau)\right) \tag{11g}$$

with a defined by (1b). For the proof it is sufficient to show that if (11c,d,e,f,g) hold for $s_0 < s < S$ then also for $s = S$, and also to show that these relations hold for $s = s_0$.

Relations (10h), (11b,c,d,f,g) imply that

$$L(s,\tau) < \frac{2\tau s}{s_0}\left(\frac{s}{s_0}\right)^{2a\mu-1} < \frac{2\tau s}{s_0} < 2\delta s \tag{12a}$$

$$M_2,\ n,\ N = O(\delta^3) \quad . \tag{12b}$$

On C_s for $s < t < s+L(s,\tau)$ we have

$$s < t < s+L < s(1+2\delta) \tag{12b*}$$

$$r = s+1 - \int_s^{s+L} (1+p)dt = s+1 - L(1+O(\delta^3)) = s(1+O(\delta)) \quad . \tag{12c}$$

In addition by (4a,b), (12b)

$$P(p) = \frac{a}{2}(1+O(\delta^3));\ Q(p) = \frac{p}{a}(1+O(\delta^3)) \quad . \tag{12d}$$

By (5b) along C_s

$$\frac{dp}{dt} = \frac{aw_1(1+O(\delta)) + \frac{1}{a}p(1+O(\delta))}{s} \quad .$$

Hence

$$-\frac{1}{s}\left(am_1 + \frac{n}{a}\right)(1+O(\delta)) < \frac{dp}{dt} < \frac{1}{s}\left(aM_1 + \frac{N}{a}\right)(1+O(\delta)) \quad . \tag{12e}$$

It follows that

$$-\frac{L}{s}\left(am_1+\frac{n}{a}\right)(1+O(\delta)) < p < \frac{L}{s}\left(aM_1+\frac{N}{a}\right)(1+O(\delta)) \quad . \tag{12f}$$

Since here by (12a)

$$\frac{L}{as} = O(\delta)$$

we find that for $0 < \tau < \tau_3$, $s_0 < s < S$

$$n < \frac{aLm_1}{s}(1+O(\delta)) < \frac{aL\mu}{s}(1+O(\delta));\ N < \frac{aLM_1}{s}(1+O(\delta)) \quad .$$

This implies (11d,e) for $s = S$ and sufficiently small δ.

For the intersection of Γ_τ and C_s we have (see Figure 1)

$$t = s+L \tag{13a}$$

and

$$r = s+1+\int_{C_s}\frac{dr}{dt}\,dt = s+1-\int_{C_s}(1+p)dt = s+1-L-\int_{C_s}p\,dt$$

$$= s_0+1-\tau-\int_{C_{s_0}}p\,dt+\int_{\Gamma_\tau}\frac{dr}{dt}\,dt = s_0+1-\tau-\int_{C_{s_0}}p\,dt+\int_{\Gamma_\tau}(1+p)dt$$

$$= s+1+L-2\tau-\int_{C_{s_0}}p\,dt+\int_{\Gamma_\tau}p\,dt \quad .$$

Thus

$$\begin{aligned} L &= \tau-\frac{1}{2}\int_{C_s}p\,dt+\frac{1}{2}\int_{C_{s_0}}p\,dt-\frac{1}{2}\int_{\Gamma_\tau}p\,dt \\ &< \tau+\frac{1}{2}nL+\frac{1}{2}N^0\tau+\frac{1}{2}\int_{s_0+\tau}^{s+L}n\,dt \end{aligned} \tag{13b}$$

where the last integral is taken over the curve Γ_τ. On that curve by (11c,d), (12a), (12b*)

$$n < \frac{3a\mu L}{2s} < \frac{3a\mu\tau}{s_0^{2a\mu}} s^{2a\mu-1} < \frac{3a\mu\tau}{s_0^{2a\mu}} t^{2a\mu-1}(1+0(\delta))$$

$$\frac{1}{2}\int_{s_0+\tau}^{s+L} n\ dt < \frac{3\tau}{4s_0^{2a\mu}}(s+L)^{2a\mu}(1+0(\delta)) < \frac{3}{4}\tau\left(\frac{s}{s_0}\right)^{2a\mu}(1+0(\delta)) \quad .$$

Then (13b) yields for $s_0 < s < S$

$$L(1 - \frac{1}{2}n) < (1 + \frac{1}{2}N^0)\tau + \frac{3}{4}\tau\left(\frac{s}{s_0}\right)^{2a\mu}(1+0(\delta))$$

$$< (1 + \frac{3}{4})\left(\frac{s}{s_0}\right)^{2a\mu}(1+0(\delta))\tau$$

$$L < \frac{7}{4}\tau\left(\frac{s}{s_0}\right)^{2a\mu}(1+0(\delta)) \quad .$$

This implies (11c) for $s = S$.

For $s = s_0$ relation (11c) holds trivially for $s = s_0$, since then $L = \tau$, and relations (11d,e) follow in the same way as for $s > s_0$.

We turn to the proof of (11f). Along C_s by (5d)

$$\frac{dw_2}{dt} = \frac{2Pw_2^2 - w_1(2Pw_2+Q) - 3Qw_2}{2r} \quad . \tag{14a}$$

Here $2Pw_2 + Q > 0$ unless

$$w_2 < -\frac{Q(p)}{2P(p)} = -\frac{1}{a^2}p(1+0(\delta)) < \frac{n}{a^2}(1+0(\delta)) < \frac{3n(s,\tau)}{2a^2} \quad .$$

Thus for any point of C_s with $s_0 < t < s_0+L(s,\tau)$ either

$$w_2 < \frac{3n(s,\tau)}{2a^2} \tag{14b}$$

or $2Pw_2 + Q > 0$, $w_2 > 0$, and

$$\frac{dw_2}{dt} < \frac{aM_2^2 + m_1(aM_2 + \frac{1}{a}N) + \frac{3}{a}nM_2}{2s}(1+0(\delta)) \quad . \tag{14c}$$

Now either (14b) holds for all t with $s_0 < t < s_0+L(s,\tau)$, in which case

$$M_2(s,\tau) < \frac{3n(s,\tau)}{2a^2} \tag{14d}$$

or there is a t^* in the interval for which

$$w_2 > \frac{3n}{2a^2} .$$

In the latter case we can find a t^{**} with $s_0 \leq t^{**} < t^*$ such that

$$w_2 = \frac{3n}{2a^2} \quad \text{for} \quad t = t^{**} \ ; \ w_2 > \frac{3n}{2a^2} \quad \text{for} \quad t^* < t < t^{**} .$$

Then for $t = t^*$

$$w_2 = \frac{3n}{2a^2} + \int_{t^{**}}^{t^*} \frac{dw_2}{dt} dt$$

where the integrand satisfies the inequality (14c). It follows that

$$M_2 < \frac{3n}{2a^2} + \frac{L}{2s} (M_2(aM_2+am_1+ \frac{3}{a} n) + \frac{1}{a} Nm_1)(1+0(\delta))$$

$$< \frac{9\mu L}{4as} + \frac{L}{2s} (M_2(aM_2+a\mu+ \frac{3}{a} n)+ \frac{1}{a} N\mu)(1+0(\delta)) .$$

Here

$$\frac{nL}{s} = 0(\delta) \ ; \ \frac{\mu L}{s} = 0(\delta) \ ; \ \frac{N\mu L}{s} = 0(\frac{\delta\mu L}{s})$$

and hence

$$M_2 < \frac{9L\mu}{4as} (1+0(\delta)) + 0(\delta M_2^2) .$$

If here

$$M_2 = 0(1) \tag{14e}$$

it would follow that

$$M_2 < \frac{9L}{4as} (1+0(\delta)) = 0(\delta) . \tag{14f}$$

Now (14e) and hence (14f) certainly holds for small τ, since $w_2 = 0$ for $\tau = 0$. By continuity then for all $\tau < \tau_3$. Thus (14f) holds for $s_0 < s < S$, which implies (11f) for $s = S$. The same argument yields (11f) for $s = s_0$.

Finally along Γ_τ by (5a), (11d,f,c), (12b*,c)

$$\frac{dp}{dt} < \frac{aM_2 + \frac{1}{a} n}{s}(1+0(\delta)) < \frac{4\mu L}{s^2}(1+0(\delta))$$

$$< \frac{8\mu\tau}{s_0^{2a\mu}} s^{2a\mu-2}(1+0(\delta)) < \frac{8\mu\tau}{s_0^{2a\mu}} t^{2a\mu-2}(1+0(\delta)) \quad .$$

Hence integrating along Γ_τ

$$N < N^0 + \int_{s_0+\tau}^{s+L} \frac{dp}{dt} dt < N_0 + \frac{8\mu\tau}{s_0^{2a\mu}(1-2a\mu)s_0^{1-2a\mu}}(1+0(\delta))$$

$$< N^0 + \frac{8\mu\tau}{s_0}(1+0(\delta)) < \frac{3aM_1^0\tau}{2s_0} + \frac{8\mu\tau}{s_0}(1+0(\delta)) \quad . \tag{14g}$$

This implies (11g) for $s = S$.

Having established (11c,d,e,f,g) under the assumptions (11a,b) we now turn to estimates for m_2 and μ in terms of M_1. Assume that

$$M_1(s,\tau) < a \quad \text{for} \quad s_0 \leq s < S \; , \; 0 \leq \tau \leq \tau_2 < 1 \quad . \tag{15a}$$

Then, with a specific choice of μ,

$$m_2 < \frac{L}{s}(aM_1^0+5\mu)\delta = 0(\delta^3 \frac{L}{s}) \tag{15b}$$

$$m_1 < \mu = 2m_1^0 + \delta^3 = 0(\delta^2) \tag{15c}$$

for $s_0 \leq s < S$, $0 \leq \tau \leq \tau_2$, provided δ is sufficiently small.

It is again sufficient to prove these assertions for $s = S$ if they hold for $s < S$ and to prove them for $s = s_0$.

Along C_s we have by (5d)

$$-\frac{dw_2}{dt} \leq \frac{(-w_2)(-2Pw_1-3Q)+Qw_1}{2r} \quad . \tag{16a}$$

Again for a point (r^*,t^*) on C_s with $w_2 < 0$

$$-w_2 = \int_{t^{**}}^{t^*} \left(-\frac{dw_2}{dt}\right)dt \tag{16b}$$

where the integral is taken along C_s, where $s \leq t^{**} < t^*$ and $w_2 = 0$ at $t = t^{**}$, $w_2 < 0$ for $t^{**} < t \leq t^*$. It follows from (16a) that in the interval of integration

$$-\frac{dw_2}{dt} < \frac{m_2(am_1 + \frac{3}{a}n) + \frac{1}{a}(NM_1 + nm_1)}{2s}(1+0(\delta)) \tag{16c}$$

making use of (10b,c,d,e), (12c). Hence

$$m_2 < \frac{L}{2s}\left[m_2(am_1 + \frac{3}{a}n) + \frac{1}{a}(NM_1 + nm_1)\right](1+0(\delta)) \quad . \tag{16d}$$

Here by (12a), (15c), (11d), (15a,b,c,), (11g)

$$\frac{L}{s}m_1 = 0(\frac{L}{s}\mu) = 0(\delta^3) \; ; \; \frac{L}{s}n = 0(\frac{L^2}{s^2}\mu) = 0(\delta^4)$$

$$\frac{1}{a}(NM_1 + nm_1) < N + \frac{3L}{2s}\mu^2 < 3\delta(\frac{a}{2}M_1^0 + 3\mu) + 3\mu^2\delta$$

$$< 3\delta(\frac{a}{2}M_1^0 + 3\mu)(1+0(\delta)) \quad .$$

It follows from (16d) that

$$m_2 < \frac{3L}{2s}(\frac{a}{2}M_1^0 + 3\mu)\delta(1+0(\delta))$$

for $s_0 < s < S$. This implies (15b) for $s = S$ and sufficiently small δ. The same argument yields (15b) for $s = s_0$, while (15c) is trivial for $s = s_0$.

Along Γ_τ with $0 < \tau < \tau_2$ we find from (5c) that

$$-\frac{dw_1}{dt} < \frac{2Pw_1w_2 - Q(3w_1 + w_2)}{2r} \quad . \tag{17a}$$

At a point (r^*, t^*) of Γ_τ with $s_0 + \tau < t^* < s+L$ we have again

$$-w_1 < m_1^0 + \int_{t^{**}}^{t^*}(-\frac{dw_1}{dt}\,dt \tag{17b}$$

where $s_0 + \tau < t^* < t^{**}$ and

$$-w_1 > m_1^0 > 0 \quad \text{for} \quad t^{**} < t < t^* \quad . \tag{17c}$$

Then in the interval of integration

$$-\frac{dw_1}{dt} < \frac{(-w_1)(-2Pw_2+3Q)-Qw_2}{2r}$$

$$< \frac{m_1(am_2+\frac{3}{a}N)+\frac{1}{a}(nM_2+Nm_2)}{2r}(1+0(\delta)) \quad . \tag{17d}$$

Here by (15a,b,c), (11e), (11c), (12b*)

$$am_2 + \frac{3}{a}N = 0(\frac{L}{s})$$

$$\int_{t^{**}}^{t^{**}} \frac{m_1(am_2+\frac{3}{a}N)}{2r}dt = 0(m_1 \int_{s_0+\tau}^{s+L} s_0^{-2a\mu} t^{2a\mu-2} dt) \tag{17e}$$

$$= 0(\frac{\tau}{s_0} m_1) = 0(\delta m_1)$$

and by (11d,f,g), (15b), (12b)

$$nM_2+Nm_2 = 0(\delta^5 \frac{L}{s})$$

and thus

$$\int_{t^{**}}^{t^{*}} \frac{1}{2ar}(nM_2+Nm_2)dt = 0(\delta^5 \int_{s_0+\tau}^{s+L} s_0^{-2a\mu} t^{2a\mu-2} dt) = 0(\delta^6) \quad . \tag{17f}$$

It follows from (17d,e,f) that on Γ_τ for $0 < \tau < \tau_2$, $s_0 < s < S$

$$-w_1 < m_1^0 + 0(\delta m_1 + \delta^6)$$

and thus also

$$m_1 < m_1^0 + 0(\delta m_1+\delta^6) \quad \text{for} \quad s_0 < s < S \quad . \tag{17g}$$

This implies (15c) for $s = S$.

We shall show that the quantity $M_1(s,\tau_1)$ tends to infinity as s approaches a certain finite value. Thus we cannot assume that $M_1(s,\tau) < a$ for $\tau_2 < \tau < \tau_1$. Nevertheless we need the estimate (15c) also in that τ-interval in order to assure the validity of (11c,d,e,f,g). This is achieved by proving that $w_1 > 0$ for $\tau_2 < \tau$, which implies that $m_1(s,\tau) = m_1(s,\tau_2)$ for $\tau > \tau_2$. More precisely we prove:

Let S be such that

$$s_0 < S \tag{18a}$$

$$t = s+L(s,\tau) < T \quad \text{for} \quad 0 < \tau < \tau_1,\ s_0 < s < S \tag{18b}$$

$$M_1(s,\tau) < a \quad \text{for} \quad 0 < \tau < \tau_2,\ s_0 < s < S \quad . \tag{18c}$$

Then

$$w_1 > K_3\delta^2,\ m_1(s,\tau) = m_1(s,\tau_2) < \mu = 2m_1^0(\tau_2) + \delta^3 \tag{18d}$$

for

$$\tau_2 < \tau < \tau_1,\ s_0 < s < S \quad . \tag{18e}$$

For the proof we observe that (18d) holds trivially for $\tau_2 < \tau < \tau_1$, $s = s_0$ by (91). If (18d) is not satisfied for all points satisfying (18e) there would be a point with the smallest s, say $s = s^*$ for which $w_1 = K_3\delta^2$, while

$$m_1(s,\tau) < \mu \quad \text{for} \quad 0 < \tau < \tau_1,\ s_0 < s < s^* \quad . \tag{18f}$$

If that point lies on a certain Γ_τ we must have $dw_1/dt < 0$ for the derivative taken along Γ_τ. Now by (5c)

$$\frac{dw_1}{dt} = \frac{(2Pw_1+3Q)w_1 - (2Pw_1-Q)w_2}{2r} \quad . \tag{18g}$$

Here by (4,a,b), (11d,g)

$$Q = O(K_3\delta^3) = O(\delta w_1)\ ;\ w_2 < M_2 = O(K_3\delta^3) = O(\delta w_1) \quad .$$

It follows that

$$2Pw_1 + 3Q = aw_1(1+O(\delta)) > 0\ ;\ 2Pw_1 - Q = aw_1(1+O(\delta)) > 0$$

$$\frac{dw_1}{dt} > \frac{aw_1^2}{2r}(1-O(\delta)) > 0 \quad . \tag{18h}$$

This completes the proof of (18d), for s, satisfying (18e).

Relation (18h) has further consequences. Let b be any number with

$$b < \frac{a}{2} \quad . \tag{19a}$$

Let (18a,b,c) be satisfied. Let τ be any value with

$$\tau_2 < \tau < \tau_1 \quad . \tag{19b}$$

Then in the points of Γ_τ with

$$s_0 + \tau < t < S+L(S,\tau) \tag{19c}$$

we have

$$\frac{dw_1}{dt} > \frac{bw_1^2}{t} \tag{19d}$$

(provided δ is sufficiently small). Then on Γ_τ

$$w_1 > \frac{w_1^0}{1-bw_1^0 \log \frac{t}{t_0}} \tag{19e}$$

where

$$t_0 = s_0+\tau \ , \ t = s+L(s,\tau) \tag{19f}$$

and w_1^0 is the value of w_1 in the point of intersection of Γ_τ and C_{s_0}. Here

$$\frac{t}{t_0} > \frac{s}{s_0}\,(1-0(\delta)) > \frac{1}{2}\,\frac{s}{s_0} \ .$$

It follows that w_1 must become infinite for some point of Γ_τ if

$$S > 2s_0 \exp\left(\frac{1}{bw_1^0}\right) \ . \tag{19g}$$

Taking here $\tau = \tau_1$ we have $w_1^0 > \delta^2 K_1$ by (9k). Thus blow-up occurs certainly for some point with

$$\tau = \tau_1 \ , \ s_0 < s < S$$

if

$$S > 2s_0 \exp\left(\frac{1}{bK_1\delta^2}\right) \tag{19h}$$

and S is such that (18c) holds.

It remains to show that there exists S satisfying both (18c) and (19h). By (18g) we have along a trajectory Γ_τ with $0 < \tau < \tau_2$

$$\frac{dw_1}{dt} = \frac{(2Pw_1+3Q-2Pw_2)w_1+Qw_2}{2r} \ . \tag{20a}$$

We shall show that

$$w_1 < a \quad \text{for} \quad 0 \leq \tau \leq \tau_2,\ s_0 \leq s < S \tag{20b}$$

if

$$S < 2s_0 \exp\left(\frac{1}{cK_2\delta^2}\right) \tag{20c}$$

where c is any number with

$$c > \frac{a}{2} \quad . \tag{20d}$$

It is sufficient to prove that (20b) implies

$$w_1 < a \quad \text{for} \quad 0 \leq \tau \leq \tau_2,\ s = S \quad . \tag{20e}$$

Let then (20b) hold. Take a τ with $0 \leq \tau \leq \tau_2$. Then along Γ_τ for $s_0+\tau \leq t < S+L(S,\tau)$ either

$$w_1 \leq K_2\delta^2$$

or there exists a point with $t = t^*$ on Γ_τ where $w_1 > K_2\delta^2$. In the latter case we can find a t^{**} with $s_0 + \tau < t^{**} < t^*$ such that $w_1 = K_2\delta^2$ at t^{**} and $w_1 > K_2\delta^2$ for $t^{**} < t < t^*$. In any case at $t = t^*$

$$w_1 \leq K_2\delta^2 + \int_{t^{**}}^{t^*} \frac{dw_1}{dt}\, dt$$

where the integration is taken along Γ_τ over some sub-interval of $s_0+\tau < t < S+L(S,\tau)$ in which

$$w_1 > K_2\delta^2 \quad . \tag{20f}$$

Then by (4a,b), (11d,f,g), (15b)

$$Q = 0(\delta^3) = 0(\delta w_1) \ ;$$

$$Pw_2 = 0(\delta^3) = 0(\delta w_1) \ ; \ Qw_2 = 0(\delta^6) = 0(\delta^2 w_1^2) \quad .$$

Hence in the interval of integration

$$\frac{dw_1}{dt} < \frac{cw_1^2}{t} \quad .$$

It follows that for $t = t^*$ with $t_0 = s_0+\tau < t^* < S+L(S,\tau)$

$$w_1 \leq K_2\delta^2 + \int_{s_0+\tau}^{t^*} \frac{c}{t}\, w_1^2\, dt$$

and thus

$$w_1 < \frac{K_2\delta^2}{1-cK_2\delta^2 \log \frac{t^*}{t_0}} < \frac{K_2\delta^2}{1-cK_2\delta^2 \log \frac{2S}{s_0}}$$

as long as the last denominator stay positive. Take now

$$S = 2s_0 \exp\left(\frac{1}{bK_1\delta^2}\right) \quad . \tag{20g}$$

Then

$$1-cK_2\delta^2 \log \frac{2S}{s_0} = 1 - cK_2\delta^2 \log 4 - \frac{cK_2}{bK_1} \quad .$$

We can choose b and c such that in addition to (19a) and (20d) the relation

$$\frac{c}{b} = \frac{1}{2}\left(1+\frac{K_1}{K_2}\right)$$

is satisfies, since $K_2 < K_1$. Then for sufficiently small δ

$$1-cK_2\delta^2 \log \frac{2S}{s_0} = \frac{1}{2}\left(1-\frac{K_2}{K_1}\right) - cK_2\delta^2 \log 4$$

and

$$w_1 < \frac{2K_1K_2\delta^2}{K_1-K_2-2cK_1K_2\delta^2 \log 4} < a \quad .$$

This completes the proof of (20b) for S given by (20g) and shows that for that S

$$S+L(s,\tau) > T$$

for some $\tau = \tau_1$, and hence that

$$T < S(1+0(\delta)) < 2S$$
$$< 4s_0 \exp\left(\frac{1}{bK_1\delta^2}\right)$$

or

$$T < \exp\left(\frac{A(\varepsilon)}{\varepsilon}\right) \quad . \tag{21a}$$

Here

$$A(\varepsilon) = \frac{1}{bK_1} + \varepsilon \log 4 + \frac{\varepsilon}{2} \log \frac{1}{\varepsilon} \quad . \tag{21b}$$

Since here K_1 can be chosen arbitrarily close to the value K defined by (7d), and b arbitrarily close to $\frac{a}{2}$, provided ε is sufficiently small, we have

$$\limsup_{\varepsilon \to 0} A(\varepsilon) < \frac{2}{aK} \quad . \tag{21c}$$

REFERENCES

1. S. Klainerman. On "almost global" solutions to quasilinear wave equations in three space dimensions. Preprint.
2. F. John. Blow-up for quasi-linear wave equations in three space dimensions. Comm. Pure and Applied Mathematics 34 (1981), 29-51.
3. F. John. Improved estimates for blow-up for solutions of strictly hyperbolic equations in three space dimensions. In preparation.
4. F. John. Lower bounds for the life span of solutions of nonlinear wave equations in three space dimensions. MRC Technical Summary Report #2393. June 1982. To appear also in the Comm. of Pure and Applied Mathematics 1983.
5. Thomas C. Sideris. Global behavior of solutions to nonlinear wave equations in three dimensions. Preprint.
6. F. John. Formation of singularities in one-dimensional nonlinear wave propagation. Comm. Pure and Applied Mathematics 17, (1974), 377-405.
7. F. John. Delayed singularity formation in solutions of nonlinear wave equations in higher dimensions. Comm. Pure and Applied Mathematics 29, (1976), 649-682.

FJ/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2493	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Blow-up of Radial Solutions of $\Box u = \frac{\partial F(u_t)}{\partial t}$ in Three Space Dimensions		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Fritz John		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE March 1983
		13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nonlinear hyperbolic equations, wave equations, radial solutions, blow-up in three space dimensions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Under consideration are strict solutions $u = u(x_1, x_2, x_3, t) = u(x,t)$ of the differential equation

$$\Box u = u_{tt} - \Delta u = \frac{\partial F(u_t)}{\partial t} \tag{*}$$

which are "radial" (or have "spherical symmetry") in the sense that u only depends on t and on $r = |x|$. We prescribe intial conditions

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (continued)

$$u(x,0) = \varepsilon f(r),\ u_t(x,0) = \varepsilon g(r) \qquad (**)$$

for u, where f and g are even functions in r of class C^∞ (for simplicity) and of compact support, and $\varepsilon > 0$ is a parameter that measures the "amplitude" of the initial data. We assume that equation (*) reduces to the linear wave equation $\Box u = 0$ for "infinitesimal" u, that is we assume that

$$F'(0) = 0 \quad .$$

In addition we postulate that (*) is "genuinely nonlinear" in the sense that

$$F''(0) \neq 0 \quad .$$

Without restriction of generality we can always assume that

$$F''(0) > 0$$

(if necessary changing u into -u) and that f and g have their support in the unit ball:

$$f(r) = g(r) = 0 \quad \text{for} \quad |r| > 1 \quad .$$

We show here that every non-trivial solution u blows up after a finite time T if ε is sufficiently small. More precisely for given f, g, F there exists a constant ε_0 and a function $A(\varepsilon)$ such that

$$T < \exp\left(\frac{A(\varepsilon)}{\varepsilon}\right) \qquad (***)$$

for all $\varepsilon < \varepsilon_0$. Here $A(\varepsilon)$ is bounded independently of ε:

$$C = \limsup_{\varepsilon\to 0} A(\varepsilon) < \infty \quad .$$

This result has to be compared with the known lower and upper bounds for T. In [4] the author showed that $T=T(\varepsilon)$ increases faster than any reciprocal power of ε, as $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} \varepsilon^N T(\varepsilon) = \infty \quad \text{for any} \quad N \quad .$$

This lower bound for T has been improved dramatically by S. Klainerman, [1], who showed that for radial solutions

$$T > \exp\left(\frac{B}{\varepsilon}\right)$$

with a positive constant B. In view of (***), Klainerman's lower bound for T is optimal in the general case. Upper bounds for T had been given previously (see [2], [3]) without requiring the initial data to be radial or ε to be small. But then certain inequalities for f and g had to be postulated, and in addition assumptions had to be made on the behavior of F for large arguments. (Results of this latter type have also been derived for other types of differential equations with spherical symmetry by Th. C. Sideris [5].)

ABSTRACT (continued)

The argument used in the present paper is based on the use of differential equations for the second derivatives of u along characteristic curves (as was done in [6] in the case of one dimension). This emphasizes blow-up as a local phenomenon. We show that for small radial initial data singularities are formed, even if the differential equation (*) is imposed on u just for small values of u_t. For the singularities in question u and its first derivatives stay small, while certain second derivatives become infinite. (This does however not exclude the possibility that other types of singularities with different behavior form earlier in other parts of the domain of u.) Blow-up takes place only after an exceedingly long time, and only after the solution has passed through a phase where the second derivatives are exceedingly small. Qualitatively the behavior of the second derivatives resembles that of the function

$$\phi(t) = \frac{\varepsilon}{t(1-\varepsilon \log t)} .$$

Setting $T = e^{1/\varepsilon}$ we have for that function

$$\phi(1) = \varepsilon$$

$$\phi(T/e) = e e^{-1/\varepsilon}$$

$$\phi(T-1) \sim 1$$

$$\phi(T) = \infty .$$

END

FILMED

6-83

DTIC