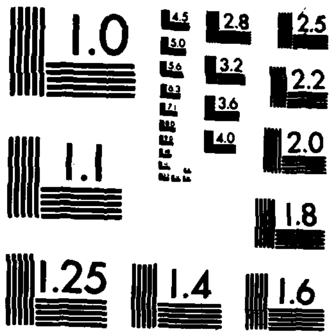
AD-A127 732 ELASTIC FIELDS UNDER A SPHERICAL INDENTER(U) CAMBRIDGE UNIV (ENGLAND) DEPT OF METALLURGY AND MATERIALS SCIENCE E H YOFFE MAR 83 DAJA37-82-C-0171 1/1

UNCLASSIFIED F/G 12/1 NL


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

(15)

AD R&D 4024-R-MS

ELASTIC FIELDS UNDER A SPHERICAL INDENTER

FIRST ANNUAL TECHNICAL REPORT

By

E.H. Yoffe

March 1983

UNITED STATES ARMY
EUROPEAN RESEARCH OFFICE OF THE U.S. ARMY
LONDON. ENGLAND

Contract Number DAJA 37-82-C-0171

Contractor: Department of Metallurgy and Materials Science, University of Cambridge, U.K.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

R&D 4024-R-MS

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A127732	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Elastic Fields Under a Spherical Indenter		5. TYPE OF REPORT & PERIOD COVERED 1st Annual Report Feb 82 - Feb 83
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) E.H. Yoffe		8. CONTRACT OR GRANT NUMBER(s) DAJA 37-82-C-0171
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Cambridge Department of Metallurgy and Materials Science Pembroke Street, Cambridge CB2 3QZ, UK		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 6.11.02A 1T16 1102BH57-04
11. CONTROLLING OFFICE NAME AND ADDRESS USARDSG-UK Box 65 FPO New York 09510		12. REPORT DATE March 1983
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Spherical indentation; elastic stress fields; Hertz

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In recent years there has been a great increase in the use of indentation techniques for testing and measuring mechanical properties of materials. The method has many practical advantages, but the interpretation of results for hard brittle materials relies heavily on the classical Hertzian theory of contact between elastic bodies. In particular, the theory provides formulae for the stress fields when a rigid spherical indenter is pressed on the flat surface of an elastic body. (cont'd)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

This theory is not exact, and since it is so widely used the present study was undertaken to form an estimate of the errors involved. The intention was to find a more accurate solution to the problem, and to compare this with the Hertzian approximation in various cases.

The method offered in the original proposal was a novel one, involving the use of certain elastic singularities or nuclei of strain, which had been developed for some previous work on defects. However, after a promising start this method was regretfully abandoned, since the complications increased faster than the accuracy.

A second method has been devised, which is much more successful. Keeping the Hertz solution as the dominant one, a second slightly different elastic solution has been added to it as a correction. The magnitude of the correction is small for light contact, and increases with the contact area for each value of the sphere radius.

Some values have been calculated and there are many more to be done before a reliable correction can be offered in every case.

It is expected that this will provide a simple and useful check on the magnitude of errors introduced by the Hertz formulae in interpreting experimental results.

AD R&D 4024-R-MS

ELASTIC FIELDS UNDER A SPHERICAL INDENTER

FIRST ANNUAL TECHNICAL REPORT

By

E.H. Yoffe

March 1983

UNITED STATES ARMY
EUROPEAN RESEARCH OFFICE OF THE U.S. ARMY
LONDON. ENGLAND

Contract Number DAJA 37-82-C-0171

Contractor: Department of Metallurgy and Materials Science, University of Cambridge, U.K.

ABSTRACT

In recent years there has been a great increase in the use of indentation techniques for testing and measuring mechanical properties of materials. The method has many practical advantages, but the interpretation of results for hard brittle materials relies heavily on the classical Hertzian theory of contact between elastic bodies. In particular, the theory provides formulae for the stress fields when a rigid spherical indenter is pressed on the flat surface of an elastic body.

This theory is not exact, and since it is so widely used the present study was undertaken to form an estimate of the errors involved. The intention was to find a more accurate solution to the problem, and to compare this with the Hertzian approximation in various cases.

The method offered in the original proposal was a novel one, involving the use of certain elastic singularities or nuclei of strain, which had been developed for some previous work on defects. However, after a promising start this method was regretfully abandoned, since the complications increased faster than the accuracy.

A second method has been devised, which is much more successful. Keeping the Hertz solution as the dominant one, a second slightly different elastic solution has been added to it as a correction. The magnitude of the correction is small for light contact, and increases with the contact area for each value of the sphere radius.

Some values have been calculated and there are many more to be done before a reliable correction can be offered in every case.

It is expected that this will provide a simple and useful check on the magnitude of errors introduced by the Hertz formulae in interpreting experimental results.

CONTENTS

1. INTRODUCTION

The theoretical investigation of the classical Hertzian theory, as applied to the indentation of a flat elastic specimen by a rigid sphere, has been extremely rewarding. A number of unexpected and significant results have been obtained, and estimates made of the magnitude of probable errors in various cases. A descriptive account of the work is given in this report, with reference to more detailed mathematical analysis which will follow later, with further results.

2. ERRORS IN THE HERTZ SOLUTION

The Hertzian theory states that the normal stress between sphere and specimen is distributed over the contact area as:

$$\sigma_z = -\frac{3P}{2\pi a^3}(a^2 - \rho^2)^{\frac{1}{2}} \tag{1}$$

where P is the applied load, ρ, z are cylindrical coordinates as in fig.1, and a is the radius of the circle of contact. There is no normal stress outside this circle, and the entire surface is free of applied shears.

For such a pressure, elastic theory shows that the normal displacement, w, of the plane surface is given by:

$$w = \frac{3P(1-\nu)}{16\,G\,a}\left(2 - \frac{\rho^2}{a^2}\right) \tag{2}$$

in the contact area. Here G is the shear modulus and ν Poissons ratio. An axial section of the indented surface is therefore a parabola for the distribution of pressure (1). The radius of the sphere, R, and the minimum radius of curvature of the parabola, is given by:

$$\frac{1}{R} = \frac{3}{8}\,\frac{P(1-\nu)}{G\,a^3}$$

But the curvature of a parabola is not constant, and this is the main source of error involved in the use of Hertzian theory. It leads always to a logical contradiction near the contact edge, since pressure is assumed between sphere and specimen there, although they are not coincident.

The magnitude of the gap, for $\nu = 1/4$, is

$$\delta = a^4/8R^3$$

This means that for moderate contact, $a/R = 0.2$ say, and a sphere of 1mm radius, the gap is 0.2μm. For smaller contact, $a/R = 0.1$, the same gap appears if the sphere radius is 16mm. A 1mm sphere under higher load, $a/R = 0.5$ say, gives a gap of nearly 8μm.

This gap may seem small, but it does indicate that the true pressure distribution for spherical indentation must differ from eqn. (1). This is unfortunate, since the Hertzian elastic field is one of the very few for which complete analytic solutions are known*, and has therefore been used extensively. Attempts to correct the error lead immediately to difficulties, because the simple form of the solution is lost. Any slight alteration of the pressure, as for example a reduction near $\rho = a$ in an attempt to lessen the gap, affects the displacement w at all points of the surface, and so the complications increase. For this reason a completely new and different approach was made, to find a new analytic solution for spherical contact.

---

* Love, A.E.H., (1929) Phil. Trans. Roy. Soc. A228, 377.

3. NEW SOLUTIONS FROM NUCLEI OF STRAIN

The new method used the fields of known elastic singularities, or nuclei of strain, situated outside the elastic specimen on the negative z axis, and therefore causing only smooth and continuous distortions at the surface. The singularities included the point force, double force, centre of pressure, line integral of the latter and derivatives of all these with respect to z. Some months of work were spent devising solutions from these singularities, combining them in various proportions to give different displacements and loads.

The method seemed very good at first, and various smoothly curving indentations were formed, with surface distributions of pressure or shear which decreased rapidly to zero at large distances.

But unfortunately this method failed in the end, because the required accuracy could not be achieved without using a large number of singularities. The initial simplicity was lost before the hoped-for improvements were gained, and so it was decided to return to the more orthodox Boussinesq approach.

The principal benefits from the first approach were the setting up of numerous valuable stress fields for future use, and an increased understanding of the indentation fields. In particular the relations governing shear stresses on the surface have become much clearer, and it is hoped to write a full account of these later.

4. BACK TO BOUSSINESQ

A fresh start was then made from the theorems of Boussinesq*. He has investigated various forms of surface pressure with axial symmetry, and found solutions in terms of two harmonic functions. This is in agreement with Neuber's** Three Function Theorem, since axial symmetry reduces the number to two.

If the shear stresses are to vanish on the surface, the two functions reduce to one only, ψ say, and the stress and displacement components can be completely expressed in terms of ψ and its derivatives. This is clearly a great advantage when ψ takes a reasonably simple analytic form, as in the case of the Hertzian distribution (1). But for most cases ψ is not known, and Boussinesq could discuss only the surface displacements, not the complete field.

Another simple pressure distribution is the parabolic one,

$$\sigma_z = -\frac{2P}{\pi a^4}(a^2 - \rho^2) \qquad (3)$$

(Boussinesq loc. cit. p.150). Like the Hertzian form (1) this pressure vanishes at the edge of contact, and therefore, as demonstrated by Love, the displacements and stresses are smooth and continuous there.

---

* Boussinesq, M.J. 1885. Mem. Soc. Sci. Agric. Lille, 13, 99.

** Neuber, H. 1937, Kerbspannungslehre. Berlin, Springer.

Boussinesq did not find the stress field for this case, but his formulae give the displacement of the surface $z = 0$ as:

$$w = \frac{8\ P\ (1 - \nu)}{9\ \pi^2\ G\ a} [\ 2(2 - k^2)\ E - (1 - k^2)\ K\ ] \qquad (4)$$

where $k^2 = \rho^2/a^2$, and E and K are complete elliptic functions of the argument k^2.

This displacement (4), like (2), gives a rough approximation to a spherical indentation if a/R is small, but diverges from the sphere even more rapidly. It then provides a useful method of correction, since in linear elasticity separate solutions may be added in any proportions to give another solution. The faults in the parabolic case being greater than those in the Hertz solution, the former is subtracted from the latter to provide improvement. It is to be expected that the modified field would be mainly Hertzian, combined with a small negative parabolic pressure (3).

For a given total load P we therefore assume that it is distributed in the forms (1) and (3) in proportions $1 + d: -d$, where d is a positive number (or zero if the system is purely Hertzian). Then d is determined so as to give the best possible fit of the surface to the sphere for each value of a/R.

This approach has proved very successful, although it has taken some time to find the best methods of fitting and evaluating.

The first attempts, trying to match the curvature of the surface to that of the sphere, were not a success. Small variations in displacement cause too large errors in the second derivative or curvature, and some approximative methods therefore diverged. Moreover the second derivative of w is discontinuous at $\rho = a$, the surface changing there from concave to convex.

However, a satisfactory method has been found, based on the displacements themselves, both w and u_ρ (the radial displacement).

5. THE METHOD OF FITTING

From the known elastic fields of the pressure distributions (1), (Hertz) and (3) (parabolic) the displaced form of the surface may be calculated. A material point initially at the point $(\rho, 0)$ is displaced to the position $(\rho + u_\rho, w)$ where u_ρ and w are composed from the two solutions in proportions $1 + d : -d$, as follows:

$$w = (1 + d)\, w_H - d w_P \qquad (5)$$

The component w_H is the Hertzian one, as in (2), and w_P for the parabolic case is given by (4). The corresponding radial displacements are:

$$u_\rho = \frac{-P\,(1 - 2\nu)}{4\pi G} \cdot \frac{a^3 - (a^2 - \rho^2)^{3/2}}{\rho\, a^3} \qquad \text{(Hertz)}$$

$$u_\rho = \frac{-P\,(1 - 2\nu)}{4\pi G} \cdot \frac{\rho\,(2 - \rho^2/a^2)}{a^2} \qquad \text{(parabolic)}$$

These are the displacements inside the contact circle. Outside it, where $\rho > a$, the radial displacement is

$$u_\rho = -P\,(1 - 2\nu)/4\pi G\rho$$

whatever the mode of pressure distribution.

The component u_ρ is usually ignored in discussions of contact problems, but it is not really negligible unless ν approaches 0.5. It is interesting to observe its effect in the Hertzian case as shown in fig. 2 for $P/Ga^2 = 1.5$, $\nu = 1/4$. The displacement w alone would give the surface paraboloid form, as mentioned above, but the radial displacement, u_ρ, brings a surface point towards the axis, actually within the surface of the sphere. Therefore if surface pressure were truly distributed as in (1), elastic theory would require interpenetration of matter between sphere and specimen, not a gap as previously predicted. In either case the value of a (the contact radius) seems open to doubt, and since this is used in many investigations of strength and toughness the question of its true value is important.

To obtain a better fit we chose as our criterion that the displaced surface should be as close as possible to spherical form, without specifying the radius of the sphere.

The method consisted of choosing a value of P/Ga^2, and using this to find the coordinates x,y of displaced surface points. These coordinates are defined by:

$$\begin{aligned} x &= \rho + u_\rho \\ y &= w(o) - w(\rho) \end{aligned} \qquad (6)$$

and the u and w are composed of Hertz and parabola terms as in (5). Now, if by adjusting the parameter d, it is possible to make the quantity

$$\frac{x^2 + y^2}{2y}$$

approximately constant for all $\rho < a$, then this constant is the radius of a sphere on which the points all lie. Moreover since the centre of the sphere must be at the point

$$x = \rho = 0 \qquad y = R,$$

the distance between a displaced surface point (x, y) and the boundary of the sphere is

$$R - ((R - y)^2 + x^2)^{\frac{1}{2}} \tag{7}$$

and therefore zero if $\frac{x^2 + y^2}{2y} = R$.

So the best possible fit was found by minimising the quantity (7), within the contact area, for each chosen value of P/Ga^2. In each case so far completed the gap or overlap was greatly reduced, the deformed surface becoming indistinguishable from the sphere on the scale of fig.2.

6. THE RESULTS

More calculations are still to be made, but the results so far, for $P/Ga^2 = 0.5$, 1, and 1.5, show that the surface of the specimen can fit very closely to the sphere when the pressure is of the composite Hertz and parabola form. For the three cases completed, the parameter d in (5) takes values approximately equal to 0.07, 0.14 and 0.22 respectively. The new solution gives the extent of contact in these cases as $a/R = 0.14$, 0.29 and 0.43 respectively, whereas Hertzian theory would give 0.14, 0.28 and 0.42.

The new combined solution therefore predicts a slightly larger contact area than that of Hertz for the same sphere and load, the difference increasing with a/R.

The new pressure distribution differs slightly from Hertz, again diverging more with increasing a/R. The pressure is slightly decreased at the centre, for example from $1.5\ P/\pi a^2$ to $1.39\ P/\pi a^2$ for $d = 0.22$. Even for $d = .07$ the maximum pressure is reduced to $1.46\ P/\pi a^2$. These slight redistributions of pressure do not appear to cause significant change in the stress trajectories, nor in the position of maximum shear stress, but there is more work to be done on this aspect.

7. APPLICATIONS

It is expected that the main application of this study will be as a check or reassurance for experimental workers. Indentation is now widely used for measuring many different properties of materials, and the interpretation of results has relied heavily on Hertz' classic work. No other solution being readily available, it has been customary to apply the Hertzian formulae beyond the recommended range of $a/R < 0.1$, and even to state that the elastic fields are all "self-similar", to be formulated in a "non-dimensional" way.

Although a little thought must show that this cannot be precisely so, it has not been possible to make a quick estimate of the error.

The present theory is expected to remedy the situation, by providing a second approximation more closely fitting the true state. Then the accuracy of a result derived from experimental data by Hertzian theory may be verified, and its accuracy assessed, by adding the proportion of parabolic solution appropriate to each load.

If the change is negligible, the work can proceed with confidence in the Hertz solution. But as the accuracy of measurement continues to improve, and the increased strength of modern materials leads to higher a/R values, it is expected that there will be more and more cases where this second approximation is needed.

8. CONCLUSIONS

The Hertz solution for the elastic stress field under a spherical indenter has been improved by the addition of a correction term, which increases with the magnitude of the load for each radius of the sphere.

The correction is made by subtracting another elastic solution, that of a parabolic pressure distribution. This solution has not previously been discussed in detail, but has now been developed and appears very well suited to the purpose. Preliminary results have been obtained which indicate the probable magnitude of the correction, but the work is not complete and will be reported more fully later.

It appears that the correction will be simple, useful, and easy to apply.

9. FURTHER WORK

As stated above, there is much more to be done on this second approximation to the indentation field. The calculations must be repeated for other values of the load, checked, tabulated and written up for publication.

It is intended also to make a thorough investigation of some cases of loading by surface shears, since some new results in this topic have emerged in the course of the work.

It is also very desirable to find some exact solutions for the elastic deformation of the sphere, but no progress has yet been made with this topic. The calculations so far have all been for a rigid sphere.

10. ACKNOWLEDGEMENTS

I am most grateful to the European Research Office of the U.S. Army for support in the course of this work.

Fig. 1. Diagram of indented surface showing axes and extent of contact

Fig. 2. Detail near boundary of contact area, showing gap between sphere and surface depressed by Hertzian displacement w only, Δ = position of points displaced by w and u_ρ.

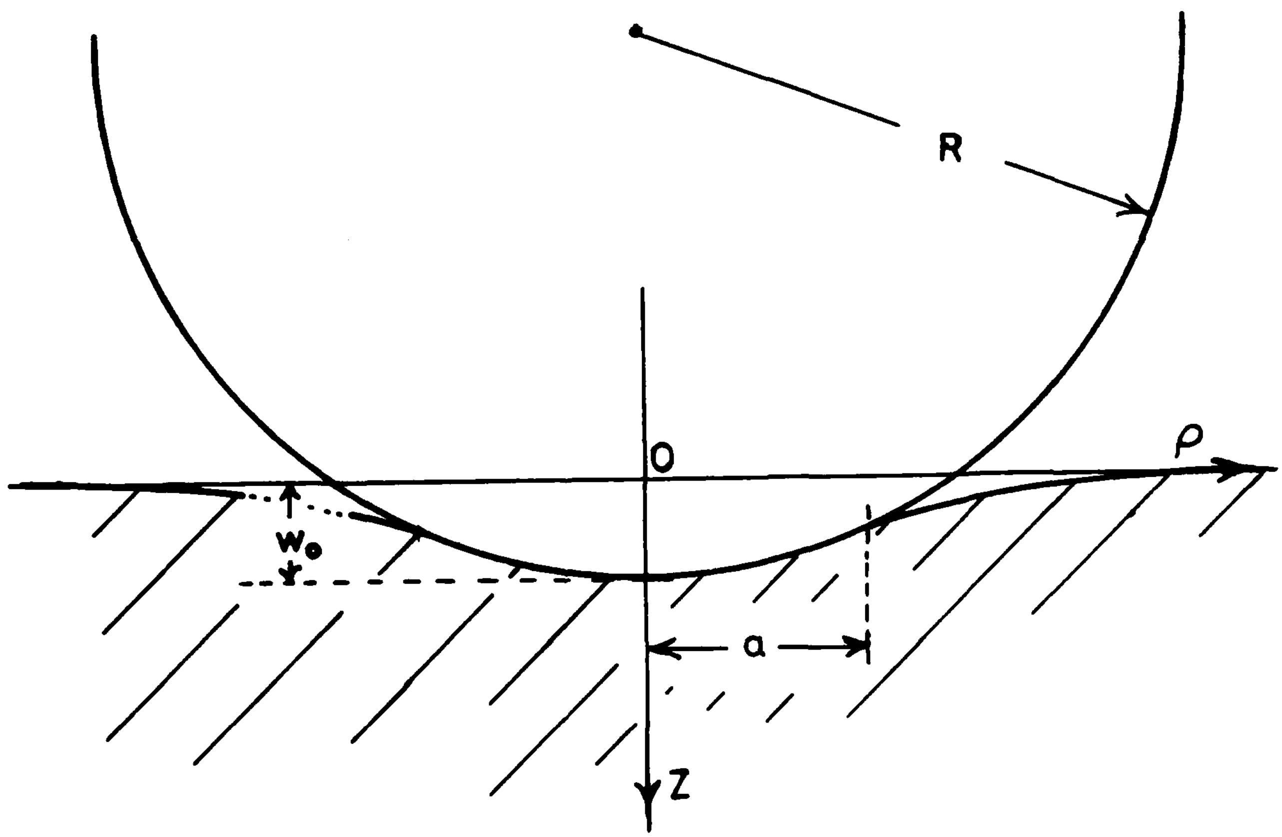


FIG.1.

Fig. 1. Diagram of indented surface showing axes and extent of contact

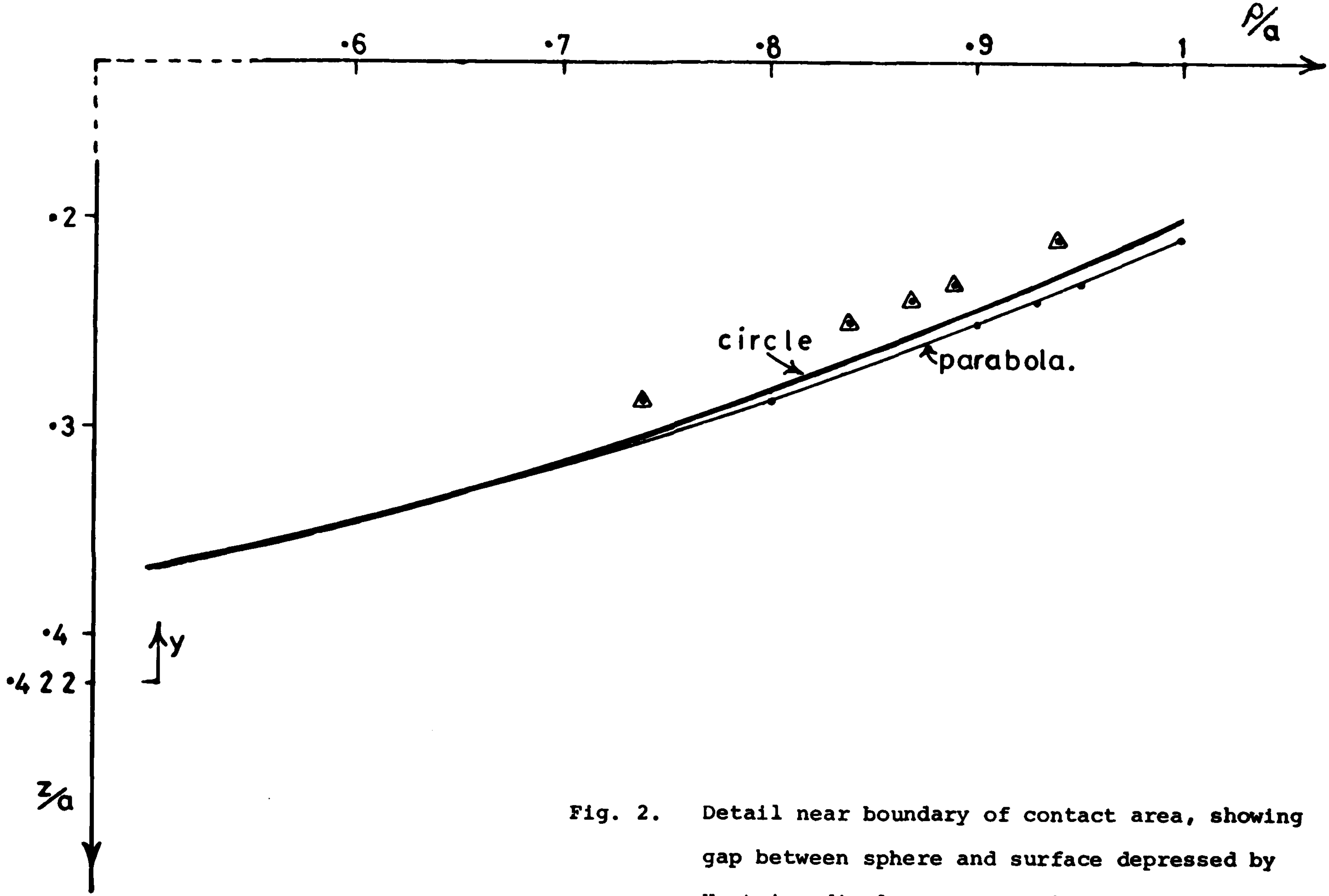


Fig. 2. Detail near boundary of contact area, showing gap between sphere and surface depressed by Hertzian displacement w only, Δ = position of points displaced by w and u_ρ.

FIG.2.

END

FILMED

6-83

DTIC