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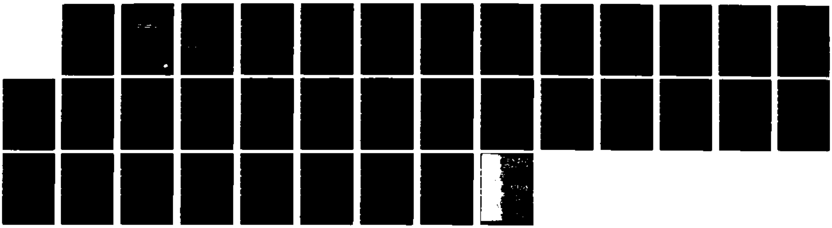

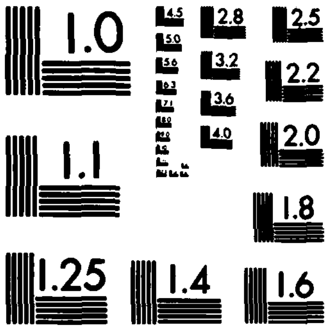

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THE USE OF DYNAMIC PROGRAMMING IN AN OCCUPATIONAL ENVIRONMENTAL PROBLEM

Arnold L. Sweet, Ph.D.

December 1982

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Aerospace Medical Division (AFSC)
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NOTICES

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Dynamic programming is applied to the problem of finding an optimal sampling strategy for a work environment in which employees are exposed to hazardous substances. The level of concentration is modeled as a stochastic process. Some numerical examples are given where the concentration level is assumed to be a Markov chain, and optimal decisions and costs are computed.

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THE USE OF DYNAMIC PROGRAMMING IN AN OCCUPATIONAL ENVIRONMENTAL PROBLEM

INTRODUCTION

The National Institute for Occupational Safety and Health (NIOSH) proposal for a sampling strategy can be used to satisfy an employer's objective to provide a work environment which can attain 95% confidence that no more than 5% of employee exposure days are over the permissible exposure limit (3, p. 29). The sampling strategy was developed using a particular stochastic model for the concentration measurements (3, p. 17). In this report, a procedure for finding an optimal sampling strategy is presented, using the technique of dynamic programming (DP). The employer's objective function used in this report is not that stated above, but instead is based on cost criteria. However, it will be shown that if an optimal sampling strategy does not satisfy the requirements that no more than a certain fraction of employee exposure days are over the permissible exposure limit, this requirement can be used as a constraint, and a nonoptimal sampling strategy can be used instead.

To apply DP, it was necessary to a) represent the concentration levels by a stochastic model, and b) introduce a cost structure for the employer. Samples taken of the concentration level (and perhaps of other related variables) should be used to identify a stochastic model, to estimate its parameters, and to help make decisions concerning control of the employer's process. The first two uses for the samples are considered to be of a statistical nature and will not be discussed in this report. Thus, it is assumed that enough samples have been taken so that a stochastic model of the concentration is known, and values for its parameters have been determined. The only use to be made of the sampled data will be to make optimal decisions to minimize the cost of the process. To define a cost structure, consider the employer's process to be operating under "steady state" conditions and consider a fixed interval of time (such as a day). The following costs are defined:

c_1 is the cost of running the process (i.e., the cost of production) over the interval of time.

c_2 is the cost of making a measurement of the concentration level during the interval of time.

c_3 is the cost of not being able to use the process in a productive way over the interval of time.

c_4 is the cost of exceeding the permissible exposure limit over the interval of time.

It is assumed that a) c_4 cannot be assessed at the end of the time interval unless a measurement has been made during the time interval, b) the process can be carried out purely for the purpose of making a measurement, with no employees subject to exposure and no penalty cost involved, and c) c_3 is greater than c_1.

Three decisions can be made to control the process:

Decision 1. The process will be ongoing during the next time interval, but no measurement is made. The cost involved is equal to c_1.

Decision 2. The process will be ongoing during the next time interval, and a measurement will be made. The expected cost is equal to $c_1 + c_2 + c_4 P(A)$,

where A is the event that the concentration level exceeds the permissible exposure limit.

Decision 3. The process will be carried out solely for the purpose of making a measurement. The expected cost is equal to $c_2 + c_3$.

While the above assumptions may be considered simple, they were chosen to illustrate the use of DP as an approach to the problem of developing sampling strategies. The statistical considerations can also be incorporated into the model, but in the author's opinion, such a development should be accomplished only after some time series data for the concentration levels are made available for investigation.

In what follows, an optimal policy will be derived, that is, rules will be stated such that for every state of the process, one of the above three decisions will be chosen and the expected cost will be a minimum. The rules will be applied to some specific numerical examples.

MARKOV DECISION PROCESSES

To illustrate the concepts involved in a DP formulation, assume that the values of the concentration level are classified as being in one of I ordered intervals. The Ith interval contains only concentration levels greater than the permissible level, and the first interval contains the lowest values of concentration levels. Let the concentration level which could be measured during time interval n be classified as being in one of the I intervals, and let X_n take the value of the interval in which the concentration level lies. Assume that X_n is an irreducible aperiodic Markov chain with (known) stationary transition probability matrix P. The (i, j)th element of P is denoted by p_{ij}, and the elements of the stationary distribution Π (a row vector) by π_j. By choosing the concentration level to be a Markov chain, use can be made of the theory of Markov Decision Processes (2; 6, pp. 739-765).

Consider finding a sequence of decisions which are optimal when the expected cost is to be minimized for an infinite horizon (6, pp. 359-392). The two cost criteria in common use are the expected cost per unit of time and the present value of the expected total cost over the infinite horizon. When choosing the latter, the equations to be solved are (2, p. 80; 6, p. 741):

$$y_i = \min_{d \in D} \left[c_{id} + \alpha \sum_{j \in S} p\,(j|i,d) y_j \right] \qquad 0 \leq \alpha < 1,\ i \in S \tag{1}$$

where S is the set of all possible states of the Markov chain, D is the set of all possible decisions, y_i is the present value of the total expected cost when the process is in state i and an optimal policy is used, c_{id} is the expected cost during one interval of time when the process is in state i and decision d is made, $p(j|i,d)$ is the probability of going to state j from state i when decision d is made, and α is the discount factor. When the process is in state i, decision d_i will always be made.

While X_n denotes the level of the concentration, the decision maker has knowledge of the concentration level only when a measurement has been made. It is thus necessary to expand the set of states S to be the pairs (X, T_n) where T_n is the number of time intervals, measured from the start of time interval n, that have passed since a measurement was made, and X is the concentration level when the last measurement was made. It will now be assumed that the decision as to whether or not to make a measurement is made at the beginning of the time interval, and that X_n represents the concentration level at the beginning of the nth time interval. When a measurement is made, the value observed is assumed to be the concentration level at the end of the time interval. Thus, if $T_n = 1$, $X = X_n$. The maximum value T_n can take will be denoted by T. It is necessary to specify a finite value for T, for otherwise (as will be seen below) the optimal policy would be never to take a measurement, and thus forever to avoid a penalty cost and a measurement cost. It is useful to think of T as the value of an interval between measurements which an inspector uses when he comes to measure the employer's compliance with regulations. When $T_n = T$, only decisions 2 and 3 will be allowed. The augmented state (X, T_n) is still a Markov chain, and expressions for the elements $p\,(j|i,d)$ to be used in equation 1 are given in Appendix A. It is possible to gain some insight from the special case where a measurement is made during every time interval (T=1). An important parameter is the ratio of costs given by

$$h = (c_3 - c_1)\ /\ c_4\ . \tag{2}$$

The following is shown in Appendix A to be valid: When the measurement $X = i$, then decision 2 is optimal for those states i for which

$$p_{iI} < h \qquad\qquad i = 1, 2, \ldots, I$$

and otherwise decision 3 is optimal. Thus, if $h > 1$, decision 2 is always optimal for all outcomes i. Similarly, when $T_n = T$, and $X = i$, then decision 2 is optimal for those states i for which

$$p_{iI}^{(T)} < h \qquad\qquad i = 1, 2, \ldots, I\ ,$$

and otherwise decision 3 is optimal, where $p_{iI}^{(T)}$ is the probability that X_n goes from state i to state I in T intervals. After having determined the optimal decisions, equation 1 can be used to compute the costs.

As T increases, $p_{iI}^{(T)}$ approaches π_I for all i. Thus, the information gained from the measurement becomes useless for the purpose of making an optimal decision. This result indicates that if the Markov chain were an independent process, it is not optimal ever to make a measurement. (This conclusion can also be extended to any time series model of the process which consists of the sum of a deterministic process and a completely random process.) Also, as T increases, it is shown in Appendix A that when $T_n < T$, decision 1 is always optimal.

It can be seen that for any particular P, the value of π_I may be larger than allowed by regulations. However, since a decision can be made to carry out the process without any employees present, it is still possible to achieve satisfactory values for the probability that an employee will be exposed to a concentration level above the permissible limit (denoted by P(B)). This goal may have to be achieved at a higher cost than would be the case if such a constraint were not present.

Some examples, solved numerically by using the policy iteration algorithm (2, 6) will follow. A description of the program is given in Appendix B. In the examples, small numbers are chosen for the costs, for as can be seen in equation 2, it is the ratio of costs which is important.

Example 1. Let $c_1 = 1$, $c_2 = .2$, $c_3 = 3.1$, $c_4 = 2$, $\alpha = .98$, and let

$$P = \begin{bmatrix} .87 & .10 & .03 \\ .60 & .25 & .15 \\ .30 & .60 & .10 \end{bmatrix} \qquad (3)$$

Using equation 2, $h = 1.05$, and thus if $T = 1$, decision 2 is always optimal. When $T > 1$, the solution was such that decision 2 is always optimal when $T_n = T$, and decision 1 is optimal otherwise: Table 1 shows how the expected future cost, C, and the probability of an employee exceeding the permissible concentration level, P(B), changes with the length of the inspection interval, T. The stationary solution of equation 3 yields $\pi_3 = .051$. If the maximum allowable value of P(B) were set to be equal to .05, then the employer would not be in compliance with the regulations. By using the nonoptimal decision vector $d' = (2, 2, 3)$ when $T = 1$, it can be shown that $C = 70.0$, and $P(B) = .046$. Similarly, when $T = 2$, if $d' = (1, 1, 1, 2, 2, 3)$, $C = 60.0$ and $P(B) = .049$. This example illustrates how the constraint $P(B) \leq .05$ increases the cost.

Example 2. The same parameters as in Example 1 are used, but the penalty cost is increased to $c_4 = 105$. By using equation 2, $h = .02$, and since $p_{i3} > .02$ for $i = 1, 2$, and 3, when $T = 1$, decision 3 is always optimal. If $T > 1$, decision 3 is always optimal when $T_n = T$, and decision 1 is optimal otherwise. Thus, when $T = 1$, the process is never used if the optimal policy is applied. By using the nonoptimal policy $d' = (2, 3, 3)$, it can be shown that the cost increased to $C = 207.0$ and $P(B) = .024$. If $T > 1$, then $P(B) \leq .05$, so that the process can be used. Thus, inspecting less often makes the process "acceptable".

Example 3. Changing the penalty cost to $c_4 = 17.5$, when $T = 1$, causes p_{13} and $p_{33} > h$, but $p_{23} < h$. Thus, low or high values of the measured concentration level yield decision 2 as optimal, but an intermediate value leads to decision 3. Also $P(B) = .029 < .05$. Increasing T to values greater than 1 leads to decision 2 being optimal when $T_n = T$, and decision 1 is optimal otherwise. However, $P(B) = .051 > .05$ for all $T > 1$. Using a nonoptimal decision equal to 3 for state (3,2) increases the cost from 77.5 to 77.7, and reduces $P(B)$ to .049.

TABLE 1. EXPECTED FUTURE COST AND PROBABILITY OF EMPLOYEE EXCEEDING PERMISSIBLE CONCENTRATION LEVEL VERSUS INSPECTION INTERVAL

	Example 1		Example 2		Example 3		Example 4	
T	C	P(B)	C	P(B)	C	P(B)	C	P(B)
1	65.1	.051	165.0	.000	101.3	.029	4.46	.009
2	57.6	.051	107.5	.026	77.5	.051	4.10	.231
3	55.0	.051	88.3	.034	68.3	.051	3.97	.178
4	53.8	.051	78.8	.039	63.7	.051	3.87	.220
5	53.0	.051	73.0	.041	61.0	.051	3.81	.218
6	52.5	.051	69.2	.043	59.2	.051	3.76	.227

Example 4. Let $c_1 = .7$, $c_2 = .01$, $c_3 = 1$, $c_4 = 2$, $\alpha = .8$ and let

$$P = \begin{bmatrix} .00 & .49 & .49 & .02 \\ .30 & .02 & .30 & .38 \\ .20 & .20 & .02 & .58 \\ .18 & .40 & .40 & .02 \end{bmatrix}$$

In this example, for $T > 1$ and $T_n < T$, decision 1 is not always optimal. When $T_n < T$ and decision 1 is not always optimal, the matrix $p(j \mid i,d)$ contains transient states. In this example, when T equals two, d' = (2, 1, 1, 2, 3, 3, 2, 3), and augmented states 5 and 8 cannot be reached because measurements are made when the system is in the augmented states 1 and 4. When evaluating C and P(B) in Table 1, the transient states were eliminated.

Notice that for the decision vector shown above, a measurement of concentration at the lowest level yields a decision to make another measurement, while a measurement at either of the next two highest levels yields a decision not to make another measurement.

DP USING ARMA PROCESSES

Instead of treating the concentration level as a discrete random variable, we can consider it to be a continuous random variable and use an autoregressive-moving average (ARMA) model (1). Such models usually have the advantage of containing less parameters to estimate than a Markov chain model. The DP equations to be solved are (4):

$$y(x,t) = \min_{d \epsilon D} \{ c_{(x,t),d} + \alpha \sum_{t'} \int y(z,t') \, f(z,t' \mid (x,t),d) \, dz \} \qquad (4)$$

where the augmented state is again defined to be (X, T_n). The variable X_n takes values x on the real line, and when $T_n = t$, $X = X_{n-t+1}$. In equation 4 $f(z,t' \mid (x,t),d)$ is the conditional probability density of (X, T_n), given that $(X = x, T_n = t)$ and decision d was made at time n. All other variables are as defined in equation 1. For the autoregressive model of order 1, defined as

$$X_n - \mu = \phi(X_{n-1} - \mu) + a_n \;, \qquad \left| \phi \right| < 1 \qquad (5)$$

where a_n is a normal white noise process with variance $\sigma_a{}^2$, expressions for f are derived in Appendix C. Note that the process defined in equation 5 is also a Markov process (5, p. 11). When T = 1, the following is shown to be valid in Appendix C:

Let h be as defined in equation 2; let L be the permissible exposure limit; and let z(h) be the solution of

$$P\ (Z > z(h)) = h \qquad o < h < 1 \tag{6}$$

where Z is a normally distributed random variable with mean equal to zero and variance equal to 1. Then when the measurement X = x, decision 2 is optimal for those states x for which

$$x\phi < L - \mu\ (1 - \phi) - z(h)\ \sigma_a\ , \tag{7}$$

and otherwise decision 3 is optimal. Thus, if h > 1, decision 2 is optimal for all outcomes x.

Further results for ARMA processes can be obtained, but one difficulty to be dealt with is the problem of forecasting when there are missing observations in the time series when ARMA processes more complex than the above example are used.

CONCLUSION

The sampling strategy proposed by NIOSH could lead to sequences of consecutive measurements being made, with the length of any particular sequence being determined by the outcomes of the previous measurements. The above properties of the sampling strategy are also true for the sampling strategies derived in this report, but with the following differences:

1. A maximum interval of time between samples must be specified,
2. It is not always true that a high concentration level leads to a decision to make a measurement, and a low concentration level leads to a decision not to make a measurement. The opposite can be the optimal strategy.

3. As the specified maximum interval between samples increases, the optimal decision is to never make a measurement unless required to.

4. A minimum cost policy is being invoked.

The applicability of DP is not limited to the cost structure and set of decisions used in this report that were chosen to illustrate in a simple but meaningful way the concepts involved in applying the technique of DP to the problem of developing an optimal sampling strategy.

To extend the DP approach to include the problem of parameter estimation, it would be necessary to derive expressions for the joint probability distribution of the parameters as a function of the number of measurements, and hence it would be necessary to use a finite horizon DP formulation. The problem of dimensionality could then cause computational difficulties (4, p. 65; 6).

It is possible to use ARMA models instead of Markov chains in a manner similar to that discussed in this report, but consideration must be given to the problem of missing observations when dealing with other than the simple autoregressive model.

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APPENDIX A

DERIVATION OF COSTS AND TRANSITION PROBABILITIES

In this Appendix, explicit expressions for use in equation 1 are derived, and some solutions are obtained.

Let the augmented states $Z_n = (X, T_n)$ be ordered lexicographically ((1,1), (2,1), ..., (1,2), (2,2), ..., (I,T)), where (1,1) is denoted as state one, and (I, T) as state IT. The costs to be used in equation 1 are

$$c_{i1} = c_1 \qquad 1 \leq i \leq I\,(T-1)\ , \tag{A1}$$

$$c_{i2} = c_1 + c_2 + c_4\, p^{(j+1)}_{i-jI,I} \qquad j\,I + 1 \leq i \leq (j+1)I\ , \quad 0 \leq j \leq T-1\ , \tag{A2}$$

and

$$c_{i3} = c_2 + c_3 \qquad 1 \leq i \leq I\,T \tag{A3}$$

where $p^{(j)}_{i,k}$ is the probability of a transition of X_n from state i to state k in j time intervals. (Note that c_{i1} is undefined if $T = 1$ or if $i > I\,(T-1)$).

If decision one is made, then the state (X, t) goes to (X, t+1) with probability one. Thus,

$$p(j \mid i, 1) = \begin{cases} 1 \text{ if } j = i + I, & 1 \leq i \leq I(T-1), \\ \text{undefined} & I\,(T-1) < i \leq I\,T \\ 0 & \text{otherwise}\ . \end{cases} \tag{A4}$$

If decisions two or three are made, the state (X_{n-t+1}, t) goes to (X_{n+1}, t) in t time intervals. Thus,

$$p(j \mid i,d) = \begin{cases} p_{i-kI,j}^{(k+1)} & kI+1 \leq i \leq (k+1)I,\ 0 \leq k \leq T-1, \\ & 1 < j < I \\ 0 & \text{otherwise} . \end{cases} \tag{A5}$$

Equation 1 can be solved using the value iteration or the policy iteration algorithms (2, 6), and a computer program was written using the policy iteration algorithm. (See Appendix B for the listing.)

Consider the special case where a measurement is made during every interval of time ($T = 1$). Then only decisions 2 or 3 are allowed. Since equation A5 applies whichever decision is made, equations A2 and A3 show that decision 2 is the optimal policy if $p_{i,I} < h$ for all outcomes $X = i$, $i = 1, 2, \ldots, I$ (see equation 2), and decision 3 is optimal otherwise. For $T \geq 1$, when $T_n = T$ and the result of the last measurement was $X = i$, decision two is optimal if $p_{i,I}^{(T)} < h$, and decision 3 is optimal otherwise. If the optimal decisions when in the augmented state j are found to be

$$d_j = \begin{cases} 1 & 1 \leq j \leq (T-1)\ I \\ 2 \text{ or } 3 & (T-1)\ I + 1 \leq j \leq IT , \end{cases} \tag{A6}$$

then the resulting equations for the minimum costs have a simple form which can be found by using equations A1 through A6 in

$$y_i = c_{id} + \alpha \sum_{j \in S} p(j \mid i, d) y_j \qquad 0 \leq \alpha < 1,\ i \in S . \tag{A7}$$

Letting u be a column vector with components

$$u_i = y_{i+(T-1)I} \qquad 1 \leq i \leq I \tag{A8}$$

and v a column vector with components

$$v_i = \frac{c_1\, \alpha(1-\alpha^{T-1})}{1-\alpha} + c_2 + \min \{ c_1 + c_4\, p_{i,I}^{(T)} ; c_3 \} \qquad 1 \leq i \leq T , \tag{A9}$$

the solution of equation A7 is given by

$$u = (I - \alpha^T P^T)^{-1} v \tag{A10}$$

where I is the identity matrix. The other costs are given by

$$y_i = \alpha^{T-k-1} u_{i-kI} + \frac{c_1\, (1-\alpha^{T-k-1})}{1-\alpha} \qquad 1+kI \leq i \leq (k-1)I,\ 0 \leq k \leq T-2,$$

$$1 \leq i \leq (T-1)\, I \quad . \tag{A11}$$

If it is assumed a priori that equation A6 is the solution to equation 1, then equations A8 though A11 can be used on the right-hand side of equation 1. If the solution of equation 1 is identical to that given by equations A8 through A11, then equation A6 is the solution (5, p. 128). It can be shown that equation A6 is the solution if all of the following inequalities are satisfied:

$$\alpha^{T-i}\,(I - \alpha^i P^i)(I - \alpha^T P^T)^{-1} \min\{c_3\underline{1};\ c_1\underline{1} + c_4\, P^{T-1} P_I\} <$$

$$\min\{c_3\underline{1};\ c_1\underline{1} + c_4 P^{i-1} P_I\} + c_2\,\frac{(1-\alpha^{T-i})}{1-\alpha^T}\,\underline{1} \qquad i = 1, 2, \ldots, T-1 \tag{A12}$$

where $\underline{1}$ is a column vector all of whose I elements equal one, P_I is the Ith column of P, and P^o is the identity matrix. Equation A12 is satisfied if

$$c_3\underline{1} < c_1\underline{1} + c_4\, P^{T-1}\, P_I$$

for all i = 1, 2, 3, ..., T-1 or vice versa. Other combinations of parameters for which equation A12 is satisfied are more difficult to find. As T becomes large, equation A12 will be satisfied, and equation A6 is the solution.

Let the probability transition matrix of the augmented states Z_n under the optimal policy be denoted by P*, and let the stationary solution for P* be denoted by the row vector π^*. Then the present value of the total expected cost using an optimal policy is given by

$$C = \sum_{i=1}^{IT} y_i\,\pi_i^* \quad . \tag{A13}$$

The probability that the process exceeds the permissible exposure limit is given by π_I. However, since it is possible not to have employees present when the process exceeds the permissible exposure limit, the probability that the process exceeds the permissible exposure limit when employees are present is given by

$$P\,(B) = \sum_{j=o}^{T-1} \sum_{\substack{i=1+jI \\ i\epsilon F}}^{(j+1)I} \pi_i^* \; p_{i-jI,I}^{(j+1)} \tag{A14}$$

where F is the set of augmented states which does not yield the optimal decision which is equal to 3.

In the special case given in equation A6, use of equations A4 and A5 yields

$$\Pi^* = (\Pi, \Pi, \ldots, \Pi) / T \quad . \tag{A15}$$

Finally, we note that as T becomes large, 1) u approaches v, where the components of v approach

$$\lim_{T\to\infty} v_i = \frac{c_1\alpha}{1-\alpha} + c_2 + \min \{c_1 + c_4 \pi_I ; c_3\} \tag{A16}$$

and 2)

$$\lim_{T\to\infty} P^T = \begin{bmatrix} \Pi' \\ \Pi' \\ \cdot \\ \cdot \\ \cdot \\ \Pi' \end{bmatrix} \quad . \tag{A17}$$

APPENDIX B

DESCRIPTION OF THE POLICY ITERATION PROGRAM

This is a description of the capabilities of the program entitled "Policy Iteration," a listing of which appears below. The program computes the optimal policy and the present value of the total expected costs y_i by solving equation 1 using the policy iteration algorithm (2, 6). The probability matrices $p(j|i,d)$ are given by equations A4 and A5, and the expected costs c_{id} by equations A1 through A3. In addition, the stationary solution of P and of P*, and the present value of the total expected cost C, given by equation A11 and P(B), given by equation A12, are also computed. If the Markov chain defined by P* is not irreducible, an irreducible chain is found by eliminating the transient states, and C* and P(B) are computed for this irreducible chain.

The output consists of a listing of the input data c_1, c_2, c_3, c_4, α, P and its dimensions, T, and the initial policy vector. The computed results displayed are $p\ (j|i,d)$ (if desired), c_{id}, the optimal policy vector, P*, Π, Π*, C, P(B), and the number of iterations needed for converging (ITN). A sample output appears in this Appendix.

Table B1 lists the order of the data necessary to run the program.

TABLE B1. ORDER OF INPUT DATA FOR THE PROGRAM

Order	Symbol	Description	Format
1.	IST;	Number of states in Markov chain (IST = 0 stops execution.)	I2
2.	IPRINT;	Print option for matrices $p(J \mid i,d)$ IPRINT = 0, no print. IPRINT = 1, print.	I2
3.	COST(4);	Costs (c_1, c_2, c_3, c_4).	4E 8.0
4.	ALPHA;	discount factor.	E 8.0
5.	P(I,J);	Transition matrix P, of dimension (IST) x (IST), one row per record, IST records.	(IST)E 8.0
6.	TINT;	Maximum value of T. If TINT = 0, program reads new IST, IPRINT, etc.	I2
7.	DEC(I);	Initial policy vector of length (IST) x (TINT) = IT.	(IT) I2

The source code is written for use on a VAX 11/780. The input file for the example problem is:

```
02
01
0.D0    .5D0    1.D0    10.D0
.95D0
.85D0   .15D0
.80D0   .20D0
02
03030303
00
00
```

```
C       FILE PITD.FOR
C --- POLICY ITERATION,A.L.SWEET,AUGUST 1980
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER TINT,DEC,TDEC
      DIMENSION COST(4),P(30,30),DEC(30),CM(30,3),R(30,30),
     -  PK(30,30), NC(30),PKT(30,30),PM(3,30,30),Y(30),WYE(30),
     -  TDEC(30),PI(30)
      OPEN(UNIT=09,NAME='PITDIN.DAT',TYPE='OLD',DISP='KEEP')
      OPEN(UNIT=10,NAME='PITDOUT.DAT',TYPE='NEW',DISP='KEEP')
C --- READ INPUT
C --- READ NO OF STATES IN MARKOV CHAIN
 1000 READ(09,100) IST
  100 FORMAT(30I2)
      IF(IST.EQ.0) GO TO 999
      READ(09,100) IPRINT
C --- READ COSTS
      READ(09,101)COST
  101 FORMAT(10D8.0)
C --- READ DISCOUNT FACTOR
      READ(09,101)ALPHA
C --- READ TRANSITION MATRIX FOR MARKOV CHAIN
      DO 1 I=1,IST
    1 READ(09,101)(P(I,J),J=1,IST)
C --- READ MAXIMUM INTERVAL BETWEEN MEASUREMENTS
 2000 READ(09,100) TINT
      IF(TINT.EQ.0) GO TO 1000
      IT=IST*TINT
C --- READ INITIAL POLICY COLUMN VECTOR,LENGTH IT
      READ(09,100) (DEC(I),I=1,IT)
C --- PRINT INPUT
      WRITE(10,200)IST,TINT
  200 FORMAT(1H1,4X,29HNO OF STATES IN MARKOV CHAIN=,I4/
     -            5X,38HMAXIMUM INTERVAL BETWEEN MEASUREMENTS=,I4)
      WRITE(10,201)COST
  201 FORMAT(1H0,4X,19HCOST OF PRODUCTION=,F7.2/
     -    5X,20HCOST OF MEASUREMENT=,F7.2/
     -    5X,28HCOST WHEN NOT IN PRODUCTION=,F7.2/
     -    5X,38HCOST OF EXCEEDING MAX POLLUTION LEVEL=,F7.2)
      WRITE(10,202)
  202 FORMAT(1H0,5X,34HTRANSITION MATRIX FOR MARKOV CHAIN,/)
      DO 2 I=1,IST
    2 WRITE(10,203)I,(P(I,J),J=1,IST)
  203 FORMAT(1H0,I4,6G12.4/(1H ,4X,6G12.4))
      WRITE(10,206)
  206 FORMAT(1H0,4X,36HDECISION 1 -PLANT RUN FOR PRODUCTION/
     -5X,46HDECISION 2 -PLANT RUN FOR BOTH PRODUCTION AND ,
     -11HMEASUREMENT/
     -5X,46HDECISION 3 -PLANT RUN FOR MEASUREMENT ONLY,NO ,
     -10HPRODUCTION)
C --- COMPUTE R VECTORS AND POWERS OF P
C --- THERE ARE TINT R VECTORS OF DIMENSION IST EACH
C --- IN R(I,J),I IS VECTOR NO,I=1,TINT AND J IS ROW,J=1,IST
C --- PK CONTAINS POWERS OF P
```

```
      DO 4 I= 1,IST
      DO 4 J= 1,IST
      PK(I,J)=0.D0
      IF(I.EQ.J) PK(I,J)=1.D0
    4 CONTINUE
      C12=COST(1)+COST(2)
      C4=COST(4)
      IC2=0
      DO 20 KLOOP=1,TINT
        DO 6 I=1,IST
        SUM=0.D0
          DO 5 J=1,IST
    5     SUM=SUM+PK(I,J)*P(J,IST)
    6   R(KLOOP,I)=C12+C4*SUM
      IC1=IC2+1
      IC2=KLOOP*IST
        DO 7 I=1,IST
    7   CM(IC1+I-1,2)=R(KLOOP,I)
        DO 10 I=1,IST
          DO 9 J=1,IST
          SUM=0.D0
            DO8 K=1,IST
    8       SUM=SUM+P(I,K)*PK(K,J)
    9     PKT(I,J)=SUM
   10   CONTINUE
        DO 11 I=1,IST
        DO 11 J=1,IST
        Z=PKT(I,J)
        PK(I,J)=Z
        PM(2,IC1+I-1,J)=Z
   11   PM(3,IC1+I-1,J)=Z
   20 CONTINUE
C --- STORE REST OF COST MATRIX,CM
      C23=COST(2)+COST(3)
      IT1=IT-IST
      IF(IT1.EQ.0) GO TO 251
      DO 25 I=1,IT1
      CM(I,3)=C23
   25 CM(I,1)=COST(1)
  251 IT2=IT1+1
      DO 26 I=IT2,IT
      CM(I,3)=C23
   26 CM(I,1)=1.D06
C --- STORE REST OF P2,P3
      IT3=IST+1
      IF(IT3.GT.IT) GO TO 271
      DO 27 I=1,IT
      DO 27 J=IT3,IT
      PM(2,I,J)=0.D0
   27 PM(3,I,J)=0.D0
C --- MAKE UP P1
  271 DO 28 I=1,IT
      DO 28 J=1,IT
```

```
   28 PM(1,I,J)=0.DO
      IT2=IT-IST
      IF(IT2.EQ.0) GO TO 30
      DO 29 I=1,IT2
   29 PM(1,I,IST+I)=1.DO
C --- PRINT P1, IF IPRINT = 1
   30 IF(IPRINT.EQ.0) GO TO 421
      WRITE(10,212)
  212 FORMAT(1HO,5X,29HTRANSITION MATRIX, DECISION 1)
      DO 41 I=1,IT
   41 WRITE(10,203)I,(PM(1,I,J),J=1,IT)
C --- PRINT P2 AND P3, IF IPRINT = 1
      WRITE(10,214)
  214 FORMAT(1HO,5X,36HTRANSITION MATRIX, DECISIONS 2 AND 3)
      DO 42 I=1,IT
   42 WRITE(10,203)I,(PM(2,I,J),J=1,IT)
C --- PRINT COST MATRIX
  421 WRITE(10,215)
  215 FORMAT(1HO,4X,11HCOST MATRIX,/)
      DO 43 I=1,IT
   43 WRITE(10,210)(CM(I,J),J=1,3)
  210 FORMAT(1H ,4X,6G12.4)
C --- PRINT INITIAL POLICY VECTOR AND DISCOUNT FACTOR
      WRITE(10,204)
  204 FORMAT(1HO,4X,21HINITIAL POLICY VECTOR,/)
      WRITE(10,209)(DEC(I),I=1,IT)
  209 FORMAT(1HO,10X,30I3)
      WRITE(10,205)ALPHA
  205 FORMAT(1HO,4X,6HALPHA=,F6.4)
C --- ITERATION LOOP
      DO 70 ITN=1,20
C --- COMPUTE COEFFICIENT MATRIX FOR ITERATION
      DO 51 I=1,IT
      DO 51 J=1,IT
      Z=-ALPHA*PM(DEC(I),I,J)
      IF(I.EQ.J) Z=Z+1.DO
   51 PKT(I,J)=Z
C --- INVERT PKT
      CALL GJR(PKT,IT,30,1.D-10,MFLG)
      IF(MFLG.NE.0) GO TO 99
C --- FIND Y(I),I=1,IT
      DO 53 I=1,IT
      SUM=0.DO
      DO 52 J=1,IT
   52 SUM=SUM+PKT(I,J)*CM(J,DEC(J))
   53 Y(I)=SUM
C --- FIND WYE(I),I=1,IT
      DO 56 I=1,IT
      TW=1.D06
      TDEC(I)=DEC(I)
      DO 55 N=1,3
      SUM=0.DO
      DO 54 J=1,IT
```

```
   54 SUM=SUM+PM(N,I,J)*Y(J)
      TWYE=ALPHA*SUM+CM(I,N)
      IF(TWYE.GE.TW) GO TO 55
      TW=TWYE
      DEC(I)=N
   55 CONTINUE
   56 WYE(I)=TW
      ISUM=0
      DO 57 I=1,IT
   57 ISUM=ISUM+IABS(DEC(I)-TDEC(I))
      IF(ISUM.EQ.0) GO TO 71
   70 CONTINUE
C --- END ITERATION LOOP
   71 WRITE(10,216)ITN
  216 FORMAT(1H0,4X,14HOPTIMUM POLICY,5X,4HITN=,I2//5X,5HSTATE,
     -       4X,8HDECISION,3X,10HMIN. COSTS)
      DO 58 I=1,IT
      NC(I)=I
   58 WRITE(10,219)I,DEC(I),WYE(I)
  219 FORMAT(6X,I2,8X,I2,8X,F7.3)
C --- OPTIMUM POLICY TRANSITION MATRIX
      ITT=IT
      DO 61 I=1,ITT
      DO 61 J=1,ITT
      PKT(I,J)=PM(DEC(I),I,J)
   61 CONTINUE
  611 WRITE(10,220)
  220 FORMAT(1H0,4X,32HOPTIMUM POLICY TRANSITION MATRIX)
      DO 62 I=1,ITT
   62 WRITE(10,203)I,(PKT(I,J),J=1,ITT)
C --- FIND STATIONARY SOLUTION OF OPTIMUM POLICY MATRIX
      DO 63 I=1,ITT
      DO 63 J=1,ITT
      R(I,J)=PKT(J,I)
      IF(I.EQ.J) R(I,I)=R(I,I)-1.D0
   63 CONTINUE
      IT1=ITT-1
      DO 64 I=1,IT1
   64 Y(I)=-R(I,ITT)
      CALL GJR(R,IT1,30,1.D-10,MFLG)
      IF(MFLG.NE.0) GO TO 680
      DO 66 I=1,IT1
      SUM=0.D0
      DO 65 J=1,IT1
   65 SUM=SUM+R(I,J)*Y(J)
   66 PI(I)=SUM
      PI(ITT)=1.D0
      SUM=0.D0
      DO 67 I=1,ITT
   67 SUM=SUM+PI(I)
      DO 68 I=1,ITT
   68 PI(I)=PI(I)/SUM
      WRITE(10,221)
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  221 FORMAT(1H0,4X,35HSTATIONARY SOLUTION,OPTIMUM POLICY ,
     -  6HMATRIX,/)
      WRITE(10,210)(PI(I),I=1,ITT)
      GO TO 681
C --- MAKE UP REDUCED OPTIMUM POLICY MATRIX
  680 WRITE(10,217)
  640 ITOG=0
      DO 642 J=1,IT
      IF(NC(J).EQ.0) GO TO 642
      SUM=0.D0
      DO 641 I=1,IT
      IF(NC(I).EQ.0) GO TO 641
      SUM = SUM+PKT(I,J)
  641 CONTINUE
      IF(SUM.NE.0.D0) GO TO 642
      ITOG=1
      NC(J)=0
  642 CONTINUE
      IF(ITOG.NE.0) GO TO 640
      JN=0
      DO 644 J=1,IT
      IF(NC(J).EQ.0) GO TO 644
      JN=JN+1
      IN=0
      DO 643 I=1,IT
      IF(NC(I).EQ.0) GO TO 643
      IN=IN+1
      PKT(IN,JN)=PKT(I,J)
  643 CONTINUE
  644 CONTINUE
      ITT=JN
      GO TO 611
C --- MINIMUM EXPECTED COST
  681 Z=0.D0
      IN=0
      DO 69 I=1,IT
      IF(NC(I).EQ.0) GO TO 69
      IN=IN+1
      Z=Z+WYE(I)*PI(IN)
   69 CONTINUE
      WRITE(10,222)Z
  222 FORMAT(1H0,4X,22HMINIMUM EXPECTED COST=,F7.3)
C --- PROBABILITY OF WORKER BEING EXPOSED TO POLLUTION LEVELS
C ---      ABOVE MAXIMUM PERMISSIBLE
      DO 82 I=1,IST
      DO 82 J=1,IST
      PK(I,J)=0.D0
      IF(I.EQ.J) PK(I,J)=1.D0
   82 CONTINUE
      B=0.D0
      IN=0
      DO 90 JT=1,TINT
      DO 86 I=1,IST
```

```
      DO 85 J=1,IST
      SUM=0.D0
      DO 84 K=1,IST
84 SUM=SUM+P(I,K)*PK(K,J)
85 PKT(I,J)=SUM
86 CONTINUE
   DO 87 I=1,IST
   DO 87 J=1,IST
87 PK(I,J)=PKT(I,J)
   SUM=0.
   I1=1+(JT-1)*IST
   I2=JT*IST
   DO 89 I=I1,I2
   IF(DEC(I).EQ.3) GO TO 89
   IF(NC(I).EQ.0) GO TO 89
   IN=IN+1
   I3=I-(JT-1)*IST
   SUM=SUM+PI(IN)*PK(I3,IST)
89 CONTINUE
   B=B+SUM
90 CONTINUE
   WRITE(10,224)B
24 FORMAT(1H0,4X,21HEXPOSURE PROBABILITY=,F6.3)
-- STATIONARY SOLUTION OF TRANSITION MATRIX
   DO 73 I=1,IST
   DO 73 J=1,IST
   R(I,J)=P(J,I)
   IF(I.EQ.J) R(I,I)=R(I,I)-1.D0
73 CONTINUE
   IT1=IST-1
      DO 74 I=1,IT1
   74 Y(I)=-R(I,IST)
      CALL GJR(R,IT1,30,1.D-10,MFLG)
      IF(MFLG.NE.0) GO TO 99
      DO 76 I=1,IT1
      SUM=0.D0
      DO 75 J=1,IT1
   75 SUM=SUM+R(I,J)*Y(J)
   76 PI(I)=SUM
      PI(IST)=1.D0
      SUM=0.D0
      DO 77 I=1,IST
   77 SUM=SUM+PI(I)
      DO 78 I=1,IST
   78 PI(I)=PI(I)/SUM
      WRITE(10,223)
  223 FORMAT(1H0.4X,37HSTATIONARY SOLUTION,TRANSITION MATRIX,/)
      WRITE(10,210)(PI(I),I=1,IST)
      GO TO 2000
  999 STOP
   99 WRITE(10,217)
  217 FORMAT(1H0,5X,15HSINGULAR MATRIX)
      STOP
      END
```

```
C     *****
C     *GJR*
C     *****
      SUBROUTINE GJR(A,N,MM,EPS,MFLG)
      IMPLICIT REAL*8 (A-H,O-Z)                                          PC20805
C     GAUSS-JORDAN-RUTISHAUSER MATRIX INVERSION WITH DOUBLE PIVOTING.    PC20806
      DIMENSION B(30),C(30),IP(30),IQ(30),A(MM,MM)
      MFLG=0
      DO 55 K=1,N                                                        PC20810
C     DETERMINATION OF THE PIVOT ELEMENT                                 PC20811
      ABSPIV=0.DO                                                        PC20812
      DO 5 I=K,N                                                         PC20813
      DO 5 J=K,N                                                         PC20814
      ABSAIJ=DABS(A(I,J))                                                PC20815
      IF (ABSPIV.GT.ABSAIJ) GO TO 5                                      PC20816
      PIVOT=A(I,J)                                                       PC20817
      ABSPIV=ABSAIJ                                                      PC20818
      IP(K)=I                                                            PC20819
      IQ(K)=J                                                            PC20820
    5 CONTINUE                                                           PC20821
      IF (ABSPIV.GT.EPS) GO TO 15                                        PC20822
      MFLG=1
      RETURN                                                             PC20826
C     EXCHANGE OF THE PIVOTAL ROW WITH THE KTH ROW                       PC20827
   15 L=IP(K)
      IF(L.EQ.K) GO TO 25
      DO 20 J=1,N                                                        PC20829
      Z=A(L,J)                                                           PC20831
      A(L,J)=A(K,J)                                                      PC20832
   20 A(K,J)=Z                                                           PC20833
C     EXCHANGE OF THE PIVOTAL COLUMN WITH THE KTH COLUMN                 PC20834
   25 L=IQ(K)                                                            PC20837
      IF(L.EQ.K) GO TO 35
      DO 30 I=1,N                                                        PC20836
      Z=A(I,L)                                                           PC20838
      A(I,L)=A(I,K)                                                      PC20839
   30 A(I,K)=Z                                                           PC20840
C     JORDAN STEP                                                        PC20841
   35 DO 50 J=1,N                                                        PC20842
      IF (J.EQ.K) GO TO 40                                               PC20843
      B(J)=-A(K,J)/PIVOT                                                 PC20844
      C(J)=A(J,K)                                                        PC20845
      GO TO 45                                                           PC20846
   40 B(J)=1.DO/PIVOT
      C(J)=1.DO                                                          PC20848
   45 A(K,J)=0.DO                                                        PC20849
   50 A(J,K)=0.DO                                                        PC20850
      DO 55 I=1,N                                                        PC20851
      DO 55 J=1,N                                                        PC20852
   55 A(I,J)=A(I,J)+C(I)*B(J)                                            PC20853
C     REORDERING THE MATRIX                                              PC20854
      DO 75 M=1,N                                                        PC20855
```

```
      K=N-M+1                                                   PC20856
      L=IP(K)                                                   PC20859
      IF(L.EQ.K) GO TO 65
      DO 60 I=1,N                                               PC20858
      Z=A(I,L)                                                  PC20860
      A(I,L)=A(I,K)                                             PC20861
   60 A(I,K)=Z                                                  PC20862
   65 L=IQ(K)                                                   PC20865
      IF(L.EQ.K) GO TO 75
      DO 70 J=1,N                                               PC20864
      Z=A(L,J)                                                  PC20866
      A(L,J)=A(K,J)                                             PC20867
   70 A(K,J)=Z                                                  PC20868
   75 CONTINUE                                                  PC20869
      RETURN                                                    PC20870
      END                                                       PC20871
```

```
NO OF STATES IN MARKOV CHAIN=   2
MAXIMUM INTERVAL BETWEEN MEASUREMENTS=   2

COST OF PRODUCTION=   0.00
COST OF MEASUREMENT=   0.50
COST WHEN NOT IN PRODUCTION=   1.00
COST OF EXCEEDING MAX POLLUTION LEVEL=  10.00

 TRANSITION MATRIX FOR MARKOV CHAIN


1  0.8500      0.1500

2  0.8000      0.2000

DECISION 1 -PLANT RUN FOR PRODUCTION
DECISION 2 -PLANT RUN FOR BOTH PRODUCTION AND MEASUREMENT
DECISION 3 -PLANT RUN FOR MEASUREMENT ONLY,NO PRODUCTION

 TRANSITION MATRIX, DECISION 1

1  0.0000E+00  0.0000E+00   1.000      0.0000E+00

2  0.0000E+00  0.0000E+00  0.0000E+00   1.000

3  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00

                                    38
4  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00

 TRANSITION MATRIX, DECISIONS 2 AND 3

1  0.8500      0.1500      0.0000E+00  0.0000E+00

2  0.8000      0.2000      0.0000E+00  0.0000E+00

3  0.8425      0.1575      0.0000E+00  0.0000E+00

4  0.8400      0.1600      0.0000E+00  0.0000E+00

COST MATRIX

  0.0000E+00  2.000      1.500
  0.0000E+00  2.500      1.500
  0.1000E+07  2.075      1.500
  0.1000E+07  2.100      1.500

INITIAL POLICY VECTOR


       3  3  3  3

ALPHA=0.9500
```

OPTIMUM POLICY ITN= 2

STATE	DECISION	MIN. COSTS
1	1	14.615
2	1	14.615
3	3	15.385
4	3	15.385

OPTIMUM POLICY TRANSITION MATRIX

1	0.0000E+00	0.0000E+00	1.000	0.0000E+00
2	0.0000E+00	0.0000E+00	0.0000E+00	1.000
3	0.8425	0.1575	0.0000E+00	0.0000E+00
4	0.8400	0.1600	0.0000E+00	0.0000E+00

STATIONARY SOLUTION,OPTIMUM POLICY MATRIX

0.4211	0.7895E-01	0.4211	0.7895E-01

MINIMUM EXPECTED COST= 15.000

EXPOSURE PROBABILITY= 0.079

STATIONARY SOLUTION,TRANSITION MATRIX

0.8421	0.1579

APPENDIX C

DERIVATION OF COSTS WHEN USING CONTINUOUS VARIABLES

In this Appendix, explicit expressions for use in equation 4 are derived. The costs to be used in equation 4 are

$$c_{(x,t),1} = c_1 \qquad 1 \leq t < T \qquad \text{(C1)}$$

$$c_{(x,t),2} = c_1 + c_2 + c_4 P(A) \qquad 1 \leq t \leq T \qquad \text{(C2)}$$

where

$$P(A) = P(X_{n+1} > L \mid X_{n-t+1} = x,\ T_n = t) \qquad 1 \leq t \leq T \qquad \text{(C3)}$$

and

$$c_{(x,t),3} = c_2 + c_3 \qquad 1 \leq t \leq T\ . \qquad \text{(C4)}$$

In the above, $c_{(x,t),1}$ is undefined if $T = 1$ or if $t = T$. To derive an expression for $P(A)$, note that the solution of equation 5, subject to an initial value of X_n equal to x, is

$$X_n = x\phi^n + \mu(1-\phi^n) + \sum_{i=1}^{n} \phi^{n-i}\ a_i \qquad n \geq 1\ . \qquad \text{(C5)}$$

Since the a_n are normal, X_n is normal with expectation given by the first two terms on the right-hand side of equation C5, and variance

$$V(X_n) = \sigma_a^2\ (1-\phi^{2n})\ /\ (1-\phi^2)\ . \qquad \text{(C6)}$$

Thus,

$$P(A) = P\left(Z > \frac{L - x\phi^t - \mu(1-\phi^t)}{V(X_t)^{1/2}} \right) \tag{C7}$$

where Z is a normal (0, 1) random variable.

If T = 1, it is only necessary to compare equations C2 and C4, and thus equations 6 and 7 follow, since decision 2 is optimal if P(A) < h.

FILM

5-8