AD-A126 289 THEORIES OF NUTATION AND POLAR MOTION III(U) OHIO STATE UNIV COLUMBUS DEPT OF GEODETIC SCIENCE AND SURVEYING H MORITZ DEC 82 342 AFGL-TR-82-0387 F19628-82-K-0017 1/2

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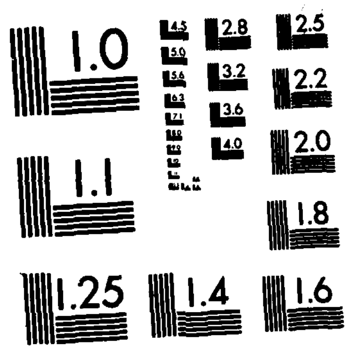

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THEORIES OF NUTATION AND POLAR MOTION III

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COLUMBUS, OHIO 43210

DECEMBER 1982

SCIENTIFIC REPORT NO. 2

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFGL-TR-82-0387	2. GOVT ACCESSION NO.: AD-A126289
3. RECIPIENT'S CATALOG NUMBER:	
4. TITLE (and Subtitle): THEORIES OF NUTATION AND POLAR MOTION III	5. TYPE OF REPORT & PERIOD COVERED: Scientific Report No.2
	6. PERFORMING ORG. REPORT NUMBER: Report No. 342
7. AUTHOR(s): HELMUT MORITZ	8. CONTRACT OR GRANT NUMBER(s): F19628-82-K-0017
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Geodetic Science & Surveying, The Ohio State University, Columbus, Ohio 43210	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 62101F, 760003AM
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Geophysics Laboratory, Hanscom AFB, Massachusetts 01731, Contract Monitor - Christopher Jekeli/LWG	12. REPORT DATE: December 1982
	13. NUMBER OF PAGES: 116 pages
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Earth rotation
Geodynamics
Nutation
Polar Motion

(AD-A108[illegible] and AD-A099015).

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The present report is a continuation and conclusion of two previous OSU-DGSS reports ~~"Theories of~~ Nutation and Polar Motion I ~~and II"~~, No. 309 ~~(1980) and 318 (1981)~~. It treats the solution of Sasao, Okubo and Saito (1980), which is the simplest and most straightforward solution of Molodensky's liquid core problem (Part A). In Part B, a relatively little known variational principle of

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) cont → Poincaré is derived and applied to a unified deduction of the equations of Sasao et al. Part C provides explicit expression for lunisolar nutation and polar motion for the principal terrestrial axes: the rotation axis, the figure axis, the mean Tisserand axis, the angular momentum axis, and the axis associated with the Celestial Ephemeris Pole as adopted by the IAU in 1979, taking into account liquid core effects.

FOREWORD

This report was prepared by Dr. Helmut Moritz, Professor, Technical University at Graz and Adjunct Professor, Department of Geodetic Science and Surveying of The Ohio State University, under Air Force Contract No. F19628-82-K-0017, The Ohio State University Research Foundation, Project No. 714255, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, MA with Dr. Christopher Jekeli, Contract Manager.

CONTENTS

INTRODUCTION

The present report is the third and last of a series of reports on the theoretical description of polar motion and nutation. The two previous reports, (Moritz, 1980), henceforth referred to as TNP I, and (Moritz, 1981), referred to as TNP II, will serve as a basis.

In TNP I, we have considered the earth as a rigid body, an elastic solid, and as a "Poincaré model" consisting of a rigid mantle and a fluid homogeneous core. The report TNP II then treated the earth as composed of an elastic mantle and fluid core, the so-called Molodensky model.

The present report considers in detail the simplest theory of the Molodensky model due to Sasao, Okubo and Saito (1980). An application of Poincaré's equations of motion on a Lie group, following (Moritz, 1982 a), will provide a unified derivation.

Finally, on the basis of this model and on the eigenvalue theory described in TNP I, secs. 10 and 11, we shall present expressions for lunisolar effects on polar motion and nutation for the various axes: the rotation axis, the figure axis, the "mean Tisserand figure axis", the angular momentum axis, and the axis corresponding to the Celestial Ephemeris Pole as adopted by the IAU in 1979.

PART A

THE SOLUTION OF SASAO, OKUBO AND SAITO (SOS)

1. The Four Basic SOS Equations

As we already briefly mentioned in TNP II[1], pp.131-133, Sasao et al. (1980) gave a particularly simple and elegant formulation of Molodensky's problem in terms of four complex equations, generalizing Poincaré's (1910) equations for a rigid mantle and a liquid core.

The first two equations may be written:

$$A\dot{u} - i(C-A)\Omega u + A_c(\dot{v} + i\Omega v) + \Omega(\dot{c} + i\Omega c) = L \ , \qquad (1-1)$$
$$A_c\dot{u} + A_c\dot{v} + iC_c\Omega v + \Omega\dot{c}_c = 0 \ . \qquad (1-2)$$

Here, A, A, C are the (average) principal moments of inertia of the earth, supposed rotationally symmetric, and

1) This notation is explained in the introduction: TNP I denotes (Moritz, 1980), and TNP II denotes (Moritz, 1981).

A_c, A_c, C_c are the corresponding principal moments of inertia for the core; Ω denotes the (average) angular velocity of the earth's rotation; and the usual mathematical symbols are used: $i^2 = -1$ and the dot denoting differentiation with respect to time. The variables entering in these equations have the following meaning:

$$u = \omega_1 + i\omega_2 \ , \qquad v = \chi_1 + i\chi_2 \ , \qquad L = L_1 + iL_2 \ , \tag{1-3}$$

$$c = c_{13} + ic_{23} \ , \qquad c_c = c^c_{13} + c^c_{23} \ . \tag{1-4}$$

They are complex combinations of components of the vectors

$$\underline{\omega} = (\omega_1, \omega_2, \omega_3) \ , \qquad \underline{\chi} = (\chi_1, \chi_2, \chi_3) \ , \qquad \underline{L} = (L_1, L_2, L_3) \ , \tag{1-5}$$

where $\underline{\omega}$ denotes the vector of angular velocity of the rotation of the earth with respect to inertial space, $\underline{\chi}$ the vector of angular velocity of the rotation of the core with respect to the earth's mantle, and $\underline{L}$ denotes the lunisolar torque, due to the attraction of sun and moon. The coordinate system ($x_1 = x$, $x_2 = y$, $x_3 = z$) used is mantle-fixed in the sense that the mantle is at rest (on the average) in this system; more precisely, it is a Tisserand frame for the mantle (cf. Munk and Macdonald, 1960, p.10; TNP II, pp.140-143). The x_3 axis is directed to the (average) North Pole. Finally, c_{13} is the x_1x_3 component of the inertia tensor of the earth, and similarly for c_{23} ; c^c_{13} and c^c_{23} are the corresponding quantities for the core.

If the mantle is rigid and if the coordinate axes are principal axes of inertia, then the inertia tensor is diagonal, so that the quantities (1 - 4) are zero. Then (1 - 1) and (1 - 2) reduce to Poincaré's (1910) well-known equations for a rigid mantle, as they should. What is surprising, however, is the discovery by Sasao et al. (1980) that the generalization to Molodensky's problem, for an elastic mantle, is so simple.

The quantities (1 - 3) and (1 - 4) are related by an equation of the form

$$c = D_{11}(u - w) + D_{12}v \quad , \tag{1-6}$$

$$c_c = D_{12}(u - w) + D_{22}v \quad , \tag{1-7}$$

where the coefficients D_{11}, D_{12}, D_{22} are real constants which depend only on the elastic properties of the mantle. They are easily expressed in terms of the Love number k and of the well-known functions $y_5(r)$ and $y_6(r)$ in standard notation introduced by Alterman et al. (1959), or of the equivalent functions $R(r)$ and $P(r)$ used in TNP II, Part B. Such expressions are represented by eqs. (54) to (57) of (Sasao et al., 1980); they will be derived in the next section. Essential is the fact that the matrix

$$\underline{D} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix} \tag{1-8}$$

is symmetric; this is a consequence of a reciprocity theorem (Sasao et al., 1980, eq. (61)).

The complex quantity

$$w = w_1 + iw_2 \tag{1-9}$$

is related to the lunisolar (tidal) potential V_e , which we assume to have the form

$$V_e = \kappa(xz\cos\sigma t + yz\sin\sigma t) \quad , \qquad (1\text{-}10)$$

κ being a constant coefficient. Such a form correspo... to a tesseral tidal potential, being a spherical harmon... of degree 2 and order 1 , which is responsible for f...ced nutation and polar motion (TNP I, p.30; TNP II, p.23 ... rather to the contribution of a certain frequency σ ..., the tidal potential, which is the sum of all such contributions. Then w_1 and w_2 are defined by

$$w_1 = \Omega^{-1}\kappa\cos\sigma t \quad ,$$
$$w_2 = \Omega^{-1}\kappa\sin\sigma t \quad , \qquad (1\text{-}11)$$

which is the desired relation between w and V_e . Let it also be mentioned that the angular momentum components are given by

$$L_1 = \kappa M J_2 \sin\sigma t \quad ,$$
$$L_2 = -\kappa M J_2 \cos\sigma t \quad , \qquad (1\text{-}12)$$

in view of well-known relations (Melchior, 1978, sec. 2.3; TNP I, p.30); here κ is the same as in (1 - 10), and

$$J_2 = \frac{C - A}{Ma^2}$$

is the well-known zonal harmonic coefficient, a denoting the earth's semimajor axis and M its mass; C and A are (mean) principal moments of inertia as before.

Eqs. (1 - 1), (1 - 2), (1 - 6), and (1 - 7) constitute a set of 4 equations for the 4 complex unknowns u, v, c, c_c, equivalent to 8 equations for 8 real unknowns; they are identical (apart from notation) to eqs.(37), (32), (54), (56) of (Sasao et al., 1980); they will be called the SOS equations. Now (1 - 6) and (1 - 7) are simply linear algebraic equations with constant coefficients, whereas (1 - 1) and (1 - 2) are linear differential equations with constant coefficients, which can be reduced to algebraic linear equations by seeking solutions of the form

$$u = u_0 e^{i\sigma t} \quad , \qquad v = v_0 e^{i\sigma t} \tag{1 - 13}$$

as usual. Thus, a linear system of 4 ordinary equations for 4 complex unknown results.

This formulation of Molodensky's problem is probably the simplest given so far. It is particularly remarkable that it is valid for arbitrary earth models with an elastic mantle, a liquid core, and even an elastic inner core. The detailed structure of mantle and inner core enters only into (1 - 6) and (1 - 7), since the coefficients D_{ij} depend on the elasticity functions R(r) and P(r) ; it does not enter into (1 - 1) and (1 - 2).

Equations (1 - 1), (1 - 6), and (1 - 7) are relatively easy to get. Eq. (1 - 1) is a consequence of Euler's equation as generalized to a non-rigid body by Liouville; it is equivalent to eq. (10 - 6) of TNP II. Equations (1 - 6) and (1 - 7) will be derived in the next section.

The derivation of (1 - 2) is considerably more difficult. Sasao et al. (1980) obtain it by means of the hydrodynamic equations, which is complicated and far from transparent. Also, the similarity between (1 - 1) and (1 -2) remains unexplained. This is the more regrettable as it is the great merit of Sasao et al. to have found eq. (1 - 2), which is simple and similar to (1 - 1); an equivalent equation by Molodensky (1961; his eq. (39)) is much more complicated.

For the simpler rigid-mantle model, Poincaré (1910) has given two different methods for obtaining his equivalent to our eq. (1 - 2). One method uses the hydrodynamic equations, corresponding to the approach by Sasao et al. (1980). Much more interesting, however, is Poincaré's other method, which uses a variational principle and exploits symmetries expressed by group theory. This method is not only simpler and more elegant, but also explains the similarity of (1 - 1) and (1 - 2): both are effects of rotating groups, the first describing the rotation of the earth with respect to inertial space, and the second expresses the rotation of the core with respect to the mantle.

A similar group-theoretic derivation of the SOS equations will be given in Part B of the present report.

2. Derivation of Two SOS Equations

Equation (1 - 6). This equation describes the change of the inertia tensor (elements c_{13} and c_{23}) by the deformation of the earth, due to centrifugal force, expressed by the rotation vector u , and to lunisolar tidal force, expressed by w according to (1 - 10) and (1 - 11). It is essentially the same as eq. (63) of (Molodensky, 1961) and constitutes a generalization of well-known expressions for the rotational deformation of an elastic earth; cf. eq. (14) of (Jeffreys, 1970, sec. 7.04) or TNP I, eq. (4 - 1).

We derive (1 - 6) here from eq.(10 - 32) of TNP II:

$$\begin{aligned} c_{13} &= -\frac{1}{3} G^{-1} k(\kappa - \Omega^2 \varepsilon)\cos\sigma t \quad , \\ c_{23} &= -\frac{1}{3} G^{-1} k(\kappa - \Omega^2 \varepsilon)\sin\sigma t \quad , \end{aligned} \tag{2 - 1}$$

which can be combined into one complex equation using (1 - 4):

$$c = -\frac{1}{3} G^{-1} k(\kappa - \Omega^2 \varepsilon) e^{i\sigma t} \quad . \tag{2 - 2}$$

Here G denotes the Newtonian gravitational constant and k the potential Love number as usual; also the symbol Ω for the (average) rotational velocity of the earth is standard.

We put

$$m_1 = \Omega^{-1}\omega_1 \quad , \qquad m_2 = \Omega^{-1}\omega_2 \quad , \tag{2 - 3}$$

so that m_1 and m_2 are the x_1 and x_2 components of the unit vector of the instantaneous rotation axis (TNP I, sec.3; TNP II, sec.10), and

$$m_1+im_2 = \varepsilon e^{i\sigma t} \tag{2 - 4}$$

by eq. (1 - 28) of TNP II. With (1 - 3) this gives

$$u = \Omega\varepsilon e^{i\sigma t} \quad , \tag{2 - 5}$$

and from (1 - 11) we get

$$w = \Omega^{-1}\kappa e^{i\sigma t} \quad . \tag{2 - 6}$$

Thus (2 - 2) may be written

$$c = -\frac{1}{3} G^{-1}k\ \Omega(w-u) \quad . \tag{2 - 7}$$

In this equation we have taken the radius of the mean terrestrial sphere, denoted by a, to be our unit of length. If $a \neq 1$, then we must replace (2 - 7) by

$$c = -\frac{1}{3} G^{-1}ka^5\Omega(w-u) \quad , \tag{2 - 8}$$

in order to get the dimensions correct. For $w = 0$ this reduces to eq. (4 - 1) of TNP I, as it should.

Following (Sasao et al., 1980), we decompose the Love number k as follows:

$$k = k_0 + \frac{v}{w-u} k_1 \,, \qquad (2-9)$$

with constant k_0 and k_1 . Note that, although u, v, w are complex numbers, the quotient v/(w-u) is real since the common complex factor $e^{i\sigma t}$ cancels. The constant k represents the Love number in the absence of core rotation (v = 0) , and k_1 expresses the effect of core rotation. The decomposition (2 - 9) will be justified later in this section.

The substitution of (2 - 9) into (2 - 8) finally yields

$$c = D_{11}(u-w) + D_{12}v \qquad (2-10)$$

with

$$D_{11} = \frac{1}{3} G^{-1} a^5 \Omega k_o \,,$$

$$D_{12} = -\frac{1}{3} G^{-1} a^5 \Omega k_1 \,. \qquad (2-11)$$

This completes the derivation of (1 - 6).

Equation (1 - 7). This equation is the equivalent for the core of (1 - 6) or (2 - 10). Its derivation, however, is considerably more laborious and may be skipped by the reader who is not interested in this detail.

Denote by C_o the undeformed core

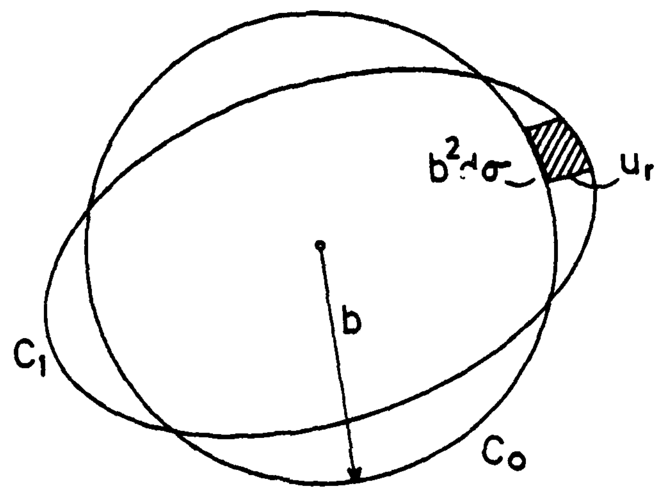

FIGURE 2.1. The core before (C_o) and after deformation (C_1)

and by C_1 the core after deformation (Fig.2.1) . For the present purpose it is sufficient to consider C_o a sphere of radius b ; this is the usual spherical approximation.

The xz product of inertia is given by the well-known formula (cf. Heiskanen and Moritz, 1967, p.62):

$$D^c_{xz} = \iiint_{C_1} xz dM \quad . \tag{2 - 12}$$

The integral is extended over the deformed kernel C_1 .

Before deformation, this integral is zero for reasons of symmetry (for the sphere as well as for an ellipsoid of revolution if the x_3 axis is the axis of symmetry):

$$\iiint_{C_o} \rho xz dv = 0 \ , \tag{2 - 13}$$

where ρ is the density before deformation, which need not be constant but can be a function of the radius vector r in the spherical approximation:

$$\rho = \rho(r) \ ; \tag{2 - 14}$$

dv denotes the volume element.

Since by definition

$$C^c_{13} = -D^c_{xz} \tag{2 - 15}$$

(TNP I, p.9; TNP II, p.113), (2 - 12) becomes

$$-C^c_{13} = \iiint_{C_1} (\rho+\rho_1) \ xz dv \ , \tag{2 - 16}$$

ρ_1 denoting the change of density due to deformation. This integral may be split up by

$$\iiint_{C_1} = \iiint_{C_o} + \iiint_{C_1 - C_o} \tag{2 - 17}$$

as follows:

$$-c^c_{13} = \iiint_{C_o} \rho xz dv + \iiint_{C_o} \rho_1 xz dv + \iiint_{C_1-C_o} \rho xz dv + \iiint_{C_1-C_o} \rho_1 xz dv \ . \tag{2 - 18}$$

Here $C_1 - C_o$ denotes the layer of thickness u (radial component of displacement) by which C_1 differs from C_o. By Fig. 2.1 we have

$$\iiint_{C_1-C_o} dv = b^2 \iint_\sigma u_r d\sigma \ , \tag{2 - 19}$$

$d\sigma$ denoting the element of solid angle, or the surface element of the unit sphere σ.

The first integral in (2 - 18) is zero by (2 - 13) and the last integral can be neglected because it is a second-order quantity, u_r and ρ_1 being small of first order.

Thus there remains

$$c^{c}_{13} = -\iiint_{C_o} \rho_1\, xzdv \;-\; b^2 \iint_{\sigma} \rho\, xzu_r d\sigma \quad ; \qquad (2-20)$$

here we have used (2 - 19).

By the well-known Poisson equation we express the anomalous density ρ_1 in terms of the corresponding potential disturbance V_1 :

$$\Delta V_1 = -4\pi G \rho_1 \quad ; \qquad (2-21)$$

cf. TNP II, eq. (6 - 30) . Thus the first integral of (2 - 20) may be written

$$-\iiint_{C_o} \rho_1 xzdv = \frac{1}{4\pi G} \iiint_{C_o} \Delta V_1 \; xzdv \quad . \qquad (2-22)$$

This integral can be transformed by means of Green's second identity:

$$\iiint_{v} U \Delta V dv = \iiint_{v} V \Delta U dv + \iint_{S} \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) dS \quad ; (2-23)$$

cf. (Heiskanen and Moritz, 1967, p.11). Here v denotes a volume bounded by a closed surface S , and $\partial/\partial n$ is a symbol for the derivative along the surface normal. In the present case, v is the spherical core C_o, the boundary S is the sphere $r = b$, the normal derivative is taken along the radius vector r :

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r} \quad , \qquad (2-24)$$

and the surface element is

$$dS = b^2 d\sigma \qquad (2-25)$$

by Fig. 2.1; furthermore we take in (2 - 23)

$$V = V_1 \quad , \qquad (2-26)$$

$$U = xz = r^2 \sin\theta\cos\theta\cos\lambda \qquad (2-27)$$

in spherical coordinates r, θ, λ .

Thus (2 - 23) gives

$$\iiint_{C_o} \Delta V_1 xz dv = b^2 \iint_\sigma \left(xz \frac{\partial V_1}{\partial r} - V_1 \frac{\partial (xz)}{\partial r} \right) d\sigma \; ; \qquad (2-28)$$

the first integral on the right-hand side is zero since $U = xz$ is a harmonic function $(\Delta U = 0)$.

The differentiation of (2 - 27) shows that

$$\frac{\partial (xz)}{\partial r} = 2r\sin\theta\cos\theta\cos\lambda = \frac{2xz}{r} \quad .$$

Hence by (2-22) and (2-28) we may transform (2-20) into

$$c_{13}^c = \frac{b^2}{4\pi G} \iint_\sigma xz \left(\frac{\partial V_1}{\partial r} - \frac{2V_1}{r} - 4\pi G\rho u_r \right) d\sigma \; . \qquad (2-29)$$

For the final transformation we must now use the formulas for spherical elasticity derived in TNP II, Part B; they may also be applied to a liquid core (homogeneous or with a density law (2 - 14)) since the equations of hydrodynamics may formally be considered a special case of the equations of elasticity (TNP II, p.63).

By TNP II, pp.82, 85, and 94 we put

$$u_r = H(r)S(\theta,\lambda) \;, \tag{2 - 30}$$

$$V_e + V_1 = R(r)S(\theta,\lambda) \;, \tag{2 - 31}$$

where $S(\theta,\lambda)$ is defined as the value of the lunisolar potential V_e at the earth's surface $(r = a)$:

$$S(\theta,\lambda) = V_e(a,\theta,\lambda) \;. \tag{2 - 32}$$

We write (1 - 10) as

$$\begin{aligned} V_e &= \kappa(xz\cos\sigma t + yz\sin\sigma t) = \\ &= \kappa r^2 \sin\theta\cos\theta(\cos\lambda\cos\sigma t + \sin\lambda\sin\sigma t) \;, \end{aligned} \tag{2 - 33}$$

or

$$V_e = \frac{1}{3}\kappa r^2 \left[R_{21}(\theta,\lambda)\cos\sigma t + S_{21}(\theta,\lambda)\sin\sigma t\right] \;, \tag{2 - 34}$$

using Legendre surface harmonics in the notation of (Heiskanen and Moritz, 1967, p.29).

From (2 - 34) we get immediately

$$\frac{\partial Ve}{\partial r} - \frac{2Ve}{r} = 0 ,$$

so that by (2 - 31)

$$\frac{\partial V_1}{\partial r} - \frac{2V_1}{r} = \left[R'(r)-2r^{-1}R(r)\right] S(\theta,\lambda) , \qquad (2 - 35)$$

putting $R'(r) = dR/dr$. Thus the expression between parentheses in (2 - 29) becomes

$$\frac{\partial V_1}{\partial r} - \frac{2V_1}{r} - 4\pi G\rho u_r = \left[R'(r)-2r^{-1}R(r)-4\pi G\rho H(r)\right] S(\theta,\lambda) =$$

$$= \left[r^{-2}P(r)-2r^{-1}R(r)\right] S(\theta,\lambda) , \qquad (2 - 36)$$

introducing the function $P(r)$ defined in TNP II, p.88, eq. (7 - 40).

We now put

$$xz = \frac{1}{3} r^2 R_{21}(\theta,\lambda) \qquad (2 - 37)$$

(cf.(2 - 33) and (2 - 34)) and note that the integral (2 - 29) is extended over the sphere $r = b$. Substituting (2 - 36) and (2 - 37), with $r = b$, into (2 - 29) we have

$$c_{13}^{c} = \frac{b^2}{12\pi G}\left[P(b)-2bR(b)\right]\iint_{\sigma} R_{21}(\theta,\lambda)S(\theta,\lambda)d\sigma \quad . \qquad (2-38)$$

By (2 - 32) we get $S(\theta,\lambda)$ by putting $r = a$ in (2 - 34). We have

$$\iint_{\sigma}\left[R_{21}(\theta,\lambda)\right]^2 d\sigma = \frac{12\pi}{5} \quad ,$$

$$\iint_{\sigma} R_{21}(\theta,\lambda)S_{21}(\theta,\lambda)\, d\sigma = 0$$

(Heiskanen and Moritz, 1967, p.29). Thus

$$\iint_{\sigma} R_{21}(\theta,\lambda)S(\theta,\lambda)d\sigma = \frac{4\pi}{5}a^2\kappa\cos\sigma t \quad ,$$

and (2 -38) takes the final form

$$c_{13}^{c} = \frac{a^2b^2}{15G}\left[P(b)-2bR(b)\right]\kappa\cos\sigma t \quad . \qquad (2-39)$$

The quantity c_{23}^{c} is obviously given by the same expression, with $\cos\sigma t$ replaced by $\sin\sigma t$.

In the expression (2 - 33), V_e denotes the potential of all perturbing forces acting on the body. This includes the lunisolar tidal potential and the disturbing centrifugal potential. If we want to restrict κ to the effect of the lunisolar potential only, we must add the effect of the centrifugal perturbation (TNP II, p.115), replacing κ by $\kappa-\Omega^2\varepsilon$. Thus from (2 - 39) we have

$$c_c = c^c_{13} + c^c_{23}$$

$$= \frac{a^2b^2}{15G} \left[P(b)-2bR(b)\right] (\kappa-\Omega^2\varepsilon)e^{i\sigma t} \qquad (2-40)$$

or, by (2 - 5) and (2 - 6), finally

$$c_c = \frac{a^2b^2}{15G} \left[P(b)-2bR(b)\right] \Omega(w-u) \ . \qquad (2-41)$$

This formula expresses the elements c^c_{13} *and* c^c_{23} of the inertia tensors of the core deformed by the effect of lunisolar (w) and centrifugal (u) perturbation.

As a check we replace b by a ; then the core is replaced by the whole earth, and (2 - 41) should reduce to *(2 - 7). In fact, putting* $a = b = 1$ *in (2 - 41) we have*

$$c = \frac{1}{15G} \left[P(a)-2R(a)\right] \Omega(w-u) \ . \qquad (2-42)$$

By TNP II, eqs. (8 - 20) and (8 - 24), we have

$$P(a) = 2 - 3k , \quad R(a) = 1 + k , \tag{2 - 43}$$

which on substitution into (2 - 42) gives (2 - 7) as it should be.

As a final step we perform a decomposition analogous to (2 - 9) :

$$\begin{aligned} P(b) &= P_o(b) + \frac{v}{w-u} P_1(b) , \\ R(b) &= R_o(b) + \frac{v}{w-u} R_1(b) . \end{aligned} \tag{2 - 44}$$

The possibility of this decomposition is a consequence of the linear character of the partial differential equations of elasticity and of the boundary conditions. In fact, by eqs. (8 - 25) of TNP II, the elastic functions $P(r)$ and $R(r)$ depend linearly on the Love numbers h, k, l , which by (10 - 44) of TNP II are the solution of three linear equations of the form

$$\begin{aligned} a_1 h + a_2 k + a_3 l &= a_o , \\ b_1 h + b_2 k + b_3 l &= b_o + b_4 \frac{v}{w-u} , \\ c_1 h + c_2 k + c_3 l &= c_o . \end{aligned} \tag{2 - 45}$$

In (10 - 44) of TNP II we have written the second equation in a slightly different form, making them dependent on the Molodensky parameter β. However, as eqs.(10 - 43), ibid., show, the dependence is rather on $\kappa^{-1}\beta$. If we have both tidal and centrifugal perturbation, we must replace this factor by

$$\frac{\beta}{\kappa-\Omega^2\varepsilon} \quad . \qquad (2 - 46)$$

By eq.(4 - 60) TNP II, β is proportional to our v, (this also holds for an elastic mantle), so that the quotient (2 - 46) is proportional to

$$\frac{v}{w-u} \qquad (2 - 47)$$

(a real quantity, since the complex factor $e^{i\sigma t}$ cancels!), which proves (2 - 45). Of course, the coefficient b_4 is now different, but all other coefficients are equal in (2 - 45) and eq.(10 - 44) of TNP II .

The solution of (2 - 45) now does give h,k,l as linear functions of the ratio (2 - 47), of the form (2 - 9), whereupon eqs. (8 - 25) of TNP II give $P(b)$ and $R(b)$ in the form (2 - 44).

The substitution of (2 - 44) into (2 - 41) thus gives

$$c_c = D_{21}(u-w) + D_{22}v \qquad (2 - 43)$$

where

$$D_{21} = -\frac{1}{15} a^2 b^2 G^{-1} \Omega \left[P_o(b) - 2bR_o(b) \right] ,$$

$$D_{22} = \frac{1}{15} a^2 b^2 G^{-1} \Omega \left[P_1(b) - 2bR_1(b) \right] . \qquad (2 - 49)$$

Equations (2 - 48) and (2 - 49) are equivalent to equations (56) and (57) of (Sasao et al., 1980).

We finally mention the symmetry property

$$D_{21} = D_{12} . \qquad (2 - 50)$$

Comparing the corresponding expressions given by (2 - 11) and (2 - 49), the equality (2 - 50) is far from evident. In fact, this equality is a consequence of a deep theorem of elasticity (Betti's reciprocity theorem). We shall not prove it here, referring the reader to (Sasao et al., 1980, sec.5).

Eq. (2 - 48), with the symmetry (2 - 50), is indeed equivalent to (1 - 7).

PART B

APPLICATION OF POINCARE'S VARIATIONAL PRINCIPLE

3. The Rotation Group

The elementary properties of the rotation group will play a basic role in Poincaré's variational principle, as already indicated at the end of sec.1. They will therefore be dicussed in this section, following (Moritz, 1982b).

Let a rotation in R^3 (threedimensional Euclidian space) be represented by

$$\underline{x}' = \underline{A}\underline{x} \quad , \tag{3 - 1}$$

$\underline{x}$ and $\underline{x}'$ being vectors and $\underline{A}$ denoting a 3 x 3 matrix.

Rotation matrices have the properties

1. The product of two rotation matrices $\underline{A}$ and $\underline{B}$ is again a rotation matrix $\underline{C} = \underline{A}\,\underline{B}$.
2. For the unit matrix $\underline{I}$ we have

$$\underline{A}\,\underline{I} = \underline{I}\,\underline{A} = \underline{A} \quad . \tag{3 - 2}$$

3. Every rotation matrix $\underline{A}$ has an inverse $\underline{A}^{-1}$ which again is a rotation matrix.

This properties characterize the mathematical structure of a group; we therefore speak of the rotation group.

Another important property of rotation matrices $\underline{A}$ is that the inverse is simply the transposed matrix:

$$\underline{A}^{-1} = \underline{A}^{T} \quad , \tag{3-3}$$

so that we have

$$\underline{A}\,\underline{A}^{T} = \underline{A}^{T}\underline{A} = \underline{I} \quad . \tag{3-4}$$

Denote the elements of $\underline{A}$ by a_{ij} :

$$\underline{A} = [a_{ij}] \quad . \tag{3-5}$$

Any rotation can be fully described by means of the three variables q_r , for which we may take the Euler angles ϕ, θ, ψ (TNP I, p.51). Thus the a_{ij} will be functions of q_r :

$$a_{ij} = a_{ij}(q_1, q_2, q_3) = a_{ij}(q_r) \quad . \tag{3-6}$$

These functions are easily seen to be continuous and differentiable. Thus the rotation group is a continuous group, or Lie group.

An infinitesimal change of q_r changes the matrix $\underline{A}$ by

$$d\underline{A} = \frac{\partial \underline{A}}{\partial q_r} dq_r \tag{3-7}$$

(summation convention!).

The matrix

$$d\underline{\Pi} = \underline{A}^{-1}\, d\underline{A} = \underline{A}^T d\underline{A} \tag{3-8}$$

will be skew-symmetric, which is an immediate consequence of differentiating (3-4) :

$$\underline{A}^T d\underline{A} + d\underline{A}^T \underline{A} = \underline{A}^T d\underline{A} + (\underline{A}^T d\underline{A})^T = 0 \quad .$$

Thus it has the form

$$d\underline{\Pi} = \begin{bmatrix} 0 & -d\pi_3 & d\pi_2 \\ d\pi_3 & 0 & -d\pi_1 \\ -d\pi_2 & d\pi_1 & 0 \end{bmatrix} . \tag{3-9}$$

On introducing the matrices

$$\underline{E}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} , \quad \underline{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} , \quad \underline{E}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{3-10}$$

this may be written

$$d\underline{\Pi} = \underline{E}_1\, d\pi_1 + \underline{E}_2\, d\pi_2 + \underline{E}_3\, d\pi_3 = \underline{E}_i\, d\pi_i \quad . \qquad (3-11)$$

The geometrical interpretation of the term $\underline{E}_1 d\pi_1$ is clear: it represents a rotation by the infinitesimal angle $d\pi_1$ around the x_1 axis, and similar for the other terms.

As (3 - 11) shows, infinitesimal rotations are commutative (finite rotations are not: there is $\underline{A}\,\underline{B} \neq \underline{B}\,\underline{A}$ in general).

The angular velocity component ω_1 may be considered a change of $d\pi_1$ with respect to time:

$$\omega_1 = \frac{d\pi_1}{dt} \quad , \qquad (3-12)$$

and similarly for ω_2 and ω_3 . Denoting $d\underline{\Pi}/dt$ by $\underline{\Omega}$, we have from (3 - 11):

$$\underline{\Omega} = \underline{E}_i \omega_i \quad . \qquad (3-13)$$

The matrices $\underline{E}_i$ satisfy the basic <u>commutation relations</u>

$$[\underline{E}_1, \underline{E}_2] = \underline{E}_3 \quad ,$$
$$[\underline{E}_2, \underline{E}_3] = \underline{E}_1 \quad , \qquad (3-14)$$
$$[\underline{E}_3, \underline{E}_1] = \underline{E}_2 \quad .$$

Here the commutation symbol [ ] stands for

$$[\underline{E}_i, \underline{E}_j] = \underline{E}_i\underline{E}_j - \underline{E}_j\underline{E}_i \quad , \qquad (3 - 15)$$

$\underline{E}_i\underline{E}_j$ being the usual matrix product of $\underline{E}_i$ and $\underline{E}_j$.

The matrix $d\underline{\Pi}$ is skew-symmetric and directly expressed in terms of three infinitesimal parameters $d\pi_i$. It is thus simpler than $d\underline{A}$. The latter can be obtained by (3 - 8),

$$d\underline{A} = \underline{A}d\underline{\Pi} \quad , \qquad (3 - 16)$$

which expresses $d\underline{A}$ also in terms of the basic "infinitesimal group variables" $d\pi_1$, $d\pi_2$, $d\pi_3$.

It is clear that the matrix

$$\underline{I} + d\underline{\Pi}$$

represents a small rotation which is close to the unit matrix. Thus we may interpret (3 - 8) by saying that the multiplication by $\underline{A}^{-1} = \underline{A}^T$ transforms an arbitrary small rotation $d\underline{A}$ into a matrix $d\underline{\Pi}$ "in the neighbourhood of the unit matrix".

For a general Lie group, the commutation relations (3 - 14) are replaced by

$$[\underline{E}_i \ , \underline{E}_j] = c_{ijk}\underline{E}_k \quad , \qquad (3 - 17)$$

where the c_{ijk} are constants, called the <u>structure constants</u> of the group. For a general group the indices i, j, k run from 1 to n , n being again the number of <u>degrees</u>

of freedom. By (3 - 15), the interchange of i and j means a change of sign . Hence,

$$c_{jik} = -c_{ijk} \quad . \tag{3 - 18}$$

For the rotation group we have in particular

$$\begin{aligned} c_{123} &= c_{231} = c_{312} = 1 \quad , \\ c_{213} &= c_{321} = c_{132} = -1 \quad , \\ &\text{all other } c_{ijk} = 0 \quad . \end{aligned} \tag{3 - 19}$$

Invariant differential forms. By substituting (3 - 7) into (3 - 8) we find

$$d\underline{\Pi} = \underline{A}^T \frac{\partial \underline{A}}{\partial q_r} dq_r \quad . \tag{3 - 20}$$

In view of (3 - 9) this has the form

$$d\pi_i = \alpha_{ri} dq_r \quad , \tag{3 - 21}$$

expressing $d\pi_i$ as a linear combination of dq_r ; the summation convention holds as usual.

If we replace $\underline{A}$ by $\underline{B}\,\underline{A}$, with an arbitrary constant rotation matrix $\underline{B}$, the right-hand side of (3-20) becomes

$$(\underline{B}\,\underline{A})^T \underline{B} \frac{\partial \underline{A}}{\partial q_r} dq_r = \underline{A}^T \underline{B}^T \underline{B} \frac{\partial \underline{A}}{\partial q_r} ,$$

identical to (3 - 20) since $\underline{B}^T\underline{B} = \underline{I}$ for a rotation $\underline{B}$. Thus a multiplication of all matrices $\underline{A}$ by the same constant rotation matrix $\underline{B}$ from the left ("left translation of $\underline{A}$ by $\underline{B}$") leaves the form (3 - 20) or (3 - 21) invariant. Hence we speak of an invariant (more precisely, left-invariant) differential form for the group under consideration.

The coefficients α_{ri} are functions of q_r. Note particularly that, in general, the differentials $d\pi_i$ cannot be integrated to give new coordinates π_i. Mathematically speaking, the $d\pi_i$ are not, in general, "perfect differentials". One also speaks of nonholonomic coordinates, which make sense only in the infinitesimal domain; cf.(Grafarend, 1975).

Briefly, the q_r are holonomic but not group invariant, whereas the $d\pi_i$ are group-invariant but not holonomic. The property of being group-invariant is so important, however, that the $d\pi_i$ are basic in the theory of continuous groups.

Dividing by dt and noting (3 - 12) we get from (3 - 21)

$$\omega_i = \alpha_{ri}\dot{q}_r . \qquad (3 - 22)$$

The inverse relation may be found by solving (3 - 22) for $\dot{q}_r$. It has the form

$$\dot{q}_r = \beta_{ri}\omega_i \tag{3 - 23}$$

or, corresponding to (3 - 21),

$$dq_r = \beta_{ri} d\pi_i \quad . \tag{3 - 24}$$

It is clear that the matrix $[\beta_{ri}]$ is the (transposed) inverse of the matrix $[\alpha_{ri}]$, so that the relations

$$\alpha_{ri}\beta_{rj} = \delta_{ij} \quad , \qquad \alpha_{ri}\beta_{si} = \delta_{rs} \tag{3 - 25}$$

hold; the Kronecker deltas δ_{ij} and δ_{rs} denote the elements of the unit matrix.

To fill these abstract formulas with a concrete meaning, let us note that (3 - 22) is simply a general form of Euler's kinematical equations (TNP I, p.51), so that for the rotation group the matrix $[\alpha_{ri}]$ has the form

$$[\alpha_{ri}] = \begin{bmatrix} -\sin\theta\sin\phi & -\cos\phi & 0 \\ -\sin\theta\cos\phi & \sin\phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \quad , \tag{3 - 26}$$

taking $q_1 = \psi$, $q_2 = \theta$, $q_3 = \phi$. The explicit computation of the matrix $[\beta_{ri}]$ by inverting (3 - 26) is left as an exercise to the reader.

<u>The generators of a Lie group</u>. Let us form the differential of a function

$$f = f(q_r)$$

of the group variables. We have

$$df = \frac{\partial f}{\partial q_r} dq_r \quad . \tag{3 - 27}$$

The substitution of (3 - 24) gives

$$df = \frac{\partial f}{\partial q_r} \beta_{ri} d\pi_i \quad .$$

This may be written in the symbolic form

$$df = \frac{\partial f}{\partial \pi_i} d\pi_i \tag{3 - 28}$$

with

$$\frac{\partial f}{\partial \pi_i} = \beta_{ri} \frac{\partial f}{\partial q_r} \quad . \tag{3 - 29}$$

Note, however, that the notation $\partial f/\partial \pi_i$ has only a formal character, it is not a true partial derivative of f with respect to π_i since for nonholonomic coordinates only $d\pi_i$ but not π_i exists.

In Lie group theory it is customary to write $X_i f$ instead of $\partial f/\partial \pi_i$, so that symbolically,

$$X_i = \frac{\partial}{\partial \pi_i} = \beta_{ri}\frac{\partial}{\partial q_r} \qquad (3 - 30)$$

by (3 - 29). The differential operators $X_i (i = 1, 2, 3)$ are called the generators of the group.

The torque. If we consider the rotation group, and if the function f is the potential energy U , then the generators X_i have an important physical meaning. In fact,

$$L_i = X_i U = \frac{\partial U}{\partial \pi_i} \qquad (3 - 31)$$

is nothing else than the x_i component of the torque $\underline{L}$. Let us consider the work dW_{rot} done by a small rotation about the x_1 axis. We have

$$dW_{rot} = L_1 d\pi_1 \quad ; \qquad (3 - 32)$$

this expression is the rotational analogue for the work done by a small translation dx_1 along the x_1 axis:

$$dW_{trans} = K_1 dx_1 \quad ,$$

K_1 being the force component along the x_1 axis. The change of potential energy dU is equal to the work dW_{rot}. Taking into account also small rotations about the other coordinates axes x_2 and x_3, (3 - 32) gives

$$dU = L_1 d\pi_1 + L_2 d\pi_2 + L_3 d\pi_3 = L_k d\pi_k \quad .$$

On the other hand, (3 - 28) and (3 - 30) yield

$$dU = \frac{\partial U}{\partial \pi_k} d\pi_k = X_k U d\pi_k \quad .$$

The comparison of these two expressions gives (3 - 31), which was to be shown.

Commutation relations and structure constants again. Consider the differential of a rotation matrix $\underline{A}$. By (3 - 28), (3 - 30), (3 - 16), and (3 - 11) we have

$$d\underline{A} = X_i \underline{A} d\pi_i = \underline{A}\,\underline{E}_i d\pi_i \quad ,$$

wnence

$$X_i \underline{A} = \underline{A}\,\underline{E}_i \quad . \tag{3 - 33}$$

This means that applying the differential operator X to a rotation matrix $\underline{A}$ is equivalent to multiplying $\underline{A}$ by the matrix $\underline{E}_i$.

A straightforward consequence of this fact (please verify!) is the basic theorem that the generators X_i satisfy the same commutation relations (3 - 17) as the matrices $\underline{E}_i$:

$$[X_i, X_j] = c_{ijk}X_k \quad , \tag{3-34}$$

where

$$[X_i, X_j] = X_iX_j - X_jX_i \quad . \tag{3-35}$$

This must be interpreted according to (3 - 30):

$$[X_i, X_j]f = X_iX_jf - X_jX_if$$

$$= \beta_{ri}\frac{\partial}{\partial q_r}\left(\beta_{sj}\frac{\partial f}{\partial q_s}\right) - \beta_{rj}\frac{\partial}{\partial q_r}\left(\beta_{si}\frac{\partial f}{\partial q_s}\right) \quad . \tag{3-36}$$

The differentiation gives

$$[X_i, X_j]f = \left(\beta_{ri}\frac{\partial\beta_{sj}}{\partial q_r} - \beta_{rj}\frac{\partial\beta_{si}}{\partial q_r}\right)\frac{\partial f}{\partial q_s}$$

since the second derivatives $\partial^2 f/\partial q_r\partial q_s$ cancel. In view of (3 - 25) this may be written

$$[X_i, X_j]f = \alpha_{sk}\left(\beta_{ri}\frac{\partial\beta_{sj}}{\partial q_r} - \beta_{rj}\frac{\partial\beta_{si}}{\partial q_r}\right)\beta_{pk}\frac{\partial f}{\partial q_p} \quad .$$

The last two factors are X_kf , by (3 - 30). Hence the comparison with (3 - 34) gives the structure constants:

$$c_{ijk} = \alpha_{sk}\left(\beta_{ri}\frac{\partial\beta_{sj}}{\partial q_r} - \beta_{rj}\frac{\partial\beta_{si}}{\partial q_r}\right) \quad , \qquad (3 - 37)$$

expressed in terms of the functions α_{ri} and β_{ri} .

An alternative expression is obtained by differentiating the first equation of (3 - 25) written in the form

$$\alpha_{sk}\beta_{sj} = \delta_{jk} \quad ,$$

obtaining

$$\alpha_{sk}\frac{\partial\beta_{sj}}{\partial q_r} + \beta_{sj}\frac{\partial\alpha_{sk}}{\partial q_r} = 0 \quad .$$

By means of this relation, (3 - 37) is easily transformed into

$$c_{ijk} = \beta_{ri}\beta_{sj}\left(\frac{\partial\alpha_{rk}}{\partial q_s} - \frac{\partial\alpha_{sk}}{\partial q_r}\right) \quad . \qquad (3 - 38)$$

As an exercise, the reader is invited to compute the structure constants (3 - 19) of the rotation group from (3 - 38), using (3 - 26).

It is surprising that the result of evaluating (3 - 38), using the <u>functions</u> α_{ri} and β_{ri} , is a <u>constant</u>. In fact, nonholonomic coordinates may be introduced by an equation of form (3 - 21), with α_{ri} being arbitrary functions of q_r , also for general dynamical systems not related to Lie groups. Then, of course, the c_{ijk} computed by (3 - 38) will not in general be constants.

The fact that c_{ijk} are constants if the $d\pi_i$ are invariant forms for a Lie group reflects an essential symmetry of the group: the relation (3 - 33) is the same at every point of the group space.

Let it finally be mentioned that the structure constants are all zero if the group is commutative (Abelian); then, by (3 - 38) we have

$$\frac{\partial \alpha_{rk}}{\partial q_s} = \frac{\partial \alpha_{sk}}{\partial q_r} \quad . \tag{3 - 39}$$

These are the integrability conditions for $d\pi_i$, which can then be integrated to give true (holonomic) coordinates π_i .

More about Lie groups can be found, e.g., in (Smirnow, 1971) or (Choquet-Bruhat et al., 1977).

4. Poincaré's Variational Principle

According to Hamilton's principle, well known from classical mechanics, a dynamical system moves from time t_o to time t_1 in such way that

$$\int_{t_o}^{t_1} (T - U)dt = \text{extremum} \quad , \qquad (4 - 1)$$

T being the kinetic energy and U the potential energy. Here U is assumed to depend on n parameters (generalized coordinates) q_i $(i = 1,2,\ldots\ldots,n)$, whereas T depends on q_i and on the generalized velocities

$$\dot{q}_i = \frac{dq_i}{dt} \quad .$$

If we introduce the Lagrangian function

$$\mathcal{L} = T - U \qquad (4 - 2)$$

which is a function of q_i and $\dot{q}_i$, then the variational principle (4 - 1) leads to Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad . \qquad (4 - 3)$$

All this can be found in any text on classical mechanics, such as (Goldstein, 1980), (Lanczos, 1970), or (Whittaker, 1961).

Now following Poincaré (1901), we shall introduce non-holonomic "velocities" ω_i by (3 - 22), considering $\mathcal{E}$ a function of q_r and ω_i :

$$\mathcal{E} = \mathcal{E}(q_r, \omega_i) \quad . \tag{4 - 4}$$

To make a distinction, we shall denote $\mathcal{E}$ as a function of q_i and $\dot{q}_i$ by $\overline{\mathcal{E}}$:

$$\mathcal{E} = \mathcal{E}(q_r, \omega_i) = \overline{\mathcal{E}}(q_r, \dot{q}_r) \quad . \tag{4 - 5}$$

Thus the Lagrangian equations (4 - 3) must now be written

$$\frac{d}{dt}\left(\frac{\partial \overline{\mathcal{E}}}{\partial \dot{q}_r}\right) - \frac{\partial \overline{\mathcal{E}}}{\partial q_r} = 0 \quad ; \tag{4 - 6}$$

as we did in the preceding section, we shall use the subscripts $i, j, k, \ldots$ for the $d\pi$'s and the ω's and $r, s, \ldots$ for the q's and $\dot{q}$'s ; both sets of subscripts run from 1 to n , n being called the number of degrees of freedom for the motion.

The differentiation of (4 - 5), using (3 - 22), gives

$$\frac{\partial \overline{\mathcal{E}}}{\partial \dot{q}_r} = \frac{\partial \mathcal{E}}{\partial \omega_k} \frac{\partial \omega_k}{\partial \dot{q}_r} = \alpha_{rk} \frac{\partial \mathcal{E}}{\partial \omega_k} \quad ,$$

so that

$$\frac{d}{dt}\left(\frac{\partial \bar{\mathcal{E}}}{\partial \dot{q}_r}\right) = \alpha_{rk}\frac{d}{dt}\left(\frac{\partial \mathcal{E}}{\partial \omega_k}\right) + \frac{\partial \mathcal{E}}{\partial \omega_k}\frac{d\alpha_{rk}}{dt} \quad . \qquad (4 - 7)$$

Now we multiply (4 - 6) by β_{ri} , defined by (3 - 23), and substitute (4 - 7). Noting that

$$\beta_{ri}\alpha_{rk} = \delta_{ik}$$

by (3 - 25), we obtain

$$\frac{d}{dt}\left(\frac{\partial \mathcal{E}}{\partial \omega_i}\right) + \beta_{ri}\frac{\partial \mathcal{E}}{\partial \omega_k}\frac{d\alpha_{rk}}{dt} - \beta_{ri}\frac{\partial \bar{\mathcal{E}}}{\partial q_r} = 0 \quad . \qquad (4 - 8)$$

Now

$$\frac{d\alpha_{rk}}{dt} = \frac{\partial \alpha_{rk}}{\partial q_s}\frac{dq_s}{dt} = \frac{\partial \alpha_{rk}}{\partial q_s}\dot{q}_s = \beta_{sj}\frac{\partial \alpha_{rk}}{\partial q_s}\omega_j \qquad (4 - 9)$$

and

$$\frac{\partial \bar{\mathcal{E}}}{\partial q_r} = \frac{\partial \mathcal{E}}{\partial q_r} + \frac{\partial \mathcal{E}}{\partial \omega_k}\frac{\partial \omega_k}{\partial q_r}$$

$$= \frac{\partial \mathcal{E}}{\partial q_r} + \frac{\partial \mathcal{E}}{\partial \omega_k}\frac{\partial \alpha_{sk}}{\partial q_r}\dot{q}_s \qquad (4 - 10)$$

by (3 - 22). The substitution of (4 - 9) and (4 - 10) into (4 - 8) gives

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \omega_i}\right) + \beta_{ri}\left(\frac{\partial \alpha_{rk}}{\partial q_s} - \frac{\partial \alpha_{sk}}{\partial q_r}\right)\dot{q}_s\frac{\partial \mathcal{L}}{\partial \omega_k} - \beta_{ri}\frac{\partial \mathcal{L}}{\partial q_r} = 0 \quad . \qquad (4 - 11)$$

Finally we substitute $\dot{q}_s$ from (3 - 23):

$$\dot{q}_s = \beta_{sj}\omega_j \quad ,$$

obtaining

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \omega_i}\right) + \beta_{ri}\beta_{sj}\left(\frac{\partial \alpha_{rk}}{\partial q_s} - \frac{\partial \alpha_{sk}}{\partial q_r}\right)\omega_j\frac{\partial \mathcal{L}}{\partial \omega_k} - \beta_{ri}\frac{\partial \mathcal{L}}{\partial q_r} = 0 \ . (4 - 12)$$

If the anholonomic velocities ω_i and the corresponding anholonomic "coordinates" $d\pi_i$ are group variables in the sense of the preceding section, then we may introduce the structure constants of the group by (3 - 38) and the generators $X_i = \partial/\partial\pi_i$ by (3 - 30). Then (4 - 12) reduces to

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \omega_i}\right) + c_{ijk}\omega_j\frac{\partial \mathcal{L}}{\partial \omega_k} - \frac{\partial \mathcal{L}}{\partial \pi_i} = 0 \quad . \qquad (4 - 13)$$

These are Poincaré's (1901) equations of motion on a Lie group. We shall also speak of Poincaré's variational principle although it is a new formulation of the classical principle (4 - 1) rather than a new variational principle. It will be seen to be fundamental for all questions of earth rotation involving a liquid core; in fact, Poincaré's mathematical investigations were motivated precisely by the liquid-core problem.

Most classical textbooks on theoretical mechanics do not give it. An exception is (Whittaker, 1961, pp.42-43), who considers general anholonomic coordinates (not necessarily on a group, so that the c_{ijk} are not necessarily constants) and whose derivation we have been following.

Nowadays, this topic, motion on a Lie group, is quite fashionable, mainly due to the work of Arnold (1978, Appendix 2); see also (Hermann, 1968, ch.16 and 33) and (Abraham and Marsden, 1978, ch.4). All these treatments have a considerable level of sophistication; most accessible is Hermann, who on pp.171-172 gives a modern derivation of (4 - 13) in terms of external differential forms, and most difficult is Abraham. None of these authors, not even Whittaker, however, mentions Poincaré!

We note that if the structure constants c_{ijk} are zero, then the integrability conditions (3 - 39) are satisfied: we have true (holonomic) coordinates π_i with $\omega_i = \dot{\pi}_i$. Then Poincaré's equations (4 - 13) reduce to the Lagrangian equations (4 - 3), with π_i instead of q_i, as it should be.

To return to the general case, we see that Poincaré's equations (4 - 13) differ from the Lagrangian equations (4 - 3) only by the second term in (4 - 13), which expresses the nonholonomity of the group variables and involves the structure constants of the group.

Taking into account (4 - 2) and the fact that U does not depend on $\dot{q}_r$ and hence is independent of ω_i, we may write (4 - 13) in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_i}\right) + c_{ijk}\omega_j\frac{\partial T}{\partial \omega_k} = L_i \quad ; \qquad (4 - 14)$$

we are from now on using the opposite sign convention for L_i:

$$-L_i = \frac{\partial U}{\partial \pi_i} = X_i U \qquad (4 - 15)$$

by (3 - 31). This is the form derived by Poincaré(1901) and used by him (Poincaré, 1910) in his elegant treatment of the liquid-core problem for a rigid mantle (his c_{ijk} have different sign).

Application to Euler's equations. Let us consider the rotation of a rigid body. Then the group under consideration is the rotation group in R^3, whose structure constants are given by (3 - 19). The kinetic energy, for principal axes of inertia, is well known to be

$$T = \frac{1}{2}\left(A\omega_1^{\,2} + B\omega_2^{\,2} + C\omega_3^{\,2}\right) \quad , \qquad (4 - 16)$$

as any textbook on analytical mechanics shows. Then (4 - 14) gives immediately

$$A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = L_1 \quad ,$$
$$B\dot{\omega}_2 + (A - C)\omega_3\omega_1 = L_2 \quad ,$$
$$C\dot{\omega}_3 + (B - A)\omega_1\omega_2 = L_3 \quad ,$$

that is, Euler's equations for the rotation of a rigid body (cf. TNP I, p.10).

The following sections will apply (4 - 14) to more realistic earth models: the Poincaré model (rigid mantle and liquid core), and the Molodensky model (elastic mantle and liquid core), leading us to the theory of (Sasao et al., 1980).

5. Rigid Shell and Liquid Core

This model, the Poincaré model, has been treated in TNP I , secs.12 and 13. We shall here derive the basic equations (12 - 8) and (12 - 9) of TNP I, which were given there without proof.

We recall some basic mathematical features of the Poincaré model, following sec.12 of TNP I .

Let us refer the ellipsoidal shell to principal axes xyz ; then the inner ellipsoidal surface, which encloses a liquid-filled cavity, has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad . \tag{5 - 1}$$

By the change of variables

$$x' = \frac{x}{a} \quad , \qquad y' = \frac{y}{b} \quad , \qquad z' = \frac{z}{c} \quad , \tag{5 - 2}$$

this surface is transformed into the unit sphere

$$x'^2 + y'^2 + z'^2 = 1 \quad . \tag{5 - 3}$$

Poincaré considers a motion of the liquid such that, by the transformation (5 - 2), it is transformed into a rotation of the sphere (5 - 3). Thus the velocity in the auxiliary x'y'z' system is

$$\underline{\chi} \times \underline{x} = \begin{bmatrix} \chi_2 z' - \chi_3 y' \\ \chi_3 x' - \chi_1 z' \\ \chi_1 y' - \chi_2 x' \end{bmatrix} \tag{5 - 4}$$

if the corresponding rotation vector is $\underline{\chi}$. Going back to the actual system xyz by (5 - 2), the relative velocity $\underline{w}_c$ of the core with respect to the shell is

$$\underline{w}_c = \begin{bmatrix} \frac{a}{c}\chi_2 z - \frac{a}{b}\chi_3 y \\ \frac{b}{a}\chi_3 x - \frac{b}{c}\chi_1 z \\ \frac{c}{b}\chi_1 y - \frac{c}{a}\chi_2 x \end{bmatrix} . \tag{5 - 5}$$

For a rigid body the velocity $\underline{w}$ with respect to inertial space is obtained by (cf. TNP I, p.8):

$$\underline{w} = \underline{\omega} \times \underline{x} = \begin{bmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{bmatrix} . \tag{5 - 6}$$

Since the mantle is rigid, the velocity of any particle of the mantle with respect to inertial space is given by (5 - 6):

$$\underline{v}_{mantle} = \underline{w} = \underline{\omega} \times \underline{x} . \tag{5 - 7}$$

For a particle of the liquid core, the relative velocity (5 - 5) must be added to (5 - 6):

$$\underline{v}_{core} = \underline{w} + \underline{w}_c \quad . \tag{5 - 8}$$

Now the kinetic energy T can be evaluated. We have

$$\begin{aligned} T &= \frac{1}{2}\iiint_{earth} \underline{v}^2 dM = \frac{1}{2}\iiint_{earth} \underline{v}\cdot\underline{v}dM \\ &= \frac{1}{2}\iiint_{mantle} \underline{v}\cdot\underline{v}dM + \frac{1}{2}\iiint_{core} \underline{v}\cdot\underline{v}dM \\ &= T_{mantle} + T_{core} \quad . \end{aligned} \tag{5 - 9}$$

The substitution of $\underline{w} = \underline{v}_{mantle}$, eq.(5 - 7), gives

$$T_{mantle} = \frac{1}{2}\iiint_{mantle} \underline{w}\cdot\underline{w}dM \quad , \tag{5 - 10}$$

whereas with $\underline{v} = \underline{v}_{core}$, eq.(5 - 8), we get

$$T_{core} = \frac{1}{2}\iiint_{core} (\underline{w}\cdot\underline{w} + 2\underline{w}\cdot\underline{w}_c + \underline{w}_c\cdot\underline{w}_c)dM \quad . \tag{5 - 11}$$

The sum of these two equations finally gives

$$T = \frac{1}{2}\iiint_{earth} \underline{w}\cdot\underline{w}\ dM + \frac{1}{2}\iiint_{core} (2\underline{w}\cdot\underline{w}_c + \underline{w}_c\cdot\underline{w}_c)dM \quad . \tag{5 - 12}$$

The first term on the right-hand side of (5 - 12) is the kinetic energy of a totally rigid earth since $\underline{w}$ is given by (5 - 6). Thus, by (4 - 16),

$$\iiint\limits_{earth} \underline{w} \cdot \underline{w} dM = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \qquad (5 - 13)$$

where

$$A = \iiint\limits_{earth} (y^2+z^2) dM \ , \qquad (5 - 14)$$

$$B, C \text{ by cyclic permutation} \ ,$$

are the earth's principal moments of inertia as usual.

Note now that the third term in (5 - 12) differs from (5 - 13) by $\underline{w}$ being replaced by $\underline{w}_c$. The comparison of (5 - 5) and (5 - 6) shows that this difference consists in ω_i being replaced by χ_i, and also in the factors a/b, a/c, b/c, etc. Thus it is not difficult to see that the analogue of (5 - 13) is

$$\iiint\limits_{core} \underline{w}_c \cdot \underline{w}_c dM = A_c\chi_1^2 + B_c\chi_2^2 + C_c\chi_3^2 \qquad (5 - 15)$$

where

$$A_c = \frac{c^2}{b^2} \iiint\limits_{core} y^2 dM + \frac{b^2}{c^2} \iiint\limits_{core} z^2 dM \ , \qquad (5 - 16)$$

B_c and C_c following by cyclic permutation of x, y, z and a, b, c.

Similarly the second term in (5 - 12) becomes

$$\iiint_{core} \underline{w} \cdot \underline{w}_c dM = F\omega_1 \chi_1 + G\omega_2 \chi_2 + H\omega_3 \chi_3 \qquad (5 - 17)$$

where

$$F = \frac{c}{b} \iiint_{core} y^2 dM + \frac{b}{c} \iiint_{core} z^2 dM \quad , \qquad (5 - 18)$$

G, H by cyclic permutation.

The core being a homogeneous ellipsoid of axes a, b, c, the integrals are easily evaluated (actually, it suffices to evaluate one of them; the others follow by cyclic permutation), and we get

$$\begin{aligned} A_c &= \tfrac{1}{5} M_c (b^2 + c^2) \quad , & F &= \tfrac{2}{5} M_c bc \quad , \\ B_c &= \tfrac{1}{5} M_c (c^2 + a^2) \quad , & G &= \tfrac{2}{5} M_c ca \quad , \\ C_c &= \tfrac{1}{5} M_c (a^2 + b^2) \quad , & H &= \tfrac{2}{5} M_c ab \quad . \end{aligned} \qquad (5 - 19)$$

Thus the kinetic energy (5 - 12) becomes finally

$$T = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + A_c\chi_1^2 + B_c\chi_2^2 + C_c\chi_3^2) +$$

$$+ F\omega_1\chi_1 + G\omega_2\chi_2 + H\omega_3\chi_3 \quad . \qquad (5 - 20)$$

Now we are ready to apply Poincaré's equations (4 - 13). We obviously have 6 degrees of freedom (n=6) . The six possible infinitesimal transformations are:

1. a rotation of the whole earth with respect to inertial space, with angular velocity components ω_1, ω_2, ω_3 and generators denoted by X_1, X_2, X_3 and

2. a rotation of the unit sphere (5 - 3) representing the motion of the core with respect to the mantle, with angular velocity components χ_1, χ_2, χ_3 and generators denoted Y_1, Y_2, Y_3 .

Thus the whole group relevant for the present problem consists of two independent rotation groups. The structure equations (3 - 14) for rotation groups, for generators X_i instead of matrices $\underline{E}_i$, give

$$\begin{aligned} [X_1, X_2] &= X_3 \quad , & [Y_1, Y_2] &= -Y_3 \quad , \\ [X_2, X_3] &= X_1 \quad , & [Y_2, Y_3] &= -Y_1 \quad , \\ [X_3, X_1] &= X_2 \quad , & [Y_3, Y_1] &= -Y_2 \quad . \end{aligned} \qquad (5 - 21)$$

Note the difference in sign due to the fact that the second rotation is with respect to the mantle whereas the first is a rotation of the mantle with respect to inertial space. Any rotation X_i commutes with any rotation Y_j since the two rotation groups are independent of each other, whence

$$[X_i, Y_j] = 0 \qquad (i \text{ and } j = 1, 2, 3). \qquad (5-22)$$

The six quantities $\omega_1, \omega_2, \omega_3, x_1, x_2, x_3$ may be identified with $\omega_1, \omega_2, \ldots, \omega_6$, and $X_1, X_2, X_3, Y_1, Y_2, Y_3$ with $X_1, X_2, \ldots, X_6$ according to sec. 3. The corresponding constants of structure c_{ijk} are all 0, 1, or -1, by (5 - 21) and (5 - 22).

Thus (4 - 13), i, j, k running from 1 to 6, gives

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) - \omega_3\frac{\partial T}{\partial \omega_2} + \omega_2\frac{\partial T}{\partial \omega_3} &= L_1 , \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) - \omega_1\frac{\partial T}{\partial \omega_3} + \omega_3\frac{\partial T}{\partial \omega_1} &= L_2 , \qquad (5-23)\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) - \omega_2\frac{\partial T}{\partial \omega_1} + \omega_1\frac{\partial T}{\partial \omega_2} &= L_3 ;
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial x_1}\right) + x_3\frac{\partial T}{\partial x_2} - x_2\frac{\partial T}{\partial x_3} &= 0 , \\
\frac{d}{dt}\left(\frac{\partial T}{\partial x_2}\right) + x_1\frac{\partial T}{\partial x_3} - x_3\frac{\partial T}{\partial x_1} &= 0 , \qquad (5-24)\\
\frac{d}{dt}\left(\frac{\partial T}{\partial x_3}\right) + x_2\frac{\partial T}{\partial x_1} - x_1\frac{\partial T}{\partial x_2} &= 0 .
\end{aligned}$$

The right-hand side of (5 - 24) is zero since the lunisolar torque $\underline{L}$ acts on the whole earth; there is no external torque which would effect a relative motion of the core with respect to the mantle. This relative motion is caused purely by the rotation of the mantle which, through the elliptical core-mantle interface, acts on the core through "inertial coupling".

Thus we have derived eqs. (12 - 8) and (12 - 9) of TNP I, where their solution and further implications are discussed.

This application already indicates the power and usefulness of Poincaré's principle for the mathematical study of the earth's rotation.

6. Elastic Mantle and Liquid Core -- Kinetic Energy

After this digression to a rigid mantle we shall now continue the discussion of the solution of (Sasao et al., 1980) from the point of view of Poincaré's variational principle. Thus the mantle is again considered elastic.

The velocity vector $\underline{v}$ of any point of the mantle with respect to inertial space may be written in the following way:

$$\underline{v} = \underline{v}_{mantle} = \underline{\omega} \times \underline{x} + \underline{v}_m = \underline{w} + \underline{v}_m \ . \qquad (6-1)$$

Here

$$\underline{w} = \underline{\omega} \times \underline{x} \qquad (6-2)$$

represents a rigid rotation of the coordinate system x_1, x_2, x_3, expressed as the vector product of the angular velocity vector $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ and the position vector $\underline{x} = (x_1, x_2, x_3)$, referred to body-fixed axes; and $\underline{v}_m$ represents a small residual velocity describing the rate of elastic deformation.

The meaning of the decomposition (6 - 1) is easily understood. If the earth is not rigid, then there is no coordinate system at which all particles of the earth are at rest. Thus the particles of the mantle move with respect to our system x_1, x_2, x_3 with velocity $\underline{v}_m$ which is considered small since it is zero for a rigid body.

In the core we similarly have

$$\underline{v} = \underline{v}_{core} = \underline{\omega} \times \underline{x} + \underline{\chi} \times \underline{x} + \underline{v}_c = \underline{w} + \underline{w}_c + \underline{v}_c \quad . \qquad (6-3)$$

Now there is a small rotation

$$\underline{w}_c = \underline{\chi} \times \underline{x} \qquad (6-4)$$

of the core with respect to the mantle, described by the angular velocity vector $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$, superimposed on $\underline{w}$; in addition, there are very small residual motions $\underline{v}_c$, which describe deviations of the fluid core motion from a rotation of the core as a whole, including effects of non-sphericity.

The decomposition (6 - 3) follows (Sasao et al., 1980, p.167). It assumes that the hydrodynamic core motion is essentially a rotation $\underline{w}$, with small perturbations $\underline{v}_c$. This assumption is valid not only for the Poincaré model with a homogeneous liquid core (Poincaré, 1910, sec.I.2) but very generally: it corresponds to a toroidal oscillation T_1^1 proportional to the distance r from the earth's center, which holds for practically all models, even with a heterogeneous core; cf. TNP II, pp.135-136.

The present decomposition (6 - 3) is similar to (5 - 8), but there are two differences. First, there is an additional term $\underline{v}_c$, and second, $\underline{w}_c$ in (6 - 4) is an exact rotation, whereas (5 - 5) deviates from an exact rotation by terms on the order of the core flattening. These terms, together with deviations from the simple "Poincaré motion", are now incorporated into the residual velocity $\underline{v}_c$.

The usual expression for the kinetic energy T is split up as follows:

$$T = \frac{1}{2} \iiint_{\text{mantle}} \underline{v} \cdot \underline{v} dM + \frac{1}{2} \iiint_{\text{core}} \underline{v} \cdot \underline{v} dM = T_{\text{mantle}} + T_{\text{core}} \quad . \qquad (6 - 5)$$

Using (6 - 1), the kinetic energy of the mantle becomes

$$T_{\text{mantle}} = \frac{1}{2} \iiint_{\text{mantle}} (\underline{w} \cdot \underline{w} + 2\underline{w} \cdot \underline{v}_m + \underline{v}_m \cdot \underline{v}_m) dM \quad . \qquad (6 - 6)$$

We neglect the second-order term $\underline{v}_m \cdot \underline{v}_m$ and put

$$\iiint_{\text{mantle}} \underline{w} \cdot \underline{v}_m dM = t_m \quad . \qquad (6 - 7)$$

Thus (6 - 6) reduces to

$$T_{\text{mantle}} = \frac{1}{2} \iiint_{\text{mantle}} \underline{w} \cdot \underline{w} dM + t_m \quad . \qquad (6 - 8)$$

Similarly we transform T_{core} using (6 - 3), obtaining

$$T_{\text{core}} = \frac{1}{2} \iiint_{\text{core}} (\underline{w} \cdot \underline{w} + 2\underline{w} \cdot \underline{w}_c + \underline{w}_c \cdot \underline{w}_c) dM + t_c \qquad (6 - 9)$$

with

$$t_c = \iiint_{\text{core}} (\underline{w} + \underline{w}_c)\cdot\underline{v}_c \, dM \quad . \qquad (6-10)$$

Hence (6 - 5) becomes

$$T = \frac{1}{2}\iiint_{\text{earth}} \underline{w}\cdot\underline{w}\,dM + \frac{1}{2}\iiint_{\text{core}} (2\underline{w}\cdot\underline{w}_c + \underline{w}_c\cdot\underline{w}_c)\,dM + t_m + t_c . \qquad (6-11)$$

For the term t_m we obtain from (6 - 7) by means of (6 - 2):

$$t_m = \iiint_{\text{mantle}} \underline{w}\cdot\underline{v}_m \, dM$$

$$= \iiint_{\text{mantle}} (\underline{\omega}\times\underline{x})\cdot\underline{v}_m \, dM$$

$$= \iiint_{\text{mantle}} \underline{\omega}\cdot(\underline{x}\times\underline{v}_m)\, dM$$

$$= \underline{\omega}\cdot\iiint_{\text{mantle}} \underline{x}\times\underline{v}_m \, dM \quad , \qquad (6-12)$$

using a well-known vector identity and the fact that $\underline{\omega}$ is a constant with respect to integration.

Now

$$\iiint_{mantle} \underline{x} \times \underline{v}_m \, dM = \underline{h}_m \qquad (6 - 13)$$

is nothing else than the relative moment of inertia for the mantle; cf. TNP I, pp.13-14; TNP II, pp.110-111. Substituting (6 - 13) into (6 - 12) we have

$$t_m = \underline{\omega} \cdot \underline{h}_m \quad ; \qquad (6 - 14)$$

in a similar way we find

$$t_c = (\underline{\omega} + \underline{\chi}) \, \underline{h}_c \quad , \qquad (6 - 15)$$

where $\underline{h}_c$ is the relative angular momentum for the core.

It is now of basic importance that both $\underline{h}_m$ and $\underline{h}_c$ can be made zero. For the mantle, $\underline{h}_m$ vanishes if we take <u>Tisserand axes for the mantle</u> as body axes $x_1 x_2 x_3$;cf. TNP II, p.111. For the core, $\underline{h}_c$ can be made zero by an appropriate definition of the core rotation $\underline{\chi}$ in (8 - 3): if $\underline{h}_c$ is not zero for a certain $\underline{\chi}$, replace $\underline{\chi}$ by $\underline{\chi} + \delta\underline{\chi}$ and determine the three components of $\delta\underline{\chi}$ by the three conditions $\underline{h}_c = 0$. Then also t_c will vanish.

Thus t_m and t_c can always be considered zero, so that (6 - 11) reduces to

$$T = \frac{1}{2} \iiint_{earth} \underline{w} \cdot \underline{w} \, dM + \frac{1}{2} \iiint_{core} (2\underline{w} \cdot \underline{w}_c + \underline{w}_c \cdot \underline{w}_c) \, dM \quad . \qquad (6 - 16)$$

By manipulations familiar from sec.5 we obtain

$$T = \tfrac{1}{2}C_{ij}\omega_i\omega_j + C^c_{ij}\omega_i\chi_j + \tfrac{1}{2}C^c_{ij}\chi_i\chi_j \ , \qquad (6-17)$$

C_{ij} denoting the inertia tensor for the whole earth, and C^c_{ij}, for the core only.

In view of $F \doteq A_c$ and similar relations following from (5 - 19), this reduces to (5 - 20) for a rigid mantle, in spite of a slightly different definition of $\underline{\chi}$. This provides a check for the decomposition (6 - 3) which treats $\underline{w}_c$ as an exact rotation (6 - 4) and incorporates residual ellipsoidal effects into $\underline{v}_c$.

Following (Munk and Macdonald, 1960, p.37; TNP I, pp.15-16) we shall split up the inertia tensor as follows:

$$\underline{C} = \underline{C}_o + \underline{c} \qquad (6-18)$$

where

$$\underline{C} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} , \quad \underline{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} . \qquad (6-19)$$

Thus, $\underline{C}_o$ corresponds to the model of an undeformed earth whose principal axes of inertia coincide with the coordinate axes, which has rotational symmetry $(B = A)$ and whose principal moments of inertia A and C are constant in time.

The tensor $\underline{c}$ takes into account the deviation of the actual earth from this simplified model.

We thus split up the inertia tensor C_{ij} and do the same for C^c_{ij}. Of the components of the residual inertia tensors we only retain c_{13}, c_{23}, c^c_{13} and c^c_{23} which are related to nutation and polar motion, cf. TNP I, sec.3. Other terms do not influence these phenomena and can be disregarded without harm. Then (6 - 17) takes the final form

$$T = \tfrac{1}{2}A(\omega_1^2+\omega_2^2) + \tfrac{1}{2}C\omega_3^2 + c_{13}\omega_1\omega_3 + c_{23}\omega_2\omega_3 + \qquad (6 - 20)$$

$$+ \tfrac{1}{2}A_c(2\omega_1 x_1 + x_1^2 + 2\omega_2 x_2 + x_2^2) + \tfrac{1}{2}C_c(2\omega_3 x_3 + x_3^2) +$$

$$+ c^c_{13}(\omega_1 x_3 + \omega_3 x_1 + x_1 x_3) + c^c_{23}(\omega_2 x_3 + \omega_3 x_2 + x_2 x_3) \quad .$$

7. Elastic Mantle and Liquid Core -- Potential Energy

For an ellipsoidal earth with a solid mantle (solid earth, Poincaré model), we have a gravitational potential energy U_g which depends on the orientation of the body with respect to the directions to sun and moon and hence on the Euler angles ϕ, θ, ψ or the nonholonomic variables $d\pi_1, d\pi_2, d\pi_3$; its derivatives with respect to these variables are the torque components

$$-L_i = \frac{\partial U_g}{\partial \pi_i} \quad ; \tag{7 - 1}$$

cf.sec.3.

For an elastic mantle, we have to add a potential energy U_{ed} related to the elastic deformation, so that the total potential energy is

$$U = U_g + U_{ed} \quad . \tag{7 - 2}$$

The potential energy U_{ed} arises from two causes: from the elastic forces which are a reaction of the earth (including its liquid core) to the external forces, and from a change of the gravitational energy due to the elastic deformation. We thus have

$$U_{ed} = U_e + U_d \quad , \tag{7 - 3}$$

where U_e represents the elastic energy in the narrow sense and U_d will be called the deformation energy. Thus the total potential energy becomes

$$U = U_g + U_e + U_d \quad . \tag{7 - 4}$$

Because of the smallness of U_e and U_d and because of the smallness of the earth's flattening f, the effect of f on U_{ed} is of second order and will be neglected. Thus, for the purpose of computing U_{ed}, we may use the spherical approximation, formally replacing the earth by a sphere.The dynamical effects of the earth's rotation on U_{ed} are also of the same second order (this is easily understood because the flattening is an effect of the earth's rotation) and will be neglected. Thus again for the purpose of computing U_{ed}, we formally consider the earth nonrotating and without other kinetical effects (with one exception discussed below); this is the static approximation. Both approximations are standard, from (Jeffreys, 1949) to (Sasao et al., 1980); only (Wahr, 1981, a, b, c , 1982) is more accurate.

The gravitational energy of a material particle of mass m in a field of potential V is $-m_i V_i$ and that of a system of particles thus

$$-\Sigma m_i V_i \quad . \tag{7 - 5}$$

The minus sign comes from the fact that the potential V in geodesy is defined with opposite sign as compared to the usage in physics. For a continuous mass distribution, (7 - 5) must be replaced by

$$- \iiint V dM \quad , \tag{7-6}$$

dM denoting the element of mass.

Thus U_d, the change of gravitational energy because of deformation, becomes

$$U_d = - \iiint_{S_1} V_e dM + \iiint_{S_o} V_e dM \quad , \tag{7-7}$$

S_1 denoting the deformed earth surface. It is essential in this context that the undeformed earth surface is considered a sphere S_o, so that the deviation of S_1 from the sphere S_o represents the elastic deformation only: the ellipticity of the earth (the "equatorial bulge" causing the torque L_i) has already been incorporated into U_g ; cf.(7 - 1).

With

$$U_d = \kappa(xz\cos\sigma t + yz\sin\sigma t) \tag{7-8}$$

as usual (cf.(1 - 10)), eq.(7 - 7) becomes

$$U_d = -\kappa\cos\sigma t \iiint_{S_1} xzdM - \kappa\sin\sigma t \iiint_{S_1} yzdM \quad ; \tag{7-9}$$

the second integral in (7-7) is zero because of symmetry; cf.(2-...

By the usual definition of the products of inertia we have

$$c_{13} = -D_{xz} = -\iiint_{S_1} xz\,dM \qquad (7-10)$$

and similarly for c_{23} ; cf. the analogous expressions for the core, (2 - 12) and (2 - 15). Thus we simply have

$$U_d = c_{13}\kappa\cos\sigma t + c_{23}\kappa\sin\sigma t \ . \qquad (7-11)$$

Using (1 - 11) this may be written

$$U_d = \Omega(c_{13}w_1 + c_{23}w_2) \ . \qquad (7-12)$$

This is the deformation energy.

The elastic energy U_e can be readily found by the following consideration. According to the static approximation, external forces are counteracted by elastic forces in such a way that both systems of forces are in equilibrium. By the very basic principles of elasticity, the equilibrium is reached when the body is deformed in such a way that the potential energy is a minimum. According to the spherical approximation, the total potential energy reduces to U_{ed} (since $U_g = 0$ for a sphere: if the earth were a sphere, there would be no torques L_i and no precession and nutation!). Hence, for equilibrium,

$$U_{ed} = \text{minimum} \qquad (7-13)$$

or

$$dU_{ed} = 0 \quad . \tag{7 - 14}$$

By (7 - 3) this means

$$dU_e = -dU_d \quad . \tag{7 - 15}$$

As far as the external gravitational potential is concerned, U_d is given by (7 - 12). To this we have to add, however, a term

$$-\Omega(c^c_{13}x_1 + c^c_{23}x_2) \quad , \tag{7 - 16}$$

*which is to be explained as follows. In the static approxi*mation used so far we have neglected the earth's rotation as expressed by complex number u ; cf.(1 - 3). It is known, however, that the core rotation, as expressed by the complex number v , can be considerably larger than u . (In fact, the table on p.122 of TNP II shows that the ratio $|v/u| \doteq \beta/\varepsilon$ can reach values of 200 and more). Thus even when we neglect u , we cannot neglect v .

Now the incremental centrifugal potential of the earth's rotation is, by TNP II, p. 115,

$$= -\Omega^2\varepsilon(xz\cos\sigma t + yz\sin\sigma t)$$

$$= -\Omega(\omega_1 xz + \omega_2 yz) \quad , \tag{7 - 17}$$

by (2 - 3) and (2 - 4). For core rotation, we must replace

ω_i by χ_i . Thus

$$\psi_{core} = -\Omega(\chi_1 xz + \chi_2 yz) \quad . \tag{7 - 18}$$

This expression is now substituted for V in (7 - 6), the integral being extended over the core. The result is

$$-\Omega(c^c_{13}\chi_1 + c^c_{23}\chi_2) \quad , \tag{7 - 19}$$

as the comparison of (7 - 8) and (7 - 18) shows: there correspond $-\Omega\chi_1$ to $\kappa\cos\sigma t$ and $-\Omega\chi_2$ to $\kappa\sin\sigma t$, and we get c^c_{13} and c^c_{23} for the core since the integral is extended over the core.

Now we finally have to add (7 - 19) to (7 - 12) to get the expression for U_d for the present purpose, denoted by U_d':

$$U_d' = \Omega(c_{13}w_1 + c_{23}w_2 - c^c_{13}\chi_1 - c^c_{23}\chi_2) \quad . \tag{7 - 20}$$

It may be asked why U_d , and not U_d' , is used in the potential energy (7 - 4). The reason is that the term (7 - 19) is alredy incorporated in the kinetic energy, so that the Lagrangian function $\mathcal{L} = T - U$ does contain it, and it can contain it only once. (The reader may find a similar reasoning in (Jeffreys, 1949).)

For the present purpose, the computation of the elastic energy U_e from the equilibrium condition (7 - 13), does however require the use of (7 - 20), so that (7 - 15) is to be replaced by

$$dU_e = -dU_d' \quad . \tag{7 - 21}$$

By (7 - 20) we have

$$-dU_d' = \Omega(-w_1 dc_{13} - w_2 dc_{23} + \chi_1 dc^c_{13} + \chi_2 dc^c_{23}), \tag{7 - 22}$$

since the variations dU_e and dU_d' refer to variations of the shape of the body, which is expressed by changes dc_{ik} and dc^c_{ik} in its inertia tensor.

Now we shall use (2 - 10) and (2 - 48) with $u = 0$ (static case):

$$\begin{aligned} c &= -D_{11}w + D_{12}v \quad , \\ c_c &= -D_{21}w + D_{22}v \quad . \end{aligned} \tag{7 - 23}$$

Denoting by E_{ij} the elements of the matrix $(D_{ij})^{-1}$, we have

$$\begin{aligned} -w &= E_{11}c + E_{12}c_c \quad , \\ v &= E_{21}c + E_{22}c_c \quad . \end{aligned} \tag{7 - 24}$$

With $w_1 = w_1 + iw_2$, $v = \chi_1 + i\chi_2$ this gives

$$\begin{aligned} -w_1 &= E_{11}c_{13} + E_{12}c^c_{13} \quad , \\ \chi_1 &= E_{21}c_{13} + E_{22}c^c_{13} \quad , \end{aligned} \tag{7 - 25}$$

and similarly for w_2, χ_2, c_{23}, c^c_{23} . We substitute this into

(7 - 22) and take (7 - 21) into account. The result is

$$dU_e = \Omega\Big[E_{11}(c_{13}dc_{13} + c_{23}dc_{23})$$

$$+ E_{12}(c^c_{13}dc_{13} + c^c_{23}dc_{23})$$

$$+ E_{21}(c_{13}dc^c_{13} + c_{23}dc^c_{23})$$

$$+ E_{22}(c^c_{13}dc^c_{13} + c^c_{23}dc^c_{23})\Big] \quad . \qquad (7 - 26)$$

This expression is a complete differential if and only if

$$E_{21} = E_{12} \quad . \qquad (7 - 27)$$

This, however, is the case since from the symmetry of the matrix D , expressed by (2 - 50), there follows the symmetry of the inverse matrix E and hence (7 - 27).

Thus (7 - 26) represents the complete differential of the function

$$U_e = \frac{1}{2}\Omega\Big[E_{11}(c_{13}{}^2 + c_{23}{}^2) +$$

$$+ 2E_{12}(c_{13}c^c_{13} + c_{23}c^c_{23}) +$$

$$+ E_{22}(c^c_{13}{}^2 + c^c_{23}{}^2)\Big] \quad , \qquad (7 - 28)$$

which constitutes the desired elastic potential.

Now the tidal potential energy related to elastic deformation is given by (7 - 3) as the sum of (7 - 28) and (7 - 12):

$$U_{ed} = \frac{1}{2}\Omega\left[E_{11}(c_{13}^{\,2} + c_{23}^{\,2}) + 2E_{12}(c_{13}c_{13}^{c} + c_{23}c_{23}^{c}) + \right.$$

$$\left. + \; E_{22}(c_{13}^{c\,2} + c_{23}^{c\,2})\right] + \Omega(c_{13}w_1 + c_{23}w_2) \quad . \quad (7-29)$$

The great simplicity of this expression as compared to analogous formulas in (Jeffreys, 1949) and (Jeffreys and Vicente, 1957) is another indication of the extremely fortunate choice of c_{13}, c_{23}, c_{13}^{c}, and c_{23}^{c} as variables characterizing the elastic deformation of the earth; this is the basic discovery of (Sasao et al., 1980) .

In (Moritz, 1982 a) we have found (7 - 29) by an ad-hoc reasoning (choosing U_{ed} so that the right result comes out); the present derivation thus provides a physical verification.

An independent check is obtained by using a theorem concerning the potential energy of deformation given on p.173 of (Love, 1927): "The potential energy of deformation of a body, which is in equilibrium under a given load, is equal to half the work done by the external forces, acting through the displacements from the unstressed state to the state of equilibrium." It is not difficult to see that this implies that U_e is equal to half of $-U_d'$. Now in fact, inserting (7-25) into (7-20) and dividing by -2 we get (7-28).

8. Application of Poincaré's Principle

Just as in sec.5, we have two rotation groups, with generators X_1, X_2, X_3 and Y_1, Y_2, Y_3. We have used the same symbols X_1, X_2, X_3 ; Y_1, Y_2, Y_3, for the core rotation as in sec.5 although the conceptual meaning is slightly different: now the core rotates, whereas in sec.5 we had a rotation of the auxiliary sphere (5 - 3).

In addition to ω_i and χ_i, the kinetic energy (6 - 20) also contains the variable products of inertia $c_{13}, c_{23}, c^c_{13}, c^c_{23}$; also the potential energy (7 - 29) depends on these variables. These four quantities constitute four additional variables (four additional degrees of freedom), which describe the elastic deformation. They are ordinary (holonomic) variables q_7, q_8, q_9, q_{10}, so that (4 - 3) holds for them. (This also fits into the group theoretic scheme, with π_i instead of q_i for $7 \leq i \leq 10$, the corresponding subgroup being Abelian with zero c_{ijk}.)

The Poincaré equations (5 - 23) and (5 - 24) remain finally the same since we have two independent rotation groups as in sec.5. In addition to these six equations we have the equations $\partial \mathcal{L}/\partial q_i = 0$ for $i = 7,8,9,10$, which give

$$\frac{\partial T}{\partial c_{13}} = \frac{\partial U_{ed}}{\partial c_{13}} , \qquad \frac{\partial T}{\partial c_{23}} = \frac{\partial U_{ed}}{\partial c_{23}} , \qquad (8 - 1)$$

$$\frac{\partial T}{\partial c^c_{13}} = \frac{\partial U_{ed}}{\partial c^c_{13}} , \qquad \frac{\partial T}{\partial c^c_{23}} = \frac{\partial U_{ed}}{\partial c^c_{23}} . \qquad (8 - 2)$$

The 10 equations (5 - 23), (5 - 24), (8 - 1) and (8 - 2) relate and determine the 10 quantities $\omega_1, \omega_2, \omega_3, \chi_1, \chi_2, \chi_3, c_{13}, c_{23}, c^c_{13}$, and c^c_{23} .

We note that the torque components L_i are the same as in (5 - 23), namely purely gravitational; in fact, by (7 - 1) and (7 - 2) we have

$$\frac{\partial U}{\partial \pi_i} = \frac{\partial U_g}{\partial \pi_i} = -L_i$$

since (7 - 29) does not depend on rotation so that $\partial U_{ed}/\partial \pi_i = 0$. On the other hand, the right-hand sides of (8 - 1) and (8 - 2) depend only on the energy of elastic deformation since

$$\frac{\partial U}{\partial c_{13}} = \frac{\partial U_{ed}}{\partial c_{13}} ,$$

as U_g does not contain c_{13} , and similarly for the other elastic variables.

Let us now substitute (6 - 20) into the third equation of (5 - 24). This gives

$$\frac{d}{dt} (C_c\omega_3 + C_c\chi_3 + c^c_{13}\omega_1 + c^c_{13}\chi_1 + c^c_{23}\omega_2 + c^c_{23}\chi_2) +$$

$$+ \chi_2(A_c\omega_1 + A_c\chi_1 + c^c_{13}\omega_3 + c^c_{13}\chi_3) -$$

$$- \chi_1(A_c\omega_2 + A_c\chi_2 + c^c_{23}\omega_3 + c^c_{23}\chi_3) = 0 . \qquad (8 - 3)$$

Now we have

$$\omega_1, \omega_2 << \omega_3 ; \quad c_{13}, c_{23} << A, C \quad , \qquad (8 - 4)$$

(if we take absolute values), and similarly for the core; because of the smallness of the core rotation, also χ_3 will be small. Thus

$$\omega_1, \omega_2, \chi_1, \chi_2, \chi_3, c_{13}, c_{23}, c_{13}^c, c_{23}^c \qquad (8 - 5)$$

can be regarded as small quantities of the first order; their squares and products are then of the second order and will be neglected.

Then, and since C_c is constant by definition, (8 - 3) reduces to

$$\frac{d}{dt}(C_c\omega_3 + C_c\chi_3) = 0 \quad . \qquad (8 - 6)$$

If we treat the third equation of (5 - 23) the same way, substituting (6 - 20) and neglecting second-order quantities, we get

$$\frac{d}{dt}(C\omega_3 + C_c\chi_3) = L_3 = 0 \quad , \qquad (8 - 7)$$

since $L_3 = 0$ because of rotational symmetry. The subtraction of (8 - 6) from (8 - 7) gives, C and C_c being constant by definition,

$$(C - C_c)\frac{d\omega_3}{dt} = 0 \quad ,$$

so that ω_3 = const. We may put

$$\omega_3 = \Omega \quad , \tag{8 - 8}$$

identifying ω_3 with the average rotational velocity Ω of the earth; this is possible since ω_1 and ω_2 are very small.

Now (8 - 6) yields $d\chi_3/dt = 0$, so that χ_3 = const. The simplest choice for this constant is zero, giving

$$\chi_3 = 0 \quad . \tag{8 - 9}$$

This is the solution we shall take.

If we substitute (6 - 20) into the first two equations of (5 - 23), neglect second-order quantities and use (8 - 8) and (8 - 9), we obtain

$$\begin{aligned} A\dot{\omega}_1 + (C - A)\Omega\omega_2 + A_c(\dot{\chi}_1 - \Omega\chi_2) + \Omega(\dot{c}_{13} - \Omega c_{23}) &= L_1 \quad , \\ A\dot{\omega}_2 - (C - A)\Omega\omega_1 + A_c(\dot{\chi}_2 + \Omega\chi_1) + \Omega(\dot{c}_{23} + \Omega c_{13}) &= L_2 \quad . \end{aligned} \tag{8 - 10}$$

If we do the same with the first two equations of (5 - 24), we get

$$\begin{aligned} A_c\dot{\omega}_1 + A_c\dot{\chi}_1 - C_c\Omega\chi_2 + \Omega\dot{c}^c_{13} &= 0 \quad , \\ A_c\dot{\omega}_2 + A_c\dot{\chi}_2 + C_c\Omega\chi_1 + \Omega\dot{c}^c_{23} &= 0 \quad . \end{aligned} \tag{8 - 11}$$

Let us finally substitute (6 - 20) and (7 - 29) into (8 - 1) and (8 - 2), again using $\omega_3 = \Omega$ and $\chi_3 = 0$. The result is

$$\omega_1 = E_{11}c_{13} + E_{12}c^c_{13} + w_1 \ ,$$
$$\omega_2 = E_{11}c_{23} + E_{12}c^c_{23} + w_2 \ , \qquad (8-12)$$

and

$$\chi_1 = E_{12}c_{13} + E_{22}c^c_{13} \ ,$$
$$\chi_2 = E_{12}c_{23} + E_{22}c^c_{23} \ . \qquad (8-13)$$

Now the complex combination of (8 - 10) and (8 - 11) gives

$$A\dot{u} - i(C-A)\Omega u + A_c(\dot{v}+i\Omega v) + \Omega(\dot{c}+i\Omega c) = L \ , \qquad (8-14)$$

$$A_c\dot{u} + A_c\dot{v} + iC_c\Omega v + \Omega\dot{c}_c = 0 \ , \qquad (8-15)$$

and the complex combination of (8 - 12) and (8 - 13) becomes

$$u = E_{11}c + E_{12}c_c + w \ ,$$
$$v = E_{12}c + E_{22}c_c \ . \qquad (8-16)$$

By means of the matrix $[D_{ij}]$ inverse to the matrix $[E_{ij}]$ the last two equations may be written

$$c = D_{11}(u-w) + D_{12}v \ , \qquad (8-17)$$

$$c_c = D_{12}(u-w) + D_{22}v \ . \qquad (8-18)$$

Our equations (8 - 14), (8 - 15), (8 - 17), and (8 - 18) are identical to the basic SOS equations (1 - 1), (1 - 2), (1 - 6), and (1 - 7) of sec.1.

Concluding remarks. The present variational method furnishes probably the most direct derivation of equations (1 - 1) and (1 - 2), or (8 - 14) and (8 - 15). This is particularly true for (1 - 2) which has been derived by Sasao et al.(1980) using both the decomposition (6 - 3) and the hydrodynamic equations. The present approach uses only (6 - 3) and provides a unified deduction of (1 - 1) and (1 - 2) which also explains the similarity of these two equations that in the complicated hydrodynamical derivation comes out almost as a miracle. In fact, the similarity of (1 - 1) and (1 - 2) is now seen to be due to the fact that each equation essentially reflects the action of a rotation group.

On the other hand, equations (1 - 6) and (1 - 7) are better obtained directly, in the way described in sec.2. In fact, the derivation of the elastic energy given in sec.7 already uses (1 - 6) and (1 - 7).

To summarize: of the four SOS equations, (1 - 1) and (1 - 2) are obtained more easily from the variational method, whereas (1 - 6) and (1 - 7) are better derived directly. Therefore, in the first presentation of the method described here (Moritz, 1982a) we have followed a "hybrid" approach, considering (1 - 6) and (1 - 7) given (from a direct derivation), and choosing the elastic deformation energy U_{ed}, eq.(7 - 29), simply in such a way that the variational principle not only furnishes the new equations (1 - 1) and (1 - 2), but also correctly reproduces the given equations (1 - 6) and (1 - 7)

This ad-hoc choice of U_{ed} was frankly pragmatic and motivated by reasons of mathematical simplicity. It has been criticized for providing no physical interpretation of U_{ed}. Therefore, we have, in the present sec.7, derived U_{ed} by physical considerations in such a way that the physical meaning becomes evident. See, in particular, the decomposition (7 - 3) of the elastic deformation energy U_{ed} as the sum of an elastic energy U_e in a narrower sense, and a deformation energy U_d. The latter, U_d, could be determined immediately, and then U_e was obtained as the elastic equilibrium response to U_d. Still, eqs. (1 - 6) and (1 - 7) served as a basis also for the physical derivation of U_{ed} in sec.7.

Thus there is no doubt that even the use of the present variational principle does not supersede a direct derivation of (1 - 6) and (1 - 7). Also, eq.(1 - 1) could have been found rather easily by other considerations, using the Euler - Liouville equation (TNP II, sec.10). There remains (1 - 2), for which the present variational principle furnishes indeed a derivation of incomparable simplicity.

But does this fact alone justify the use of the whole mathematical machinery of Poincaré's variational principle? The answer is the same as in many other applications of analytical mechanics: the results could also be found by elementary Newtonian methods, but it is the application of Lagrangian or Hamiltonian methods which makes us better understand the mathematical and physical structure.

It may be helpful to again summarize the basic logical structure of the present approach which may have been obscured by the computational details. We start from (1 - 6) and (1 - 7) as derived in sec.2:

$$c = D_{11}(u-w) + D_{12}v \quad ,$$

$$c_c = D_{12}(u-w) + D_{22}v \quad .$$

Then the kinetic energy (6 - 17)

$$T = \tfrac{1}{2}C_{ij}\omega_i\omega_j + C^c_{ij}\omega_i\chi_j + \tfrac{1}{2}C^c_{ij}\chi_i\chi_j \quad ,$$

used in the form (6 - 20), together with the gravitational energy U_g, is substituted in Poincaré's equation (4 - 14),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\omega_i}\right) + c_{ijk}\,\omega_j\frac{\partial T}{\partial\omega_k} = L_i \quad ,$$

or more explicitly (5 - 23) and (5 - 24), which directly gives (1 - 1) and (1 - 2), or (8 - 14) and (8 - 15):

$$A\dot{u} - i(C-A)\Omega u + A_c(\dot{v}+i\Omega v) + \Omega(\dot{c}+i\Omega c) = L \quad ,$$

$$A_c\dot{u} + A_c\dot{v} + iC_c\Omega v + \Omega\dot{c}_c = 0 \quad .$$

The gravitational energy U_g is not used directly, but only through the well-known expressions (1 - 12) for the torque components L_i. As we have seen above (after eq.(8 - 2)), we have $\partial U_{ea}/\partial\pi_i = 0$, so that the elastic deformation energy does not contribute to the torque. Hence the simple expressions (1 - 12) hold independently of the internal constitution of the body.

We also see that the left-hand sides of both equations

(1 - 1) and (1 - 2) only depend on the kinetic energy T through Poincaré's equation (4 - 14), no Euler - Liouville condition and no hydrodynamics are needed!

So far everything has been rather analogous to the simple Poincaré model (rigid mantle and homogeneous core) as outlined in sec.5. To take care of elasticity, we have only to add a term U_{ed} to the gravitational potential energy to get the total potential energy

$$U = U_g + U_{ed} \cdot$$

U_{ed} is chosen in such a way that Poincaré's principle, through (8 - 1) and (8 - 2) (which do not depend on U_g !) gives equations (1 - 6) and (1 - 7); the expression (7 - 29),

$$U_{ed} = \frac{1}{2}\Omega \left[E_{11}(c_{13}{}^2 + c_{23}{}^2) + 2E_{12}(c_{13}c^c_{13} + c_{23}c^c_{23}) + \right.$$
$$\left. + E_{22}(c^{c\,2}_{13} + c^{c\,2}_{23}) \right] + \Omega(c_{13}w_1 + c_{23}w_2) \quad ,$$

is determined uniquely (up to an additive constant) by this condition.

The present variational approach may be compared to that of Jeffreys (1949) and Jeffreys and Vincente (1957). Common to all three approaches is the reduction of a problem of continuum mechanics (infinitely many degrees of freedom) to a problem of analytical mechanics with only a finite number (here, 10) of degrees of freedom. This is brought about by restricting ourselves to second-degree harmonic perturbations and disregarding all the infinitely many other harmonics.

The essentially greater simplicity of the present method is brought about by the use of anholonomic Poincaré's variables related to the two rotation groups considered, and of the corresponding angular velocities ω_i and χ_i (Jeffreys uses holonomic rotational variables), as well as by the fortunate choice of c_{13}, c_{23}, c^c_{13}, and c^c_{23} as elastic variables due to Sasao, Okubo and Saito.

The resulting SOS equations (1 - 1), (1 - 2),(1-6),(1 - 7) are equivalent to Molodensky's equations both with respect to accuracy (spherical approximation for the mantle) and to applicability to rather general earth models (heterogeneous mantle, heterogeneous core, even elastic inner core). They not only give the simplest formulation of Molodensky's liquid-core problem, but also clearly show its logical structure.

Their practical usefulness will be seen in the following sections.

PART C

EXPRESSIONS FOR NUTATION AND POLAR MOTION INCLUDING LIQUID-CORE EFFECTS

9. Transformation of Eq. (1 - 1)

This equation

$$A\dot{u} - i(C-A)\Omega u + A_c(\dot{v}+i\Omega v) + \Omega(\dot{c}+i\Omega c) = L \qquad (9 - 1)$$

will be basic for the investigation of the movement of various axes (rotation axis, angular momentum axis, figure axis, etc.) for the Molodensky-SOS model to be performed in Part C.

Let us recall the notations: A and C denote the moments of inertia of the whole earth, A_c and C_c those of the core, Ω denotes the (average) angular velocity of the earth, u and v are the complex numbers describing small rotations (u of the whole earth with respect to space, v of the core with respect to the mantle), and L is a complex number describing the lunisolar torque; these complex numbers are defined by (1 - 3) and (1 - 5). The complex number c combines anomalous elements of the inertia tensor, given by (1 - 4). It is related to u by (2 - 8)

$$c = -\frac{1}{3}G^{-1}ka^5\Omega(w-u) \quad , \qquad (9 - 2)$$

where G is the gravitational constant, a can be considered the radius of a spherical earth, k denotes the potential Love number depending on frequency (we keep in mind the possibility of the decomposition (2 - 9) but shall not use it), and w is related to the lunisolar (tidal) potential by (1 - 11). Thus the anomalous elements of the inertia tensor, described by c, depend on u (rotational deformation) and w (tidal deformation); cf. TNP I, p. 27.

We seek solutions that are functions of time t of form

$$u = u_o e^{i\sigma t} \quad ,$$
$$v = v_o e^{i\sigma t} \quad , \qquad (9 - 3)$$

with real constants u_o, v_o, taking

$$w = w_o e^{i\sigma t} \quad ; \qquad (9 - 4)$$

this is in agreement with (1 - 11):

$$w = \Omega^{-1}\kappa e^{i\sigma t} \quad . \qquad (9 - 5)$$

These particular solutions have the well-known advantage that differentiation is simply multiplicating by $i\sigma$:

$$\dot{u} = \frac{du}{dt} = i\sigma u_o e^{i\sigma t} = i\sigma u \quad . \tag{9 - 6}$$

They depend on the frequency σ ; but, our equations being linear, we can obtain a rather general solution by adding the contributions from different frequencies. On substituting (9 - 2), assuming (9 - 3) and considering (9 - 6), the basic equation (9 - 1) becomes after dividing by i :

$$\left[A\sigma - (C-A)\Omega\right]u + A_c(\sigma+\Omega)v - \frac{1}{3}G^{-1}ka^5\Omega^2(w-u)(\sigma+\Omega) = -iL \quad . \tag{9 - 7}$$

It is now convenient to introduce the dimensionless constant

$$k_s = \frac{3G(C-A)}{a^5\Omega^2} \quad , \tag{9 - 8}$$

called <u>secular Love number</u> (TNP I, p. 21), and express the torque L in terms of the complex potential coefficient w by

$$L = -i(C-A)\Omega w \quad ; \tag{9 - 9}$$

this is an immediate consequence of (1 - 11) and (1 - 12) (putting $a = 1$, which is of no relevance). Thus (9 - 7) becomes

$$\left[A\sigma - (C-A)\Omega + \frac{k}{k_s}(\sigma+\Omega)(C-A)\right]u + A_c(\sigma+\Omega)v =$$

$$= -(C-A)\left[1 - \frac{k}{k_s}\frac{\sigma+\Omega}{\Omega}\right]\Omega w \quad . \qquad (9-10)$$

We now introduce the Euler frequency defined by

$$\sigma_E = \frac{C-A}{A}\Omega \quad , \qquad (9-11)$$

cf. TNP I, p. 10, and divide by A . The result is

$$\left[\sigma - \sigma_E + \frac{k}{k_s}(\sigma+\Omega)\frac{\sigma_E}{\Omega}\right]u + \frac{A_c}{A}(\sigma+\Omega)v =$$

$$= -\sigma_E\left[1 - \frac{k}{k_s}\frac{\sigma+\Omega}{\Omega}\right]w \quad . \qquad (9-12)$$

In agreement with TNP I, p. 121, we now introduce the tidal frequency ω_j by

$$\sigma = -\omega_j \quad ; \qquad (9-13)$$

then

$$\Omega + \sigma = -(\omega_j - \Omega) = -\Delta\omega_j \qquad (9-14)$$

where

$$\Delta\omega_j = \omega_j - \Omega \qquad (9-15)$$

is the corresponding <u>nutational frequency</u> (TNP I, p. 36). Thus we can put

$$\Omega + \sigma = -\Delta\omega \ , \qquad \sigma = -(\Omega+\Delta\omega) \qquad (9-16)$$

omitting the subscript. Dividing by $-\Omega$ we then get

$$\left(1 + \frac{\sigma_E}{\Omega} + \frac{\Delta\omega}{\Omega} + \frac{k}{k_s}\frac{\sigma_E\Delta\omega}{\Omega^2}\right)u + \frac{A_c}{A}\frac{\Delta\omega}{\Omega}v =$$

$$= \frac{\sigma_E}{\Omega}\left(1 + \frac{k}{k_s}\frac{\Delta\omega}{\Omega}\right)w \ . \qquad (9-17)$$

Since the usual tidal frequencies ω_j are very close to Ω, the angular motions of sun and moon being so much smaller than the earth's rotation, we have

$$\frac{\Delta\omega}{\Omega} \ll 1 \ . \qquad (9-18)$$

Similarly,

$$\frac{\sigma_E}{\Omega} \doteq \frac{1}{300} \ll 1 \ , \qquad (9-19)$$

so that the last term between parentheses in the factor of

u in (9 - 17) will be of second order and can be neglected. Thus we finally obtain

$$\left(1 + \frac{\sigma_E}{\Omega} + \frac{\Delta\omega}{\Omega}\right) u = \frac{\sigma_E}{\Omega}\left(1 + \frac{k}{k_s}\,\frac{\Delta\omega}{\Omega}\right) w - \frac{A_c}{A}\,\frac{\Delta\omega}{\Omega}\, v \ . \qquad (9 - 20)$$

This equation expresses u in terms of the tidal potential characterized by w and of the core rotation v. Let us recall what u means. It is defined by

$$u = \omega_1 + i\omega_2 \qquad (9 - 21)$$

as the complex combination of the x and y components of the rotation vector $\underline{\omega}$. If we divide these components by the length Ω of this vector, we get the components of the unit vector of the (instantaneous) axis of rotation. The x and y components of this vector, m_1 and m_2, can by combined into a complex number

$$m = m_1 + im_2 \qquad (9 - 22)$$

which describes the polar motion of the rotation axis R :

$$p_R = m = \frac{u}{\Omega} \ ; \qquad (9 - 23)$$

the notations are the same as in TNP I (cf. pp. 17 and 93) and in TNP II (cf. pp. 128 and 138):

As we did in TNP II (p. 9), we may put

$$m_1 = \varepsilon \cos\sigma t \quad ,$$
$$m_2 = \varepsilon \sin\sigma t \quad , \tag{9 - 24}$$

or

$$m = \varepsilon e^{i\sigma t} \quad , \tag{9 - 25}$$

so that ε is the amplitude of polar motion for the frequency σ under consideration. Then

$$u = \Omega\varepsilon e^{i\sigma t} \quad . \tag{9 - 26}$$

We likewise put

$$v = \Omega\zeta e^{i\sigma t} \quad , \tag{9 - 27}$$

$$w = \Omega\tau e^{i\sigma t} \quad . \tag{9 - 28}$$

The numbers ε, ζ, and τ are dimensionless real constants.

The substitution of (9 - 26), (9 - 27), and (9 - 28) into (9 - 20) gives

$$\left(1 + \frac{\sigma_E}{\Omega} + \frac{\Delta\omega}{\Omega}\right)\varepsilon = \frac{\sigma_E}{\Omega}\left(1 + \frac{k}{k_s}\frac{\Delta\omega}{\Omega}\right)\tau - \frac{A_c}{A}\frac{\Delta\omega}{\Omega}\zeta \quad . \tag{9 - 29}$$

For a rigid earth without liquid core ($k=0$, $A_c = 0$) this gives

$$\varepsilon_o = \left(1 + \frac{\sigma_E}{\Omega} + \frac{\Delta\omega}{\Omega}\right)^{-1} \frac{\sigma_E}{\Omega}\tau \quad . \tag{9 - 30}$$

This equation gives the amplitude of polar motion forced by a tidal potential of frequency σ and amplitude τ . In terms of this ε_o we may write (9 - 29) as

$$\varepsilon = \varepsilon_o\left(1 + \frac{k}{k_s}\frac{\Delta\omega}{\Omega}\right) - \lambda\frac{A_c}{A}\frac{\Delta\omega}{\Omega}\zeta \tag{9 - 31}$$

where we have put

$$\lambda = \left(1 + \frac{\sigma_E}{\Omega} + \frac{\Delta\omega}{\Omega}\right)^{-1} \doteq 1 \quad . \tag{9 - 32}$$

In fact, in view of (9 - 18) and (9 - 19), the factor λ differs only very little from unity.

<u>Special cases</u>. For the <u>rigid earth</u> ($k=0$, $\zeta=0$), eq.(9 - 31) reduces to (9 - 30), which we can also write, using (9 - 15):

$$\varepsilon_o = \frac{\sigma_E}{\Omega+\Delta\omega+\sigma_E}\tau = \frac{\sigma_E}{\omega_j+\sigma_E}\tau \quad , \tag{9 - 33}$$

which is the same as the amplitude of TNP I, eq.(11 - 45), with $\tau = iB_j$.

For the <u>elastic earth</u> without fluid core ($k \doteq 0.3$, $\zeta=0$), eq.(9 - 31), together with (9 - 33), gives

$$\varepsilon = \left(1 + \frac{k}{k_s}\frac{\Delta\omega}{\Omega}\right)\frac{\sigma_E}{\omega_j + \sigma_E}\tau \quad . \tag{9 - 34}$$

On the other hand, we have McClure's expression (TNP I, p.36), apart from the factor i (see next section):

$$\begin{aligned} \varepsilon &= \frac{C-A}{A}\kappa_j B_j \\ &= \frac{C-A}{A}\frac{\Omega}{\omega_j + \sigma_C}\frac{1 + \frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}} B_j \quad , \end{aligned} \tag{9 - 35}$$

where

$$\sigma_C = \frac{1 - \frac{k}{k_s}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}}\sigma_E \tag{9 - 36}$$

is the Chandler frequency. Are (9 - 34) and (9 - 35) equivalent?

We have by (9 - 11),

$$\frac{C-A}{A}\frac{\Omega}{\omega_j + \sigma_C} = \frac{\sigma_E}{\omega_j + \sigma_C} \quad ,$$

so that (9 - 34) and (9 - 35) will be equal provided

$$(\omega_j + \sigma_c)\left(1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}\right) = \omega_j + \sigma_E \ . \qquad (9 - 37)$$

In fact, by (9 - 36),

$$(\omega_j + \sigma_c)\left(1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}\right) =$$

$$= \omega_j\left(1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}\right) + \left(1 - \frac{k}{k_s}\right)\sigma_E$$

$$= \omega_j + \sigma_E + \frac{k}{k_s}\left(\frac{\omega_j}{\Omega} - 1\right)\sigma_E$$

$$= \omega_j + \sigma_E + \frac{k}{k_s}\frac{\Delta\omega_j}{\Omega}\sigma_E \ , \qquad (9 - 38)$$

which differs from $\omega_j + \sigma_E$ only by the last term which is small of second order since $\Delta\omega_j$ and σ_E are both small, by (9 - 18) and (9 - 19), and which we can neglect.

For the Poincaré model (rigid mantle, liquid core) we have $k=0$ and $\zeta \neq 0$, so that (9 - 31) reduces to

$$\varepsilon = \varepsilon_o - \lambda\frac{A_c}{A}\frac{\Delta\omega}{\Omega}\zeta \qquad (9 - 39)$$

with $\lambda \doteq 1$. Since $\Delta\omega/\Omega$ is very small for tidal perturbations, this is almost the same as (9 - 31).

From this it would be tempting to conclude that the effect of elasticity on ε is very small, at least for nearly-diurnal frequencies which we have been considering in this section (this is plainly not true for the Chandler period which is lengthened due to elasticity from 305 days for $k=0$ to 403 days for $k \doteq 0.3$). Thus we might think that the simple Poincaré model (rigid mantle) might be used to calculate liquid-core effects on ε , still getting basically the same numerical results as for the Molodensky model with an elastic mantle.

Such a conclusion would be wrong, however, as the comparison of the tables in TNP I, pp 122-123, and TNP II, p. 122, shows. The reason is that, though the <u>direct</u> effect of k on (9 - 31) is negligible, there is also an <u>indirect</u> effect: elasticity affects core rotation, hence ζ is different for a rigid and an elastic mantle, and elasticity enters indirectly in (9 - 31) through ζ .

This shows that (9 - 31) shows very well the effect of mantle elasticity and core rotation if we assume that the core rotation parameter ζ is known. If only the lunisolar potential (coefficient τ) is given, then we must proceed differently, in a way to be outlined now.

<u>Solution for u and v in terms of w</u> . In addition to (1 - 1), let us now also consider (1 - 2):

$$A_c \dot{u} + A_c \dot{v} + iC_c \Omega v + \Omega \dot{c}_c = 0 . \qquad (9 - 40)$$

We express c_c in terms of u and w by (2 - 41) and substitute into (9 - 40). With (9 - 26), (9 - 27), and (9 - 28), differentiation is replaced by multiplication by $i\sigma$, according to (9 - 6). This reduces the differential equation (9 - 40) into a linear equation of the form

$$a_{21}u + a_{22}v = c_2 w \quad . \qquad (9 - 41)$$

This is in complete analogy to the procedure by which we have obtained (9 - 17) from (9 - 1); we may also write (9 - 17) in the form

$$a_{11}u + a_{12}v = c_1 w \quad . \qquad (9 - 42)$$

The solution of the two linear equations (9 - 42) and (9 - 41) for u and v (say, by means of determinants) gives

$$u = b_1 w \quad ,$$

$$v = b_2 w \quad .$$

Since it is not difficult to see that the coefficients a_{ij} and c_i are all real, also the coefficients b_i will be real. In terms of the real constants ε, ζ, τ we have the equations

$$a_{11}\varepsilon + a_{12}\zeta = c_1\tau \quad ,$$
$$a_{21}\varepsilon + a_{22}\zeta = c_2\tau \quad , \qquad (9 - 43)$$

with the solution

$$\varepsilon = b_1\tau \quad , \qquad \zeta = b_2\tau \quad , \qquad (9 - 44)$$

with the same coefficients as above.

Thus both ε and ζ can be expressed in terms of τ only. The computation is easy, but the resulting expressions are not very elegant. Therefore they will not be given here. The corresponding calculations for the Poincaré model (k=0) can be found in TNP I, pp. 120-122.

This reduction of a system of differential equations to a system of linear equations (9 - 43) has been made possible by using exponentials (9 - 3) to (9 - 5). This is nothing else than a transformation from the time domain to the frequency domain which is known to simplify matters on such occasions. As a matter of fact, the coefficients a_{ij} and c_i, and hence b_i in (9 - 44), will depend on the frequency $\sigma = -\omega_j$.

<u>Relation to the Molodensky coefficient β</u>. In TNP II, p.56, we have, for the Poincaré model, found the relation

$$v_o = \Omega(\beta - \varepsilon) \qquad (9 - 45)$$

whence, by (9 - 27),

$$\zeta = \beta - \varepsilon \qquad (9 - 46)$$

or

$$\beta = \zeta + \varepsilon \quad , \qquad (9 - 47)$$

β being the Molodensky parameter used in TNP II. In the interesting cases we have $\beta \gg \varepsilon$, so that approximately

$$\zeta \doteq \beta \quad . \qquad (9 - 48)$$

AD-A126 289 THEORIES OF NUTATION AND POLAR MOTION III(U) OHIO STATE UNIV COLUMBUS DEPT OF GEODETIC SCIENCE AND SURVEYING H MORITZ DEC 82 342 AFGL-TR-82-0387 F19628-82-K-0017 2/2

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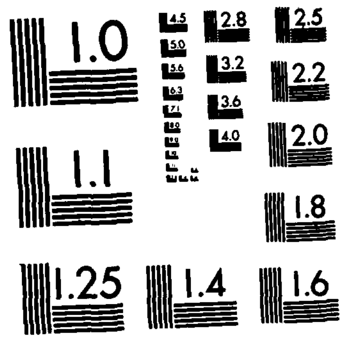

MICROCOPY RESOLUTION TEST CHART
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It can be shown that these relations also hold for an elastic mantle. In fact, substituting (9 - 47) into eq.(10-41) of TNP II it is not difficult, though somewhat laborious, to derive (9 - 29). Substituting (9 - 26) and

$$v = \Omega(\beta-\varepsilon)e^{i\sigma t}$$

into the second SOS equation (9 - 40) and taking a homogeneous core, one obtains eq. (9 - 34) of TNP II. (Hint: show that

$$c_c = \frac{4\pi G}{15}\rho e^2 b^5 \gamma e^{i\sigma t} ,$$

notations as in TNP II, sec.2.)

10. The Kinematic Axes

In this section we shall investigate the motion of the rotation axis, the figure axis, and the "mean Liouville figure axis". These axes will be called kinematic axes, in contrast to the angular momentum axis which, being dynamically defined, behaves differently and will be considered in the next section.

Reference frame. As in the previous reports TNP I and TNP II, we shall use two basic reference frames, the "nutation frame" $x_1^o x_2^o x_3^o$ which is the natural frame for describing nutation, and the "body frame" $x_1 x_2 x_3$ to which polar motion refers; cf. TNP II, pp.5-6 and 140-143.

The nutation frame x_i^o is connected to the inertial system in a prescribed way: the x_3^o axis has a fixed direction in inertial space, and the system $x_1^o x_2^o x_3^o$ rotates with constant angular velocity Ω around the x_3^o axis. Being space-fixed, the x_3^o axis is a natural reference for nutation.

The body frame x_i represents a system of Liouville axes for the mantle (TNP II, pp.140-143) with respect to which the mantle, in the absence of elastic deformations, would be at rest. The $x_3 = z$ axis represents the figure axis of the undeformed ellipsoidal earth and is the "mean Liouville figure axis" mentioned above. It is the natural origin of polar motion and corresponds to the point 0 in Fig. 6.1 on p.38 of TNP I.

The basic trick. These two frames are related by a small rotation (TNP II, p. 6):

$$\underline{x} = (\underline{I} + \underline{\Theta})\underline{x}^o \qquad (10-1)$$

where

$$\Theta = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \qquad (10-2)$$

is an "infinitesimal rotation matrix". In terms of the vector

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} , \qquad (10-3)$$

this may also be written

$$\underline{x} = \underline{x}^o - \underline{\theta} \times \underline{x}^o , \qquad (10-4)$$

the cross denoting the vector product as usual.

Now these relations between coordinate systems are geometrically the same as the formulas for an infinitesimal rotation of a rigid body. In fact, the body axes $x_1 x_2 x_3$ can be assumed to be fixed with respect to an undeformed, or rigid, fictitious "reference earth". Thus we may use, with some care, the simple formulas for the rotation of a rigid body (TNP I, secs.10 and 11) even to describe nutation and polar motion for an elastic earth with or without a liquid core. This is the basic trick which will be used here, following Wahr(1981c) and indications in TNP II, pp.136-138.

Basic relations. Denote the unit vector of the rotation axis R by $\underline{e}_P$, the unit vector of the instantaneous figure axis F (which is defined as the axis of maximum inertia for the deformed earth) by $\underline{e}_F$, and the "mean Liouville figure axis" (which is nothing else than the x_3=z axis of the the body frame) by $\underline{e}_z$. Denote further the unit vector of the x_3^o axis (of the nutation frame) by $\underline{e}_3$. Then polar motion and nutation of the various axes can by characterized by the vectors

$$
\begin{aligned}
\underline{p}_R &= \underline{e}_R - \underline{e}_z \,, & \underline{n}_R &= \underline{e}_R - \underline{e}_3 \,, \\
\underline{p}_F &= \underline{e}_F - \underline{e}_z \,, & \underline{n}_F &= \underline{e}_F - \underline{e}_3 \,, \\
\underline{p}_z &= \underline{e}_z - \underline{e}_z = 0, & \underline{n}_z &= \underline{e}_z - \underline{e}_3 \,,
\end{aligned}
\qquad (10-5)
$$

cf. Fig. 10.1.

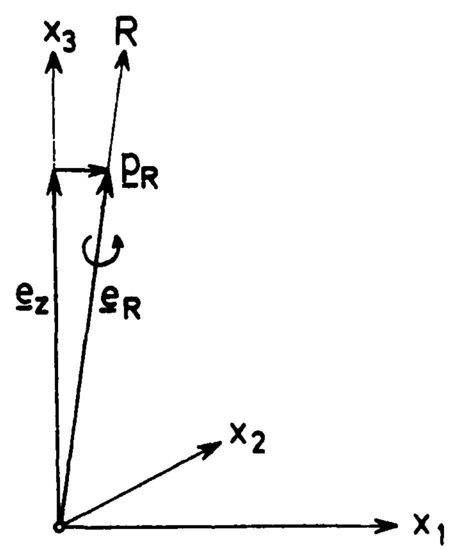

FIGURE 10.1. Polar motion of the rotation axis

Since all vectors $\underline{e}_R$, $\underline{e}_F$, $\underline{e}_z$, and $\underline{e}_3$ are almost parallel to each other, the polar motion vectors $\underline{p}$ and nutation vectors $\underline{n}$ will be small and only have a negligible component along the x_3 axis (Fig, 10.1). Thus we can put

$$\underline{p} = \begin{bmatrix} p_1 \\ p_2 \\ 0 \end{bmatrix} , \qquad \underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} , \tag{10 - 6}$$

and characterize polar motion and nutation by the complex numbers

$$p = p_1 + ip_2 \quad , \qquad n = n_1 + in_2 \tag{10 - 7}$$

instead of the three-dimensional vectors $\underline{p}$ and $\underline{n}$. More details about this can be found in TNP I, sec.11.

First we note the basic relation between nutation and polar motion: from (10 - 5) there follows for any axis

$$\underline{p} = \underline{n} + (\underline{e}_3 - \underline{e}_z) = \underline{n} - \underline{n}_z$$

or for the corresponding complex numbers,

$$\begin{aligned} p &= n - n_z \quad , \\ n &= p + n_z \quad . \end{aligned} \tag{10 - 8}$$

n_z may be expressed by the second equation of (11 - 37) of TNP I, noting that the figure axis F for the rigid "reference earth" is the z-axis:

$$n_z = -i\theta \quad , \tag{10-9}$$

θ being defined by the complex number

$$\theta = \theta_1 + i\theta_2 \tag{10-10}$$

combining the first two components of the vector (10 - 3). The first equation of (11 - 39) of TNP I gives for a frequency σ :

$$p_R = i\,\frac{\sigma+\Omega}{\Omega}\,\theta \quad . \tag{10-11}$$

By eqs. (9 - 23) and (9 - 26) of the present report we have

$$p_R = m = \frac{u}{\Omega} = \varepsilon e^{i\sigma t} \quad . \tag{10-12}$$

Then it follows from (10 - 9) and (10 - 11) that

$$n_z = -\frac{\Omega}{\sigma+\Omega}\,p_R = -\frac{\Omega}{\sigma+\Omega}\,\varepsilon e^{i\sigma t} \quad . \tag{10-13}$$

Then (10 - 8) gives the nutation of the rotation axis:

$$n_R = p_R + n_z = \frac{\sigma}{\sigma+\Omega}\,p_R \tag{10-14}$$

or

$$n_R = \frac{\sigma}{\sigma+\Omega} \varepsilon e^{i\sigma t} \quad . \qquad (10-15)$$

The obvious relation

$$p_z = 0 \qquad (10-16)$$

following from (10 - 5) concludes the formulas for precession and nutation of the rotation axis and the z-axis.

For the instantaneous figure axis F we have by TNP I, eq. (3 - 27):

$$p_F = f = \frac{c}{C-A} \quad . \qquad (10-17)$$

We express $c = c_{13} + ic_{23}$ by (9 - 2), introducing the secular Love number k_s defined by (9 - 8). This gives

$$p_F = \frac{k}{k_s} \frac{u-w}{\Omega} \quad . \qquad (10-18)$$

By (9 - 26) and (9 - 28) this becomes

$$p_F = \frac{k}{k_s} (\varepsilon-\tau) \; e^{i\sigma t} \quad . \qquad (10-19)$$

The nutation of the axis F then follows from (10 - 8) and (10 - 13):

$$n_F = p_F + n_z = \left[\frac{k}{k_s}(\varepsilon - \tau) - \frac{\Omega}{\sigma+\Omega}\,\varepsilon\right] e^{i\sigma t} \quad . \qquad (10-20)$$

We finally state the relation which connects the nutation number n (for any axis) with the usual expression in terms of the Euler angles ψ and θ, the latter being the obliquity of the ecliptic:

$$\Delta\theta + i\Delta\psi \sin\theta = -ine^{i\Omega t} \quad . \qquad (10-21)$$

This is eq.(11 - 55) of TNP I. The factor $-i$ expresses a rotation by -90° and has no deeper significance since it characterizes only the choice of coordinate axes; the factor $e^{i\Omega t}$ expresses the uniform rotation of the $x_1^{o}x_2^{o}x_3^{o}$ system to which n refers, with respect to the inertial system to which ψ and θ refer.

Forced polar motion. For the lunisolar torque we take the expression (11 - 40) of TNP I:

$$L = (C-A)\Omega^2 \sum_j B_j e^{-i(\omega_j t + \beta_j)} \qquad (10-22)$$

as the sum of contributions

$$L_j = (C-A)\Omega^2 B_j e^{-i(\omega_j t + \beta_j)} \qquad (10-23)$$

of different frequencies ω_j with phase β_j. We put, as usual,

$$\sigma = -\omega_j \quad , \tag{10-24}$$

then

$$e^{-i(\omega_j t+\beta_j)} = e^{i\sigma t-i\beta_j} = e^{-i\beta_j}e^{i\sigma t} \quad . \tag{10-25}$$

So far we have disregarded the phase factor $e^{-i\beta_j}$ because, being common also to u, v, w, it cancels in relations such as (9 - 20) or (9 - 42). We may also take care of this phase factor by allowing u_o, v_o, and w_o in (9 - 3) and (9 - 4) to by complex constants. We shall now use the subscripts j to indicate that we deal with contributions of the frequency ω_j.

We thus can write (9 - 28) as

$$w_j = \Omega\tau_j e^{-i(\omega_j t+\beta_j)} \quad , \tag{10-26}$$

whence (9 - 9) gives

$$\begin{aligned} L_j &= -i(C-A)\Omega w_j \\ &= -i(C-A)\Omega^2\tau_j e^{-i(\omega_j t+\beta_j)} \quad . \end{aligned} \tag{10-27}$$

The comparison with (10 - 23) shows that

$$\tau_j = iB_j \quad . \qquad (10 - 28)$$

This relation is not more mysterious than a rotation by 90° since iz is a complex number obtained by rotating z by the angle of 90° :

$$iz = e^{i\frac{\pi}{2}} z \quad . \qquad (10 - 29)$$

It expresses the fact that w and hence, by (9 - 42), u and v are normal to L . Since, as we have seen above, the polar motions and nutations, for a given frequency, are all proportional to u and w , all quantities p_R, p_F, n_R , n_z, and n_F

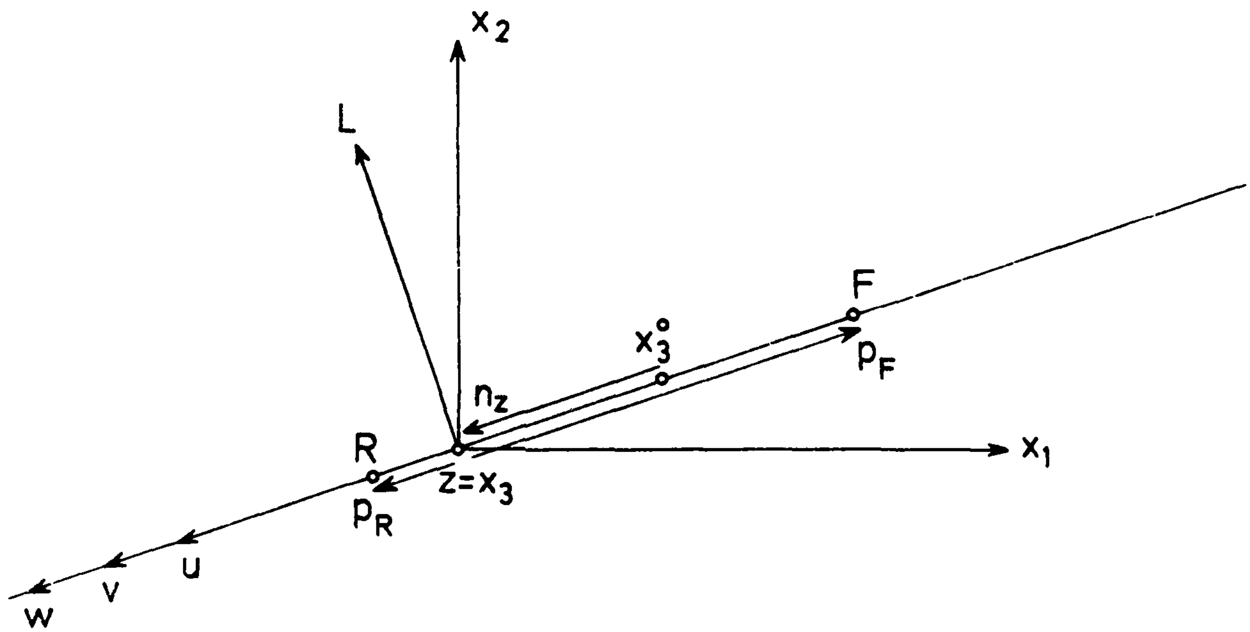

FIGURE 10.2 Forced nutation and polar motion for a fixed frequency

have the same directions as u, v, w, namely normal to L , as Fig. 10.2 schematically shows. Thus the points x_3^o, x_3, R and F (the end points of the respective spatial unit vectors $\underline{e}_3$, $\underline{e}_z$, $\underline{e}_R$, and $\underline{e}_F$) lie on a straight line.

Following (9 - 26) and (9 - 27), we similarly write

$$u_j = \Omega \varepsilon_j e^{-i(\omega_j t + \beta_j)} \quad , \qquad (10 - 30)$$

$$v_j = \Omega \zeta_j e^{-i(\omega_j t + \beta_j)} \quad . \qquad (10 - 31)$$

Then (9 - 29) gives

$$\varepsilon_j = \left(1 + \frac{\sigma_E}{\Omega} + \frac{\Delta\omega_j}{\Omega}\right)^{-1} \left[\frac{\sigma_E}{\Omega}\left(1 + \frac{k_j}{k_s}\Delta\omega_j\right)\tau_j - \right.$$

$$\left. - \frac{A_c}{A}\frac{\Delta\omega_j}{\Omega}\zeta_j\right] ; \qquad (10 - 32)$$

note that we have put $k = k_j$ since the elastic Love number k also depends on frequency.

Now (10-12), (10-16), and (10-19) give immediately, after summing the contributions of the individual frequencies:

$$p_R^{\text{forced}} = \sum_j \varepsilon_j e^{-i(\omega_j t+\beta_j)} \quad ,$$

$$p_z^{\text{forced}} = 0 \quad , \tag{10 - 33}$$

$$p_F^{\text{forced}} = \sum_j \frac{k_j}{k_s}(\varepsilon_j - \tau_j) e^{-i(\omega_j t+\beta_j)} \quad .$$

Here, ε_j has the same direction as $\tau_j = iB_j$.

Forced nutation. With

$$\Delta\omega_j = \omega_j - \Omega = -(\sigma+\Omega) \quad ,$$

eqs. (10 - 13), (10 - 15), and (10 - 20), combined with (10 - 21), give

$$(\Delta\theta + i\Delta\psi\sin\theta)_R^{\text{forced}} = -i\sum_j \frac{\omega_j}{\Delta\omega_j} \varepsilon_j e^{-i(\Delta\omega_j t+\beta_j)} \quad ,$$

$$(\Delta\theta + i\Delta\psi\sin\theta)_z^{\text{forced}} = -i\sum_j \frac{\Omega}{\Delta\omega_j} \varepsilon_j e^{-i(\Delta\omega_j t+\beta_j)} \quad , \tag{10 - 34}$$

$$(\Delta\theta + i\Delta\psi\sin\theta)_F^{\text{forced}} = -i\sum_j \left[\frac{k_j}{k_s}(\varepsilon_j - \tau_j) + \frac{\Omega}{\Delta\omega_j} \varepsilon_j\right] e^{-i(\Delta\omega_j t+\beta_j)} \quad .$$

For a purely elastic earth (no liquid core) we have (9 - 34) which in view of (9 - 37) may also be written

$$\varepsilon_i^{elastic} = \frac{C-A}{A} \kappa_j \tau_j = i \frac{C-A}{A} \kappa_j B_j \quad , \qquad (10-35)$$

using (10 - 28) and the abbreviation κ_j according to TNP I, eq. (6 - 8). The substitution of (10 - 35) and of $\tau_j = iB_j$ then immediately gives the forced (lunisolar) parts of eqs. (6 - 9), (6 - 10), (8 - 7), and (8 - 11) of TNP I, which have been obtained there in a considerably more complicated way.

Free motion. There are two proper frequencies, σ_1 corresponding to the Chandler period, and σ_2 responsible for the "nearly diurnal free wobble" (NDFW); cf. TNP II, pp.123-126. Thus p_R^{free} will have the form of a linear combination of both frequencies:

$$p_R^{free} = m_1 e^{i\sigma_1 t} + m_2 e^{i\sigma_2 t} \quad ; \qquad (10-36)$$

a phase factor of the form $e^{i\delta}$ can be incorporated in the complex constants m_1 and m_2. Then (10 - 16) and (10 - 19) give

$$p_z^{free} = 0 \quad ,$$

$$p_F^{free} = \frac{k_1}{k_s} m_1 e^{i\sigma_1 t} + \frac{k_2}{k_s} m_2 e^{i\sigma_2 t} \quad , \qquad (10-37)$$

k_1 and k_2 being the Love numbers k for the frequencies σ_1 and σ_2 ; there is $\tau = 0$ for free motion.

Free nutation is given by (10 - 13), (10 - 15), and (10 - 20), with (10 - 21):

$$(\Delta\theta + i\Delta\psi\sin\theta)_R^{free} = -i\left[\frac{\sigma_1}{\sigma_1+\Omega}\, m_1 e^{i(\sigma_1+\Omega)t} + \frac{\sigma_2}{\sigma_2+\Omega}\, m_2 e^{i(\sigma_2+\Omega)t}\right] ,$$

$$(\Delta\theta + i\Delta\psi\sin\theta)_z^{free} = i\left[\frac{\Omega}{\sigma_1+\Omega}\, m_1 e^{i(\sigma_1+\Omega)t} + \frac{\Omega}{\sigma_2+\Omega}\, m_2 e^{i(\sigma_2+\Omega)t}\right] , \qquad (10 - 38)$$

$$(\Delta\theta + i\Delta\psi\sin\theta)_F^{free} = -i\left[\left(\frac{k_1}{k_s} - \frac{\Omega}{\sigma_1+\Omega}\right) m_1 e^{i(\sigma_1+\Omega)t} + \left(\frac{k_2}{k_s} - \frac{\Omega}{\sigma_2+\Omega}\right) m_2 e^{i(\sigma_2+\Omega)t}\right] .$$

The nearly diurnal free wobble (NDFW) does not, so far, seem to have been confirmed by observation, neither in the polar motion (amplitude $|m_2|$) nor in nutation. For nutations, formulas (10 - 38) show that this amplitude would even be increased by a factor on the order of

$$\left|\frac{\Omega}{\sigma_2+\Omega}\right| = \left|\frac{1}{-1.002+1}\right| = 500 \quad ,$$

according to TNP II, p.167!

Thus we can safely disregard m_2, and consider only the Chandler frequency σ_1. Then these formulas for free polar motion and nutation reduce to the free terms in the corresponding expressions for an elastic earth, TNP I, eqs. (6-9), (6 - 10), (8 - 7), and (8 - 11), with $\sigma_1 = \sigma_C$ and k being the value k_1 for the Chandler frequency.

Also the expressions (10 - 33) and (10 - 34) are formally the same as for a purely elastic earth, liquid-core effects entering only indirectly: weakly through the frequency-dependent Love numbers k_j and more strongly through the ε_j which depend on core rotation ζ_j through (10 - 32). This is not surprising since the basic relation for these "geometric" or "kinematic" axes depend only on the surface form of our earth.

Thus the qualitative picture of polar motion for an elastic earth with a liquid core remains basically the same as for a purely elastic earth; cf. Fig. 6.1 on p.38 of TNP I. Only the amplitudes of forced polar motion are somewhat affected by the liquid core. The orders of magnitude remain the same: 6 m for free polar motion, 60 cm for forced motion of the rotation axis, and 60 m (!) for the forced motion of the instantaneous figure axis; cf. TNP I, pp. 39-40.

Finally we note that, as far as free polar motion is concerned, the elastic earth model, even with a liquid core, gives a highly schematic and unrealistic picture. The actual free polar motion is so irregular that it can only be determined by actual observations. On the other hand, the forced polar motion and nutation is represented well by the elastic mantle-liquid core model. Thus the formulas (10 - 33)and (10 - 34) describe them with sufficient accuracy.

11. The Dynamic Axes

These axes comprise the angular momentum axis and the related "Celestial Ephemeris Pole".

Angular momentum axis. It represents the direction of the angular momentum vector $\underline{H}$.

Since the relation between angular momentum and torque,

$$\underline{L} = \frac{d\underline{H}}{dt} , \qquad (11 - 1)$$

is independent of the internal structure of the body (cf. TNP I, pp. 48 and 128), we have for the nutation n_H the same form

$$n_H = - \frac{iL}{C\Omega(\sigma+\Omega)} \qquad (11 - 2)$$

as for a rigid body or for the Poincaré model; this is eq. (13 - 37) of TNP I.

The free nutation of the H axis is zero since, in absence of external forces, the vector $\underline{H}$ remains unchanged in inertial space. For the forced motion we have, summing over the different frequences,

$$n_H = \frac{i}{C\Omega} \sum_j \frac{L_j}{\Delta\omega_j} , \qquad (11 - 3)$$

or by (9 - 11), (10 - 21), and (10 - 27):

$$(\Delta\theta+i\Delta\psi\sin\theta)_H=-i\frac{A}{C}\sum_j \frac{\sigma_E}{\Delta\omega_j}\tau_j e^{-i(\Delta\omega_j t+\beta_j)} \quad . \qquad (11-4)$$

Using (10 - 28) we have the alternative form

$$(\Delta\omega+i\Delta\psi\sin\theta)_H = \frac{A}{C}\sum_j \frac{\sigma_E}{\Delta\omega_j} B_j e^{-i(\Delta\omega_j t+\beta_j)} \quad . \qquad (11-5)$$

This is identical to the nutational part of eq.(7 - 14) of TNP I, but the derivation is much simpler in the present way. It is also identical to the nutation of H as given by TNP I, eq.(13 - 69) for the Poincaré model.

Polar motion for the H axis may be obtained from (10 - 8):

$$p_H = n_H - n_z \quad . \qquad (11-6)$$

Using (10 - 13) and (11 - 4) and summing over different frequencies we get

$$p_H = \sum_j \left(\frac{A}{C}\frac{\sigma_E}{\Delta\omega_j}\tau_j - \frac{\Omega}{\Delta\omega_j}\epsilon_j\right) e^{-i(\omega_j t+\beta_j)} \quad . \qquad (11-7)$$

A <u>thorough check</u> is provided by computing p_H from an expression of the vector $\underline{H}$ in the body frame. The angular momentum equation

$$\underline{H} = \underline{C}\,\underline{\omega} + \underline{h} \tag{11-8}$$

(TNP I, p.13) holds for an arbitrary nonrigid body (even with a liquid core), provided the relative angular momentum $\underline{h}$ is taken into account besides the main term $\underline{C}\,\underline{\omega}$, $\underline{C}$ being the inertia tensor and $\underline{\omega}$ the rotation vector as usual.

For Tisserand axes referred to the mantle, the relative angular momentum $\underline{h}$ is due to core rotation. Thus

$$\underline{h} = \underline{C}_c\underline{\chi} = \begin{bmatrix} A_c\chi_1 \\ A_c\chi_2 \\ 0 \end{bmatrix} \tag{11-9}$$

since $\chi_3 = 0$ by (8-9); we have disregarded the second-order products of the anomalous tensor of inertia for the core, c^c_{ij}, with χ_i as usual.

The linearization of $\underline{C}\,\underline{\omega}$ is provided by eq.(3-22) of TNP I; by (11-8) we have to add $\underline{h}$ as given by (11-9). The result is

$$H = A\Omega m + \Omega c + A_c(\chi_1 + i\chi_2) \ .$$

By (1-3) this is

$$H = A\Omega m + \Omega c + A_c v \ . \tag{11-10}$$

Here

$$H = H_1 + iH_2 \qquad (11-11)$$

is the complex number combining the x_1 and x_2 components of the angular momentum vector $\underline{H}$. The components of the corresponding unit vector are obtained by dividing by its (approximate) length $C\Omega$, whence

$$p_H = \frac{A}{C} m + \frac{c}{C} + \frac{A_c}{\Omega C} v \; ; \qquad (11-12)$$

we obtain indeed p_H since everything is referred to the body frame $x_1 x_2 x_3$. With (9 - 23) and (10 - 17) this may be written

$$p_H = \frac{A}{C} p_R + \frac{C-A}{C} p_F + \frac{A_c}{\Omega C} v \; . \qquad (11-13)$$

After substituting p_R from (10 - 12), p_F from (10 - 19) and v from (9 - 20), and neglecting second-order terms we get (11 - 7), which completes the check.

This equation can also be used to compute the <u>free polar motion</u> of H . Core rotation only plays a role in the tidal frequency range, that is, for $\omega_j \doteq \Omega$; its effect is rather small for the Chandler frequency, cf.TNP I, p.133. Thus, since the calculation of free motion is of little practical value anyway, we may neglect v in (11 - 13), obtaining

$$p_H^{free} = \frac{A}{C} p_R^{free} + \frac{C-A}{C} p_F^{free} \; . \qquad (11-14)$$

Using (10 - 36) and (10 - 37) and neglecting the hardly observable NDFW $(m_2 = 0)$, we get

$$p_H^{free} = \left(\frac{A}{C} + \frac{C-A}{C}\frac{k_1}{k_s}\right)m_1 e^{i\sigma_1 t} , \qquad (11 - 15)$$

the same as for a purely elastic earth (TNP I, p.37).

The Celestial Ephemeris Pole. The corresponding axis has been adopted at the General Assembly of the International Astronomical Union in Montreal in 1979 to define the celestial pole for reference purposes, according to Commission Resolution (3):

"Commissions 4, 8, 19 and 31

endorse the recommendations given in the Report of the Working Group on Nutation, as set out below, and recommend that they shall be used in the national and international ephemerides for the years 1984 onwards, and in all other relevant astronomical work.

Recommendations of the Working Group on Nutation

Whereas the complete theory of the general nutational motion of the Earth about its centre of mass may be described as the sum of two components, (i) astronomical nutation, commonly referred to as nutation, which is motion with respect to a space-fixed coordinate system, and (ii) polar motion, which is motion with respect to a body-fixed coordinate system, it is recommended that:
(a) astronomical nutation be computed for the "Celestial Ephemeris Pole" using a non-rigid model of the Earth

such that there are no nearly diurnal motions of this celestial pole with respect to either space-fixed or body-fixed coordinates, which can be calculated from torques external to the Earth and its atmosphere ,

(b) the numerical values given in Table 1 of the complete report be used for computing astronomical nutation of the "Celestial Ephemeris Pole". "

Cf. (IAU, 1980, pp. 40-41). For the scientific background cf. (Leick and Mueller, 1979; Moritz, 1979; Mueller, 1981) and also TNP I, pp. 58-59.

The Celestial Ephemeris Pole C thus corresponds to the angular momentum axis freed from the lunisolar diurnal motion, seen from the earth-fixed body frame, that is with respect to polar motion. With respect to the inertial frame, it shares with H the property that its free nutation (which has a nearly-diurnal period) is zero. Thus in fact, C contains no diurnal motions, neither in the body frame nor in the inertial frame, as the above resolution requires.

Hence the free polar motion of the Celestial Ephemeris Pole C equals the free part of p :

$$p_C = p_H^{free} \quad . \tag{11 - 16}$$

An analytical expression is (11 - 15), which is of little practical use, however, since free polar motion is too irregular to be analytically predicted. So p_C must come from observation.

As far as the nutation is concerned we have seen above that its nearly-diurnal part is zero. There remains the lunisolar analytical part which can be analytically described and predicted well. Free motion playing no part, the pole C does not depend on polar motion. Its nutation is therefore the same whether polar motion exists or is zero. In the latter case, C coincides with the z-axis. Cf. TNP I, Fig. 6.1 on p.38: if there is no polar motion, then $C = H_o$ coincides with the origin O representing the z-axis. See also TNP I, pp. 59-60.

Thus the nutation of C equals the forced (lunisolar) nutation of the z- axis as given by (10 - 34):

$$(\Delta\theta + i\Delta\psi)_C = -i \sum_j \frac{\Omega}{\Delta\omega_j} \varepsilon_j e^{-i(\Delta\omega_j t + \beta_j)} \tag{11 - 17}$$

Concluding Remark. Only the forced (lunisolar) parts of polar motion and nutation can be accurately described by the elastic mantle-liquid core model and hence accurately predicted. Therefore, the main practical result of the present report are the formulas (10 - 33), (10 - 34), (11 - 5), (11-7), (11-17). They contain the coefficients ε_j and τ_j which are given by (10 - 23), (10 - 28), and (10 - 32).

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