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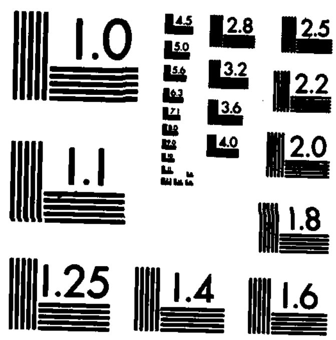

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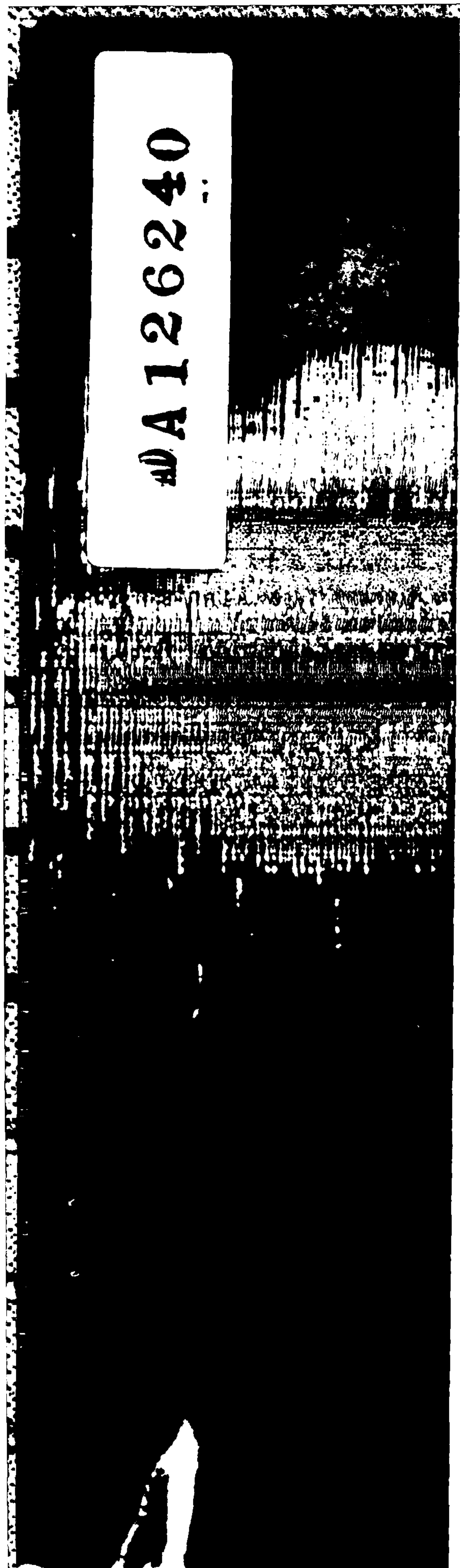

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Research Report CCS 450

A SUCCESSIVE LINEAR PROGRAMMING METHOD AND ITS CONVERGENCE ON NONLINEAR PROBLEMS

by

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ABSTRACT

Since Griffith and Stewart firstly proposed as successive linear programming method for solving general nonlinear programming problems, such methods have been widely used in practice because of their ease of implementation and their ability to deal with large scale problems. However, neither the original version, nor a more recent one contain convergence proofs possibly because of non-robustness of their algorithms. Using exact penalty functions and Levenberg-Marquardtlike steps, an improved algorithm has been recently devised. In this paper, we give this modified SLP method a theoretical analysis and convergence proof, and thereby provide a sound basis for it.

KEY WORDS

Constrained nonlinear programming

Exact penalty function

Nonsmooth optimization

Stationary points

Generalized gradient

Linear subproblem

Restricted step length

Convergence

A Successive Linear Programming Method and Its Convergence on Nonlinear Problems

by

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§1 Introduction

In this paper we discuss the following type of general nonlinear programming problem

$$\begin{aligned} \min\ & f(x_1) + px_2 \\ \text{s.t.}\ & g_i(x_1) + q^{(i)}x_2 = b_i\ ,\quad i = 1,\dots,k \\ & g_i(x_1) + q^{(i)}x_2 \leq b_i\ ,\quad i = k+1,\dots,h \\ & (A_1,A_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq c \end{aligned} \qquad (1.1)$$

where $x_1 \in R^n$, $x_2 \in R^m$, the vectors p, $q^{(i)} \in R^m$, A_1 and A_2 are respectively $l \times n$ and $l \times m$ matrices, $c \in R^l$, f and g_i $(i = 1,\dots,h)$ are continuously differentiable functions in R^n. For simplicity, we write the scalar product of vectors x and y as xy. When there is no need to differentiate between the nonlinear part and the linear part of the problem functions, we write

$$F(x) \triangleq f(x_1) + px_2$$

$$G_i(x) \triangleq g_i(x_1) + q^{(i)}x_2 - b_i\ ,\ i = 1,\dots,h$$

Here $x = (x_1,x_2) \in R^{n+m}$.

In 1961, Griffith and Stewart [1] proposed a successive linear programming method for the above problem (1.1). In their method, linear approximations for all nonlinear functions f and g_i were made and then the resulting linear programming problem was solved in a bounded region. Because of its ease of implementation and its ability to deal with large scale problems, this method has been widely used in various organizations. However, although this method has worked in many examples, its convergence has not been established in general cases.

Since its appearance, some modified and improved versions have appeared, such as [2], [3]. For example, in [3] using penalty functions, another SLP method has been presented. Computational results for it compare favorably with the well-known Generalized Reduced Gradient Method. However, the algorithm of [3] is rather complicated and its convergence was only proved (in an unpublished dissertation) for linearly constrained problems. Using the basic idea of [3], Lasdon, Kim and this author improved the SLP algorithm further in [4]. The new SLP method has worked efficiently in extensive sets of test examples. In this paper we give the method a theoretical analysis and convergence proof and thereby provide a sound basis for it.

In the next section, §2, we discuss the exact penalty function of problem (1.1). In §3 we introduce the improves SLP algorithm, and in §4 we solve the problem of convergence of this method.

§2 Some Relative Results on Exact Penalty Function

For problem (1.1), we construct the exact penalty function

$$P(z) \triangleq F(x) + \sum_{i=1}^{k} \omega_i |G_i(x)| + \sum_{i=k+1}^{h} \omega_i \max(0, G_i(x)) \qquad (2.1)$$

where $\omega = (\omega_1, \ldots, \omega_h)$ and $_i$ are suitable positive numbers ($i = 1, \ldots, h$).

Now we consider the problem

$$\min_{Ax \leq c} P(x) \qquad (2.2)$$

There are many papers devoted to expounding the close relation between the optimal solution of the exact penalty function problem and that of the original problem. According to the results of [5], and considering the linear inequality constraints, which still remain in (2.2), we can easily find the relation between problem (2.2) and (1.1).

Set

$$\Omega = \{x : G_i(x) = 0 \ , \ i = 1,\ldots,k \ ;$$
$$G_i(x) \leq 0 \ , \ i = k+1,\ldots,h \ ;$$
$$Ax \leq c\}$$

and

$$\Omega' = \{x : Ax \leq c\} \ .$$

We can obtain the following:

Proposition 2.1 If there exists a $\overline{\omega} \geq 0$ such that for all $\omega \geq \overline{\omega}$, $\overline{x}$ is a local minimum point of the penalty function $P(x,\omega)$ in $N(\overline{x}) \cap \Omega'$, which contains at least one feasible point of original problem (here, $N(\overline{x})$ is an open neighborhood of $\overline{x}$), then $\overline{x} \in \Omega$ and is a local optimal solution of problem (1.1). Especially, if for some ω, the local optimal solution $\overline{x}$ is a feasible point of (1.1), then $\overline{x}$ must be a local optimal solution of the original problem.

For the original problem (1.1), if we write the last constraint as $x \in \Omega'$, then under any common K-T type constraint qualification (for instance, the qualifications of Mangasarian-Fromovitz or Kuhn-Tucker in [6]), we can state the necessary condition for an extremum point $\overline{x}$ as follows:

There are $\overline{u} = (\overline{u}_1,\ldots,\overline{u}_k)$

$\overline{v} = (\overline{u}_{k+1},\ldots,\overline{u}_h)$

satisfying

$$\overline{u}_i \geq 0 \ , \ \overline{u}_i G_i(\overline{x}) = 0 \ , \ i = k+1,\ldots,h \tag{2.3}$$

and for $\forall s \in T(\Omega',\overline{x})$, we always have

$$(\nabla f(\overline{x}) + \sum_{i=1}^{h} \overline{u}_i \ \nabla G_i(\overline{x})) \ s \geq 0 \tag{2.4}$$

where $T(\Omega',\bar{x})$ is the tangent cone of linear constraint set Ω' at $\bar{x}$:

$$T(\Omega',\bar{x}) = \{s : a_i s \leq 0 \ , \ i \in I'(\bar{x})\} \tag{2.5}$$

and

$$I'(\bar{x}) \triangleq \{i : i \in \{1,2,\dots,l\} \text{ and } a_i x = c_i\} \tag{2.6}$$

a_i is the i-th row vector of matrix A.

In fact, (2.3) and (2.4) form an optimality criteria of the minimum principle type, see [6], [7]. Of course, we can also rewrite (2.4) as

$$[\nabla f(\bar{x}) + \sum_{i=1}^{k} \bar{u}_i \nabla G_i(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{u}_i \nabla G_i(\bar{x})] \, s \geq 0$$

where

$$I(\bar{x}) \triangleq \{i : i \in \{k+1,\dots,h\} \text{ and } G_i(\bar{x}) = 0\}.$$

Furthermore, it is easy to modify the standard Fiacco-McCormick second order sufficient condition [8] as follows:

If the feasible point $\bar{x}$ of problem (1.1) satisfies the necessary conditions (2.3) and (2.4), and for any $s \neq 0$ with the following properties:

$$\begin{cases} \nabla F_i(\bar{x})s \leq 0 \\ \nabla G_i(\bar{x})s = 0 \ , \ i = 1,\dots,k \\ \nabla G_i(\bar{x})s \leq 0 \quad i \in I(\bar{x}) \\ s \in T(\Omega',x) \end{cases}$$

The following strict inequality always holds:

$$s[\nabla^2 f + \sum_{i=1}^{h} \bar{U}_i \nabla^2 G_i]_{\bar{x}} s > 0 \ , \tag{2.7}$$

then $\bar{x}$ is a strict local optimal solution of the original problem (1.1).

Consulting the proof of lemma 4.5 and theorem 4.6 of [5], it is not difficult to obtain:

<u>Proposition 2.2</u> If the feasible point $\bar{x}$ of problem (1.1) satisfies second order sufficient conditions (2.3), (2.4) and (2.7), and $\omega = (\omega_1,\ldots,\omega_h)$ has the property:

$$\min_{i=1,\ldots,h} \{\omega_i\} \quad > \quad \max_{i=1,\ldots,h} \{|\bar{u}_i|\}$$

then $\bar{x}$ must be a strict local minimum solution of exact penalty function $P(x,\omega)$ in Ω'.

From the above, we know that except in some particular cases (for example, see [9]), usually by solving problem (2.2), we can get the optimal solution of the original problem (1.1).

Now we turn to the optimality criteria for problem (2.2). Obviously, $P(x)$ is a nonsmooth function. However, since $F(x)$ and $G_i(x)$ are all continuously differentiable, $P(x)$ is a locally Lipschitz continuous function and so, according to [10], we know that along every direction s, function $P(x)$ has generalized directional derivative

$$P^\circ(x;s) \triangleq \limsup_{\substack{x'\to x \\ \delta \downarrow 0}} \frac{P(x'+\delta s) - P(x')}{\delta} \tag{2.8}$$

and by virtue of it, Clarke defined the generalized gradient of $P(x)$:

$$\partial P(x) \triangleq \{\xi : \forall v \in R^{n+m}, \; P^\circ(x;v) \geqslant v\zeta\} \tag{2.9}$$

and furthermore, we know that $\partial P(x)$ is a compact convex set of R^{n+m} and

$$P^\circ(x;s) = \max_{\zeta \in \partial P(x)} \zeta s \tag{2.10}$$

Because the constraints in problem (2.2) are all linear, obviously the problem satisfies common constraint qualifications for nonsmooth extremum problems and therefore, according to [11], [12], [13], we have

Proposition 2.3 The necessary condition for $\overline{x}$ being a local optimal solution of extremum problem (2.2) is that for $i = 1,2,\ldots,l$ and $i \in I'(\overline{x})$ (see (2.6)), there exist $\lambda_i \geq 0$ such that

$$0 \in \partial P(x) + \sum_{i \in I'(\overline{x})} \lambda_i a_i \, .$$

Or, if we denote

$$D(x) \triangleq \partial P(x) + \sum_{\substack{i \in I'(x) \\ \lambda_i \geq 0}} \lambda_i a_i \qquad (2.11)$$

$$\Omega^* \triangleq \{x : 0 \in D(x)\}$$

and call Ω^* the set of stationary points of non-differentiable programming (2.2), then Proposition 2.3 implies that

$\overline{x}$ is a local optimal solution of (2.2) $\Rightarrow \overline{x} \in \Omega^*$.

In the last part of this section, we are going to discuss the structure of the generalized gradient $\partial P(x)$. First, we have

Proposition 2.4 If $\varphi(x,u^{(k)})$ is a finite family of functions, in which $x \in R^{n+m}$, $u^{(k)} \in R$, $k = 1,\ldots,s$; and for every $u^{(k)}$, $\varphi(x,u^{(k)})$ is a continuously differentiable function of x, then the function

$$\psi(x) = \max_{k=1,\ldots,s} \varphi(x,u^{(k)})$$

is a locally Lipschitz continuous function; and for every direction v, there is a directional derivative $\psi'(x;v)$ under the usual meaning, which equals $\psi^\circ(x;v)$; and furthermore,

$$\partial\psi(x) = \text{co } \{\nabla_x \varphi(x,u^{(k)}) : u^{(k)} \in M(x)\} \qquad (2.12)$$

where

$$M(x) = \{u^{(k)} : \varphi(x,u^{(k)}) = \psi(x)\}$$

and co(·) denotes the convex hull of set (·).

In fact, this proposition is a particular case of [14], Ch. III, wherein Pshenichnyi has some general results about quasi-differentiable functionals involving extensive arguments. Here we give a straightforward proof for our case.

Proof

Using the locally Lipschitz property of differentiable function φ, it is not difficult to see that $\psi(x)$ is also locally Lipschitz continuous.

Now, for every $u^{(k)} \in M(x)$,

$$\nabla\varphi(x,u^{(k)})v = \lim_{\delta\downarrow 0} \frac{\varphi(x + \delta v , u^{(k)}) - \varphi(x,u^{(k)})}{\delta}$$

$$\leq \liminf_{\delta \downarrow 0} \frac{\psi(x + \delta v) - \psi(x)}{\delta} \qquad (\text{since } \varphi(x + \delta v,u^{(k)}) \leq \psi(x + \delta v$$

$$\leq \limsup_{\delta \downarrow 0} \frac{\psi(x + \delta v) - \psi(x)}{\delta}$$

$$\leq \psi^\circ(x;v)$$

$$= \limsup_{\substack{x' \to x \\ \delta \downarrow 0}} \frac{\psi(x' + \delta v) - \psi(x')}{\delta}$$

$$= \limsup_{\substack{x' \to x \\ \delta \downarrow 0}} \frac{\max\limits_{u^{(k)} \in M(x)} \varphi(x' + \delta v,u^{(k)}) - \max\limits_{u^{(k)} \in M(x)} \varphi(x',u^{(k)})}{\delta}$$

(when x' nears x, δ nears 0, we have

$M(x') \subseteq M(x)$ and $M(x' + \delta v) \subseteq M(x)$)

$$\leq \limsup_{\substack{x' \to x \\ \delta \downarrow 0}} \max_{u^{(k)} \in M(x)} \left\{ \frac{\varphi(x' + \delta v, u^{(k)}) - \varphi(x', u^{(k)})}{\delta} \right\}$$

$$= \limsup_{\substack{x' \to x \\ \delta \downarrow 0}} \max_{u^{(k)} \in M(x)} \{\nabla\varphi(x', u^{(k)})v + o(1)\}$$

$$= \max_{u^{(k)} \in M(x)} \nabla\varphi(x, u^{(k)})v \tag{2.13}$$

$$= \max_{w \in co\{\nabla\varphi(x, u^{(k)}) \,:\, u^{(k)} \in M(x)\}} wv$$

Obviously, at the very beginning of the series of expressions above, we can select the particular $u^{(k)}$, which makes the expression $\nabla\varphi(x,u^{(k)})v$ in (2.13) reach its maximum, and it means that all these inequalities must be held as strict equality. So, there exists $\psi'(x,v)$ and

$$\lim_{\delta \downarrow 0} \frac{\psi(x + \delta v) - \psi(x)}{\delta}$$

$$= \psi'(x;v)$$

$$= \psi^{\circ}(x;v)$$

$$= \max_{w \in co\{\nabla\varphi(x, u^{(k)}) \,:\, u^{(k)} \in M(x)\}} wv \tag{2.14}$$

On the other hand, by virtue of (2.10),

$$\psi^\circ(x;v) = \max_{\zeta\in\partial\psi(x)} \zeta v \tag{2.15}$$

Because both sets appeared respectively in (2.14) and (2.15) are closed convex sets, so according to Corollary 13.1.1 of [15],

$$\partial\psi(x) = \text{co } \{\nabla\varphi(x,u^{(k)}) : u^{(k)}\in M(x)\}$$

From the above result, now it is easy to express the structure of $\partial P(x)$.

Proposition 2.5 For every positive vector ω, the generalized gradient set of function $P(x)$ defined by (2.1) is

$$\partial P(x) = \{\nabla F(x) + \sum_{i=1}^{h} u_i\omega_i \nabla G_i(x)\} \tag{2.16}$$

and for $i=1,\ldots,k$,

$$u_i = \begin{cases} \text{sgn } G_i(x) & \text{if } G_i(x) \neq 0 \\ [-1,1] & \text{if } G_i(x) = 0 \end{cases} \tag{2.17}$$

for $i=k+1,\ldots,h$,

$$u_i = \begin{cases} [1 + \text{sgn } G_i(x)]/2 & \text{if } G_i(x) \neq 0 \\ [0,1] & \text{if } G_i(x) = 0 \end{cases} \tag{2.18}$$

Proof

Since $|G_i(x)| = \max \{G_i(x)\cdot 1 , G_i(x)\cdot -1\}$

$\max (G_i(x),0) = \max \{G_i(x)\cdot 1 , G_i(x)\cdot 0\}$

$$\therefore \sum_{i=1}^{k} \omega_i|G_i(x)| + \sum_{i=k+1}^{h} \omega_i \max (G_i(x),0)$$

$$= \max_{u \in U} \{ \sum_{i=1}^{h} \omega_i G_i(x) u_i \}$$

where $U \triangleq \{u = (u_1,\ldots,u_k,u_{k+1},\ldots,u_h) : \begin{matrix} u_i = \pm 1 & \text{when } i=1,\ldots,k; \\ u_i = 0 \text{ or } 1 & \text{when } i=k+1,\ldots,h \end{matrix} \}$.

We set

$$\varphi(x,u) \triangleq F(x) + \sum_{i=1}^{h} \omega_i G_i(x) u_i$$

then $P(x) = \max_{u \in U} \varphi(x,u)$

and by virtue of Proposition 2.4, we have

$$\partial P(x) = \text{co} \{\nabla\varphi(x,u) : u \in M(x)\}$$

$$= \text{co} \{\nabla F(x) + \sum_{i=1}^{h} \omega_i u_i \nabla G_i(x) : \begin{matrix} u_i \text{ are determined} \\ \text{by (2.17) and (2.18)} \end{matrix} \}$$

and which is just the same as (2.16).

Incidently, Coleman and Conn has obtained nearly the same result, see [9], Theorem 1 and its Corollary 1, but in their proof, they assumed that vectors $\{\nabla G_i(x) \mid G_i(x) = 0, i=1,2,\ldots,h\}$ are linearly independent, which is redundant according to the proof given here.

As a matter of fact, according to Propositions 2.3 and 2.5, it is obvious that there exists another relation between the original problem (1.1) and the penalty problem (2.2).

Proposition 2.6 If $\bar{x}$ is a Kuhn-Tucker point of problem (1.1) with the corresponding multipliers $\bar{u}_i$ $(i=1,\ldots,h)$ and $\bar{\lambda}_i$ $(i=1,\ldots,l)$, and the coefficients ω_i of the penalty function P satisfy the condition that

$$\omega_i \geq |\bar{u}_i| \quad , \quad i=1,\ldots,h$$

then $\bar{x}$ must be a stationary point of penalty problem (2.2).

Proof

By K-T conditions, we have

$$0 = \nabla F(\bar{x}) + \sum_{i=1}^{h} \bar{u}_i \nabla G_i(\bar{x}) + \sum_{i\in I'(\bar{x})} \bar{\lambda}_i a_i$$

$$(\bar{u}_i \geq 0 \ \text{ and } \ \bar{u}_i G_i(\bar{x}) = 0 \ \text{ for } i=k+1,\ldots,h;$$

$$\bar{\lambda}_i \geq 0 \ \text{ for } \ i\in I'(\bar{x}))$$

$$\in \nabla F(\bar{x}) + \sum_{i=1}^{k} \omega_i \{u_i \nabla G_i(\bar{x}) : u_i \in [-1,1]\}$$

$$+ \sum_{i=k+1}^{h} \omega_i \left\{ u_i \nabla G_i(\bar{x}) : \begin{array}{ll} u_i \in [-1,1] , & \text{if } G_i(\bar{x}) = 0 \\ u_i = 0 & \text{if } G_i(\bar{x}) < 0 \end{array} \right\}$$

$$+ \sum_{i\in I'(\bar{x})} \bar{\lambda}_i a_i$$

$$= \partial P(\bar{x}) + \sum_{i\in I'(\bar{x})} \bar{\lambda}_i a_i$$

Here, the last equation is because of expressions (2.17), (2.18) and the fact that $\bar{x}$ is a feasible point of the original problem (1.1).

Now, by virtue of Propositon 2.3, we know that

$$0 \in D(\bar{x})$$

which means $\bar{x}$ is a stationary point of (2.2).

§3 Successive Linear Programming Algorithm

Now we present the modified SLP algorithm of [4] which we shall employ in our proof. Taking a linear approximation for penalty function $P(x)$, that means for $\bar{x} = (\bar{x}_1,\bar{x}_2)$ and $d = (d_1,d_2)$, where $d_1 \in R^n$, $d_2 \in R^m$, we approximately have

$$P(\bar{x} + d)$$

$$\approx R(\bar{x},d)$$

$$\triangleq (F(\bar{x}) + \nabla F(\bar{x})d) + \sum_{i=1}^{k} \omega_i |G_i(x) + \nabla G_i(\bar{x})d|$$

$$+ \sum_{i=k+1}^{h} \omega_i \max (0, G_i(\bar{x}) + \nabla G_i(\bar{x})d) \qquad (3.1)$$

In fact, because the second parts of F and G_i themselves are linear functions, only the approximation for functions f and g_i is needed.

The problem (2.2) has now been changed approximately to a linear programming problem:

$$\min \quad R(\bar{x},d)$$

$$\text{s.t.} \quad \|d_1\|_\infty \leq \alpha$$

$$\|d_2\|_\infty \leq M \qquad (3.2)$$

$$A(\bar{x} + d) \leq c$$

Here, the reason for restricting the norm of vector d_1 is clear, because it is only when we do so that the linear approximation can be reasonable, and furthermore, the value of α should be adjusted according to how better the linear approximation was performed. Positive number M can be taken as any large amount. From the approximate point of view, it is unnecessary to restrict the norm of vector d_2; however, in order to guarantee that the subproblem (3.2) must have a finite optimal solution, we need to restrict d in a bounded area.

Using L_∞ norm can make (3.2) easily become a linear programming problem, and for simplicity, from now on we just use $\|\cdot\|$ instead of sign $\|\cdot\|_\infty$. Now we state the whole algorithm as follows:

SLP Algorithm

1° Select initial point $x^{(1)}$ which satisfies linear constraint $Ax \leq c$, positive α_1 and M (large enough), $0 < \rho_1 < \rho_2 < 1$, $\gamma > 1$, set $k = 1$.

2° Solve subproblem

$$\begin{aligned} &\min \ R(x^{(k)},d) \\ \text{s.t.} \ &\|d_1\| \leq \alpha_k \\ &\|d_2\| \leq M \qquad (3.3) \\ &A(x^{(k)} + d) \leq c \end{aligned}$$

and obtain solution $d^{(k)}$ (if the optimal solutions are not unique, choose any one of them as $d^{(k)}$).

3° Calculate

$$\Delta P^{(k)} \triangleq P(x^{(k)}) - P(x^{(k)} + d^{(k)})$$

$$\Delta R^{(k)} \triangleq P(x^{(k)}) - R(x^{(k)}, d^{(k)})$$

if $\Delta R^{(k)} = 0$, then stop; otherwise calculate

$$\sigma_k = \Delta P^{(k)}/\Delta R^{(k)}$$

4° Set

$$\alpha_{k+1} = \begin{cases} \alpha_k/\gamma & \sigma_k < \rho_1 \\ \gamma\alpha_k & \sigma_k > \rho_2 \\ \alpha_k & \text{otherwise} \end{cases}$$

and

$$x^{(k+1)} = \begin{cases} x^{(k)} & \sigma_k \leq 0 \\ x^{(k)} + d^{(k)} & \text{otherwise} \end{cases}$$

Set $k \leftarrow k+1$, and return to 2°.

The main part of this algorithm is successively solving linear programming (3.3). We can rewrite the objective function $R(x^{(k)},d)$ as

$$\nabla f(x_1^{(k)})\, d_1 + pd_2 + \sum_{i=1}^{k} \omega_i (S_i^+ + S_i^-) + \sum_{i=k+1}^{h} \omega_i S_i^+ ,$$

and at the same time supplement a set of constraint conditions:

$$\begin{cases} g_i(x_1^{(k)}) + \nabla g_i(x_1^{(k)}) d_1 + q^{(i)}(x_2^{(k)} + d_2) - b_i = S_i^+ - S_i^- \\ S_i^+ \geq 0 \ , \ S_i^- \geq 0 \\ i = 1,2,\ldots,h \end{cases}$$

Now the subproblems in the algorithm become ordinary LP problems.

We use the ratio of $\Delta P^{(k)}$ and $\Delta R^{(k)}$ to assess whether the last linearization is a good approximation or not, and then to decide the maximum step length for next iteration. This is a common technique in Levenberg-Marquardt type methods for nonlinear least square problems (see [16] and references therein).

§4 Convergence

In this section, we are going to prove the convergence of this modified SLP algorithm. The main idea of our proof, especially the logic order of the following Lemma 4.3, Lemma 4.4 and Theorem 4.5, to a large extent, come from Zhu Meifang and author's paper [16], in which we gave a nearly uniform proof to several versions of Levenberg-Marquardt algorithm for nonlinear least square problems. We find that the way of dealing with the convergence of restricted step algorithms used in [16] is also very useful in the following argument and the only difference is that now we have to notice the nondifferentiable nature of the present problem.

Theorem 4.1 The $\Delta R^{(k)}$ defined in above algorithm $(k=1,2,\ldots)$ are always non-negative, and if $\Delta R^{(k)} = 0$, the corresponding $x^{(k)}$ is a stationary point of programming (2.2), i.e., $x^{(k)} \in \Omega^*$, and it is the case of finite convergence of this algorithm.

Proof

Obviously $d = 0$ is a feasible solution of problem (3.3), hence

$$R(x^{(k)},0) \geq R(x^{(k)},d^{(k)}) \tag{4.1}$$

By definition of function R, $R(x^{(k)},0) = P(x^{(k)})$ and so, (4.1) implies

$$\Delta R^{(k)} = P(x^{(k)}) - R(x^{(k)},d^{(k)}) \geq 0$$

Furthermore, $\Delta R^{(k)} = 0$ if and only if $d = 0$ is an optimal solution of problem (3.3). Because the constraints appeared in (3.3) are all linear inequalities, they meet usual constraint qualifications and the first two of them are obviously inactive. So according to the Proposition 2.3, it is certain that there exist $\lambda_i \geq 0$, $i \in I'(x^{(k)})$, such that

$$0 \in \partial_d R(x^{(k)},d) \big|_{d=0} + \sum_{i \in I'(x^{(k)})} \lambda_i a_i \tag{4.2}$$

where a_i is the i-th row vector of A.

Using Proposition 2.4 and Proposition 2.5, but instead of x with d and taking values at $d = 0$, we get

$$\partial_d R(x^{(k)},d) \big|_{d=0} = \nabla F(x^{(k)}) + \sum_{i=1}^{h} u_i \omega_i \nabla G_i(x^{(k)})$$

and where, for $i = 1,\dots,k$,

$$u_i = \begin{cases} \operatorname{sgn} G_i(x^{(k)}) & \text{if } G_i(x^{(k)}) \neq 0 \\ [-1,1] & \text{otherwise} \end{cases}$$

and for $i = k+1,\dots,h$,

$$u_i = \begin{cases} [1 + \operatorname{sgn} G_i(x^{(k)})]/2 & \text{if } G_i(x^{(k)}) \neq 0 \\ [0,1] & \text{otherwise} \end{cases}$$

The above result indicates that

$$\partial_d R(x^{(k)},d)\ \Big|_{d=0} = \partial P(x^{(k)}) \tag{4.3}$$

and hence from (4.2) and (4.3) we know

$$x^{(k)} \in \Omega^*.$$

In order to prove the convergence when $\{x^{(k)}\}$ is an infinite sequence, we need several lemmas at first.

Lemma 4.2 For any x, we have

$$P(x + d) = R(x,d) + o(\|d_1\|) \tag{4.4}$$

and if the nonlinear part x_1 of variable x is restricted in a bounded closed set $\Lambda \subset R^n$, then the $o(\|d_1\|)$ is independent of x on set $\Lambda x R^m$.

Proof

From the calculus, we know that if function f is continuously differentiable in an open set containing Λ, then for any $x_1 \in \Lambda$, we have

$$f(x_1 + d_1) = f(x_1) + \nabla f(x_1)d_1 + o(\|d_1\|)$$

and the remainder $o(\|d_1\|)$ of it is independent of x_1 on Λ.

For the linear part of the objective function in problem (1.1), Px_2, obviously we have

$$P(x_2 + d_2) = Px_2 + Pd_2\ ,\quad \forall x_2, \forall d_2$$

So,

$$F(x + d) = F(x) + \nabla F(x)d + o(\|d_1\|).$$

And for the terms related to $G_i(x + d)$ in exact penalty function $P(x + d)$, we have similar results and so we get (4.4).

<u>Lemma 4.3</u> If the functions f and g_i are continuously differentiable, then for any $\hat{x} \in \Omega' \setminus \Omega^*$, any $\rho_1 < 1$, there must exist $\hat{\alpha} = \alpha(\hat{x}) > 0$ and $\hat{\varepsilon} = \varepsilon(\hat{x}) > 0$, such that when $x \in N(\hat{x},\hat{\varepsilon}) \cap \Omega'$ and $0 < \alpha \leq \hat{\alpha}$, the optimal solution $\tilde{d} = d(\alpha,x)$ of the subproblem

$$\begin{aligned} &\min \quad R(x,d) \\ \text{s.t.} \quad &\|d_1\| \leq \alpha \\ &\|d_2\| \leq M \\ &A(x+d) \leq c \end{aligned} \tag{4.5}$$

must satisfy the inequality

$$\sigma(\alpha,x) = \frac{P(x) - P(x + \tilde{d})}{P(x) - R(x,\tilde{d})} \geq \rho_1 \tag{4.6}$$

where $N(\hat{x},\hat{\varepsilon}) \triangleq \{x : \|x - \hat{x}\| < \hat{\varepsilon}\}$

<u>Proof</u>

Since $\hat{x} \notin \Omega^*$, that means

$$0 \notin D(\hat{x}). \tag{4.7}$$

Because $\partial P(\hat{x})$ is a compact convex set and $\{\Sigma\lambda_i a_i \mid i \in I'(\hat{x}),\ \lambda_i \geq 0\}$ is a finitely generated convex cone, and it is a closed cone ([15], Th. 19.1). As the sum set of these two sets we know $D(\hat{x})$ is also a closed convex set [17]. According to the separation theorem for convex sets, (4.7) implies that there exists a vector t, $\|t\| = 1$, and a number $\tau > 0$, such that for all $s \in D(\hat{x})$

$$st \leq -\tau < 0 \tag{4.8}$$

Especially for $\forall s \in \partial P(\hat{x}) \subset D(\hat{x})$ the above inequality holds. So we have

$$\max_{s \in \partial p(\hat{x})} st \leq -\tau < 0$$

By virtue of (2.10), the above inequality means

$$P^\circ(\hat{x},t) = \limsup_{\substack{x\to\hat{x}\\ \alpha\downarrow 0}} \frac{P(x+\alpha t) - P(x)}{\alpha} \leq -\tau$$

Hence there exist positive numbers $\bar{\varepsilon}$ and $\bar{\alpha}$, such that when $x \in N(\hat{x},\bar{\varepsilon})$, $0 < \alpha \leq \bar{\alpha}$, we have

$$\frac{P(x+\alpha t) - P(x)}{\alpha} < -\frac{\tau}{2} \tag{4.9}$$

On the other hand, (4.8) also implies that

$$a_i t \leq 0 \ , \ \forall i \in I'(\hat{x}) \tag{4.10}$$

Otherwise, if $a_i t > 0$ for some $i \in I'(\hat{x})$, then taking $s' = \lambda_i a_i + d \in D(\hat{x})$ $(d \in \partial P(\hat{x}))$ and letting the corresponding $\lambda_i \to \infty$, we shall get a result which contradicts (4.8).

For every $i \notin I'(\hat{x})$, if we set

$$-\hat{\delta}_i \triangleq a_i\hat{x} - c_i < 0$$

then there must exist an ε_i such that when $x \in N(\hat{x},\varepsilon_i)$, we have

$$a_i x - c_i < -\frac{\hat{\delta}_i}{2}$$

By selecting $\alpha_i > 0$ such that when $0 < \alpha \leq \alpha_i$

$$\alpha a_i t \leq \frac{\hat{\delta}_i}{2}$$

now for $\forall i \notin I'(\hat{x})$, if $x \in N(\hat{x},\varepsilon_i)$ and $0 < \alpha \leq \alpha_i$, we have

$$a_i(x+\alpha t) - c_i \leq (a_i x - c_i) + \alpha a_i t < 0 \tag{4.11}$$

Now we take

$$\hat{\varepsilon} \triangleq \min_{i\notin I'(\hat{x})} \{\bar{\varepsilon},\varepsilon_i\} \ , \ \tilde{\alpha} \triangleq \min_{i\notin I'(\hat{x})} \{\bar{\alpha},\alpha_i\}$$

and for any $x \in N(\hat{x},\hat{\varepsilon}) \cap \Omega'$, any $0 < \alpha \leq \tilde{\alpha}$, because of (4.10) and (4.11), we obtain

$$a_i(x + \alpha t) \leq c_i \quad , \quad i = 1,\dots,l \tag{4.12}$$

and at the same time, inequality (4.9) holds, too.

For such x and α let us consider the optimal solution $\tilde{d}(\alpha,x)$ and corresponding optimal value $q(x,\tilde{d})$ of the subproblem (4.5).

From Lemma 4.2, it is clear that

$$\begin{aligned} \Delta P(x,d) &= P(x) - P(x + \tilde{d}) \\ &= P(x) - R(x,\tilde{d}) + o(\|\tilde{d}_1\|) \\ &= \Delta R(x,\tilde{d}) + o(\alpha) \end{aligned} \tag{4.13}$$

(Since $\|\tilde{d}_1\| \leq \alpha$) and so,

$$\sigma(\alpha,x) = \frac{\Delta P(x,\tilde{d})}{\Delta R(x,\tilde{d})} = 1 + \frac{o(\alpha)}{\Delta R(x,\tilde{d})} \tag{4.14}$$

Now we are going to get an asymptotic estimation for $\Delta R(x,\tilde{d})$.

Let $\hat{d} \,\underline{\Delta}\, \alpha t$, then according to expresssion (4.12), we know that $\hat{d}$ is a feasible solution of problem (4.5), so we have

$$R(x,\tilde{d}) \leq R(x,\hat{d})$$

and

$$\Delta R(x,\tilde{d}) \geq \Delta R(x,\hat{d}) \tag{4.15}$$

By (4.9),

$$\Delta P(x,\hat{d}) = P(x) - P(x + \hat{d}) > \frac{\tau}{2}\alpha$$

Similarly to (4.13) ,we can have

$$\Delta R(x,\hat{d}) = \Delta P(x,\hat{d}) + o(\alpha) > \frac{\tau}{2}\alpha + o(\alpha)$$

(Since $\|\hat{d}_1\| = \|\alpha t_1\| \leq \alpha\|t\| = \alpha$.) Substituting above expression for the right side of (4.15) ,we get

$$\Delta R(x,\tilde{d}) > \frac{\tau}{2}\alpha + o(\alpha)$$

and hence the right side of (4.14) tends to 1 when α approaches to zero, and which means that for every $\rho_1 < 1$, there exists $\hat{\alpha} = \alpha(\hat{x})$ $(<\tilde{\alpha})$, such that when $0 < \alpha \leq \hat{\alpha}$, for every $x \in N(\hat{x},\hat{\varepsilon}) \cap \Omega'$, we have (4.6).

<u>Lemma 4.4</u> If bounded closed set $R \subset \Omega' \setminus \Omega^*$, then for every $\rho_1 \in (0,1)$, there must exist $\alpha^* > 0$, such that for any $\alpha \in (0,\alpha^*]$ and arbitrary $x \in R$, the quotient $\sigma(\alpha,x)$ determined by (4.6) satisfies the condition:

$$\sigma(\alpha,x) \geq \rho_1 \tag{4.16}$$

<u>Proof</u>

According to Lemma 4.3, for every $\hat{x} \in R$, there are $\varepsilon(\hat{x}) > 0$ and $\alpha(\hat{x}) > 0$ such that

$$\begin{cases} x \in N(\hat{x},\varepsilon(\hat{x})) \cap R \\ 0 < \alpha \leq \alpha(\hat{x}) \end{cases} \Longrightarrow \sigma(\alpha,x) \geq \rho_1$$

All these $N(\hat{x},\varepsilon(\hat{x}))$ cover the bounded and closed set R, so there must exist finite number of neighborhoods $N(\hat{x}_1,\hat{\varepsilon}_1), \ldots, N(\hat{x}_j,\hat{\varepsilon}_j)$, such that

$$R \subset \bigcup_{i=1}^{j} N(\hat{x}_i,\hat{\varepsilon}_i).$$

From the finite number of $\hat{\alpha}_i = \alpha(\hat{x}_i)$, which correspond to those neighborhoods, we take their minimum value

$$\alpha^* = \min_{i=1,\ldots,j} \{\hat{\alpha}_i\}$$

and it is clear that when $0 < \alpha \leq \alpha^*$, (4.16) holds for all $x \in R$.

<u>Theorem 4.5</u> If the constrained level set of exact penalty function $P(x)$ about the initial point $x^{(1)}$ of the algorithm,

$$L(x^{(1)}) \triangleq \{x : x \in \Omega' \text{ and } P(x) \leq P(x^{(1)})\}$$

is bounded, and the sequence of iterative points $\{x^{(k)}\}$ obtained from SLP algorithm is infinite, then $\{x^{(k)}\}$ must have limit points, and furthermore, at least one of them is a constrained stationary point of $P(x)$ in Ω'.

<u>Proof</u>

According to the rule of the algorithm, we have $P(x^{(k+1)}) \leq P(x^{(k)})$, and hence $\{x^{(k)}\} \subset L(x^{(1)})$, which means $\{x^{(k)}\}$ must have at least one limit point because of the boundness of the level set $L(x^{(1)})$.

We now prove the second conclusion by contradiction. Suppose none of the limit points of $\{x^{(k)}\}$ were constrained stationary points, then for every $x \in L(x^{(1)}) \cap \Omega^*$, there exists an $\varepsilon = \varepsilon(x) > 0$, such that neighborhood $N(x,\varepsilon)$ contains no point of $\{x^{(k)}\}$. We denote the union of all these neighborhoods as Σ:

$$\Sigma = \bigcup_{x \in L(x^{(1)}) \cap \Omega^*} N(x,\varepsilon(x))$$

then

$$\{x^{(k)}\} \subset R \triangleq L(x^{(1)}) \setminus \Sigma \subset L(x^{(1)}) \setminus \Omega^*.$$

Since R defined in this way is obviously a closed bounded set, according to Lemma 4.4 and the algorithm, if $\alpha_k \leq \alpha^*$, then we must have $\sigma(\alpha_k, x^{(k)}) \geq \rho_1$ and α_{k+1} will no longer be less than α_k: $\alpha_{k+1} \geq \alpha_k$, and it means that

$$\alpha_k \geq \min \{\alpha_1, \frac{\alpha^*}{\mu}\} \triangleq \beta \quad (k=1,2,\ldots) \qquad (4.17)$$

From (4.17) we know that there must exist a subsequence $\{x^{(k_i)}\}$ of $\{x^{(k)}\}$, satisfying

$$\sigma(\alpha_{k_i}, x^{(k_i)}) \geq \rho_1 \ . \qquad (4.18)$$

Otherwise, if there were a k_0, such that when $k \geq k_0$,

$$\sigma(\alpha_k, x^{(k)}) < \rho_1$$

then by the rule of SLP algorithm,

$$\alpha_{k+1} = \frac{\alpha_k}{\mu} \quad , \quad (k \geq k_0)$$

which leads to $\alpha_k \to 0$, contradicting (4.17).

In the bounded sequence $\{x^{(k_i)}\}$, there must contain a convergent subsequence of $\{x^{(k)}\}$. For simplicity, without loss of generality we assume

$$x^{(k_i)} \to x^* \in R \qquad (i \to \infty)$$

From (4.18) and algorithm, we know that

$$x^{(k_i+1)} = x^{(k_i)} + \tilde{d}^{(k_i)}$$

and

$$P(x^{(k_i+1)}) - P(x^{(k_i)}) \geq \rho_1 \, \Delta R(\alpha_{k_i}, x^{(k_i)}) \ , \ i=1,2,\ldots \qquad (4.19)$$

Now we estimate $\Delta R(\alpha_{k_i}, x^{(k_i)})$. Consider the following supplementary problem

$$\min \ R(x^*, d)$$

$$\text{s.t.} \quad \|d_1\| \leq \frac{\beta}{2}$$

$$\|d_2\| \leq M - \frac{\beta}{2} \qquad (4.20)$$

$$A(x^* + d) \leq C$$

and assume the optimal solution of it is d^* (because M can be taken as large as one likes, and β, if necessary, can be decreased from the value defined in

(4.19), we can always regard $M - \frac{\beta}{2}$ as a positive number). Since $x^* \notin \Omega^*$, by Theorem 4.1, we have

$$\Delta R(\tfrac{\beta}{2}, x^*) = P(x^*) - R(x^*, d^*) \triangleq \theta > 0$$

Set $\hat{x} = x^* + d^*$, and in virtue of continuity, there is an $i_o > 0$ such that when $i \geqslant i_o$,

$$P(x^{(k_i)}) - R(x^{(k_i)}, \hat{x} - x^{(k_i)}) > \frac{\theta}{2} \tag{4.21}$$

and

$$\|x^{(k_i)} - x^*\| < \frac{\beta}{2}$$

Now, from the above inequality and (4.17), when $i \geqslant i_o$, we have

$$\|(x - x^{(k_i)})_1\| \leqslant \|(\hat{x} - x^*)_1\| + \|(x^* - x^{(k_i)})_1\|$$

$$< \beta \leqslant \alpha_{k_i}$$

Similarly, we know that

$$\|(\hat{x} - x^{(k_i)})_2\| < M$$

and clearly,

$$A[x^{(k_i)} + (\hat{x} - x^{(k_i)})] = A(x^* + d^*) \leqslant c$$

The above three inequalities indicate that vector

$$\hat{d}^{(k_i)} \triangleq \hat{x} - x^{(k_i)}$$

is a feasible point of the k_i-th subproblem (4.5) ($i \geqslant i_o$). Hence,

$$R(x^{(k_i)}, \tilde{d}^{(k_i)}) \leqslant R(x^{(k_i)}, \hat{x} - x^{(k_i)})$$

and

$$\Delta R(\alpha_{k_i}, x^{(k_i)}) \geq P(x^{(k_i)}) - R(x^{(k_i)}, \hat{x} - x^{(k_i)})$$

$$> \frac{\theta}{2} \qquad \text{(See (4.23))}$$

combining above inequality and (4.19), we obtain

$$P(x^{(k_i+1)}) - P(x^{(k_i)}) > \frac{\rho_1\theta}{2} \qquad (i \geq i_o)$$

which contradicts the convergence of the following series of positive terms:

$$\sum_{i=1}^{\infty} [P(x^{(k_i+1)}) - P(x^{(k_i)})] \leq \sum_{i=1}^{\infty} [P(x^{(k_{i+1})}) - P(x^{(k_i)})]$$

$$= P(x^*) - P(x^{(k_1)}) < +\infty .$$

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Since Griffith and Stewart firstly proposed as successive linear programming method for solving general nonlinear programming problems, such methods have been widely used in practice because of their ease of implementation and their ability to deal with large scale problems. However, neither the original version, nor a more recent one contain convergence proofs possibly because of non-robustness of their algorithms. Using exact penalty functions and Levenberg-Marquardtlike steps, an improved

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20. ABSTRACT (continued)

algorithm has been recently devised. In this paper, we give this modified SLP method a theoretical analysis and convergence proof, and thereby provide a sound basis for it.