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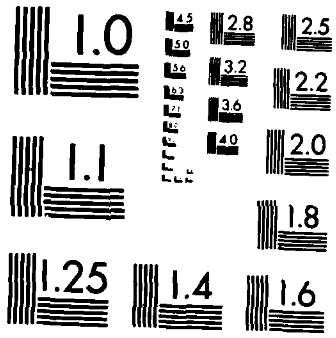

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ABSTRACT

This ~~note~~ discusses some simple properties of digital triangles, whose vertices are lattice points and whose side lengths are measured using city block distance. Many of the familiar theorems of geometry break down in this situation.

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1. Introduction

Digital geometry is the study of geometric properties of sets of lattice points in the plane, or equivalently, subsets of digital pictures. The discrete nature of the space gives rise to substantial differences between digital geometry and ordinary (continuous) plane geometry. A few areas of digital geometry have been extensively studied. One of them is digital topology, which deals with such concepts as adjacency, connectedness, and surroundedness, and with general properties of digital arcs and curves; for introductions to this topic see [1,2]. Another is digital convexity, which is concerned with characterizing digital subsets that are digitizations of convex objects, and digital arcs that are digitizations of straight line segments; for a review of this area see [3].

One of the factors that gives rise to basic differences between digital and Euclidean geometry is the nature of the metrics used to measure distances. In digital geometry, it is very natural to measure distance using integer-valued metrics, known as city block distance and chessboard distance, since simple iterative algorithms exist for computing these metrics, e.g., for computing the distances from all points of a digital picture to a given subset, whereas in the case of Euclidean distance this can only be done approximately. The city block distance between two digital points (x,y),(u,v) is defined as

$$d_4((x,y),(u,v)) = |x-u|+|y-v|$$

while the chessboard distance is defined as

$$d_8((x,y),(u,v)) = \max(|x-u|,|y-v|)$$

It is easily verified that the city block distance is the length of a shortest path from (x,y) to (u,v) in which only horizontal and vertical moves (from lattice point to neighboring lattice point) are allowed, while in the chessboard distance diagonal moves to neighbors are also allowed. (The notations "d_4" and "d_8" are used because a point has four city-block neighbors (at distance 1 from it) and eight chessboard neighbors.) An early study of discrete metrics on digital pictures can be found in [4]. A delightful treatment of city block (or "taxicab") geometry, emphasizing loci defined in terms of distances (perpendicular bisectors, conic sections, etc.) is presented in [5].

This note discusses some simple properties of digital triangles, i.e., triangles whose vertices are lattice points and whose side lengths are measured in terms of city block distance. (Analogous results would be obtained if chessboard distance were used; their investigation is left as an exercise to the reader.) The subject is complicated by the fact that city block distance is direction-sensitive; it is equal to Euclidean distance in the horizontal or vertical direction, but exceeds it by a factor of $\sqrt{2}$ in the diagonal directions. In addition, many of the familiar theorems about triangles break down in the city block case.

2. Distance and angle measurement

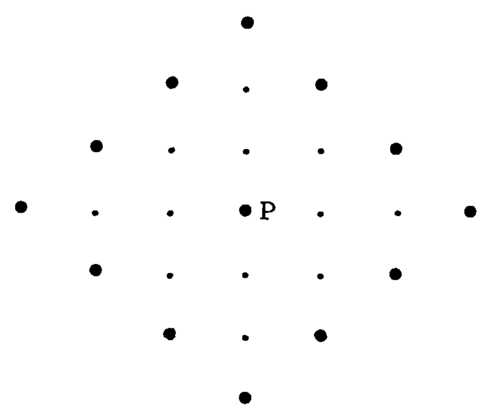
We define the length of a side of a digital triangle as the city block distance between the endpoints. Since city block distance is a metric [4], the lengths of the three sides of a triangle satisfy the triangle equality. The proof of this fact is given here for the sake of completeness.

Proposition 1. The length of any side of a triangle does not exceed the sum of the lengths of the other two sides.

Proof: Since the lengths do not change when the triangle is shifted, we may assume without loss of generality that the vertex opposite the side in question is (0,0). Let the other two vertices be (a,b) and (c,d); then we must prove that $|c-a|+|b-d| \leq (|a|+|b|)+(|c|+|d|)$, which is an immediate consequence of the fact that $|x+y| \leq |x|+|y|$ for all x,y. ||

Note that in Euclidean geometry, the inequality is strict unless the triangle is degenerate (its vertices are collinear), but in city block geometry it is non-strict whenever two of the vertices lie in opposite quadrants relative to the third (in the proof: whenever a and c, b and d both have opposite signs).

As regards angle measurement, one possibility would be to simply use the Euclidean definition. (Note, incidentally, that since the vertices are lattice points, the slopes of the sides must be rational numbers, so that all angles are rational multiples of 2π and have rational tangents.) Alternatively, we can give a city-block definition based on the concept that the angle between two radii of a circle is proportional to the area of the circular sector defined by the radii. Now in city block geometry, a "circle" (i.e., the locus of points at a given distance from a fixed point) turns out to be a square with diagonally oriented sides; for example, the points at distance 3 from p are



If the "radius" is r, the square has (Euclidean) side length $r\sqrt{2}$. We shall define the angle between two rays emanating from P as the fraction of the area of a diagonally oriented square (centered at P) defined by these rays. Strictly speaking, we should measure the area by counting lattice points; but as the square becomes large, the result tends toward the fraction of the Euclidean area of a Euclidean square defined by the rays regarded as Euclidean lines, so for simplicity we shall use Euclidean area measurement.

When we use the city block definition of angle, a given (Euclidean) angle can vary in value depending on its orientation. Nevertheless, we have

<u>Proposition 2</u>. All right angles are equal.

<u>Proof</u>: When we divide a diagonally oriented square into four quadrants by drawing two perpendicular lines through its center, no matter how the lines are oriented, the quadrants are all congruent; hence each of them has $\frac{1}{4}$ of the total area. Thus a (Euclidean) right angle in any orientation has (city-block) angular measure $\frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$. ||

On the other hand, angles that are not multiples of 90° have values that vary with their orientation. To see this, consider a 45° angle emanating from the center of a diagonally oriented square. When one side of the angle is vertical or horizontal and the other is diagonal, it evidently cuts off $\frac{1}{8}$ of the area of the square, so that its city block angular measure is $\frac{\pi}{4}$. On the other hand, when the bisector of the angle is vertical, it cuts off a larger area (fraction $\frac{3}{8}\tan\frac{\pi}{8}$), and when the bisector is diagonal, the area is smaller (fraction $\frac{1}{4}\tan\frac{\pi}{8}$); since $\tan\frac{\pi}{8}=\sqrt{2}-1$, the angular measures in the two cases are thus $\frac{3\pi}{4}(\sqrt{2}-1)$ and $\frac{\pi}{2}(\sqrt{2}-1)$, respectively. In spite of this, we have

Proposition 3. The sum of the angles of a triangle is π.

Proof: No matter how the triangle is oriented, if we put one vertex of the triangle at the center of a diagonally oriented square and draw a line through it parallel to the opposite side of the triangle, we have three angles at the vertex whose sides are the same as or parallel to those of the angles in the triangle, and whose sum cuts off half the area of the square. ||

3. Some theorems that fail to hold

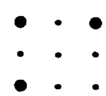
The results obtained in Section 2 are not typical; most of the familiar properties of Euclidean triangles break down for city block triangles (even when angles that differ in measure because they differ in orientation are not involved). In this section we present examples that illustrate the extent of this breakdown.

3.1 Congruence

Example 1 [5]. Two triangles that have two pairs of corresponding sides equal, and all corresponding angles equal, are not necessarily congruent.

Illustration: Consider the triangles

(a)

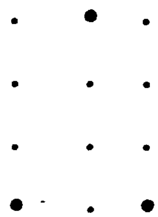


(b)

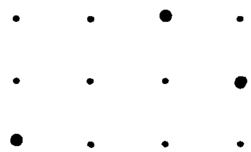
In (a), the angles are $\frac{\pi}{2}$, $\frac{\pi}{4}$, and $\frac{\pi}{4}$, and the sides all have length 2; in (b), the angles are the same, and the sides adjacent to the right angle have length 2, but the third side has length 4. Note that this counterexample is not a result of differences in angle measure due to orientation.

Example 2. Two triangles that have all three pairs of corresponding sides equal are not necessarily congruent.

Illustration: Consider the triangles



(a)



(b)

In both (a) and (b), the side lengths are 4, 4, and 2, but the angles all differ.

3.2 Equilaterality

Example 3. The base angles of an isoscles triangle are not necessarily equal; in fact, an equilateral triangle is not necessarily equiangular.

Illustration: The triangle of Example 1a is equilateral, but its angles are $\frac{\pi}{2}$, $\frac{\pi}{4}$, and $\frac{\pi}{4}$.

3.3 Altitudes and area

The altitude of a triangle, as measured from a given vertex, is just the city block distance from that vertex to the opposite side. (On the city block distance from a point to a line see [5], p. 34 ff.) In Euclidean geometry, the altitude times the length of the opposite side is twice the area, and is the same for all three vertices. In the city block case, on the other hand, we have

Example 4. The altitudes to the equal sides of an isoscles triangle are not necessarily equal; in fact, the altitudes of an equilateral triangle are not necessarily equal.

Illustration: In Example 1a, the altitudes are 1, 2, and 2.

Corollary. The "areas" measured by multiplying each altitude by the length of its opposite side are not necessarily equal.

Example 5. If two altitudes of a triangle are equal, the opposite sides are not necessarily equal; in fact, if all three altitudes are equal, the triangle is not necessarily equilateral.

Illustration: In Example 2a, all three altitudes are 2.

3.4 Right triangles

The Pythagorean theorem does not hold for city block right triangles. In fact, if the legs of a right triangle are horizontal and vertical, the hypotenuse is equal to the sum of the legs; if the legs are diagonal, the hypotenuse is equal to the larger leg. This also makes it difficult to define the trigonometric functions; for example, in Examples 1a-b the acute angles are all $\frac{\pi}{4}$, but (a) gives $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=1$, while (b) gives $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{2}$. It does not seem possible to develop a "digital trigonometry" based on city block distance.

4. Concluding remarks

Many of the standard concepts of Euclidean geometry are not even defined in the digital case; for example, a line segment may have no midpoint (e.g., if it has an even number of points), and two line segments may cross each other without intersecting. For this reason we have not discussed concepts such as medians or perpendicular bisectors, or lengths of angle bisectors; for more on these concepts see [5]. As we have seen, even concepts that are well defined do not have their expected properties. Digital triangles are simple objects, but if we use city block distance to measure their sides (and angles), most of the Euclidean properties no longer hold.

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