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DYNAMICS OF A PROJECTILE IN A CONCENTRIC FLEXIBLE TUBE

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Each of the four main sections in this report deals with the motion, the forces, and the moments experienced by a gun tube with a straight axis and an offset breech, and with certain simple types of support. Section I presents a development of the kinematics of a projectile in a concentric moving flexible tube. Section 2 treats the forces and moments that act on a projectile in terms of kinematic variables of the tube. Section 3 treats the motion of a tapered cantilever tube that is actuated by the projectile and by prescribed motion at

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the breech. In Section 4, the problem of a rigid curve tube that is hinged at the breech so that it can swing sideways is treated.

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GENERAL INTRODUCTION

Because of the high acceleration of a projectile in a gun tube, and because of the high precision that is sought, gun dynamics requires consideration of secondary effects that are negligible in ordinary structural analysis. Slight deflections of the tube are aggravated by centrifugal action of the projectile. Vertical oscillations of the tube are coupled with sidewise oscillations because of gyroscopic action of the projectile. These effects are studied in this report.

Section 1 is a development of the kinematics of a projectile in a concentric moving flexible tube. Aside from the assumptions that there is no balloting, that the central axis of the bore is inextensional, and that plane cross sections of the tube remain plane, unstrained, and normal to the deflected axis of the bore, the basic theory in Sections 1 and 2 is exact. Insofar as kinematics is concerned, dynamic unbalance of the projectile, the Bourdon effect, axial inertia of the tube, and axial friction of the projectile are irrelevant.

Although deflections and twist of the tube are small, the engineering approximations in beam theory are precluded. The Kirchhoff-Clebsch theory of large deflections of thin rods is a natural starting point for the analysis. This theory is presented in A. E. H. Love's "Mathematical Theory of Elasticity," but in a manner that Love acknowledges "is not without difficulty." Gibbs' vector analysis helps to clarify arguments of this kind. It is used in this report. However, with a view to computer programming, the results are expanded in scalar notation.

The Kirchhoff-Clebsch theory treats statics of a deformed rod. For problems of gun dynamics, it must be extended to admit time-dependent deflections and twist of the tube. A part of Section 1 deals with this problem. On the basis of kinematics of the tube, kinematic relations for the projectile are derived. Section 2 treats the forces and moments that act on a projectile in terms of kinematic variables of the tube.

Section 3 treats the motion of a tapered cantilever tube that is actuated by the projectile and by prescribed motion at the breech. Bending of the tube caused by the Bourdon effect, axial inertia of the tube, and axial friction of the projectile are neglected, but the theory can be generalized to include these effects. The deflections and twist of the tube are represented as series of flexural and torsional modes of a uniform cantilever beam. The coefficients in these series are generalized coordinates of the tube. They are functions of time that are determined by Lagrange's equations. The theory exhibits gyroscopic action of the projectile that causes coupling between vertical and horizontal oscillations of the tube. However, quantitative studies of this phenomenon must await numerical computer analysis. A much simpler problem that displays the same characteristics is treated in Section 4, namely, a rigid curved tube that is hinged at the breech so that it can swing sideways.

SECTION 1
KINEMATICS OF A PROJECTILE IN A CONCENTRIC FLEXIBLE TUBE

1.1 INTRODUCTION

Expressions for the velocity, the acceleration, the angular velocity, the kinetic energy, and the virtual work of a rigid, spinning projectile in a concentric flexible tube are derived in this section. Approximations are deferred to the last article (Art. 1.14). The general theory is exact, aside from the assumptions that there is no balloting, that the axis of the bore is inextensional, and that plane cross sections of the tube remain plane, unstrained, and normal to the central axis of the bore when the tube is bent and twisted. The axis of symmetry of the projectile is assumed to be tangent to the deflected axis of the bore at the location of the centroid of the projectile.

Gibbs' vector analysis is used. A brief development of vector analysis that suffices for the present applications is presented in Appendix C of Ref. 1. A scalar formulation of the theory is presented in Arts. 1.11, 1.12, and 1.13.

1.2 NOTATIONS

A bar over a letter denotes a vector.

A caret over a letter denotes a unit vector.

A dot over a letter denotes the derivative with respect to time t.

d/dt denotes the total (or substantial) derivative with respect to time (Eq. (1.20)).

An asterisk * denotes the deformed state of the tube.

Subscripts s and t denote partial derivatives with respect to arc length s and time t, respectively.

For the next several notations, refer to Figures 1, 2, and 3.

C is the undeformed axis of the bore.

C* is the deflected axis of the bore.

[1] "*Dynamics of Rigid Guns with Straight Tubes,*" *BLM-AMC Final Report DAAK-11-80-C-0039-Task 2, Army Research and Development Command, BRL, Aberdeen Proving Ground, Maryland.*

$\hat{i}'$, $\hat{j}'$, $\hat{k}'$ are orthogonal unit vectors, in the directions of the principal normal, the binormal, and the tangent to curve C, respectively.

s is arc length on curves C and C*.

$\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$ are unit vectors that coincide with lines in the deformed tube which initially have directions $\hat{i}'$, $\hat{j}'$, $\hat{k}'$.

$\hat{t}$ is the unit tangent vector of curve C*; ($\hat{t} = \hat{k}^*$), Figure 1.

$\hat{n}$ is the principal unit normal of curve C*, Figure 3.

$\hat{b}$ is the unit binormal of curve C*, Figure 3.

$1/R$ is the curvature of curve C*.

$1/\Sigma$ is the tortuosity (usually called "torsion" in differential geometry) of curve C*.

τ_0 is the tortuosity of Curve C.

$\overline{w}$ is a vector, such that the vector triad ($\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$) issuing from a point P* on C* is brought parallel to the orthogonal triad ($\hat{i}^* + \delta\hat{i}^*$, $\hat{j}^* + \delta\hat{j}^*$, $\hat{k}^* + \delta\hat{k}^*$) at a neighboring point P_1^* on C* by the infinitesimal rotation $\overline{w}\,ds$, where ds is the distance $P^*P^*_1$, Figure 2.

κ, κ', τ are components of $\overline{w}$, defined by $\overline{w} = \hat{i}^*\kappa + \hat{j}^*\kappa' + \hat{k}^*\tau$.

α is the angle between vectors $\hat{j}^*$ and $\hat{n}$ at a point P* on curve C*, Figure 3.

$\overline{r}(s,t)$ is the radius vector from a designated fixed origin to a point P* on curve C*; ($\partial\overline{r}/\partial s = \hat{t}$), Figure 1.

$\overline{\omega}(s,t)$ is the angular velocity of a cross section of the tube.

ω_1, ω_2, ω_3 are components of $\overline{\omega}$, defined by[#]

$$\overline{\omega} = \hat{i}^*\omega_1 + \hat{j}^*\omega_2 + \hat{k}^*\omega_3 \ , \ \omega_3 = \omega_a \ .$$

$\xi(t)$ is the value of s locating the centroid of the projectile at time t.

$\overline{V} = d\overline{r}/dt$ is the velocity of the centroid of the projectile (Eq. (1.19)).

$\dot{\xi}(t)$ is the speed of the projectile relative to the tube.

$\overline{a}$ is the acceleration of the centroid of the projectile.

ω is the angular velocity (spin) of the projectile relative to the tube. (It is not the magnitude of vector $\overline{\omega}$.)

[#] *The subscript "a", denoting "axial" is used rather than the subscript "t", denoting "tangential" to avoid confusion with the subscript "t" denoting the partial derivative with respect to time.*

$\bar{\Omega}$ is the absolute angular velocity of the projectile (Eq. (1.30)).

Ω_n, Ω_b, Ω_a are components of $\bar{\Omega}$, defined by Eq. (1.31).

m is the mass of the projectile.

i_1 is the moment of inertia of the projectile about a transverse axis through its center of mass.

i_3 is the moment of inertia of the projectile about its longitudinal axis.

T_p is the kinetic energy of the projectile.

F is the axial frictional force on the projectile.

P_1 is the pressure on the base of the projectile.

P_2 is the resisting pressure ahead of the projectile.

A is the cross-sectional area of the bore.

M_r is the rifling torque.

$$\chi = \int_0^t \omega dt$$

δW is the virtual work of the forces associated with the projectile.

C** is a varied curve, lying infinitesimally close to curve C*.

ω_n, ω_b, ω_a are components of $\bar{\omega}$, defined by

$$\bar{\omega} = \hat{n}\omega_n + \hat{b}\omega_b + \hat{t}\omega_a \; ; \; \omega_a = \omega_3 \; .$$

V_n, V_b, V_a are the components of the velocity $\bar{V}$ on the principal normal, the binormal, and the tangent of curve C*.

a_n, a_b, a_a are the components of the acceleration $\bar{a}$ on the principal normal, the binormal, and the tangent of curve C*.

$\psi(s,t)$ is the angular displacement of a cross section of the tube in its plane (see Eq. (1.44)).

x, y, z are rectangular coordinates attached to a Galilean reference frame.

$\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along the axes x, y, z.

$\bar{F}$ denotes the net force on the projectile.

F_n, F_b, F_a denote the components of $\bar{F}$ in the $\hat{n}$, $\hat{b}$, $\hat{t}$ directions.

$\bar{H}$ denotes the angular momentum of the projectile with respect to its center of mass.

H_n, H_b, H_a denote the components of $\bar{H}$ in the $\hat{n}$, $\hat{b}$, $\hat{t}$ directions.

$\overline{M}$ denotes the moment about the center of mass of the projectile of all the forces that act on the projectile.

M_n, M_b, M_a denote the components of $\overline{M}$ in the $\hat{n}$, $\hat{b}$, $\hat{t}$ directions.

J denotes the mass moment of inertia about the hinge line of a rigid tube and attached breech.

M_g is the gyroscopic couple that the projectile exerts on a rigid curved immovable tube (Figure 7).

$$\lambda = \omega_a + \frac{\dot{\xi}}{\Sigma} .$$

u(t), v(t) are the x and y components of displacement of the tube at the breech (Eq. (3.13)).

$\theta(t)$, $\phi(t)$, $\zeta(t)$ are the x, y, and z components of rotation of the tube at the breech. Also, θ is the angle of the tangent to the center line of a rigid curved tube (Figs. 6, 7, and 8).

ρ is the mass density of the tube.

I(s) is the moment of inertia of a cross section of the tube about a diameter.

S(s) is the cross-sectional area of the tube, excluding the bore.

g is the acceleration of gravity.

ℓ is the length of the tube.

E is Young's modulus.

G is the shear modulus.

α_n, β_n are constants defined by Eqs. (3.7) and (3.8), and by Table 1.

$f_n(s)$ is the n'th natural bending mode of a uniform elastic cantilever beam (Eq. (3.6)).

$\psi_n(s)$ is the n'th natural torsional mode of a uniform straight tube that is fixed at one end and free at the other (Eq. (3.11)).

$X_n(t)$, $Y_n(t)$, $Z_n(t)$ are coefficients in the modal expansions of the deflection and twist of the tube (Eq. (3.13)). They are generalized coordinates of the tube.

L = T - U is the Lagrangian function.

1.3 THE KIRCHHOFF-CLEBSCH THEORY OF THIN FLEXIBLE RODS

The Kirchhoff-Clebsch theory of bending and twisting of thin rods is presented in References 2, 3, and 4. In the present work, the rod is taken to be the tube of a gun. The undeformed axis C of the bore is an arbitrary curve. The principal normal, the binormal, and the tangent to curve C are orthogonal unit vectors, denoted respectively by $(\hat{i}', \hat{j}', \hat{k}')$. If the undeformed tube is straight, $(\hat{i}', \hat{j}', \hat{k}')$ may be any constant orthogonal unit vectors such that $\hat{k}'$ coincides with line C. When the tube is deformed, curve C passes into another curve C*. Nearby points, P and P_1, on C pass into points P* and $P*_1$ on C*. The lengths PP_1 and $P*P*_1$ are both taken to be ds; i.e., extensionality of the axis of the bore is neglected. Love and Basset (Refs. 2 and 3) also assumed inextensionality.

Lines in the tube, issuing from point P on C in the directions $\hat{i}'$, $\hat{j}'$, $\hat{k}'$ pass into lines in the directions $\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$ (Figure 1). Vector $\hat{k}^*$ is the unit tanger of C*. Since plane cross sections of the tube are assumed to remain plane, unstr ned, and normal to the centroidal axis C*, the vectors $\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$ are mutually perpendicular. Love called straight lines coinciding with vectors $\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$ the "principal torsion-flexure axes" of the rod.[#] Although $\hat{k}^* = \hat{t}$, where $\hat{}$'s the unit tangent of curve C*, the vectors $\hat{i}^*$ and $\hat{j}^*$ generally do coincide with the principal normal $\hat{n}$ and the binormal $\hat{b}$ of curve C*.

The vectors $\hat{t}$, $\hat{n}$, $\hat{b}$ are to be a right-handed system; i.e., the thumb, the forefinger, and .e finger of the right hand can be simultaneously pointed ections $\hat{t}$, $\hat{n}$, and $\hat{b}$. Consequently, with the right-hand con .e vector product,

$$\hat{b} = \hat{t} \times \hat{n}\ , \quad \hat{n} = \hat{b} \times \hat{t}\ , \quad \hat{t} = \hat{n} \times \hat{b} \qquad (1.1)$$

With the approximation ds = ds*, Frenet's formulas in the differential

[2] A. E. H. Love, The Mathematical Theory of Elasticity, 4th ., Cambridge University Press, 1934, Chap. XVIII, pp. 381-398.

[3] A. B. Bas et, "On the Deformation of Thin Elastic Wires," American Journal of Mathematics, Vol. 17, 1895, pp. 28 7.

[4] "The Kirchhoff-Clebsch Theory of Thi astic Rods," Interim Report BLM-AMC-81-2, Contract No. DAAK-11-80-C-0039, Army Research and Development Command, BRL, Aberdeen Proving Ground, Maryland.

[#] Love considered a rod of arbitrary cross section, and he took $\hat{i}'$ and $\hat{j}'$ to be along the principal axes of inertia of the cross section.

geometry of curves (Ref. 5) are

$$\frac{\partial\hat{t}}{\partial s} = \frac{\hat{n}}{R}, \quad \frac{\partial\hat{n}}{\partial s} = \frac{\hat{b}}{\Sigma} - \frac{\hat{t}}{R}, \quad \frac{\partial\hat{b}}{\partial s} = -\frac{\hat{n}}{\Sigma} \tag{1.2}$$

where $1/R$ is the curvature and $1/\Sigma$ is the tortuosity of curve C*. Partial derivatives are indicated in Eq. (1.2), because $\hat{t}$, $\hat{n}$, and $\hat{b}$ generally depend on time t as well as on the arc length s. However, in this article, t is a passive parameter, since a single configuration of the tube is considered.

If the point P* on C* moves to a neighboring point P^*_1 on C* (Figure 2), while the curve C* is unchanged, the vectors $\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$ receive increments $\delta\hat{i}^*$, $\delta\hat{j}^*$, $\delta\hat{k}^*$, such that $\hat{i}^* + \delta\hat{i}^*$, $\hat{j}^* + \delta\hat{j}^*$, and $\hat{k}^* + \delta\hat{k}^*$ are mutually orthogonal unit vectors. This transformation could be accomplished by a translation and a rigid-body rotation of the sytem ($\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$). It is shown in the kinematics of a rigid body that an infinitesimal angular displacement is a vector quantity (Ref. 6). Consequently, there is an infinitesimal vector $\bar{w}$ ds which represents the rotation that brings the system ($\hat{i}^*$, $\hat{j}^*$, $\hat{k}^*$) into parallelism with the system ($\hat{i}^* + \delta\hat{i}^*$, $\hat{j}^* + \delta\hat{j}^*$, $\hat{k}^* + \delta\hat{k}^*$), where ds is the distance $P^*P^*_1$ (Figure 2). Love defined scalars κ, κ', τ by

$$\bar{w} = \hat{i}^*\kappa + \hat{j}^*\kappa' + \hat{k}^*\tau \tag{1.3}$$

The orthogonal projection of curve C* onto the j*k* plane (or the k*i* plane) is a plane curve with curvature κ (or κ') at point P*. The tortuosity of the undeformed axis C is denoted by τ_0. It is the rate of rotation of the osculating plane of curve C with respect to arc length s. The deformational twist of the tube per unit length is accordingly $\tau - \tau_0$.

If a rigid body undergoes an infinitesimal angular displacement $\bar{w}$ ds about a fixed axis, and if $\bar{\rho}$ is a radius vector from a point on that axis to a particle Q of the body, the displacement of Q is $\bar{w} \times \bar{\rho}$ ds. Letting $\bar{\rho}$ stand successively for $\hat{i}^*$, $\hat{j}^*$, and $\hat{k}^*$, and noting Eq. (1.3), we get

---

[5] *D. Struik, Differential Geometry, Addison-Wesley Press, Cambridge, Mass., 1950.*

[6] *E. T. Whittaker, Analytical Dynamics, 4th ed., Dover Publications, New York, 1944, Chap. 1, Art. 8.*

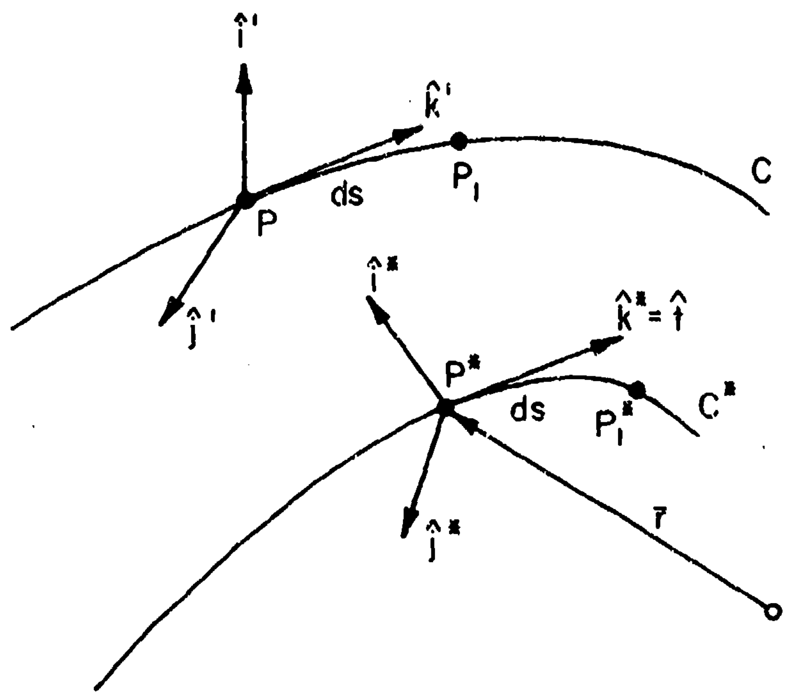

Figure 1. Notations

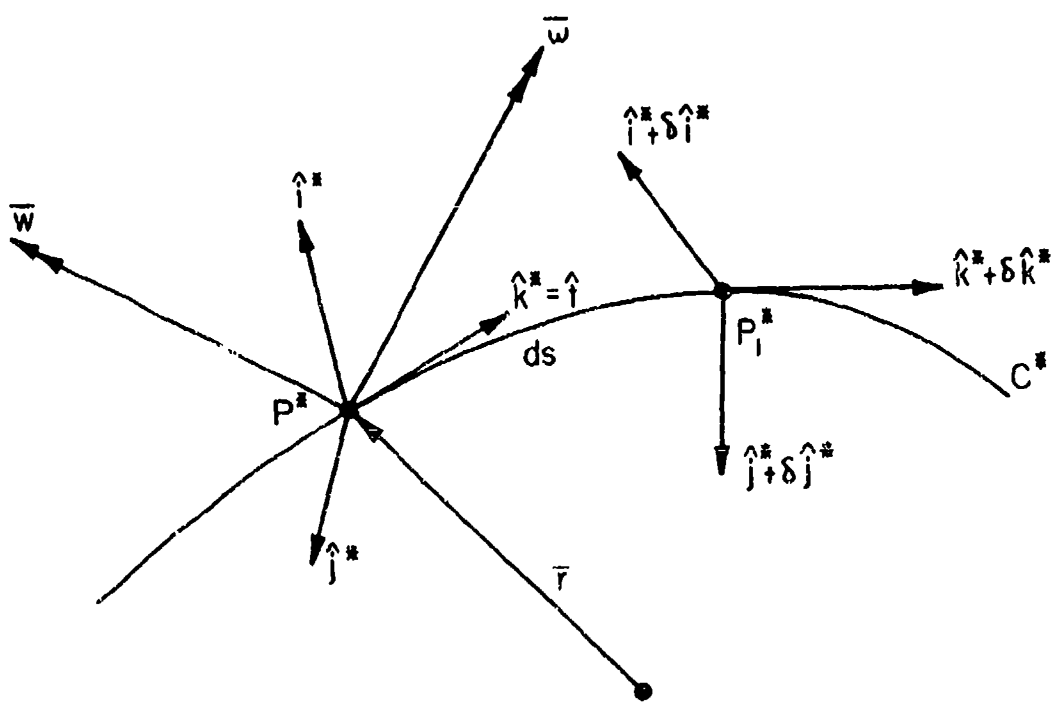

Figure 2. Illustration of Vectors

$$\frac{\partial \hat{i}^*}{\partial s} = \bar{w} \times \hat{i}^* = \hat{j}^*\tau - \hat{k}^*\kappa'$$

$$\frac{\partial \hat{j}^*}{\partial s} = \bar{w} \times \hat{j}^* = \hat{k}^*\kappa - \hat{i}^*\tau$$

$$\frac{\partial \hat{k}^*}{\partial s} = \bar{w} \times \hat{k}^* = \hat{i}^*\kappa' - \hat{j}^*\kappa \tag{1.4}$$

Figure 3 shows the vectors $\hat{i}^*$, $\hat{j}^*$, $\hat{n}$, $\hat{b}$ in the normal plane of curve C*. It follows from the geometry of the figure that

$$\hat{i}^* = \hat{n} \sin\alpha - \hat{b} \cos\alpha \ , \quad \hat{j}^* = \hat{n} \cos\alpha + \hat{b} \sin\alpha \tag{1.5}$$

Conversely,

$$\hat{n} = \hat{i}^* \sin\alpha + \hat{j}^* \cos\alpha, \quad \hat{b} = -\hat{i}^* \cos\alpha + \hat{j}^* \sin\alpha \tag{1.6}$$

Differentiation of the second of Eqs. (1.6) yields, with the help of Eq. (1.4),

$$\frac{\partial \hat{b}}{\partial s} = \hat{i}^*\left(\frac{\partial\alpha}{\partial s} - \tau\right)\sin\alpha + \hat{j}^*\left(\frac{\partial\alpha}{\partial s} - \tau\right)\cos\alpha + \hat{k}^*(\kappa' \cos\alpha + \kappa \sin\alpha) \tag{1.7}$$

Also, Eqs. (1.2) and (1.6) yield

$$\frac{\partial \hat{b}}{\partial s} = -\frac{1}{\Sigma}(\hat{i}^* \sin\alpha + \hat{j}^* \cos\alpha) \tag{1.8}$$

Equations (1.7) and (1.8) yield

$$\tau = \frac{\partial\alpha}{\partial s} + \frac{1}{\Sigma} \tag{1.9}$$

and

$$\tan\alpha = -\frac{\kappa'}{\kappa} \tag{1.10}$$

Also, since $\hat{k}^* = \hat{t}$, Eqs. (1.2) and (1.4) yield

$$\frac{\partial \hat{t}}{\partial s} = \frac{\hat{n}}{R} = \hat{i}^*\kappa' - \hat{j}^*\kappa \tag{1.11}$$

Equations (1.6) and (1.11) yield

$$\kappa = -\frac{\cos\alpha}{R}, \quad \kappa' = \frac{\sin\alpha}{R} \tag{1.12}$$

At a point where a concentrated couple is introduced into the tube, classical beam theory indicates that the curvature $1/R$ is discontinuous. Discontinuities also may appear in $1/\Sigma$, $\hat{n}$, and $\hat{b}$. The preceding theory is limited to points at which the pertinent geometric quantities are continuous.

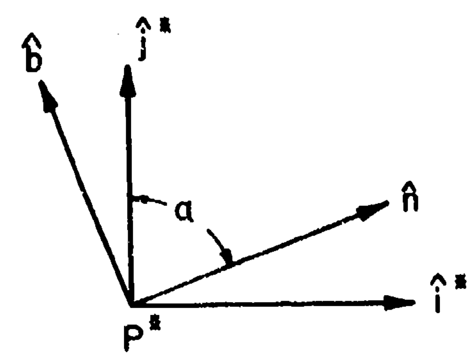

Figure 3. Vectors $\hat{i}^*$, $\hat{j}^*$, $\hat{n}$, $\hat{b}$ in a Normal Plane of Curve C^*

1.4 KINEMATICS OF THE TUBE

The curve C^* representing the deflected axis of the tube at time t is defined by the vector equation $\bar{r} = \bar{r}(s,t)$, in which s is arc length on the axis of the tube and $\bar{r}$ is a radius vector from a fixed origin to the point s on the curve. The vector $\partial\bar{r}/\partial s$ is the unit tangent $\hat{t}$ of the axis of the tube. The vector $\partial\bar{r}/\partial t$ is the velocity of the center of the cross section of the tube at point s.

The triad of unit vectors $(\hat{i}^*, \hat{j}^*, \hat{k}^*)$ may be conceived to be glued to a cross section of the tube, with its origin at the center of the cross section. As the tube deflects and twists, that cross section and the attached triad $(\hat{i}^*, \hat{j}^*, \hat{k}^*)$ rotate with angular velocity $\bar{\omega}(s,t)$. The vector $\bar{\omega}$ is resolved into components $(\omega_1, \omega_2, \omega_3)$ in the directions $(\hat{i}^*, \hat{j}^*, \hat{k}^*)$; i.e.,

$$\bar{\omega} = \hat{i}^*\omega_1 + \hat{j}^*\omega_2 + \hat{k}^*\omega_3 \qquad (1.13)$$

Since $(\hat{i}^*, \hat{j}^*, \hat{k}^*)$ are unit vectors attached to the tube, the following kinematic relations exist:

$$\frac{\partial\hat{i}^*}{\partial t} = \bar{\omega} \times \hat{i}^* = \hat{j}^*\omega_3 - \hat{k}^*\omega_2$$

$$\frac{\partial \hat{j}^*}{\partial t} = \bar{\omega} \times \hat{j}^* = \hat{k}^*\omega_1 - \hat{i}^*\omega_3$$

$$\frac{\partial \hat{k}^*}{\partial t} = \bar{\omega} \times \hat{k}^* = \hat{i}^*\omega_2 - \hat{j}^*\omega_1 \tag{1.14}$$

Equation (1.14) is similar to Eq. (1.4), but the physical interpretation is different. Since $\hat{k}^* = \hat{t}$, Eq. (1.14) yields

$$\omega_1 = -\hat{j}^* \cdot \frac{\partial \hat{t}}{\partial t}, \quad \omega_2 = \hat{i}^* \cdot \frac{\partial \hat{t}}{\partial t} \tag{1.15}$$

The components of $\bar{\omega}$ in the directions $\hat{n}$ and $\hat{b}$ are $\omega_n = \bar{\omega} \cdot \hat{n}$ and $\omega_b = \bar{\omega} \cdot \hat{b}$. By Eqs. (1.6) and (1.13),

$$\omega_n = \omega_1 \sin \alpha + \omega_2 \cos \alpha \tag{1.16}$$

By Eqs. (1.6), (1.14), and (1.16), $\omega_n = -\hat{b} \cdot \partial\hat{t}/\partial t$. Similarly, ω_b is derived. Accordingly,

$$\omega_n = -\hat{b} \cdot \frac{\partial \hat{t}}{\partial t}, \quad \omega_b = \hat{n} \cdot \frac{\partial \hat{t}}{\partial t} \tag{1.17}$$

Subsequently, ω_3 is designated as ω_a, in which the subscript "a" indicates the axial component. The subscript "t" is reserved to indicate the partial derivative with respect to time.

1.5 VELOCITY OF THE PROJECTILE

Balloting is not considered. At any instant, the axis of the projectile is assumed to be tangent to the axis of the deflected tube at the point where the centroid of the projectile lies.

The location of the centroid of the projectile at time t is specified by $s = \xi(t)$, where $\xi(t)$ is regarded as a given function. The absolute trajectory of the centroid of the projectile accordingly is represented by

$$\bar{r} = \bar{r}[\xi(t), t] \tag{1.18}$$

The absolute velocity of the centroid of the projectile is $\bar{V} = d\bar{r}/dt$, where d/dt denotes the total derivative. By the chain rule of partial differentiation,

$$\overline{V} = \frac{d\overline{r}}{dt} = \frac{\partial \overline{r}}{\partial s}\frac{d\xi}{dt} + \frac{\partial \overline{r}}{\partial t}$$

Since $\partial\overline{r}/\partial s = \hat{t}$, this yields

$$\overline{V} = \dot{\xi}\hat{t} + \frac{\partial \overline{r}}{\partial t}\bigg|^{s=\xi} = \frac{d\overline{r}}{dt} \tag{1.19}$$

Equation (1.19) signifies that the absolute velocity of the centroid of the projectile is the vector sum of the velocity relative to the contiguous part of the tube and the velocity of the center of the cross section of the tube at which the centroid of the projectile lies. This conclusion could have been anticipated from general kinematical theory.

The distinction between the total derivative and the partial derivative applies to any function of s and t; i.e.,

$$\frac{d(-)}{dt} = \frac{\partial(-)}{\partial t} + \dot{\xi}\,\frac{\partial(-)}{\partial s}\bigg|^{s=\xi} \tag{1.20}$$

where (-) denotes any function of s and t.

In view of Eq. (1.19), the square of the speed of the projectile is[#]

$$V^2 = \dot{\xi}^2 + 2\dot{\xi}\hat{t} \cdot \frac{\partial \overline{r}}{\partial t} + \left(\frac{\partial \overline{r}}{\partial t}\right)^2 \tag{1.21}$$

By Eq. (1.19), the component of velocity of the projectile tangent to the deflected axis of the tube is

$$V_a = \overline{V} \cdot \hat{t} = \dot{\xi} + \hat{t} \cdot \frac{\partial \overline{r}}{\partial t}\bigg|^{s=\xi} \tag{1.22}$$

The components of velocity of the projectile in the directions of the principal normal and the binormal to the deflected axis of the tube are

$$\overline{V} \cdot \hat{n} = \hat{n} \cdot \frac{\partial \overline{r}}{\partial t}\bigg|^{s=\xi}, \quad \overline{V} \cdot \hat{b} = \hat{b} \cdot \frac{\partial \overline{r}}{\partial t}\bigg|^{s=\xi} \tag{1.23}$$

[#] *By definition,* $\overline{A}^2 = \overline{A} \cdot \overline{A} = A^2$ *in which* $\overline{A}$ *is any vector.*

Equation (1.23) signifies that any component of velocity of the projectile normal to the axis of the tube is the same as the corresponding normal component of velocity of the contiguous tube.

1.6 ACCELERATION OF THE PROJECTILE

The acceleration of the centroid of the projectile is

$$\bar{a} = \frac{d\bar{V}}{dt} = \dot{\xi}\,\frac{\partial \bar{V}}{\partial s} + \frac{\partial \bar{V}}{\partial t}$$

By Eqs. (1.2) and (1.19),

$$\frac{\partial \bar{V}}{\partial s} = \dot{\xi}\,\frac{\hat{n}}{R} + \frac{\partial \hat{t}}{\partial t}\Bigg|^{s=\xi}$$

$$\frac{\partial \bar{V}}{\partial t} = \ddot{\xi}\hat{t} + \dot{\xi}\,\frac{\partial \hat{t}}{\partial t} + \frac{\partial^2 \bar{r}}{\partial t^2}\Bigg|^{s=\xi}$$

Consequently,

$$\bar{a} = \hat{n}\,\frac{\dot{\xi}^2}{R} + 2\dot{\xi}\,\frac{\partial \hat{t}}{\partial t} + \ddot{\xi}\hat{t} + \frac{\partial^2 \bar{r}}{\partial t^2}\Bigg|^{s=\xi} \qquad (1.24)$$

or, since $\partial\hat{t}/\partial t = \bar{\omega} \times \hat{t}$ (see Eq. (1.14)),

$$\bar{a} = \hat{n}\,\frac{\dot{\xi}^2}{R} + 2\dot{\xi}\bar{\omega} \times \hat{t} + \ddot{\xi}\hat{t} + \frac{\partial^2 \bar{r}}{\partial t^2}\Bigg|^{s=\xi} \qquad (1.25)$$

The terms on the right side of Eq. (1.25) can be identified as follows: (a) The centripetal acceleration of the centroid of the projectile relative to the momentary form of the axis of the tube. (b) The Coriolis acceleration of the centroid of the projectile. (c) The tangential acceleration of the centroid of the projectile relative to the adjacent part of the tube. (d) The absolute acceleration of the center of the cross section of the tube at which the centroid of the projectile lies. This decomposition of the acceleration could have been anticipated by general kinematical theory.

By Eq. (1.25), the components a_a, a_n, a_b of the absolute acceleration $\bar{a}$ in the directions of the tangent, the principal normal, and the binormal of the deflected axis of the tube are

$$a_a = \hat{t} \cdot \bar{a} = \ddot{\xi} + \hat{t} \cdot \left.\frac{\partial^2 \bar{r}}{\partial t^2}\right|^{s=\xi}$$

$$a_n = \hat{n} \cdot \bar{a} = \frac{\dot{\xi}^2}{R} + 2\dot{\xi}\bar{\omega} \cdot \hat{b} + \hat{n} \cdot \left.\frac{\partial^2 \bar{r}}{\partial t^2}\right|^{s=\xi}$$

$$a_b = \hat{b} \cdot \bar{a} = -2\dot{\xi}\bar{\omega} \cdot \hat{n} + \hat{b} \cdot \left.\frac{\partial^2 \bar{r}}{\partial t^2}\right|^{s=\xi} \tag{1.26}$$

since

$$\hat{n} \cdot \bar{\omega} \times \hat{t} = \bar{\omega} \cdot \hat{t} \times \hat{n} = \bar{\omega} \cdot \hat{b}$$

and

$$\hat{b} \cdot \bar{\omega} \times \hat{t} = \bar{\omega} \cdot \hat{t} \times \hat{b} = -\bar{\omega} \cdot \hat{n}$$

1.7 ANGULAR VELOCITY OF THE PROJECTILE

In the time interval dt, the cross section of the tube at which the centroid of the projectile lies undergoes the angular displacement $\bar{\omega}$ dt. In the same time interval, the projectile advances the distance $\dot{\xi}$ dt relative to the tube. The angular displacement of the projectile due to the latter displacement is $\bar{W}\dot{\xi}dt$. The spin of the projectile relative to the tube is $\omega\hat{t}$, where ω is the magnitude of the spin. (ω is not the magnitude of the vector $\bar{\omega}$.) Consequently, the relative angular displacement of the projectile during dt because of the spin is $\omega\hat{t}dt$. The absolute angular displacement of the projectile during dt is the vector sum of these components, namely,

$$\bar{\omega}dt + \dot{\xi}\bar{W}\,dt + \omega\hat{t}\,dt$$

Therefore, the absolute angular velocity of the projectile is

$$\bar{\Omega} = \bar{\omega} + \dot{\xi}\bar{W} + \omega\hat{t} \tag{1.27}$$

Equations (1.3), (1.13), and (1.27) yield

$$\overline{\Omega} = \hat{i}^*(\omega_1 + \kappa\dot{\xi}) + \hat{j}^*(\omega_2 + \kappa'\dot{\xi}) + \hat{k}^*(\omega_3 + \tau\dot{\xi} + \omega) \tag{1.28}$$

Now ω_1 and ω_2 can be eliminated from Eq. (1.28) by Eq. (1.15); κ and κ' can be eliminated by Eq. (1.12). Thus, Eq. (1.28) yields

$$\overline{\Omega} = -\hat{i}^*\hat{j}^* \cdot \frac{\partial \hat{t}}{\partial t} + \hat{j}^*\hat{i}^* \cdot \frac{\partial \hat{t}}{\partial t} + \frac{\dot{\xi}}{R}(-\hat{i}^* \cos\alpha + \hat{j}^* \sin\alpha)$$

$$+ \hat{k}^*(\omega + \omega_3 + \dot{\xi}\tau) \tag{1.29}$$

By the vector-triple-product theorem (Ref. 1, Appendix C):

$$-\hat{i}^*\hat{j}^* \cdot \frac{\partial \hat{t}}{\partial t} + \hat{j}^*\hat{i}^* \cdot \frac{\partial \hat{t}}{\partial t} = -\frac{\partial \hat{t}}{\partial t} \times (\hat{i}^* \times \hat{j}^*)$$

or, since $\hat{i}^* \times \hat{j}^* = \hat{k}^* = \hat{t}$,

$$-\hat{i}^*\hat{j}^* \cdot \frac{\partial \hat{t}}{\partial t} + \hat{j}^*\hat{i}^* \cdot \frac{\partial \hat{t}}{\partial t} = \hat{t} \times \frac{\partial \hat{t}}{\partial t}$$

Also, by Eq. (1.6), $-\hat{i}^* \cos\alpha + \hat{j}^* \sin\alpha = \hat{b}$. Consequently, Eq. (1.29) yields

$$\overline{\Omega} = \hat{t} \times \frac{\partial \hat{t}}{\partial t} + \hat{b}\,\frac{\dot{\xi}}{R} + \hat{t}(\omega + \omega_a + \dot{\xi}\tau) \tag{1.30}$$

The components of $\overline{\Omega}$ in the $\hat{n}$, $\hat{b}$, and $\hat{t}$ directions are

$$\Omega_n = \overline{\Omega} \cdot \hat{n}\ , \quad \Omega_b = \overline{\Omega} \cdot \hat{b}\ , \quad \Omega_a = \overline{\Omega} \cdot \hat{t} \tag{1.31}$$

Since the terms in the scalar triple product can be permuted cyclically,

$$\hat{n} \cdot \hat{t} \times \frac{\partial \hat{t}}{\partial t} = \frac{\partial \hat{t}}{\partial t} \cdot \hat{n} \times \hat{t} = -\hat{b} \cdot \frac{\partial \hat{t}}{\partial t}$$

$$\hat{b} \cdot \hat{t} \times \frac{\partial \hat{t}}{\partial t} = \frac{\partial \hat{t}}{\partial t} \cdot \hat{b} \times \hat{t} = \hat{n} \cdot \frac{\partial \hat{t}}{\partial t} \tag{1.32}$$

Equations (1.30), (1.31), and (1.32) yield

$$\Omega_n = -\hat{b} \cdot \frac{\partial \hat{t}}{\partial t}\ , \quad \Omega_b = \hat{n} \cdot \frac{\partial \hat{t}}{\partial t} + \frac{\dot{\xi}}{R}\ , \quad \Omega_a = \omega + \omega_a + \dot{\xi}\tau \tag{1.33}$$

Equations (1.17) and (1.33) yield

$$\Omega_n = \omega_n \ , \quad \Omega_b = \omega_b + \frac{\dot{\xi}}{R} \ , \quad \Omega_a = \omega + \omega_a + \dot{\xi}\tau \Big|^{s=\xi} \tag{1.34}$$

1.8 KINETIC ENERGY OF A BALANCED PROJECTILE

In the theory of kinematics, any reference frame is admissible. For kinetics, however, a Galilean reference frame must be introduced. Consequently, the vector $\bar{r}$ is now considered to be specified with respect to a Galilean reference frame; e.g., the earth.

The kinetic energy of translation of the projectile is

$$T_t = \frac{1}{2} m \, \bar{V}^2$$

where m is the mass of the projectile. Consequently, by Eq. (1.21),

$$T_t = \frac{1}{2} m[\dot{\xi}^2 + 2\dot{\xi}\hat{t} \cdot \frac{\partial \bar{r}}{\partial t} + (\frac{\partial \bar{r}}{\partial t})^2] = \frac{1}{2} m(\frac{d\bar{r}}{dt})^2 \tag{1.35}$$

The kinetic energy of rotation of the projectile is

$$T_r = \frac{1}{2} i_1(\Omega_n^{\ 2} + \Omega_b^{\ 2}) + \frac{1}{2} i_3 \, \Omega_a^{\ 2} \tag{1.36}$$

where i_1 is the moment of inertia of the projectile about a transverse axis through its center of mass and i_3 is the moment of inertia of the projectile about its longitudinal axis. The total kinetic energy of the projectile is $T_p = T_t + T_r$. Consequently, by Eqs. (1.33), (1.35), and (1.36)

$$\begin{aligned} T_p = \frac{1}{2} m[\dot{\xi}^2 + 2\dot{\xi}\hat{t} \cdot \frac{\partial \bar{r}}{\partial t} + (\frac{\partial \bar{r}}{\partial t})^2] + \frac{1}{2} i_1[(\hat{b} \cdot \frac{\partial \hat{t}}{\partial t})^2 + (\hat{n} \cdot \frac{\partial \hat{t}}{\partial t})^2 \\ + 2\frac{\dot{\xi}}{R}\hat{n} \cdot \frac{\partial \hat{t}}{\partial t} + \frac{\dot{\xi}^2}{R^2}] + \frac{1}{2} i_3(\omega + \omega_a + \dot{\xi}\tau)^2 \end{aligned} \tag{1.37}$$

Since $\hat{n} \cdot \partial\hat{t}/\partial t$, $\hat{b} \cdot \partial\hat{t}/\partial t$, and $\hat{t} \cdot \partial\hat{t}/\partial t$ are three orthogonal components of $\partial\hat{t}/\partial t$,

$$(\hat{b} \cdot \frac{\partial \hat{t}}{\partial t})^2 + (\hat{n} \cdot \frac{\partial \hat{t}}{\partial t})^2 + (\hat{t} \cdot \frac{\partial \hat{t}}{\partial t})^2 = (\frac{\partial \hat{t}}{\partial t})^2$$

Furthermore, since $\hat{t} \cdot \hat{t} = 1$, $\hat{t} \cdot \partial\hat{t}/\partial t = 0$. Consequently, Eq. (1.37) reduces to

$$T_p = \frac{1}{2} m[\dot{\xi}^2 + 2\dot{\xi}\hat{t} \cdot \frac{\partial\bar{r}}{\partial t} + (\frac{\partial\bar{r}}{\partial t})^2] + \frac{1}{2} i_1[(\frac{\partial\hat{t}}{\partial t})^2 + 2\frac{\dot{\xi}}{R}\hat{n} \cdot \frac{\partial\hat{t}}{\partial t} + \frac{\dot{\xi}^2}{R^2}]$$

$$+ \frac{1}{2} i_3(\omega + \omega_a + \dot{\xi}\tau)^2 \qquad (1.38)$$

Since $\partial\hat{t}/\partial s = \hat{n}/R$ and $1/R^2 = (\partial\hat{t}/\partial s)^2$, Eq. (1.38) may be expressed as follows:

$$T_p = \frac{1}{2} m[\dot{\xi}^2 + 2\dot{\xi}\hat{t} \cdot \frac{\partial\bar{r}}{\partial t} + (\frac{\partial\bar{r}}{\partial t})^2] + \frac{1}{2} i_1(\frac{\partial\hat{t}}{\partial t} + \dot{\xi}\frac{\partial\hat{t}}{\partial s})^2$$

$$+ \frac{1}{2} i_3(\omega + \omega_a + \dot{\xi}\tau)^2 \qquad (1.39)$$

or more concisely,

$$T_p = \frac{1}{2} m(\frac{d\bar{r}}{dt})^2 + \frac{1}{2} i_1(\frac{d\hat{t}}{dt})^2 + \frac{1}{2} i_3(\omega + \omega_a + \dot{\xi}\tau)^2 \qquad (1.40)$$

1.9 RELATION BETWEEN TWIST AND ANGULAR VELOCITY

Torsional vibrations of a perfectly straight tube exhibit a simple relationship between twist and angular velocity. During a time interval dt, the cross section of the tube at point s undergoes the angular displacement ω_a dt. During the same time interval, the cross section at point s + ds undergoes the angular displacement

$$(\omega_a + \frac{\partial\omega_a}{\partial s} ds)dt$$

Consequently, the increment of twist at point s during dt is $(\partial\omega_a/\partial s)dt$. Also, since the total twist at point s is $\tau - \tau_0$, the increment of twist at point s during dt is $(\partial\tau/\partial t)dt$. Therefore, <u>for a straight tube that executes torsional motion,</u>

$$\frac{\partial\tau}{\partial t} = \frac{\partial\omega_a}{\partial s} \qquad (a)$$

Equation (a) might be adopted as an approximation for a slightly curved tube, but an inconsistency arises. It is illustrated by a perfectly rigid curved tube. In this case, $\tau = \tau_0$, since there is no deformational twist of a rigid tube. Accordingly, Eq. (a) yields $\partial\omega_a/\partial s = 0$. Therefore, ω_a does not depend on s. Furthermore, since all particles of a rigid body have the same angular velocity at any instant, the vector angular velocity $\overline{\omega}$ of a rigid tube is independent of s. The axial component of angular velocity of a cross section is $\omega_a = \overline{\omega} \cdot \hat{t}$. For a curved tube, $\hat{t}$ obviously depends on s. Thus, we arrive at the contradictory conclusions that ω_a depends on s and ω_a does not depend on s. Consequently, in general, Eq. (a) must be rejected, although it is correct in certain special cases.

Since a theory of flexible tubes must be consistent with the theory of rigid-body displacements, the nature of the function $\omega_a(s,t)$ for a rigid curved tube is pertinent. In general, $\omega_a = \overline{\omega} \cdot \hat{t}$, and, for a rigid tube, $\overline{\omega} = \overline{\omega}(t)$. Therefore, for rigid tubes,

$$\frac{\partial\omega_a}{\partial s} = \overline{\omega} \cdot \frac{\partial\hat{t}}{\partial s}$$

In view of Eqs. (1.1) and (1.2), this yields

$$\frac{\partial\omega_a}{\partial s} = \frac{1}{R}\,\overline{\omega} \cdot \hat{n} = \frac{1}{R}\,\overline{\omega} \cdot \hat{b} \times \hat{t} = -\frac{1}{R}\,\overline{\omega} \cdot \quad \times \hat{b}$$

Since the vectors in the scalar triple product may be permuted cyclically, this yields

$$\frac{\partial\omega_a}{\partial s} = -\frac{1}{R}\,\hat{b} \cdot \overline{\omega} \times \hat{t}$$

Since $\overline{\omega} \times \hat{t} = \partial\hat{t}/\partial t$, this yields the following equation <u>for rigid tubes</u>:

$$\frac{\partial\omega_a}{\partial s} + \frac{1}{R}\,\hat{b} \cdot \frac{\partial\hat{t}}{\partial t} = 0 \tag{b}$$

The general theory of flexible curved tubes is now considered. An infinitesimal segment of such a tube, as viewed along the binormal of the axis C*, is shown in Figure 4. Infinitesimal angular displacements are resolved along the horizontal line L. To first-degree quantities, the component on line L of the rotation of the right-hand cross section relative to the left-hand cross section is

$$\frac{\partial \omega_a}{\partial s} ds\ dt + \omega_1\ d\theta\ dt$$

Accordingly, the increment of twist at section s during dt is

$$\left(\frac{\partial \omega_a}{\partial s} + \omega_1 \frac{\partial \theta}{\partial s}\right) dt$$

Since this is equal to $\partial \tau / \partial t\ dt$,

$$\frac{\partial \tau}{\partial t} = \frac{\partial \omega_a}{\partial s} + \omega_1 \frac{\partial \theta}{\partial s}$$

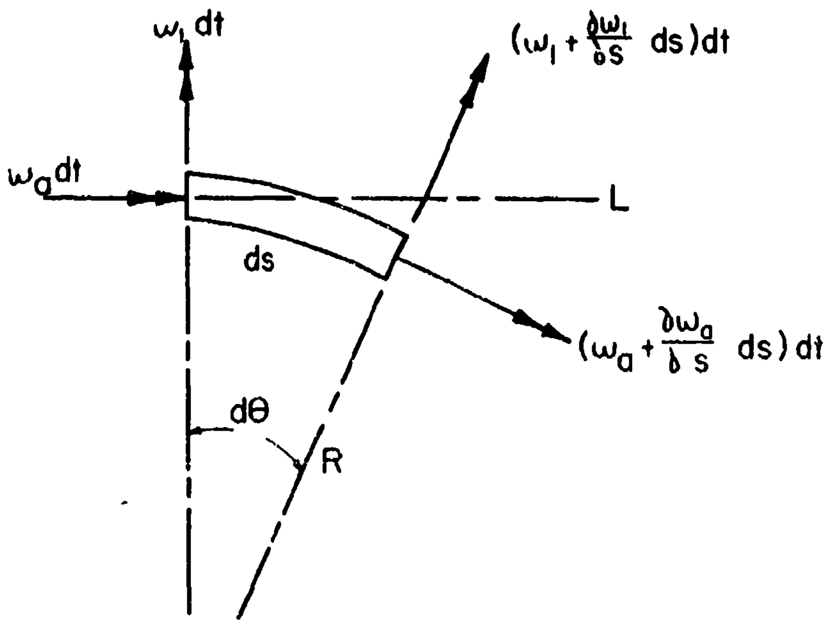

Figure 4. Resolution of Angular Velocities

With the sense of ω_1 indicated in Figure 4, $\omega_1 = -\bar{\omega} \cdot \hat{n}$. Also, $\partial\theta/\partial s = 1/R$. Therefore,

$$\frac{\partial\tau}{\partial t} = \frac{\partial\omega_a}{\partial s} - \frac{\bar{\omega} \cdot \hat{n}}{R}$$

Since $\hat{n} = \hat{b} \times \hat{t}$, this yields

$$\frac{\partial\tau}{\partial t} = \frac{\partial\omega_a}{\partial s} - \frac{1}{R}\bar{\omega} \cdot \hat{b} \times \hat{t} = \frac{\partial\omega_a}{\partial s} + \frac{1}{R}\hat{b} \cdot \bar{\omega} \times \hat{t}$$

Since $\bar{\omega} \times \hat{t} = \partial\hat{t}/\partial t$, this yields

$$\frac{\partial\tau}{\partial t} = \frac{\partial\omega_a}{\partial s} + \frac{1}{R}\hat{b} \cdot \frac{\partial\hat{t}}{\partial t} \tag{1.41}$$

For torsional motion of a straight tube, Eq. (1.41) yields Eq. (a) since $1/R = 0$. For motion of a rigid tube, it yields Eq. (b), since $\partial\tau/\partial t = 0$. If the axis of the tube bends in a plane Γ, vector $\hat{b}$ is perpendicular to Γ, and vector $\partial\hat{t}/\partial t$ lies in plane Γ. Consequently, $\hat{b} \cdot \partial\hat{t}/\partial t = 0$. Accordingly, if curves C and C* are constrained to lie in a plane, Eq. (1.41) reduces to Eq. (a). Usually, in studies of plane motion of gun tubes, τ and ω_a are taken to be zero.

Equation (1.41) can be expressed more simply, since

$$\frac{1}{R}\hat{b} \cdot \frac{\partial\hat{t}}{\partial t} = \frac{1}{R}\hat{b} \cdot \bar{\omega} \times \hat{t} = \frac{1}{R}\bar{\omega} \cdot \hat{t} \times \hat{b} = -\frac{1}{R}\bar{\omega} \cdot \hat{n} = -\bar{\omega} \cdot \frac{\partial\hat{t}}{\partial s}$$

Also,

$$\frac{\partial\omega_a}{\partial s} = \frac{\partial}{\partial s}(\bar{\omega} \cdot \hat{t}) = \bar{\omega} \cdot \frac{\partial\hat{t}}{\partial s} + \hat{t} \cdot \frac{\partial\bar{\omega}}{\partial s}$$

Therefore, Eq. (1.41) reduces to

$$\frac{\partial\tau}{\partial t} = \hat{t} \cdot \frac{\partial\bar{\omega}}{\partial s} \tag{1.42}$$

For a rigid tube, $\partial\tau/\partial t = 0$ and $\bar{\omega} = \bar{\omega}(t)$, so Eq. (1.42) is satisfied. Equation (1.42) means that $\partial\tau/\partial t$ is equal to the tangential component of $\partial\bar{\omega}/\partial s$.

In view of Eq. (1.17), Eq. (1.41) may be expressed as follows:

$$\frac{\partial\tau}{\partial t} = \frac{\partial\omega_a}{\partial s} - \frac{\omega_n}{R} \tag{1.43}$$

The general solution of Eq. (1.43) is easily derived. There is a function $\psi(s,t)$ such that $\tau = \partial\psi/\partial s$. The function ψ contains an arbitrary additive function of t. Also, there is a function $\lambda(s,t)$ such that $\omega_a = \lambda + \partial\psi/\partial t$. Equation (1.43) accordingly yields

$$\frac{\partial\lambda}{\partial s} = \frac{\omega_n}{R}$$

Consequently,

$$\lambda = \int_0^s \frac{\omega_n}{R}\, ds + q(t)$$

The arbitrary additive function of t in the function ψ may be chosen to cancel $q(t)$. Therefore, there is a function $\psi(s,t)$ such that

$$\tau = \frac{\partial\psi}{\partial s}, \quad \omega_a = \frac{\partial\psi}{\partial t} + \int_0^s \frac{\omega_n}{R}\, ds \tag{1.44}$$

The function $\psi(s,t)$ represents the angular displacement of a cross section of the tube in its plane.

Equations (1.34) and (1.44) yield

$$\Omega_a = \omega + \omega_a + \dot{\xi}\tau\Big|^{s=\xi} = \omega + \int_0^{\xi} \frac{\omega_n}{R}\, ds + \frac{d\psi}{dt}\Big|^{s=\xi} \tag{1.45}$$

Equation (1.45) may be substituted into the kinetic energy expression for the projectile (Eq. (1.39) or (1.40)).

1.10 VIRTUAL WORK ASSOCIATED WITH THE PROJECTILE

In order to apply Hamilton's principle or Lagrange's equations to problems of gun dynamics, we require the expression for the virtual work of all the non-inertial forces that operate. It is a linear expression in the infinitesimal virtual displacements. Only the part of the virtual work that explicitly involves the projectile is considered in this section.

Conceptually, the system receives an infinitesimal virtual displacement that generally does not coincide with the true course of the motion. The real motion of the system is imagined to be stopped while the virtual displacement is performed. The actual forces in the system are imagined to persist while the virtual displacement is executed. In the present case, the vector $\bar{r}(s,t)$, defining the curve C*, receives a virtual increment $\delta\bar{r}(s)$. This transforms curve C* into another curve C**, Figure 5. Since the center line of the tube is considered to be inextensional, the variation $\delta\bar{r}$ must conform with this constraint. Consequently, the mapping C* → C** must be performable by inextensional bending and twisting of curve C*. In addition to the virtual displacement $\delta\bar{r}$, the coordinate ξ of the projectile receives a virtual increment $\delta\xi$, and it also receives a virtual angular displacement about its longitudinal axis.

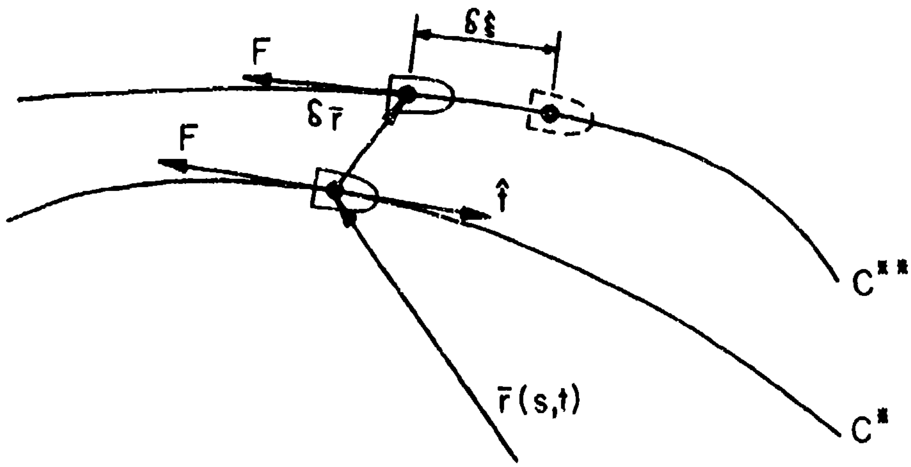

Figure 5. Virtual Displacements

The work that a force $\overline{F}$ performs on a particle during a time interval (t_0, t_1) is defined by

$$W = \int_{t_0}^{t_1} \overline{F} \cdot \overline{V}\, dt$$

where $\overline{V}$ is the velocity of the particle. Since $\overline{V}$ depends on the choice of the reference frame, so does W; i.e., work is a relative quantity. This conclusion is consistent with the fact that the work of all the forces that act on a system equals the increase of kinetic energy of the system, since kinetic energy also is a relative quantity, inasmuch as it depends on the velocities of the particles. In this analysis, work is calculated with respect to the Galilean reference frame to which the vector $\overline{r}$ is referred.

The contact forces that the projectile exerts on the bore are reacted by equal and opposite contact forces that the bore exerts on the projectile. Consequently, the normal components of all these forces perform no net work on the system. Accordingly, the gyroscopic couple of the projectile and its reaction perform no net work. An analogy is a person who lifts an object. The person performs work on the object, but gravity performs an equal amount of negative work. Together, the lifter and gravity perform no net work.

The projectile receives the virtual displacement $\delta\xi$ relative to the tube, and the contiguous tube receives the axial virtual displacement $\hat{t} \cdot \delta\overline{r}$. Consequently, the absolute axial component of the virtual displacement of the projectile is $\delta\xi + \hat{t} \cdot \delta\overline{r}$. The driving force on the base of the projectile is P_1A, where P_1 is the pressure of the gas and A is the cross sectional area of the bore. There is a resisting force F which results from axial friction and engraving of the rifling. Also, there is a resisting force P_2A resulting from pressure of the air ahead of the projectile. Consequently, the virtual work of forces that act on the projectile is

$$(P_1A - P_2A - F)(\delta\xi + \hat{t} \cdot \delta\overline{r})$$

The virtual work of friction and engraving on the tube is $F\hat{t} \cdot \delta\overline{r}$. The net virtual work δW_1, resulting from axial movement of the projectile, is the sum of these expressions. Consequently,

$$\delta W_1 = [(P_1 - P_2)A - F]\delta\xi + (P_1 - P_2)A\,\hat{t} \cdot \delta\bar{r}$$

in which $\delta\bar{r}$ is evaluated at the point $s = \xi$.

Also, there is a contribution δW_2 to the virtual work from the rifling torque. The virtual angular displacement of the projectile relative to the tube is $\delta\chi$, where $\dot{\chi} = \omega$. The angular displacement of the cross section of the tube at point $s = \xi$ is denoted by ψ. The absolute virtual angular displacement of the projectile is $\delta\chi + \delta\psi$. The rifling torque is denoted by M_r. The virtual work performed on the projectile by the rifling torque is $M_r(\delta\chi + \delta\psi)$. The virtual work performed on the tube by the rifling torque is $-M_r\,\delta\psi$. Consequently,

$$\delta W_2 = M_r(\delta\chi + \delta\psi) - M_r\,\delta\psi = M_r\,\delta\chi$$

Aside from effects of gravity, the virtual work explicitly related to the projectile is $\delta W = \delta W_1 + \delta W_2$. Hence,

$$\delta W = [(P_1 - P_2)A - F]\delta\xi + (P_1 - P_2)A\,\hat{t} \cdot \delta\bar{r} + M_r\,\delta\chi \tag{1.46}$$

in which relevant functions are evaluated at the point $s = \xi$. Additional contributions to the total virtual work of the system, coming from the action of gas pressure on the breech, effects of gravity, strain energy of the tube, and effects of the supporting structure, are not considered here.

We adopt the viewpoint that $\xi(t)$ and $\omega(t)$ are given functions. Then $\delta\xi = \delta\chi = 0$, and Eq. (1.46) is simplified accordingly. Also, $\delta\bar{r}$ is restricted by the condition of inextensionality of the tube. Since $\hat{t} \cdot \hat{t} = 1$, $\hat{t} \cdot \delta\hat{t} = 0$. Consequently, since $\hat{t} = \partial\bar{r}/\partial s$,

$$\hat{t} \cdot \delta\bar{r}_s = 0 \tag{1.47}$$

Equation (1.47) expresses the constraint on $\delta\bar{r}$.

If the tube is initially straight, $\hat{t}$ is approximately a constant vector, since the deflections are small. With this approximation, Eq. (1.47) yields

$$\frac{\partial}{\partial s}(\hat{t} \cdot \delta\bar{r}) = 0 \quad \text{or} \quad \hat{t} \cdot \delta\bar{r} = \text{constant}$$

If $\bar{r}$ is given at the breech (s = 0), $\delta\bar{r} = 0$ at the breech. Then, since $\hat{t} \cdot \delta\bar{r}$ = constant, $\hat{t} \cdot \delta\bar{r} = 0$ everywhere. Accordingly, for a gun with a straight tube, Eq. (1.46) yields $\delta W = 0$. The condition $\hat{t} \cdot \delta\bar{r} = 0$ means that $\delta\bar{r}$ must be perpendicular to the axis of the bore.

1.11 KINEMATIC AND GEOMETRIC RELATIONS IN SCALAR NOTATION

Adaptation of the foregoing theory to digital computer programming requires that the equations be expressed in scalar form. Rectangular coordinates (x, y, z) with corresponding unit vectors $(\hat{i}, \hat{j}, \hat{k})$ are attached to a Galilean reference frame, but, insofar as kinematics is concerned, the reference frame is arbitrary. The orientation of these axes with respect to the gun is not of immediate concern. The deflected axis C* of the tube at time t is defined by the equations, $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$. Since s is defined to be arc length on curve C*,

$$x_s^2 + y_s^2 + z_s^2 = 1 \tag{1.48}$$

in which the subscript denotes the partial derivative; e.g., $x_s = \partial x/\partial s$. The radius vector from the origin of the (x, y, z) coordinates to a point on curve C* is

$$\bar{r} = \hat{i}\, x + \hat{j}\, y + \hat{k}\, z \tag{1.49}$$

The unit tangent of curve C* is

$$\hat{t} = \frac{\partial \bar{r}}{\partial s} = \hat{i}\, x_s + \hat{j}\, y_s + \hat{k}\, z_s \tag{1.50}$$

By Eqs. (1.2) and (1.50),

$$\frac{\hat{n}}{R} = \hat{i}\, x_{ss} + \hat{j}\, y_{ss} + \hat{k}\, z_{ss} \tag{1.51}$$

Therefore,

$$\frac{1}{R} = \sqrt{x_{ss}^2 + y_{ss}^2 + z_{ss}^2} \tag{1.52}$$

By Eqs. (1.1) and (1.51), $\hat{b}/R$ may be expressed in determinant notation, as follows:

$$\frac{\hat{b}}{R} = \hat{i} \begin{vmatrix} y_s & z_s \\ y_{ss} & z_{ss} \end{vmatrix} + \hat{j} \begin{vmatrix} z_s & x_s \\ z_{ss} & x_{ss} \end{vmatrix} + \hat{k} \begin{vmatrix} x_s & y_s \\ x_{ss} & y_{ss} \end{vmatrix} \tag{1.53}$$

By Eq. (1.50),

$$\frac{\partial \hat{t}}{\partial t} = \hat{i}\, x_{st} + \hat{j}\, y_{st} + \hat{k}\, z_{st} \tag{1.54}$$

By Eqs. (1.17), (1.51), (1.53), and (1.54),

$$\frac{\omega_b}{R} = x_{ss}\, x_{st} + y_{ss}\, y_{st} + z_{ss}\, z_{st} \tag{1.55}$$

and

$$-\frac{\omega_n}{R} = \begin{vmatrix} x_{st} & y_{st} & z_{st} \\ x_s & y_s & z_s \\ x_{ss} & y_{ss} & z_{ss} \end{vmatrix} \tag{1.56}$$

Accordingly, ω_n and ω_b are determined, if the functions x(s,t), y(s,t), z(s,t) are known. The component ω_a is not determined solely by these functions, since it depends on the twist of the tube, as is indicated by Eq. (1.43).

The tortuosity of curve C* is determined most readily by the second of Eqs. (1.2). It yields

$$\frac{1}{\Sigma} = \hat{b} \cdot \frac{\partial \hat{n}}{\partial s} \tag{1.57}$$

By the first of Eqs. (1.2),

$$\frac{\partial^2 \hat{t}}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\hat{n}}{R}\right) = \frac{1}{R}\frac{\partial \hat{n}}{\partial s} + \hat{n}\,\frac{\partial}{\partial s}\left(\frac{1}{R}\right) = \hat{i}\, x_{sss} + \hat{j}\, y_{sss} + \hat{k}\, z_{sss} \tag{1.58}$$

Therefore,

$$\frac{\hat{b}}{R} \cdot \frac{\partial}{\partial s}\left(\frac{\hat{n}}{R}\right) = R^{-2}\hat{b} \cdot \frac{\partial \hat{n}}{\partial s} = \frac{1}{R^2\Sigma}$$

With Eqs. (1.53) and (1.58), this yields

$$\frac{1}{R^2\Sigma} = \begin{vmatrix} x_s & y_s & z_s \\ x_{ss} & y_{ss} & z_{ss} \\ x_{sss} & y_{sss} & z_{sss} \end{vmatrix} \tag{1.59}$$

It is possible to eliminate R^2 from Eq. (1.59) by means of Eq. (1.52). Thus, $1/\Sigma$ is expressed as a rational function of first, second, and third derivatives of x, y, and z with respect to s.

1.12 VELOCITY AND ACCELERATION OF THE PROJECTILE IN SCALAR NOTATION

The velocity of the centroid of the projectile is given by Eq. (1.19). Consequently, by Eqs. (1.49) and (1.50),

$$\overline{V} = \hat{i}(x_t + \dot{\xi}\, x_s) + \hat{j}(y_t + \dot{\xi}\, y_s) + \hat{k}(z_t + \dot{\xi}\, z_s)\Big|^{s=\xi}$$

or

$$\overline{V} = \hat{i}\,\frac{dx}{dt} + \hat{j}\,\frac{dy}{dt} + \hat{k}\,\frac{dz}{dt}\Big|^{s=\xi} \tag{1.60}$$

where d/dt denotes the substantial derivative. Hence,

$$\overline{V}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\Big|^{s=\xi} \tag{1.61}$$

The axial component of velocity of the projectile is

$$V_a = \overline{V} \cdot \hat{t} = x_s\,\frac{dx}{dt} + y_s\,\frac{dy}{dt} + z_s\,\frac{dz}{dt}\Big|^{s=\xi} \tag{1.62}$$

The components of $\overline{V}$ on the principal normal and the binormal of the deflected axis C* are given by Eq. (1.23). Consequently, by Eqs. (1.49), (1.51), and (1.53),

$$\frac{V_n}{R} = x_{ss}\, x_t + y_{ss}\, y_t + z_{ss}\, z_t \Big|^{s=\xi} \tag{1.63}$$

$$\frac{V_b}{R} = \begin{vmatrix} x_t & y_t & z_t \\ x_s & y_s & z_s \\ x_{ss} & y_{ss} & z_{ss} \end{vmatrix}^{s=\xi} \tag{1.64}$$

The acceleration of the centroid of the projectile is given by Eq. (1.24). Consequently, by means of Eqs. (1.49), (1.50), (1.51), and (1.54),

$$\begin{aligned} \bar{a} = {} & \hat{i}(\dot{\xi}^2\, x_{ss} + 2\,\dot{\xi}\, x_{st} + \ddot{\xi}\, x_s + x_{tt}) \\ & + \hat{j}(\dot{\xi}^2\, y_{ss} + 2\,\dot{\xi}\, y_{st} + \ddot{\xi}\, y_s + y_{tt}) \\ & + \hat{k}(\dot{\xi}^2\, z_{ss} + 2\,\dot{\xi}\, z_{st} + \ddot{\xi}\, z_s + z_{tt}) \Big|^{s=\xi} \end{aligned} \tag{1.65}$$

Therefore, by Eq. (1.26), the axial component of acceleration of the centroid of the projectile is

$$a_a = \hat{t} \cdot \bar{a} = \ddot{\xi} + x_s\, x_{tt} + y_s\, y_{tt} + z_s\, z_{tt} \Big|^{s=\xi} \tag{1.66}$$

By Eq. (1.26), the component of $\bar{a}$ on the principal normal to curve C* is determined by

$$\frac{\bar{a} \cdot \hat{n}}{R} = \frac{a_n}{R} = \frac{\dot{\xi}^2}{R^2} + 2\,\frac{\dot{\xi}}{R}\,\omega_b + \frac{\hat{n}}{R} \cdot \frac{\partial^2 \bar{r}}{\partial t^2}\Big|^{s=\xi}$$

Consequently, in view of Eq. (1.51),

$$\frac{a_n}{R} = \frac{\dot{\xi}^2}{R^2} + 2\,\frac{\dot{\xi}}{R}\,\omega_b + x_{ss}\, x_{tt} + y_{ss}\, y_{tt} + z_{ss}\, z_{tt} \Big|^{s=\xi} \tag{1.67}$$

The factor ω_b/R in Eq. (1.67) can be eliminated by means of Eq. (1.55).

Likewise, the component of $\bar{a}$ in the direction of the binormal $\hat{b}$ of curve C* is determined by Eq. (1.26). In view of Eq. (1.53),

$$\frac{a_b}{R} = -2\dot{\xi}\,\frac{\omega_n}{R} + \begin{vmatrix} x_{tt} & y_{tt} & z_{tt} \\ x_s & y_s & z_s \\ x_{ss} & y_{ss} & z_{ss} \end{vmatrix} \tag{1.68}$$

The factor ω_n/R in Eq. (1.68) can be eliminated by means of Eq. (1.56). Also, R can be eliminated by means of Eq. (1.52).

1.13 KINETIC ENERGY OF THE PROJECTILE IN SCALAR NOTATION

The kinetic energy of the projectile is given by Eq. (1.40). Consequently, by Eq. (1.50),

$$\begin{aligned} T_p = {} & \frac{1}{2}\, m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \\ & + \frac{1}{2}\, i_1\left[(x_{st} + \dot{\xi}\, x_{ss})^2 + (y_{st} + \dot{\xi}\, y_{ss})^2 + (z_{st} + \dot{\xi}\, z_{ss})^2\right] \\ & + \frac{1}{2}\, i_3(\omega + \omega_a + \dot{\xi}\tau)^2 \Big|^{s=\xi} \end{aligned} \tag{1.69}$$

1.14 APPROXIMATE THEORY FOR INITIALLY STRAIGHT TUBES

If the tube is initially straight, the z-axis is conveniently chosen to coincide with the undeflected axis of the bore. Then $x(s,t)$ and $y(s,t)$ are deflection components of the tube. By Eq. (1.48),

$$z_s = \sqrt{1 - (x_s^{\,2} + y_s^{\,2})} = 1 - \frac{1}{2}(x_s^{\,2} + y_s^{\,2}) + \ldots \tag{1.70}$$

Since the deflections of the barrel of a gun are small, it is reasonable to approximate Eq. (1.70) by $z_s = 1$. Then,

$$z = s + f(t)\ , \quad z_{ss} = z_{st} = z_{sss} = 0\ , \quad z_s = 1 \tag{1.71}$$

The function $f(t)$ represents the value of z at the point $s = 0$. This point is conveniently chosen to lie at the breech. It may vary with time, if the breech moves. It is to be noted that Eq. (1.71) signifies that

${x_s}^2 + {y_s}^2 = 0$. Therefore, the accuracy of quadratic expressions in derivatives of x and y requires study. In view of Eq. (1.71), Eq. (1.49) becomes

$$\bar{r} = \hat{i}\, x + \hat{j}\, y + \hat{k}\, s + \hat{k}\, f(t) \tag{1.72}$$

Accordingly, the unit tangent of curve C* is

$$\hat{t} = \hat{i}\, x_s + \hat{j}\, y_s + \hat{k} \tag{1.73}$$

By Eq. (1.51),

$$\frac{\hat{n}}{R} = \hat{i}\, x_{ss} + \hat{j}\, y_{ss} \tag{1.74}$$

By Eq. (1.52),

$$\frac{1}{R} = \sqrt{{x_{ss}}^2 + {y_{ss}}^2} \tag{1.75}$$

If curve C* is constrained to lie in the yz plane, x = 0 and Eq. (1.75) reduces to $1/R = \pm\, y_{ss}$, which is a well-known linear approximation in the engineering theory of beams. However, if x_{ss} and y_{ss} both differ from zero, there is no linear approximation of Eq. (1.75) available by Taylor series expansion.

The linear approximation of $\hat{b}/R$, obtained from Eqs. (1.53) and (1.71), is

$$\frac{\hat{b}}{R} = -\hat{i}\, y_{ss} + \hat{j}\, x_{ss} \tag{1.76}$$

Here, the term $\hat{k}(x_s\, y_{ss} - y_s\, x_{ss})$ has been discarded, since it presumably is small compared to the linear terms in Eq. (1.76).

In view of Eqs. (1.59) and (1.75), the tortuosity of curve C* is approximated by

$$\frac{1}{\Sigma} = \frac{x_{ss}\, y_{sss} - y_{ss}\, x_{sss}}{{x_{ss}}^2 + {y_{ss}}^2} \tag{1.77}$$

Equations (1.55) and (1.56) reduce to

$$\frac{\omega_b}{R} = x_{ss}\, x_{st} + y_{ss}\, y_{st}$$

$$\frac{\omega_n}{R} = x_{st}\, y_{ss} - y_{st}\, x_{ss} \tag{1.78}$$

Introduction of the foregoing approximations into the equations for the velocity and the acceleration of the projectile (Art. 1.12) is routine. Equation (1.69), which gives the kinetic energy of the projectile, becomes

$$\begin{aligned} T_p = {} & \frac{1}{2} m[(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\dot{\xi} + \dot{f})^2] \\ & + \frac{1}{2} i_1 [(x_{st} + \dot{\xi}\, x_{ss})^2 + (y_{st} + \dot{\xi}\, y_{ss})^2] \\ & + \frac{1}{2} i_3 (\omega + \omega_a + \dot{\xi}\tau)^2 \Big|^{s=\xi} \end{aligned} \tag{1.79}$$

By Eqs. (1.45) and (1.78),

$$\Omega_a = \omega + \omega_a + \dot{\xi}\tau \Big|^{s=\xi} = \omega + \int_0^{\xi} (x_{st}\, y_{ss} - y_{st}\, x_{ss}) ds + \frac{d\psi}{dt}\Big|^{s=\xi} \tag{1.80}$$

Consequently, Eq. (1.79) yields

$$\begin{aligned} T_p = {} & \frac{1}{2} m[x_t^{\,2} + y_t^{\,2} + 2\dot{\xi}(x_s\, x_t + y_s\, y_t) + (\dot{\xi} + \dot{f})^2] \\ & + \frac{1}{2} i_1 [x_{st}^{\,2} + y_{st}^{\,2} + 2\dot{\xi}(x_{ss}\, x_{st} + y_{ss}\, y_{st}) + \dot{\xi}^2(x_{ss}^{\,2} + y_{ss}^{\,2})] \\ & + \frac{1}{2} i_3 [\omega + \psi_t + \dot{\xi}\psi_s + \int_0^{\xi} (x_{st}\, y_{ss} - y_{st}\, x_{ss}) ds]^2 \Big|^{s=\xi} \end{aligned} \tag{1.81}$$

Since the approximation $z = s$ is used, Eq. (1.70) implies that $x_s^{\,2} + y_s^{\,2} = 0$, so this expression has been discarded from Eq. (1.81). Also, because of the approximation $z = s$, terms of third and fourth degree in Eq. (1.81) have little credibility. Consequently, Eq. (1.81) is simplified by elimination of these terms. Thus, Eq. (1.81) is reduced to the following quadratic expression in x, y, ψ and their derivatives:

$$T_p = \frac{1}{2} m[x_t^{\,2} + y_t^{\,2} + 2\dot{\xi}(x_s\, x_t + y_s\, y_t) + (\dot{\xi} + \dot{f})^2]\Big|^{s=\xi}$$

$$+ \frac{1}{2}\, i_1 [x_{st}^{\,2} + y_{st}^{\,2} + 2\dot{\xi}(x_{ss}\, x_{st} + y_{ss}\, y_{st}) + \dot{\xi}^2 (x_{ss}^{\,2} + y_{ss}^{\,2})]^{s=\xi}$$

$$+ \frac{1}{2}\, i_3 (\omega + \psi_t + \dot{\xi}\psi_s)^2\Big|^{s=\xi} + i_3\omega \int_0^{\xi} (x_{st}\, y_{ss} - y_{st}\, x_{ss})ds \qquad (1.82)$$

SECTION 2

FORCES AND MOMENTS ACTING ON A BALANCED PROJECTILE IN A FLEXIBLE TUBE

2.1 INTRODUCTION

Section 1 deals primarily with kinematics of a flexible tube containing an accelerating projectile. In this section, the theory is extended to provide formulas for the forces and moments acting on the projectile. By specialization, the forces and moments acting on a projectile in a moving rigid tube are obtained.

As in Section 1, balloting is excluded. The weight of the projectile is disregarded, but it would merely augment the forces on the projectile by the term $m\bar{g}$, where $\bar{g}$ is the vector acceleration of gravity. It would have no effect on the moment vector, except indirectly, through its dynamic effect on the deflection of the tube. The dynamic response of the tube is not treated in this section.

2.2 FORC' ON THE PROJECTILE

The net force on the projectile is $\bar{F} = m\,\bar{a}$. Consequently, if the center of mass of the projectile coincides with the centroid, Eq. (1.25) yields

$$\bar{F} = m(\hat{n}\,\frac{\dot{\xi}^2}{R} + 2\dot{\xi}\bar{\omega} \times \hat{t} + \ddot{\xi}\hat{t} + \frac{\partial^2\bar{r}}{\partial t^2}) \tag{2.1}$$

By Eq. (1.26), the $\hat{n}$, $\hat{b}$, $\hat{t}$ components of $\bar{F}$ are

$$F_a = m(\ddot{\xi} + \hat{t} \cdot \frac{\partial^2\bar{r}}{\partial t^2})$$

$$F_n = m(\frac{\dot{\xi}^2}{R} + 2\dot{\xi}\bar{\omega} \cdot \hat{b} + \hat{n} \cdot \frac{\partial^2\bar{r}}{\partial t^2})$$

$$F_b = m(-2\dot{\xi}\bar{\omega} \cdot \hat{n} + \hat{b} \cdot \frac{\partial^2\bar{r}}{\partial t^2}) \tag{2.2}$$

wherein relevant functions are evaluated at point $s = \xi$.

The axial force on the projectile due to friction and engraving is denoted by F. Its positive sense is toward the breech. The pressure on the base of the projectile is denoted by P_1, and the resisting pressure of air ahead of the projectile is denoted by P_2. Accordingly,

$$F_a = -F + (P_1 - P_2)A$$

where A is the cross sectional area of the bore. Therefore,

$$F = (P_1 - P_2)A - m(\ddot{\xi} + \hat{t} \cdot \frac{\partial^2 \bar{r}}{\partial t^2}) \tag{2.3}$$

The factor $\partial^2 \bar{r}/\partial t^2$ is the acceleration of the center of the cross section of the tube at which the center of mass of the projectile lies.

2.3 MOMENT OF FORCES ACTING ON THE PROJECTILE

Because of axial symmetry of the projectile, two of its principal moments of inertia are equal; i.e., $i_1 = i_2$. Consequently, the components of angular momentum of the projectile with respect to its center of mass are

$$H_n = i_1 \Omega_n \ , \quad H_b = i_1 \Omega_b \ , \quad H_t = i_3 \Omega_a \tag{2.4}$$

Hence,

$$\bar{H} = i_1(\hat{n} \Omega_n + \hat{b} \Omega_b) + i_3 \hat{t} \Omega_a \tag{2.5}$$

The vector $\bar{\Omega}$ is given by Eq. (1.34).

Since $\partial\hat{n}/\partial t = \bar{\omega} \times \hat{n}$, etc.,

$$\frac{\partial \hat{n}}{\partial t} = \bar{\omega} \times \hat{n} = \hat{b} \omega_a - \hat{t} \omega_b$$

$$\frac{\partial \hat{b}}{\partial t} = \bar{\omega} \times \hat{b} = \hat{t} \omega_n - \hat{n} \omega_a$$

$$\frac{\partial \hat{t}}{\partial t} = \bar{\omega} \times \hat{t} = \hat{n} \omega_b - \hat{b} \omega_n \tag{2.6}$$

Equations (2.6) are analogous to Eq. (1.14).

The moment about the center of mass of the projectile of the forces that act on the projectile is

$$\overline{M} = \frac{d\overline{H}}{dt} = \frac{\partial \overline{H}}{\partial t} + \dot{\xi}\,\frac{\partial \overline{H}}{\partial s} \tag{2.7}$$

The pressure force $(P_1 - P_2)A$ and the weight exert no moment about the center of mass of the projectile. Consequently, the moment $\overline{M}$ results entirely from contact forces between the projectile and the bore. The rifling torque is $\overline{M} \cdot \hat{t} = M_a$. The gyroscopic couple is $\overline{M} \cdot \hat{n} = M_n$.

Equations (2.5) and (2.7) yield

$$\overline{M} = i_1 \frac{d}{dt}(\hat{n}\,\Omega_n + \hat{b}\,\Omega_b) + i_3 \frac{d}{dt}(\hat{t}\,\Omega_a) \tag{2.8}$$

The derivatives $\partial\hat{n}/\partial s$, $\partial\hat{b}/\partial s$, $\partial\hat{t}/\partial s$ are given by Eq. (1.2). Since $d\hat{n}/dt = \partial\hat{n}/\partial t + \dot{\xi}\,\partial\hat{n}/\partial s$, etc., Eqs. (1.2) and (2.6) yield

$$\frac{d\hat{n}}{dt} = \hat{b}\,\omega_a - \hat{t}\,\omega_b + \dot{\xi}\left(\frac{\hat{b}}{\Sigma} - \frac{\hat{t}}{R}\right)$$

$$\frac{d\hat{b}}{dt} = \hat{t}\,\omega_n - \hat{n}\,\omega_a - \dot{\xi}\,\frac{\hat{n}}{\Sigma}$$

$$\frac{d\hat{t}}{dt} = \hat{n}\,\omega_b - \hat{b}\,\omega_n + \dot{\xi}\,\frac{\hat{n}}{R} \tag{2.9}$$

It is convenient to introduce the notation,

$$\omega_a + \frac{\dot{\xi}}{\Sigma} = \lambda \tag{2.10}$$

Equations (1.34), (2.9), and (2.10) yield

$$\frac{d\hat{n}}{dt} = \hat{b}\lambda - \hat{t}\,\Omega_b\ , \quad \frac{d\hat{b}}{dt} = \hat{t}\,\Omega_n - \hat{n}\lambda\ , \quad \frac{d\hat{t}}{dt} = \hat{n}\,\Omega_b - \hat{b}\,\Omega_n \tag{2.11}$$

Consequently,

$$\frac{d}{dt}(\hat{n}\,\Omega_n) = \hat{n}\,\frac{d\,\Omega_n}{dt} + \Omega_n(\hat{b}\lambda - \hat{t}\,\Omega_b)$$

$$\frac{d}{dt}(\hat{b}\ \Omega_b) = \hat{b}\ \frac{d\ \Omega_b}{dt} + \Omega_b(\hat{t}\ \Omega_n - \hat{n}\lambda)$$

$$\frac{d}{dt}(\hat{t}\ \Omega_a) = \hat{t}\ \frac{d\ \Omega_a}{dt} + \Omega_a(\hat{n}\ \Omega_b - \hat{b}\ \Omega_n) \qquad (2.12)$$

Equations (2.8) and (2.12) yield

$$\overline{M} = i_1[\hat{n}\ \frac{d\ \Omega_n}{dt} + \hat{b}\ \frac{d\ \Omega_b}{dt} - \lambda(\hat{n}\ \Omega_b - \hat{b}\ \Omega_n)] + i_3[\hat{t}\ \frac{d\ \Omega_a}{dt} + \Omega_a(\hat{n}\ \Omega_b - \hat{b}\ \Omega_n)] \qquad (2.13)$$

Hence,

$$M_n = i_1\ \frac{d\ \Omega_n}{dt} + \Omega_b(i_3\ \Omega_a - i_1\lambda)$$

$$M_b = i_1\ \frac{d\ \Omega_b}{dt} - \Omega_n(i_3\ \Omega_a - i_1\lambda)$$

$$M_a = i_3\ \frac{d\ \Omega_a}{dt} \qquad (2.14)$$

Equations (2.14) resemble the Euler equations (with $i_1 = i_2$), but they are not identical to them unless, by chance, $\Omega_a = \lambda$. The explanation for the difference lies in the physical reference frames to which the equations refer. Euler's axes (1, 2, 3) are the principal axes of inertia of the projectile. They are imbedded in the projectile. For example, Euler's Ω_1 is the component of $\overline{\Omega}$ on a transverse axis that is imbedded in the projectile. On the other hand, Ω_n is the component of $\overline{\Omega}$ on the principal normal of the trajectory of the center of mass of the projectile. At a given instant, these two axes may coincide, so that $\Omega_1 = \Omega_n$, but the time rate of change of Ω_1, denoted by $d\ \Omega_1/dt$, is generally not the same as the time rate of change of Ω_n. The time rate of change of the orthogonal projection of vector $\overline{\Omega}$ on a moving axis clearly depends on the motion of that axis, and the vectors $\hat{n}$, $\hat{b}$, do not have the same motions as the lateral principal axes of inertia of the projectile.

2.4 MOMENTS ACTING ON A PROJECTILE IN A MOVING RIGID TUBE OF ANY FORM

If the tube is rigid, its angular velocity is $\bar{\omega}(t)$; i.e., $\bar{\omega}$ is independent of s. Also, the deformational twist τ is zero. Furthermore, $R = R(s)$ and $\Sigma = \Sigma(s)$. Equations (1.2), (1.34), and (2.10) apply without alteration (except that $\tau = 0$). Since $\bar{\omega} = \bar{\omega}(t)$ and $\omega_n = \bar{\omega} \cdot \hat{n}$, etc.,

$$\frac{\partial \omega_n}{\partial s} = \bar{\omega} \cdot \frac{\partial \hat{n}}{\partial s} = \frac{\omega_b}{\Sigma} - \frac{\omega_a}{R} , \quad \frac{\partial \omega_b}{\partial s} = - \frac{\omega_n}{\Sigma} , \quad \frac{\partial \omega_a}{\partial s} = \frac{\omega_n}{R} \tag{2.15}$$

Therefore,

$$\frac{d\omega_n}{dt} = \frac{\partial \omega_n}{\partial t} + \dot{\xi}\left(\frac{\omega_b}{\Sigma} - \frac{\omega_a}{R}\right) , \quad \frac{d\omega_b}{dt} = \frac{\partial \omega_b}{\partial t} - \frac{\dot{\xi}}{\Sigma}\,\omega_n , \quad \frac{d\omega_a}{dt} = \frac{\partial \omega_a}{\partial t} + \frac{\dot{\xi}}{R}\,\omega_n \tag{2.16}$$

By Eq. (1.34),

$$\frac{d\Omega_n}{dt} = \frac{d\omega_n}{dt} , \quad \frac{d\Omega_b}{dt} = \frac{d\omega_b}{dt} + \frac{d}{dt}\left(\frac{\dot{\xi}}{R}\right) , \quad \frac{d\Omega_a}{dt} = \dot{\omega} + \frac{d\omega_a}{dt} \tag{2.17}$$

Introducing Eqs. (1.34), (2.10), (2.16), and (2.17) into Eq. (2.14) we get

$$M_n = i_1\left(\frac{\partial \omega_n}{\partial t} - \frac{\dot{\xi}}{R}\,\omega_a\right) + \left(\omega_b + \frac{\dot{\xi}}{R}\right)\left[i_3\omega - (i_1 - i_3)\omega_a\right] - i_1\,\frac{\dot{\xi}^2}{R\Sigma}$$

$$M_b = i_1\left[\frac{\partial \omega_b}{\partial t} + \frac{d}{dt}\left(\frac{\dot{\xi}}{R}\right)\right] - \omega_n\left[i_3\omega - (i_1 - i_3)\omega_a\right]$$

$$M_a = i_3\left(\frac{\partial \omega_a}{\partial t} + \frac{\dot{\xi}}{R}\,\omega_n + \dot{\omega}\right) \tag{2.18}$$

It is to be noted that the tortuosity $1/\Sigma$ enters these formulas only in the final term of the equation for the gyroscopic couple M_n.

If the axis of the tube is a plane curve, $1/\Sigma = 0$. The equations for a straight tube are obtained by setting $1/\Sigma = 0$ and $1/R = 0$. If the tube is immovable, $\omega_n = \omega_b = \omega_a = 0$. Then Eq. (2.18) reduces to

$$M_n = \frac{\dot{\xi}}{R}\left(i_3\omega - i_1\,\frac{\dot{\xi}}{\Sigma}\right) , \quad M_b = i_1\,\frac{d}{dt}\left(\frac{\dot{\xi}}{R}\right) , \quad M_a = i_3\dot{\omega} \tag{2.19}$$

If the tube is straight and rigid, $1/R = 0$ and $\tau = 0$. Also, $\bar{\omega}$ is independent of s; i.e., $\bar{\omega} = \bar{\omega}(t)$. The Frenet formulas (Eq. (1.2)) reduce to

$$\frac{\partial \hat{n}}{\partial s} = \frac{\hat{b}}{\Sigma} , \quad \frac{\partial \hat{b}}{\partial s} = - \frac{\hat{n}}{\Sigma} , \quad \frac{\partial \hat{t}}{\partial s} = 0 \tag{a}$$

The tortuosity of a straight line is indeterminate. Consequently, Σ should cancel from the equations for M_n, M_b, M_a. This condition provides a partial check on the theory.

Equation (1.34) yields

$$\Omega_n = \omega_n , \quad \Omega_b = \omega_b , \quad \Omega_a = \omega_a + \omega \tag{b}$$

Since $\omega_n = \bar{\omega} \cdot \hat{n}$, etc., and $\bar{\omega} = \bar{\omega}(t)$, Eq. (a) yields

$$\frac{\partial \omega_n}{\partial s} = \frac{\omega_b}{\Sigma} , \quad \frac{\partial \omega_b}{\partial s} = - \frac{\omega_n}{\Sigma} , \quad \frac{\partial \omega_a}{\partial s} = 0 \tag{c}$$

Consequently,

$$\frac{d\omega_n}{dt} = \frac{\partial \omega_n}{\partial t} + \frac{\xi}{\Sigma} \omega_b , \quad \frac{d\omega_b}{dt} = \frac{\partial \omega_b}{\partial t} - \frac{\xi}{\Sigma} \omega_n , \quad \frac{d\omega_a}{dt} = \frac{\partial \omega_a}{\partial t} \tag{d}$$

With Eqs. (b) and (d), Eqs. (2.10) and (2.14) yield

$$M_n = i_1 \frac{\partial \omega_n}{\partial t} - (i_1 - i_3)\omega_b \, \omega_a + i_3 \omega \, \omega_b$$

$$M_b = i_1 \frac{\partial \omega_b}{\partial t} - (i_3 - i_1)\omega_n \, \omega_a - i_3 \omega \, \omega_n$$

$$M_a = i_3(\dot{\omega}_a + \dot{\omega}) \tag{2.20}$$

Equations (2.20) reduce to the Euler equations if $\omega = 0$.

It appears that $\partial \omega_n/\partial t$ and $\partial \omega_b/\partial t$ depend on time rates of change of $\hat{n}$ and $\hat{b}$. However, since $\partial \hat{n}/\partial t = \bar{\omega} \times \hat{n}$, $\bar{\omega} \cdot \partial \hat{n}/\partial t = 0$. Consequently, since $\omega_n = \bar{\omega} \cdot \hat{n}$,

$$\frac{\partial \omega_n}{\partial t} = \hat{n} \cdot \frac{\partial \bar{\omega}}{\partial t}$$

Likewise,

$$\frac{\partial \omega_b}{\partial t} = \hat{b} \cdot \frac{\partial \bar{\omega}}{\partial t}$$

Consequently, $\partial \omega_n / \partial t$ and $\partial \omega_b / \partial t$ do not depend on time derivatives of $\hat{n}$ and $\hat{b}$. Therefore, in Eq. (2.20), $(\hat{n}, \hat{b}, \hat{t})$ may be any right-handed orthogonal triad of unit vectors, such that $\hat{t}$ coincides with the axis of the tube. Since all cross sections of the tube have the same angular velocity, $\bar{\omega}$ is simply the angular velocity of the tube relative to a Galilean reference frame.

SECTION 3

RESPONSE OF A TAPERED ELASTIC CANTILEVER GUN TUBE TO EXCITATION BY THE PROJECTILE AND PRESCRIBED MOTION AT THE BREECH

3.1 INTRODUCTION

In this section the theory in Section 1 is used to determine the motion of a tapered cantilever tube that is actuated by the projectile and prescribed motion at the base of the tube. The section properties of the tube are arbitrary functions of the axial coordinate s. Initial droop due to gravity is admitted. The deflections and twist of the tube are represented as series of flexural and torsional eigenfunctions of a uniform cantilever beam. The coefficients in these series are functions of time. Such series have the capacity to converge, in the least-square sense, to the exact solution of the problem, since the eigenfunctions are complete sets of functions. The coefficients in the series are generalized coordinates of the tube. By means of Lagrange's equations, they are represented as the solution of certain coupled non-homogeneous ordinary linear differential equations of second order with time-dependent coefficients.

3.2 THE LAGRANGIAN FUNCTION

The tube is considered to be horizontal, and the y-axis is directed downward. The z-axis coincides with the undeflected axis of the tube. Accordingly, the potential energy of the projectile is

$$U_p = -m\, g\, y(\xi,t) \tag{3.1}$$

A cross section of the tube is required to have the same moment of inertia I about all diametral axes. Consequently, the polar moment of inertia of a cross section of the tube is 2I. Accordingly, the kinetic energy of the tube is

$$T_{tube} = \frac{1}{2}\rho\int_0^{\ell} S(x_t^{\,2} + y_t^{\,2})ds + \rho\int_0^{\ell} I\omega_a^{\,2}\, ds \tag{3.2}$$

Equation (3.2) includes the torsional kinetic energy, but the rotary kinetic energy due to the deflections (x,y) has been neglected.

With Eqs. (1.44) and (1.78), Eq. (3.2) yields

$$T_{tube} = \frac{1}{2}\rho\int_0^\ell S(x_t^{\,2} + y_t^{\,2})ds + \rho\int_0^\ell I[\psi_t + \int_0^s (x_{st}\, y_{ss} - y_{st}\, x_{ss})ds]^2 ds$$

Discarding cubic and quartic terms in (x, y, ψ), we get

$$T_{tube} = \frac{1}{2}\rho\int_0^\ell S(x_t^{\,2} + y_t^{\,2})ds + \rho\int_0^\ell I\psi_t^{\,2}\, ds \tag{3.3}$$

Since the twist of the tube is $\tau = \psi_s$, the potential energy of the tube is

$$U_{tube} = \frac{1}{2}\int_0^\ell EI(x_{ss}^{\,2} + y_{ss}^{\,2})ds + \int_0^\ell GI\,\psi_s^{\,2}\, ds - \rho g\int_0^\ell S\, y\, ds \tag{3.4}$$

The three expressions in Eq. (3.4) respectively represent the strain energy of bending, the strain energy of torsion, and the potential energy due to gravity.

Equations (1.82), (3.1), (3.3), and (3.4) yield the Lagrangian function:

$$\begin{aligned} L = T - U = {} & \frac{1}{2} m[x_t^{\,2} + y_t^{\,2} + 2\dot{\xi}(x_s\, x_t + y_s\, y_t)]\Big|^{s=\xi} \\ & + \frac{1}{2} i_1[x_{st}^{\,2} + y_{st}^{\,2} + 2\dot{\xi}(x_{ss}\, x_{st} + y_{ss}\, y_{st}) + \dot{\xi}^2(x_{ss}^{\,2} + y_{ss}^{\,2})]\Big|^{s=\xi} \\ & + \frac{1}{2} i_3(\omega + \psi_t + \dot{\xi}\psi_s)^2\Big|^{s=\xi} + i_3\omega\int_0^\xi (x_{st}\, y_{ss} - y_{st}\, x_{ss})ds \\ & + \frac{1}{2}\rho\int_0^\ell S(x_t^{\,2} + y_t^{\,2})ds + \rho\int_0^\ell I\psi_t^{\,2}\, ds + m\, g\, y(\xi,t) \\ & - \frac{1}{2}\int_0^\ell EI(x_{ss}^{\,2} + y_{ss}^{\,2})ds - \int_0^\ell GI\psi_s^{\,2}\, ds + \rho g\int_0^\ell S\, y\, ds \end{aligned} \tag{3.5}$$

The term $(\dot{\xi} + \dot{f})^2$ has been omitted, since it contributes nothing to the Lagrange equations.

3.3 NATURAL MODES OF A CANTILEVER BEAM

The n'th natural bending mode of a uniform cantilever beam (Ref. 7) is

$$f_n(s) = \cosh \beta_n s - \cos \beta_n s - \alpha_n(\sinh \beta_n s - \sin \beta_n s) \tag{3.6}$$

in which $\beta_n \ell$ is the n'th positive root of the equation,

$$\cos \beta\ell \cosh \beta\ell = -1 \tag{3.7}$$

The dimensionless constant α_n is defined by

$$\alpha_n = \frac{\cos \beta_n \ell + \cosh \beta_n \ell}{\sin \beta_n \ell + \sinh \beta_n \ell} \tag{3.8}$$

Values of $\beta_n \ell$ and α_n are given in Table 1. If $n > 5$, $\alpha_n \approx 1$ and $\beta_n \ell \approx (2n - 1)\pi/2$, with accuracy at least to seven significant figures.

TABLE 1

Eigenvalues for a Cantilever Beam

n	$\beta_n \ell$	α_n
1	1.8751041	0.7340955
2	4.6940911	1.0184664
3	7.8547574	0.9992245
4	10.9955407	1.0000336
5	14.1371684	0.9999986

A few pertinent integrals of the functions $f_n(s)$ are given below (Ref. 8):

[7] *D. Young and R. Felgar, Tables of Characteristic Functions Representing Normal Modes of Vibration of a Beam, Engineering Research Series No. 44, Bureau of Engineering Research, The University of Texas, Austin, Texas, 1949.*

[8] *R. P. Felgar, Formulas for Integrals Containing Characteristic Functions of a Vibrating Beam, Bureau of Engineering Research, Circular No. 14, The University of Texas, Austin, Texas, 1950.*

$$\int_0^\ell f_m(s)\ f_n(s)ds = \begin{cases} 0\ , & m \neq n \\ \ell\ , & m = n \end{cases}$$

$$\int_0^\ell f_m''(s)\ f_n''(s)ds = \begin{cases} 0\ , & m \neq n \\ \beta_n^4 \ell\ , & m = n \end{cases}$$

$$\int_0^\ell f_n(s)ds = 2\,\frac{\alpha_n}{\beta_n}\ ,\quad \int_0^\xi f_n'(s)\ f_n''(s)ds = \frac{1}{2}\,f_n'^2(\xi) \tag{3.9}$$

Also, derivatives of the functions $f_n(s)$ arise. They are

$$f_n'(s) = \beta_n[\sinh \beta_n s + \sin \beta_n s - \alpha_n(\cosh \beta_n s - \cos \beta_n s)]$$

$$f_n''(s) = \beta_n^2[\cosh \beta_n s + \cos \beta_n s - \alpha_n(\sinh \beta_n s + \sin \beta_n s)]$$

$$f_n'''(s) = \beta_n^3[\sinh \beta_n s - \sin \beta_n s - \alpha_n(\cosh \beta_n s + \cos \beta_n s)] \tag{3.10}$$

The n'th torsional mode of a uniform straight tube that is fixed at one end and free at the other is

$$\psi_n(s) = \sin(2n - 1)\,\frac{\pi s}{2\ell} \tag{3.11}$$

The following integrals of these functions arise:

$$\int_0^\ell \psi_m(s)\psi_n(s)ds = \begin{cases} 0\ , & m \neq n \\ \ell/2\ , & m = n \end{cases}$$

$$\int_0^\ell \psi_m'(s)\psi_n'(s)ds = \begin{cases} 0\ , & m \neq n \\ \frac{\pi^2}{8\ell}(2n - 1)^2\ , & m = n \end{cases} \tag{3.12}$$

The function $\psi(s,t)$ represents the angular displacement of a cross section of the tube in its plane. Consequently, the boundary conditions are $\psi(0,t) = 0$, $\psi_s(\ell,t) = 0$. These boundary conditions are satisfied automatically by expansion of $\psi(s,t)$ in a truncated series of the functions $\psi_n(s)$. Also, the boundary conditions,

$$x(0,t) = x_s(0,t) = 0 \ , \quad y(0,t) = y_s(0,t) = 0$$

$$x_{ss}(\ell,t) = x_{sss}(\ell,t) = 0 \ , \quad y_{ss}(\ell,t) = y_{sss}(\ell,t) = 0$$

are satisfied automatically by expansions of $x(s,t)$ and $y(s,t)$ in truncated series of the functions $f_n(s)$.

3.4 EXPANSION OF THE LAGRANGIAN FUNCTION

Prescribed motion of the breech imposes time-dependent constraints on the tube. The Lagrange equations remain valid for such systems, provided that the kinetic and potential energies are computed with respect to a Galilean reference frame (Ref. 9).

As in Art. 1.14, rectangular coordinates (x, y, z) are set up so that the z-axis coincides with the undeflected and undisplaced axis of the tube. The y-axis is directed downward. Axes (x, y, z) are attached to a Galilean reference frame. The Lagrangian function is given by Eq. (3.5).

The rectilinear and angular displacements of the axis of the tube are represented as follows:

$$x = u(t) + s\ \phi(t) + \Sigma X_n(t)\ f_n(s)$$

$$y = v(t) - s\ \theta(t) + \Sigma Y_n(t)\ f_n(s)$$

$$\psi = \zeta(t) + \Sigma Z_n(t)\ \psi_n(s) \tag{3.13}$$

The functions $u(t)$ and $v(t)$ are the x and y components of displacement at the base of the tube where $s = 0$. In accordance with Eq. (1.71), the z-component of displacement at the base of the tube is $f(t)$, but this term is irrelevant. The x and y components of rotation at the base of the tube are $\theta(t)$ and $\phi(t)$, respectively. The z-component of rotation at the base of the tube is $\zeta(t)$. The functions $f_n(s)$ and $\psi_n(s)$ are defined by Eqs. (3.6), (3.7), (3.8), and (3.11). The functions $u(t)$, $v(t)$, $\phi(t)$, $\theta(t)$, and $\zeta(t)$ are considered to be given. The functions $X_n(t)$, $Y_n(t)$, $Z_n(t)$ are generalized coordinates of the tube. The range of the subscript n in

[9] *H. L. Langhaar, Energy Methods in Applied Mechanics, John Wiley & Sons, New York, 1962, Art. 7-4.*

Eq. (3.13) is 1, 2, 3, ... , N, but the number N is unspecified, and, for simplicity, the range of n is not indicated on the summation signs.

The Lagrangian function is obtained by substituting Eq. (3.13) into Eq. (3.5). Terms that do not contain the dependent variables X_n, Y_n, Z_n or their derivatives are omitted from the Lagrangian function, since they cancel from the Lagrange equations. The result is

$$
\begin{aligned}
L = {} & \frac{1}{2} m\Sigma\Sigma(\dot{X}_m \dot{X}_n + \dot{Y}_m \dot{Y}_n) f_m(\xi) f_n(\xi) + m\dot{\xi}\Sigma\Sigma(X_m \dot{X}_n + Y_m \dot{Y}_n) f_m'(\xi) f_n(\xi) \\
& + \frac{1}{2} i_1\Sigma\Sigma(\dot{X}_m \dot{X}_n + \dot{Y}_m \dot{Y}_n) f_m'(\xi) f_n'(\xi) + i_1\dot{\xi}\Sigma\Sigma(X_m \dot{X}_n \\
& + Y_m \dot{Y}_n) f_m''(\xi) f_n'(\xi) + \frac{1}{2} i_1\dot{\xi}^2\Sigma\Sigma(X_m X_n + Y_m Y_n) f_m''(\xi) f_n''(\xi) \\
& + \frac{1}{2} i_3\Sigma\Sigma\dot{Z}_m \dot{Z}_n \psi_m(\xi) \psi_n(\xi) + \frac{1}{2} i_3\dot{\xi}^2 \Sigma\Sigma Z_m Z_n \psi_m'(\xi) \psi_n'(\xi) \\
& + i_3\omega\Sigma\dot{Z}_n \psi_n(\xi) + i_3\dot{\xi}\omega\Sigma Z_n \psi_n'(\xi) + i_3\dot{\xi}\Sigma\Sigma Z_m \dot{Z}_n \psi_m'(\xi) \psi_n(\xi) \\
& + i_3\omega\Sigma\Sigma[(\dot{X}_m Y_n - \dot{Y}_m X_n)\int_0^{\xi} f_m' f_n'' \, ds] + \frac{1}{2} \rho\Sigma\Sigma[(\dot{X}_m \dot{X}_n \\
& + \dot{Y}_m \dot{Y}_n)\int_0^{\ell} S f_m f_n \, ds] + \rho\Sigma\Sigma[\dot{Z}_m \dot{Z}_n\int_0^{\ell} I \psi_m \psi_n \, ds] + m g \Sigma Y_n f_n(\xi) \\
& - \frac{1}{2} \Sigma\Sigma[(X_m X_n + Y_m Y_n)\int_0^{\ell} EI f_m'' f_n'' \, ds] - \Sigma\Sigma[Z_m Z_n\int_0^{\ell} GI \psi_m' \psi_n' ds] \\
& + \rho g\Sigma[Y_n\int_0^{\ell} S f_n \, ds] + m(\dot{u} + \xi\dot{\phi})\Sigma\dot{X}_n f_n(\xi) + m(\dot{v} - \xi\dot{\theta})\Sigma\dot{Y}_n f_n(\xi) \\
& + m\dot{\xi}(\dot{u} + \xi\dot{\phi})\Sigma X_n f_n'(\xi) + m\dot{\xi}\phi\Sigma\dot{X}_n f_n(\xi) - m\dot{\xi}\theta\Sigma\dot{Y}_n f_n(\xi) \\
& + m\dot{\xi}(\dot{v} - \xi\dot{\theta})\Sigma Y_n f_n'(\xi) + i_1 \dot{\phi}\Sigma\dot{X}_n f_n'(\xi) - i_1 \dot{\theta}\Sigma\dot{Y}_n f_n'(\xi) \\
& + i_1\dot{\xi}\dot{\phi}\Sigma X_n f_n''(\xi) - i_1\dot{\xi}\dot{\theta}\Sigma Y_n f_n''(\xi) + i_3\dot{\zeta}\Sigma\dot{Z}_n \psi_n(\xi)
\end{aligned}
$$

$$+ i_3\dot{\xi}\dot{\zeta}\Sigma Z_n\ \psi_n'(\xi) + i_3\omega\Sigma(\dot{\theta}\ X_n + \dot{\phi}\ Y_n)f_n'(\xi) + \rho\Sigma(\dot{u}\ \dot{X}_n$$

$$+ \dot{v}\ \dot{Y}_n)\int_0^{\ell} S\ f_n\ ds + \rho\Sigma(\dot{\phi}\ \dot{X}_n - \dot{\theta}\ \dot{Y}_n)\int_0^{\ell} S\ s\ f_n\ ds + 2\rho\dot{\zeta}\Sigma\dot{Z}_n\int_0^{\ell} I\ \psi_n\ ds \tag{3.14}$$

If the tube is uniform, the integrals over the range $(0,\ell)$ in Eq. (3.14) can be evaluated by means of Eqs. (3.9) and (3.12).

In the present case, ξ and ω are regarded as given functions. Consequently $\delta\xi$ and $\delta\chi$ are zero. Hence, by the argument in Art. 1.10, $\delta W = 0$. Therefore, the Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}_r} - \frac{\partial L}{\partial X_r} = 0\ , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{Y}_r} - \frac{\partial L}{\partial Y_r} = 0\ , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{Z}_r} - \frac{\partial L}{\partial Z_r} = 0 \tag{3.15}$$

3.5 LAGRANGE'S EQUATIONS

Since $f_n(\xi)$ is a function of t, Eqs. (3.14) and (3.15) yield

$$\sum_n [m\ f_r(\xi)\ f_n(\xi) + i_1\ f_r'(\xi)\ f_n'(\xi) + \rho\int_0^{\ell} S\ f_r\ f_n\ ds]\ddot{X}_n$$

$$+ 2\dot{\xi}\sum_n [m\ f_r(\xi)\ f_n'(\xi) + i_1\ f_r'(\xi)\ f_n''(\xi)]\dot{X}_n$$

$$+ \sum_n [m\ddot{\xi}\ f_r(\xi)\ f_n'(\xi) + m\dot{\xi}^2\ f_r'(\xi)\ f_n'(\xi) + m\dot{\xi}^2\ f_r(\xi)\ f_n''(\xi)$$

$$+ i_1\ddot{\xi}\ f_r'(\xi)\ f_n''(\xi) + i_1\dot{\xi}^2\ f_r'(\xi)\ f_n'''(\xi) + \int_0^{\ell} EI\ f_r''\ f_n''\ ds]X_n$$

$$+ i_3\omega\ f_r'(\xi)\Sigma\ f_n'(\xi)\dot{Y}_n + i_3\sum_n [\omega\dot{\xi}\ f_r'(\xi)\ f_n''(\xi) + \dot{\omega}\int_0^{\xi} f_r'\ f_n''\ ds]Y_n$$

$$= [-m\ddot{u} - m\xi\ddot{\phi} - m\ddot{\xi}\phi - 2\ m\dot{\xi}\dot{\phi}]f_r(\xi) + [-m\dot{\xi}^2\phi - i_1\ddot{\phi} + i_3\omega\dot{\theta}]f_r'(\xi)$$

$$- \rho\ddot{u}\int_0^{\ell} S\ f_r\ ds - \rho\ddot{\phi}\int_0^{\ell} S\ s\ f_r\ ds \tag{3.16}$$

$$\sum_n [m\, f_r(\xi)\, f_n(\xi) + i_1\, f_r'(\xi)\, f_n'(\xi) + \rho\int_0^{\ell} S\, f_r\, f_n\, ds]\ddot{Y}_n$$

$$+ 2\dot{\xi}\sum_n [m\, f_r(\xi)\, f_n'(\xi) + i_1\, f_r'(\xi)\, f_n''(\xi)]\dot{Y}_n + \sum_n [m\ddot{\xi}\, f_r(\xi)\, f_n'(\xi)$$

$$+ m\dot{\xi}^2\, f_r'(\xi)\, f_n'(\xi) + m\dot{\xi}^2\, f_r(\xi)\, f_n''(\xi) + i_1\ddot{\xi}\, f_r'(\xi)\, f_n''(\xi)$$

$$+ i_1\dot{\xi}^2\, f_r'(\xi)\, f_n'''(\xi) + \int_0^{\ell} EI\, f_r''\, f_n''\, ds]Y_n - i_3\omega\, f_r'(\xi)\sum_n f_n'(\xi)\dot{X}_n$$

$$- i_3\sum_n \omega\dot{\xi}\, f_r'(\xi)\, f_n''(\xi) + \dot{\omega}\int_0^{\xi} f_r'\, f_n''\, ds]X_n$$

$$= [-m\ddot{v} + 2m\dot{\xi}\dot{\theta} + m\xi\ddot{\theta} + m\ddot{\xi}\theta + m\, g]f_r(\xi) + [m\dot{\xi}^2\theta + i_1\ddot{\theta} + i_3\omega\dot{\phi}]f_r'(\xi)$$

$$- \rho\ddot{v}\int_0^{\ell} S\, f_r\, ds + \rho\ddot{\theta}\int_0^{\ell} S\, s\, f_r\, ds + \rho g\int_0^{\ell} S\, f_r\, ds \tag{3.17}$$

$$\sum_n [i_3\, \psi_r(\xi)\, \psi_n(\xi) + 2\rho\int_0^{\ell} I\, \psi_r\, \psi_n\, ds]\ddot{Z}_n + 2\dot{\xi}i_3\, \psi_r(\xi)\sum_n \psi_n'(\xi)\dot{Z}_n$$

$$+ \sum_n [i_3\ddot{\xi}\, \psi_r(\xi)\, \psi_n'(\xi) + i_3\dot{\xi}^2\, \psi_r(\xi)\, \psi_n''(\xi) + 2\int_0^{\ell} GI\, \psi_r'\, \psi_n'\, ds]Z_n$$

$$= -i_3(\dot{\omega} + \ddot{\zeta})\psi_r(\xi) - 2\rho\ddot{\zeta}\int_0^{\ell} I\, \psi_r\, ds \tag{3.18}$$

Equations (3.16), (3.17), and (3.18) are ordinary linear non-homogeneous differential equations of second order. They have the following form:

$$\sum_n \ddot{X}_n\, F_{rn}{}^1(t) + \sum_n \dot{X}_n\, F_{rn}{}^2(t) + \sum_n X_n\, F_{rn}{}^3(t) + \sum_n \dot{Y}_n\, F_{rn}{}^4(t)$$

$$+ \sum_n Y_n\, F_{rn}{}^5(t) = K_r{}^1(t)$$

$$\sum_n \ddot{Y}_n F_{rn}^{1}(t) + \sum_n \dot{Y}_n F_{rn}^{2}(t) + \sum_n Y_n F_{rn}^{3}(t) - \sum_n \dot{X}_n F_{rn}^{4}(t)$$

$$- \sum_n X_n F_{rn}^{5}(t) = K_r^{2}(t)$$

$$\sum_n \ddot{Z}_n H_{rn}^{1}(t) + \sum_n \dot{Z}_n H_{rn}^{2}(t) + \sum_n Z_n H_{rn}^{3}(t) = K_r^{3}(t) \tag{3.19}$$

Since ξ and ω are regarded as known functions of t, the coefficients F_{rn}^{j}, H_{rn}^{j}, K_r^{j} are presumably known, at least, in tabular form. Although they are rather complicated, they can be programmed for a computer. It is noteworthy that the root excitation functions $u(t)$, $v(t)$, $\theta(t)$, $\phi(t)$, $\zeta(t)$ do not enter into the functions F_{rn}^{j} or H_{rn}^{j}; they affect only the functions K_r^{j}. The case of a cantilever tube that is fixed at the root is obtained by setting $u = v = 0$ and $\theta = \phi = \zeta = 0$.

If the gun is initially at rest, the initial conditions are

$$X_n(0) = Y_n(0) = Z_n(0) = 0 \quad \text{and} \quad \dot{X}_n(0) = \dot{Y}_n(0) = \dot{Z}_n(0) = 0 \tag{3.20}$$

Equations (3.19) and (3.20) present an initial-value problem of a type for which numerical methods are available. The fact that the Z_r equations are separated from the others is helpful. However, there is coupling between the X_r and Y_r equations. Accordingly, a vertical oscillation of the tube excites a horizontal oscillation. This coupling vanishes if the spin ω is zero. The coupling manifests gyroscopic action of the projectile.

SECTION 4

GYROSCOPIC ACTION OF A BALANCED SPINNING PROJECTILE IN A MOVABLE RIGID CURVED TUBE

4.1 INTRODUCTION

The barrel of a gun is unavoidably slightly curved because of effects of gravity, temperature gradients, manufacturing imperfections, etc. The spinning projectile consequently exerts a gyroscopic couple that tends to bend the tube sideways. As Section 3 shows, the analysis of this action for an actual gun is complicated, although it appears to lie within the scope of numerical methods for differential equations. However, insight is gained by studying much simpler problems that are not without practical significance. Consequently, in this section, the motion of a rigid tube whose axis is a plane curve is analyzed for a gun that is hinged at the breech so that the tube can swing sideways. A resisting moment M that depends arbitrarily on the side sway ϕ and its time derivative $\dot{\phi}$ is introduced. The moment M that yields $\phi = 0$ is that which is experienced by a rigid immovable gun.

The motion of the projectile in the tube is considered to be prescribed. Balloting is disregarded. The projectile is considered to be perfectly balanced. The geometric axis of the projectile is accordingly tangent to the axis of the tube.

For comparative purposes, two different methods of solution are employed. The first treatment is based on the principle of angular momentum. The second treatment is based on Lagrange's equation.

4.2 LATERAL MOTION OF A HINGED, RIGID, CURVED TUBE

The vertical axis a-a (Figure 6) is taken to be a hinge line. The hinge contains a spring and a damper, which may be nonlinear. The hinge allows the tube to swing horizontally. The angular displacement of the tube about the hinge is denoted by ϕ. The angular velocity of the tube is $\dot{\phi}$, where the dot denotes the time derivative. The axis of the tube is considered to be a plane curve with curvature 1/R. The spin of the projectile relative to the tube is $\omega(t)$.

The angular velocity of the projectile has components $\dot{\phi}$ and ω in the plane of the axis of the tube. The axial and normal components of vector $\dot{\phi}$ are $-\dot{\phi}\sin\theta$ and $\dot{\phi}\cos\theta$, as shown by Figure 6. The axial component of $\dot{\phi}$

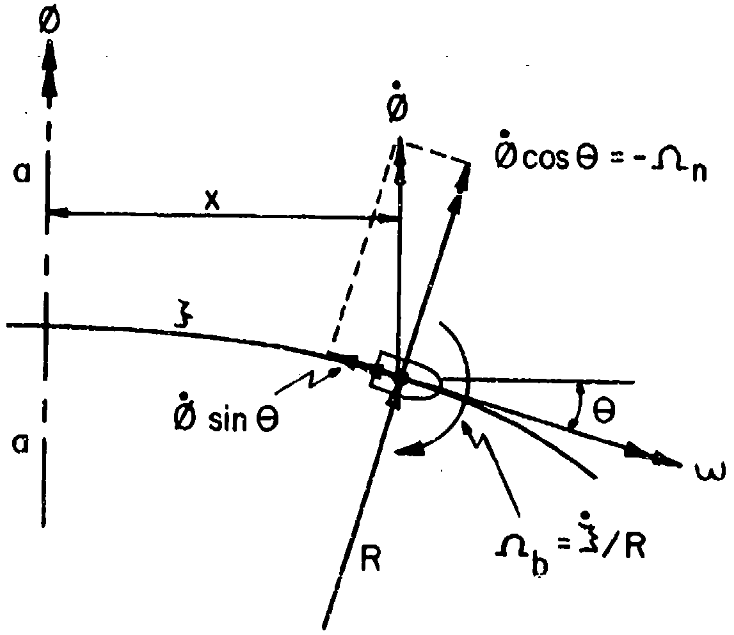

Figure 6. Components of Angular Velocity of a Projectile in a Hinged Rigid Curved Tube

detracts from the spin vector, so the net absolute axial component of angular velocity of the projectile is $\omega - \dot{\phi} \sin \theta$. The transverse component of angular velocity of the projectile is $\dot{\xi}/R$, where $\dot{\xi}$ is the speed of the projectile relative to the tube. Accordingly, the absolute angular velocity components of the projectile are

$$-\Omega_n = \dot{\phi} \cos \theta \ , \quad \Omega_b = \frac{\dot{\xi}}{R} \ , \quad \Omega_a = \omega - \dot{\phi} \sin \theta \tag{4.1}$$

Consequently, the principal components of angular momentum of the projectile about axes through the center of mass of the projectile are

$$H_1 = i_1 \dot{\phi} \cos \theta \ , \quad H_2 = i_1 \frac{\dot{\xi}}{R} \ , \quad H_3 = i_3(\omega - \dot{\phi} \sin \theta) \tag{4.2}$$

where i_1 is the moment of inertia of the projectile about the transverse axis through its center of mass, and i_3 is the moment of inertia of the projectile about its longitudinal axis. The angular momentum of the projectile about the axis of the hinge is

$$H_1 \cos \theta - H_3 \sin \theta + m x^2 \dot{\phi} \tag{4.3}$$

where m is the mass of the projectile and x is the distance from the center of mass of the projectile to the axis of the hinge. Equation (4.3) must be augmented by the angular momentum $J\dot{\phi}$ of the tube and the breech, where J is the moment of inertia of the tube and the attached rotating part of the breech about the axis of the hinge. Consequently, if the angular momentum of the charge is disregarded,* the angular momentum of the system about the hinge line is

$$H = J\dot{\phi} + H_1 \cos \theta - H_3 \sin \theta + m x^2 \dot{\phi} \tag{4.4}$$

Consequently, by Eq. (4.2),

$$H = (J + i_1 \cos^2 \theta + i_3 \sin^2 \theta + m x^2)\dot{\phi} - i_3\omega \sin \theta \tag{4.5}$$

**The angular momentum of the charge may be introduced in an empirical way by augmenting the mass of the projectile by a part of the mass of the charge.*

The angular-momentum principle is expressed by the equation

$$-M = \frac{dH}{dt} \tag{4.6}$$

where $M(\phi,\dot{\phi})$ is the resisting moment of the spring and damper in the hinge. The quantity dH/dt is the substantial derivative; i.e., it is the time rate of change of H with due regard for the time dependence of θ, x, and ω. It can be seen that $d\theta/dt = \dot{\xi}/R$, $d\omega/dt = \dot{\omega}$, and $dx/dt = \dot{\xi}\cos\theta$. Consequently, Eqs. (4.5) and (4.6) yield

$$(J + i_1 \cos^2\theta + i_3 \sin^2\theta + m\,x^2)\ddot{\phi} + 2[(i_3 - i_1)\frac{\dot{\xi}}{R}\sin\theta\cos\theta$$

$$+ m\,\dot{\xi}\,x\cos\theta]\dot{\phi} - i_3\dot{\omega}\sin\theta - i_3\frac{\dot{\xi}}{R}\omega\cos\theta + M(\phi,\dot{\phi}) = 0 \tag{4.7}$$

Equation (4.7) is a second-order ordinary differential equation that determines $\phi(t)$, if the initial values $\phi(0)$ and $\dot{\phi}(0)$ are given. The function $M(\phi,\dot{\phi})$ may be nonlinear. Otherwise, Eq. (4.7) is linear.

If $M = 0$, the gun swings freely. Then, Eq. (4.6) yields H = constant. Therefore, if $M = 0$,

$$(J + i_1 \cos^2\theta + i_3 \sin^2\theta + m\,x^2)\dot{\phi} - i_3\omega\sin\theta$$

$$= (J + i_1\cos^2\theta_0 + i_3\sin^2\theta_0 + m\,x_0^2)\dot{\phi}_0 - i_3\omega_0\sin\theta_0 \tag{4.8}$$

where θ_0, $\dot{\phi}_0$, x_0, ω_0 are initial values. Equation (4.8) determines $\dot{\phi}$ explicitly.

Since θ ordinarily is a very small angle, it is reasonable to make the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ in Eqs. (4.7) and (4.8). If these approximations are made, and if $\dot{\phi}_0 = \theta_0 = 0$, Eq. (4.8) yields

$$\phi(t) = i_3\int_0^t \frac{\omega\,\theta\,dt}{J + i_1 + m\,x^2} \tag{4.9}$$

4.3 LATERAL MOTION DERIVED FROM LAGRANGE'S EQUATION

The velocity components of the center of mass of the projectile (Figure 6) are

$$V_1 = 0 \ , \quad V_2 = x\,\dot{\phi} \ , \quad V_3 = \dot{\xi}$$

where ξ is the distance that the projectile has moved along the axis of the tube. Consequently, the kinetic energy of translation of the projectile is

$$\frac{1}{2}\, m(x^2\, \dot{\phi}^2 + \dot{\xi}^2)$$

The principal rotation components of the projectile are given by Eq. (4.1). Consequently, the kinetic energy of rotation of the projectile is

$$\frac{1}{2}\, i_1(\dot{\phi}^2 \cos^2\theta + \frac{\dot{\xi}^2}{R^2}) + \frac{1}{2}\, i_3(\omega - \dot{\phi}\sin\theta)^2$$

Accordingly, the kinetic energy of the system is

$$T = \frac{1}{2}\, J\, \dot{\phi}^2 + \frac{1}{2}\, m(x^2\, \dot{\phi}^2 + \dot{\xi}^2) + \frac{1}{2}\, i_1(\dot{\phi}^2\cos^2\theta + \frac{\dot{\xi}^2}{R^2})$$
$$+ \frac{1}{2}\, i_3(\omega - \dot{\phi}\sin\theta)^2 \tag{4.10}$$

The virtual work of the external forces is

$$\delta W = -M\delta\phi = Q\delta\phi$$

Accordingly, the generalized external force is $Q = -M$. Lagrange's equation for ϕ is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = Q \tag{4.11}$$

In the present case, $\partial T/\partial\phi = 0$. By Eq. (4.10),

$$\frac{\partial T}{\partial \dot{\phi}} = (J + i_1\cos^2\theta + i_3\sin^2\theta + m\,x^2)\dot{\phi} - i_3\omega\sin\theta \tag{4.12}$$

Consequently, in view of Eq. (4.5), $\partial T/\partial\dot{\phi}$ is identified as the angular momentum H of the system about the hinge. Therefore, Eq. (4.11) is identical to Eq. (4.6), and Lagrange's equation leads to Eq. (4.7).

4.4 MOMENTS IN A RIGID IMMOVABLE CURVED TUBE

If $\phi = 0$, Eq. (4.7) gives

$$M = i_3(\dot{\omega} \sin \theta + \frac{\omega\dot{\xi}}{R} \cos \theta) \tag{4.13}$$

Equation (4.13) gives the sidewise moment on the tube at the breech, if the tube is immovable. It acts to oppose ϕ.

The rifling torque is (Figure 7)

$$i_3\dot{\omega} = M_3 = M_r$$

This is the torque exerted on the projectile by the tube. The torque exerted on the tube by the projectile is $-M_3$. The component of this torque on the axis of the hinge is

$$i_3\dot{\omega} \sin \theta$$

The driving force of the gases exerts no moment about the hinge. There is no force on the projectile transverse to the plane of the axis of the tube. Consequently, the reaction of the forces on the projectile exerts no moment about the hinge line. The gyroscopic couple that the projectile exerts on the tube (Figure 7) is denoted by M_g. Equilibrium of moments about the hinge line yields

$$-M + M_g \cos \theta + M_3 \sin \theta = 0$$

Therefore, by Eq. (4.13),

$$-i_3(\dot{\omega} \sin \theta + \frac{\omega\dot{\xi}}{R} \cos \theta) + M_g \cos \theta + i_3\dot{\omega} \sin \theta = 0$$

This reduces to

$$M_g = i_3 \frac{\omega\dot{\xi}}{R} \tag{4.14}$$

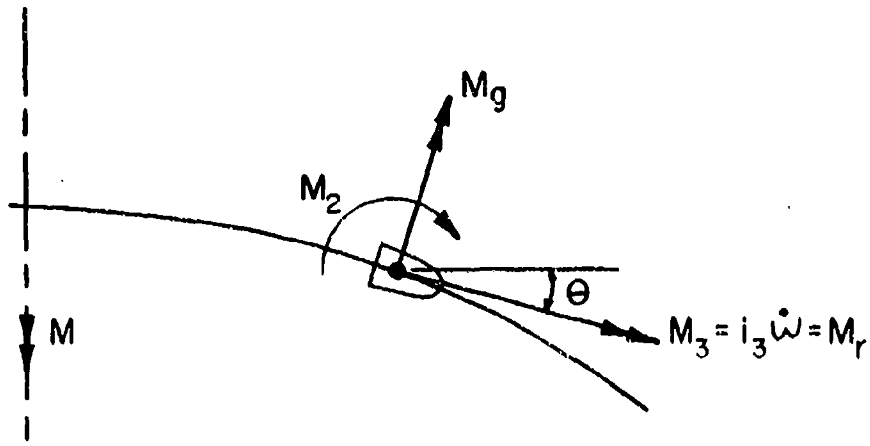

Figure 7. Components of Moments on a Projectile in a Fixed Rigid Curved Tube

The positive sense of M_g is indicated by Figure 7, if the spin of the projectile is that of a right-hand screw advancing along the tube. Although 1/R is very small, ω and v are very large. Consequently, Eq. (4.14) indicates that the gyroscopic action of the projectile might bend the tube appreciably.

The moment component about the binormal is $M_2 = i_1 \ddot{\theta}$. Hence,

$$M_2 = i_1 \frac{d}{dt}\left(\frac{\dot{\xi}}{R}\right) = i_1\left[\frac{\ddot{\xi}}{R} + \dot{\xi}^2 \frac{d}{ds}\left(\frac{1}{R}\right)\right] \tag{4.15}$$

where s is arc length on the axis of the tube.

If the axis of the rigid tube is a space curve, it can be shown that Eq. (4.14) is generalized as follows:

$$M_g = i_3 \frac{\omega\dot{\xi}}{R} - i_1 \frac{\dot{\xi}^2}{R\Sigma} \tag{4.16}$$

4.5 MOVING RIGID TUBE AS A SPECIAL CASE OF A FLEXIBLE TUBE

As an example, Eq. (1.41) is applied to the problem treated in this section. The axis of the tube is a plane curve. The tube is rigid, and the vertical axis (Figure 8) is a hinge line. As before, the angular displacement of the tube about the hinge is ϕ.

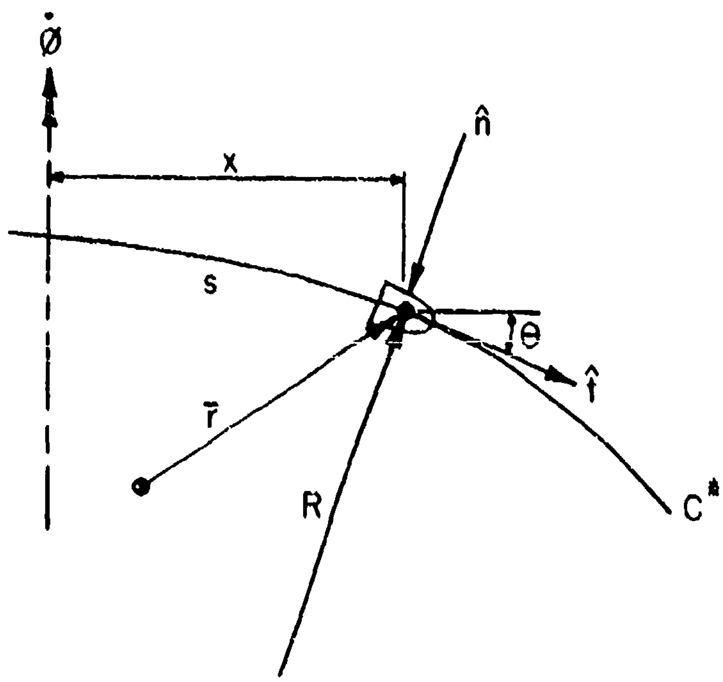

Figure 8. Rotating Rigid Tube

Since the tube is rigid, the rate of twist is zero; i.e., $\partial\tau/\partial t = 0$. The binormal $\hat{b}$ is perpendicular to the plane of Figure 8. It is directed away from the reader. The vector $\partial\hat{t}/\partial t$ has the direction of $\hat{b}$. Its magnitude is $\dot{\phi}\cos\theta$. Accordingly, $\hat{b}\cdot\partial\hat{t}/\partial t = \dot{\phi}\cos\theta$, and Eq. (1.41) yields

$$\frac{\partial \omega_a}{\partial s} + \frac{1}{R}\dot{\phi}\cos\theta = 0$$

Also, $1/R = \partial\theta/\partial s$. Consequently,

$$\frac{\partial \omega_a}{\partial s} = -\dot{\phi}\frac{\partial \theta}{\partial s}\cos\theta$$

Integration yields

$$\omega_a = -\dot{\phi}\sin\theta + f(t) \tag{4.17}$$

It has been shown in Art. 4.2 that $f(t) = 0$, but there seems to be nothing in the theory of the flexible tube that determines $f(t)$. With $f(t) = 0$, Eq. (1.33) and Eq. (4.17) yield the angular velocity components of the projectile:

$$\Omega_n = -\dot{\phi}\cos\theta\ , \quad \Omega_b = \frac{\dot{\xi}}{R}\ , \quad \Omega_a = \omega - \dot{\phi}\sin\theta$$

These results agree with Eq. (4.1).

SECTION 5
CONCLUSIONS

Vector and scalar formulas for the velocity, the acceleration, the angular velocity, and the kinetic energy of a geometrically perfect projectile in a concentric flexible tube are derived rigorously in Section 1. Approximations of these formulas for an initially straight tube also are developed. The relation between twist of the tube and angular velocities of cross sections of the tube is complicated by curvature of the axis of the tube. This matter is examined in Art. 1.9. Also, evaluation of the virtual work of the forces associated with the projectile is primarily a kinematic problem. It is investigated in Art. 1.10. Aside from the complex phenomenon of balloting, Section 1 lays a rigorous kinematic foundation for gun dynamics. Section 2 deals with the forces and moments acting on a dynamically balanced projectile in a flexible tube. It provides formulas for the rifling torque and the gyroscopic couple.

Section 3 treats the action of a spinning projectile on a tapered elastic cantilever tube that has prescribed motion at the breech. The deflections and twist of the tube are expanded in series of natural bending modes and torsional modes of a uniform cantilever beam. Since these modes constitute complete sets of functions, truncated series of them can represent the deflections and twist of the tube to any desired degree of accuracy (in the least-square sense), irrespective of variable taper of the tube. The coefficients (X_n, Y_n, Z_n) in the modal series are time-dependent generalized coordinates for the tube. Lagrange's equations provide linear, second-order, non-homogeneous, ordinary, differential equations for (X_n, Y_n, Z_n). Although the coefficients in these differential equations are complicated functions of time, the differential equations may be expected to be amenable to numerical methods that can be programmed for a digital computer. Although the theory in Section 3 is not immediately applicable to a gun for which the motion of the breech is unknown, it may be assumed tentatively that the motion of the breech complies approximately with that of a completely rigid gun with the type of support and recoil mechanism that is under consideration (Ref. 1).

Although gyroscopic action of the projectile is inherent in the behavior of a flexible tube, the phenomenon is combined with centrifugal effects of the projectile in the deflected tube and other complications. Section 4 isolates the phenomenon in a setting that has practical elements. A rigid curved tube whose axis lies in a vertical plane is hinged at the breech, so that the tube can swing sideways. A resisting moment M, that depends arbitrarily on the angle ϕ of side sway and its time derivative $\dot{\phi}$, is introduced. The moment M that yields $\phi = 0$ is that which is experienced by a rigid immovable gun. A single second-order ordinary differential equation that determines $\phi(t)$ is derived. The function $M(\phi,\dot{\phi})$ may be nonlinear. Otherwise, the differential equation is linear. A numerical study of the differential equation should be instructive.

Gyroscopic couples acting on the projectile are reacted on the tube. According to elementary beam theory, the curvature of the tube is proportional to the bending moment. Consequently, a gyroscopic couple (ideally conceived to act at the cross section of the tube where the center of mass of the projectile lies) introduces a stepwise discontinuity in the curvature. This anomaly portends a puzzling mathematical question, since the gyroscopic couple depends on the local curvature of the tube, but the curvature is indeterminate at the point where the couple is conceived to act. However, numerical methods tend to smooth over discontinuities. For example, any linear combination of natural modes of the tube is continuous. Likewise, a piecewise polynomial is continuous with all its derivatives, except at the junctions of the polynomial segments.

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