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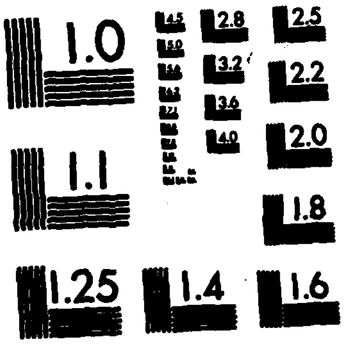

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MODULATIONAL STABILITY OF TWO-PHASE SINE-GORDON WAVETRAINS

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I. INTRODUCTION

In this note we study the modulational stability of real, two-phase sine-Gordon wavetrains. There are three classes of such waves; we find the kink-kink trains are stable, while the breather trains and kink-radiation trains are unstable to modulations.

These results continue the investigations of Flaschka, Forest, and McLaughlin [1] for the KdV equation and of Forest and McLaughlin [2, 3] for the sinh-Gordon and sine-Gordon equations. In a previous paper [2], the sine-Gordon two-phase modulation theory could only be carried to an intermediate stage. Here we use recent results of Ercolani and Forest [4] to complete this project.

References [1, 2, 3] contain detailed accounts of how inverse spectral theory can be used to prescribe and analyze the modulations of quasi-periodic wavetrains. Here we assume some familiarity with these references. In particular, we assume the sine-Gordon relevant discussions in [2] about squared eigenfunctions, conservation laws, and the $\vec{\theta}$ - and $\vec{\mu}$ - representations of two-phase solutions. We also refer to several calculations in [2] which directly apply here. Moreover, we quote two important results from [4].

II. NOTATION AND STATEMENT OF THE PROBLEM

We consider a real sine-Gordon wave which locally appears as a two-phase wavetrain, but which has physical characteristics (such as wave numbers and frequencies) that change slowly over large scales in space and time. Thus, this modulating two-phase wave locally is described by the "$\vec{\theta}$ - representation",

$$u \sim u(\theta_1, \theta_2; \vec{E}(X, T)) \quad , \qquad \text{(II.1a)}$$

where $\vec{E} = (E_1, \dots, E_4)$. The local or fast dependence is described by

$$\begin{cases} \frac{\partial}{\partial x} \theta_j = \kappa_j \ , \\ \frac{\partial}{\partial t} \theta_j = \omega_j \ ; \end{cases} \quad j = 1, 2 \qquad \text{(II.1b)}$$

the slow dependence of $\vec{E}$ on $X = \epsilon t$ models the modulations in the physical characteristics of the wave. (For example, $\vec{\kappa}$ and $\vec{\omega}$ are functions of $\vec{E}(X, T)$.)

There are three physically distinct classes of real two-phase sine-Gordon wavetrains, corresponding to the following possible configurations of $E_1, \dots, E_4$ consistent with real waves (see [2]).

Case 1. Breather Train $\quad E_1, E_2, E_3 = E_1^*, E_4 = E_2^*$, all distinct. (II.2a)

Case 2. Kink-Kink Train $\quad E_1 < E_2 < E_3 < E_4 < 0$. (II.2b)

Case 3. Kink-Radiation Train $\quad E_1 < E_2 < 0,\ E_3 = E_4^*,\ E_3 \neq E_4$. (II.2c)

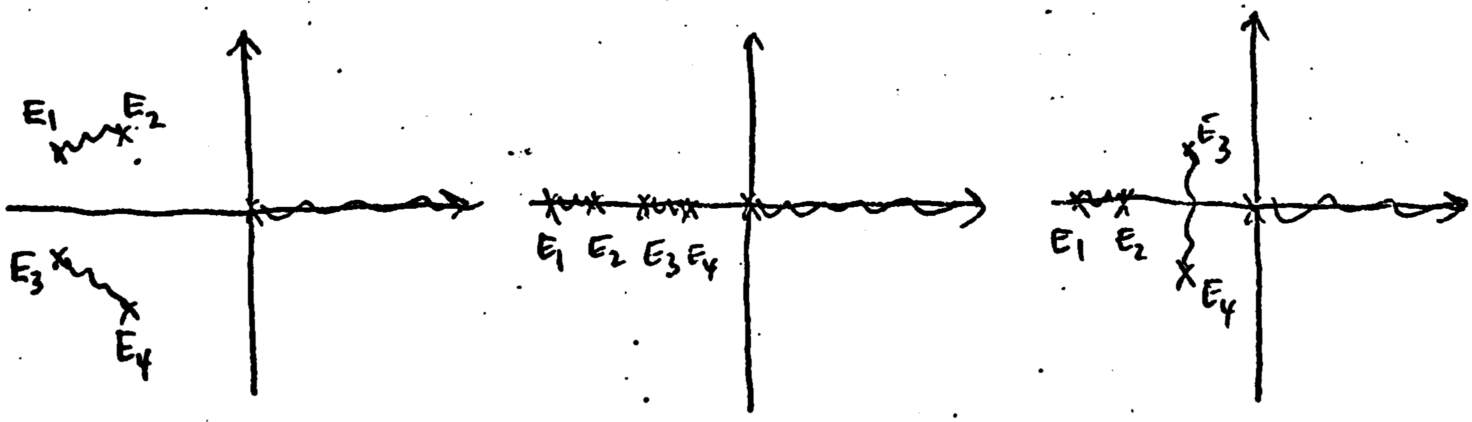

FIGURE 1

The modulational equations for such wavetrains constitute a first-order system of quasi-linear partial differential equations for $E_j(X, T)$, $j = 1, \ldots, 4$. In [2], it is shown this system admits the following representation on the underlying Riemann surface $\mathfrak{R}$ of the curve $(E, R(E))$, where

$$R^2(E) = E \prod_{j=1}^{4} (E - E_j) \quad . \qquad \text{(II.3)}$$

The modulation equations are described by

$$\frac{\partial}{\partial T} \Omega^{(-)} - \frac{\partial}{\partial X} \Omega^{(+)} = 0 \ , \qquad \text{(II.4a)}$$

$$\Omega^{(\pm)} = \Omega_{+1} \pm \frac{1}{16} \Omega_{-1} \ , \qquad \text{(II.4b)}$$

where Ω_{+1}, Ω_{-1} are well-defined differentials on $\mathfrak{R}$ given by

$$\Omega_{+1} = -\frac{1}{2} \left\langle \prod_{j=1}^{2} (E - \mu_j) \right\rangle \frac{dE}{R(E)} \ , \qquad \text{(II.5a)}$$

$$\Omega_{-1} = \frac{\sqrt{\pi E_k}}{2} \left\langle \prod_{j=1}^{2} \left(\frac{E}{\mu_j} - 1 \right) \right\rangle \frac{dE}{E\, R(E)} \ . \qquad \text{(II.5b)}$$

The symbol $\langle f(u) \rangle$ represents an average over the real isospectral manifold for that class of waves, defined with respect to the $\vec{\theta}$-representation by

$$\langle f(u(\theta_1, \theta_2)) \rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(u(\theta_1, \theta_2))\, d\theta_1\, d\theta_2 \ . \qquad \text{(II.6)}$$

The average is computed for frozen values of $\vec{E}(X, T)$.

III. DERIVATION OF AN INVARIANT REPRESENTATION

We begin with the system (II.4), (II.5) as derived in [2], and now use recent developments in [4] to compute these averages as products of single integrals. These arguments apply to all three classes of real two-phase waves; we specialize only at the end to deduce the consequences for each class.

Consider some function $f(u)$ of the modulating two-phase wave $u(\theta_1, \theta_2)$. Viewed as complex coordinates, θ_1, θ_2 parametrize a complex two-torus; the restriction $\theta_1 \in [0, 2\pi)$, $\theta_2 \in [0, 2\pi)$ identifies the real isospectral manifold $\mathcal{M}$, which is a real two-torus $\mathcal{T}^2$ [2, 4],

$$\mathcal{M} = \mathcal{T}^2 = \theta_1 \times \theta_2 = [0, 2\pi) \times [0, 2\pi) . \qquad \text{(III.1)}$$

Thus, the average over $\mathcal{M}$ is given as in (II.6),

$$\langle f(u) \rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(u(\theta_1, \theta_2))\, d\theta_1\, d\theta_2 . \qquad \text{(III.2)}$$

Next we change variables from $\vec{\theta}$ to the "$\vec{\gamma}$-coordinates" of $\mathcal{M}$ provided in [4],

$$\mathcal{M} = \mathcal{T}^2 = (\gamma_1\text{-cycle}) \times (\gamma_2\text{-cycle}) , \qquad \text{(III.3)}$$

where $\vec{\gamma} = (\gamma_1, \gamma_2) \in \mathcal{R} \times \mathcal{R}$ are the following cycles (Figure 2) for the three classes of two-phase waves.

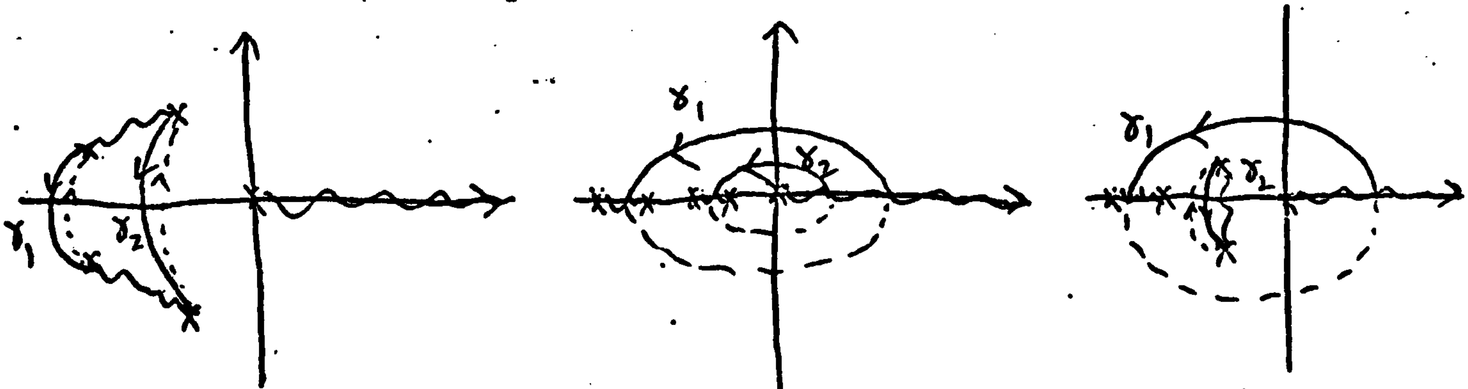

FIGURE 2

Changing variables of integration,

~~With $|\frac{\partial\vec{\theta}}{\partial\vec{\gamma}}|$ = the Jacobian of the change of coordinates,~~ (III.2) becomes

$$\langle f(u) \rangle = \frac{1}{(2\pi)^2} \iint_{\mathcal{M}} f(u(\vec{\gamma})) \, \left|\frac{\partial\vec{\theta}}{\partial\vec{\gamma}}\right| \, d\gamma_1 \, d\gamma_2 \, . \qquad \text{(III.4)}$$

We write the Jacobian as a product,

$$\left|\frac{\partial\vec{\theta}}{\partial\vec{\gamma}}\right| = \left|\frac{\partial\vec{\theta}}{\partial\vec{\mu}}\right| \left|\frac{\partial\vec{\mu}}{\partial\vec{\gamma}}\right| \, ,$$

and analyze each factor. First, by parallel arguments in [1, 2],

$$\left|\frac{\partial\vec{\theta}}{\partial\vec{\mu}}\right| = (2\pi)^2 \, \frac{\det \rho}{\det M} \, , \qquad \text{(III.5a)}$$

where

$$M_{ij} \equiv \int_{\gamma_i\text{-cycle}} \mu^{j-1} \, \frac{d\mu}{R(\mu)} \, , \quad \rho_{nj} = \frac{\mu_j^{2-n}}{R(\mu_j)} \, . \qquad \text{(III.5b)}$$

Second, it is proven in [4] that the map from $\vec{\gamma}$ to $\vec{\mu}$ variables is <u>diagonal</u> and <u>globally defined</u>, with

$$\left.\begin{aligned} &\mu_1(\vec{\gamma}) \equiv \mu_1(\gamma_1) \, , \quad \mu_2(\vec{\gamma}) \equiv \mu_2(\gamma_2) \, , \\ &\left|\frac{\partial\vec{\mu}}{\partial\vec{\gamma}}\right| = \frac{\partial\mu_1}{\partial\gamma_1}(\gamma_1) \, \frac{\partial\mu_2}{\partial\gamma_2}(\gamma_2) \neq 0 \, . \end{aligned}\right\} \qquad \text{(III.6)}$$

For $f(u)$ given by $(E-\mu_1)(E-\mu_2)$ or $(\frac{E}{\mu_1} - 1)(\frac{E}{\mu_2} - 1)$, these facts now allow us to factor the two-fold integrals (III.4). We compute one average and the other will be obvious. From (III.5), (III.6),

$$\left\langle \left(\frac{E}{\mu_1} - 1\right)\left(\frac{E}{\mu_2} - 1\right) \right\rangle \, .$$

$$= \frac{1}{\det M} \iint_{\mathcal{M}} \left(1 - \frac{E}{\mu_1(\gamma_1)}\right) \left| \rho(\vec{\mu}(\vec{\gamma})) \right| \frac{\partial \mu_1}{\partial \gamma_1}(\gamma_1) \; \frac{\partial \mu_2}{\partial \gamma_2}(\gamma_2) \, d\gamma_1 \, d\gamma_2 ,$$

and by the nature of the integrands,

$$= \frac{1}{\det M} \iint_{\mathcal{M}} \det\left[\left(1 - \frac{E}{\mu_i(\gamma_i)}\right) \frac{\mu_i^{j-1}(\gamma_i)}{R(\mu_i(\gamma_i))} \; \frac{\partial \mu_i}{\partial \gamma_i}(\gamma_i) \, d\gamma_i \right] ;$$

since $\mathcal{M} = (\gamma_1\text{-cycle}) \times (\gamma_2\text{-cycle})$ is a product space, Fubini's theorem gives

$$= \frac{1}{\det M} \det\left[\int_{\gamma_i\text{-cycle}} \left(1 - \frac{E}{\mu_i(\gamma_i)}\right) \frac{\mu_i^{j-1}(\gamma_i)}{R(\mu_i(\gamma_i))} \; \frac{\partial \mu_i}{\partial \gamma_i}(\gamma_i) \, d\gamma_i \right] . \qquad \text{(III.7)}$$

Now that the two-fold integrals have been factored into single integrals over the γ_i-cycles, we change variables by (III.6) from γ_i to μ_i, and make use of the result in [4] that the $\vec{\mu}$-cycles are <u>globally</u> homologous to the $\vec{\gamma}$-cycles of Figure 2,

$$\mu_i\text{-cycle} \sim \gamma_i\text{-cycle} , \quad i = 1, 2 . \qquad \text{(III.8)}$$

This yields

$$\left\langle \left(\frac{E}{\mu_1} - 1\right)\left(\frac{E}{\mu_2} - 1\right) \right\rangle = \frac{\det\left[\int_{\gamma_i\text{-cycle}} \left(1 - \frac{E}{\mu_i}\right) \frac{\mu_i^{j-1}}{R(\mu_i)} \, d\mu_i \right]}{\det M}$$

$$= \frac{\det\left[\int_{\gamma_i\text{-cycle}} \left(1 - \frac{E}{\mu}\right) \frac{\mu^{j-1}}{R(\mu)} \, d\mu \right]}{\det\left[\int_{\gamma_i\text{-cycle}} \frac{\mu^{j-1}}{R(\mu)} \, d\mu \right]} . \qquad \text{(III.9b)}$$

REMARK. The formulas (III.9a, b) are precisely those obtained in [2] for the two-phase sinh-Gordon modulations; the only change is the γ_i - cycles are given by Figure 2 for sine-Gordon waves. Therefore, the remaining analysis in [2] applies verbatim, with the following conclusions.

THEOREM (III.1) (AN INVARIANT REPRESENTATION)

(1) The three classes of real two-phase sine-Gordon wavetrains modulate according to

$$\Omega = \frac{\partial}{\partial T}\,\Omega^{(-)} - \frac{\partial}{\partial X}\,\Omega^{(+)} = 0 \;, \qquad \text{(III.10a)}$$

where

$$\Omega^{(\pm)} \equiv \Omega_{+1} \pm \frac{1}{16}\,\Omega_{-1} \;, \qquad \text{(III.10b)}$$

$$\Omega_{+1} = -\frac{1}{2}\,\langle (E-\mu_1)(E-\mu_2)\rangle\,\frac{dE}{R(E)}$$

$$= -\frac{1}{2}\,\Big(E^2 - \sum_{j=1}^{2} c_j^{(+)}\,E^{j-1}\Big)\,\frac{dE}{R(E)} \;, \qquad \text{(III.11a)}$$

$$\Omega_{-1} = \frac{\sqrt{\prod_{1}^{4} E_k}}{2}\,\Big\langle \Big(\frac{E}{\mu_1} - 1\Big)\Big(\frac{E}{\mu_2} - 1\Big)\Big\rangle\,\frac{dE}{E\,R(E)}$$

$$= \frac{\sqrt{\prod_{1}^{4} E_k}}{2}\,\Big(1 - \sum_{j=1}^{2} c_j^{(-)}\,E^{j}\Big)\,\frac{dE}{E\,R(E)} \;. \qquad \text{(III.11b)}$$

(2) $\Omega_{\pm 1}$ are the unique Abelian differentials of the second kind which satisfy the following criteria:

(i) Ω_{+1} is holomorphic except at $E = \infty$,

Ω_{-1} is holomorphic except at $E = 0$.

(ii) Near $E = \xi^{-2} = \infty$, respectively $E = \xi^2 = 0$,

$$\Omega_{\pm 1} \sim \frac{d\xi}{\xi^2} + \text{(a holomorphic part)} .$$

(iii) $\int_{\gamma_i\text{-cycle}} \Omega_{\pm 1} = 0, \quad i = 1, 2,$ where

$$\gamma_1 = \begin{cases} a_1 - b_1 + b_2 \\ b_1 \\ b_1 \end{cases}, \quad \gamma_2 = \begin{cases} a_2 + b_1 - b_2 \\ b_2 \\ a_2 \end{cases} \quad \text{for the} \quad \begin{cases} \text{Breather Train} \\ \text{Kink-Kink Train} \\ \text{Kink-Radiation Train} \end{cases} .$$

(3) Away from the branch points, the expansions of $\Omega_{\pm 1}$ in local coordinates are given by (III.11a, b).

IV. CONSEQUENCES OF THE INVARIANT REPRESENTATION

The three classes of real, two-phase sine-Gordon waves modulate according to

$$\Omega = \frac{\partial}{\partial T} \Omega^{(-)} - \frac{\partial}{\partial X} \Omega^{(+)} = 0 . \qquad \text{(IV.1)}$$

As in [2], expanding (IV.1) near $E = 0$ and $E = \infty$ yields the averages of the familiar "polynomial" conservation laws, such as energy and momentum. Likewise, the same arguments in Section (IV.c) of [2] yield a Hamiltonian form of these modulation equations. We do not expand on these points of view here, but rather move on to deduce the predictions for modulational stability of these three classes of waves.

As in [2], we expand (IV.1) in the local coordinate near each branch point $E = E_k$, $k = 1, 2, 3, 4,$ invoke $\Omega = 0$, and the vanishing of the leading order term in these expansions yields

THEOREM (IV.1) (Riemann Invariants for the Modulation Equations)

For all three classes of real waves, the branch points $\{E_k\}_1^4$ are Riemann invariants, with

$$\frac{\partial}{\partial T} E_k - S^{(k)} \frac{\partial}{\partial X} E_k = 0 , \quad k = 1, \dots, 4, \qquad \text{(IV.2a)}$$

with the characteristic speeds given by

$$S^{(k)} = \frac{\sum_{j=0}^{3} D_j^{(+)} E_k^{j-1}}{\sum_{j=0}^{3} D_j^{(-)} E_k^{j-1}} , \quad k=1, \dots, 4, \qquad \text{(IV.2b)}$$

where $D_j^{(\pm)}$ are given in terms of the coefficients $C_j^{(\pm)}$ of $\Omega_{\pm 1}$

$$D_j^{(\pm)} = \frac{1}{2} [C_j^{(+)} \mp \frac{1}{16} \sqrt{\Pi E_k} \; C_j^{(-)}], \quad j=1, 2,$$

$$\text{(IV.2c)}$$

$$D_0^{(\pm)} = \pm \frac{\sqrt{\Pi E_k}}{32} , \quad D_3^{(\pm)} = - \frac{1}{2} .$$

Analysis of the characteristic speeds $S^{(k)}$ for each class of waves fields.

THEOREM (IV.2) (Modulational Stability of Two-Phase Waves)

(1) Kink-Kink Train. The characteristic speeds $S^{(k)}$, $k=1, \dots, 4$ are real; modulational stability is predicted.

(2) Breather Train. The characteristic speeds $S^{(k)}$, $k=1, \dots, 4$ are complex, with $[S^{(3)}]^* = S^{(1)}$, $[S^{(4)}]^* = S^{(2)}$; modulational instability is predicted.

(3) Kink-Radiation Train. The characteristic speeds $S^{(k)}$, $k=1,\dots, 4$ are complex, with $[S^{(4)}]^* = S^{(4)}$; modulational instability is predicted.

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