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The governing equations for the nonlinear analysis of imperfect, stiffened, laminated, circular, cylindrical, thin shells, subjected to uniform axial compression and torsion, and supported in various ways, are derived and presented. Two types of formulations have been developed: one (w,F - Formulation) is based Donnell-type nonlinear kinematic relations; and the other (u,v,w - formulation) is based on Sanders'-type of nonlinear kinematic relations (small strains, moderate rotations about in-plane axes).

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A solution methodology is developed and presented. Numerical results are generated for certain special geometries, and these serve as bench marks for the solution scheme. Pamametric studies are performed for composite cylinders. The scope of these studies is to assess the effect of (a) geometric imperfections (b) lamina stacking, (c) in-plane and transverse boundary conditions, and (d) load eccentricity on the critical condition. Moreover, dynamic (suddenly applied) critical loads are obtained for certain configurations under axial compression.

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STABILITY OF LAMINATED COMPOSITE SHELLS SUBJECTED TO UNIFORM AXIAL COMPRESSION AND TORSION

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STABILITY OF LAMINATED COMPOSITE SHELLS
SUBJECTED TO UNIFORM AXIAL COMPRESSION
AND TORSION[*]

by

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I

NOMENCLATURE

$$A_{ij} = \sum_{k=1}^{N} Q_{ij}^{k}(Z_k - Z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} Q_{ij}^{k}(Z_k^2 - Z_{k-1}^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} Q_{ij}^{k}(Z_k^3 - Z_{k-1}^3)$$

F = Airy Stress Function

L = Length of Shell

M_{xx}, M_{xy}, M_{yy} = Moment Resultants

N_{xx}, N_{xy}, N_{yy} = Stress Resultants

Q_{ij} = Material Elastic Constant

$\bar{Q}_x$ = Shearing Force at Boundary

R = Radius of Shell

U_T = Total Potential

U_i = Strain Energy

U_P = Potential of External Forces

h_n, h_o = Z Coordinate of Extreme Surfaces of the Shell

q = Pressure Force in Z direction

u, v, w = Displacement Components

w^o = Initial Geometric Imperfection

x, y, z = Coordinates

δ_1 = 0 for Donell's Approximation

= 1 for Sanders' Approximation

$\epsilon_{xx}^o, \epsilon_{xy}^o, \epsilon_{yy}^o$ = Reference Surface Strain Components

NOMENCLATURE

(Continued)

$\varkappa_{xx}$, $\varkappa_{yy}$, $\varkappa_{xy}$ =   Changes of Curvatures and Torsion of Reference Surface

ξ =   Imperfection Amplitude Parameter

σ_{xx}, σ_{yy}, σ_{xy} =   Stress Components

Summary

An imperfect, laminated, circular, cylindrical, thin shell, simply supported or clamped at the boundaries, and subjected to a uniform axial compression and torsion (individually applied or in combination) is analyzed. The analysis is based on nonlinear kinematic relations, linearly elastic material behavior, and the usual lamination theory. The laminate consists of orthotropic laminae, which typically characterize fiber reinforced composites. Two types of formulation have been developed; one is referred to as the w,F-formulation, based on Donnell-type of kinematic relations. The governing equations consist of the transverse equilibrium equation and the in-plane compatibility equation. These two equations are expressed in terms of the transverse displacement, w, and an airy stress resultant function, F. The other, referred to as the u, v, w-formulation, is based on Sanders'-type of kinematic relations. The governing equations for this case consist of the three equilibrium equations. These three equations are expressed in terms of two in-plane displacement components u, v, and the transverse displacement component, w. Donnell's type of shell theory approximation can be treated as a special case in the u, v, w-formulation.

Some results are generated for certain geometries (isotropic and laminated) and these serve as bench marks for the solution scheme. Results are also generated for composite cylinders by changing several parameters. The scope of these parametric studies is to establish the effect of (a) geometric imperfections, (b) lamina stacking, (c) in-plane and transverse boundary conditions and (d) load eccentricity on the critical conditions. Moreover, dynamic critical loads are obtained for certain configurations under axial load (suddenly applied).

CHAPTER I
INTRODUCTION

Shell-like structural configurations find wide uses in complicated aerospace structural systems. Their use requires sophisticated analyses in order to answer questions associated with their behavioral response to external loads and extreme temperature environments. In the past forty years or so, numerous investigations addresses themselves to several specific questions of shell behavior, and the answers to these questions have tremendously enhanced our understanding of their behavior. All of this was done primarily for metallic construction of these configurations. In particular, attention was paid to the degree of approximation involved in the use of various kinematic relations (which led to several linear and nonlinear shell theories), to the discrepancy between theory and experiment for the buckling of shells (post-buckling analyses and imperfection-sensitivity studies), to the use of stiffening for shell configurations (including eccentricity effects) to the effect of support conditions, cutouts, foreign inclusions and others. Moreover, as the size of shell-like structures increased and as the computational capability improved, large computer codes became available, for the analysis of the configurations.

In the recent few years, the constant demand for lightweight efficient structures led the structural engineer to the use of nonconventional materials, such as fiber-reinforced composites. The correct and effective use of these materials requires good understanding of the system response characteristics to external causes (loads, properties of the environment, etc.). Several research programs have been initiated in order to evaluate the physical properties of such materials. The main emphasis in these studies is placed on the characterization of physical properties (finding the constants in the

constitutive relations and how the environment affects them). In addition, there are several efforts related to failure criteria and failure-related effects, such as scissoring and delamination.

In 1975, R. C. Tennyson (1) made a review of previous studies on the buckling of laminated cylinders. According to Tennyson's (1) review, perhaps one of the earliest stability analyses of homogeneous orthotropic cylindrical shells was published by March et al. (2) in 1945. After that time, several theoretical analyses limited to orthotropic shell configurations were performed by Schnell and Bruhl (3), Thielemann et al. (4), and Hess (5). In these studies, simply supported end conditions were partially satisfied. The general linear theoretical solutions to anisotropic cylinders were presented by Cheng and Ho (6) (7), Jones and Morgan (8), Jones and Hennemann (9) and Hirano (10). Several papers were involved in the comparison of the efficiency and accuracy between Flugge's linear shell theory, which was employed by Cheng and Ho (6) (7), and other shell theories (such as the work done by Tasi (11), Martin and Drew (12) whose theory was based on Donnell's equations, and the work done by Chao (13), whose analysis was based on Timoshenko's buckling equations). Stiffened composite cylinderical shells have been analyzed by Jones (14). Terebushdo (15) and Cheng and Card (16). Theoretical analyses of the effect of initial geometric imperfection based on anisotropic shell theory have been published for the loading cases of pure torsion (17) axial compression (18) and combined loads (19) (20). Moreover, several computer codes (21-32) (based on finite elements and/or differences) that deal with the analysis of stiffened shell configurations have been modified in order to account for laminated shell construction. These codes do serve their purpose, and that is that they are very good analytical tools. On the other hand, it is very difficult, if not

possible, to use these codes for parametric studies or for evaluating the applicability and limitations of various shell theories. In this report, the following are presented:

(1) The mathematical formulation and derivation of the governing equations, based on Donnell-type (33) nonlinear kinematic relations and in terms of the transverse displacement component and an Airy stress (resultant) function, defined in the text.

(2) The mathematical formulation and derivation of the governing equations, based on Sanders'-type (34) nonlinear kinematic relations and in terms of the three displacement components (small strains but moderate rotations about in-plane axes).

(3) Solution schemes for both formulations. The solution methodology for the first formulation includes post-limit point behavior, while the solution methodology for the second formulation refers only to the pre-limit point behavior and it is employed to estimate critical static conditions (limit point loads). The listing of the related computer codes are presented in the Appendices of this report.

(4) Some numerical results are generated (and presented herein) with two objectives in mind. (a) Some serve as bench marks for the solution schemes and (b) some limited parametric studies are performed in order to assess effects of boundary conditions and of the lamina stacking sequence, for axially-loaded laminated cylindrical shells.

In closing, this report should be viewed as the first in a series of reports dealing with the behavior of geometrically imperfect, stiffened and laminated, thin, circular, cylindrical shells, supported in various ways (all possible extreme cases of transverse and in-plane boundary conditions) and subjected to static, as well as suddenly applied, destabilizing loads.

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CHAPTER II.

MATHEMATICAL FORMULATION AND SOLUTION METHODOLOGY

The governing equations are derived, with all necessary steps shown in detail, in Appendix A. The geometry is a thin, circular, geometrically imperfect cylindrical shell. The construction consists of an orthogonally and eccentrically stiffened laminate (each lemina is orthotropic). Note that a laminated geometry, an eccentrically stiffened metallic configuration and a metallic shell are all special cases of the construction used herein. The stiffeners are uniform in geometry and with constant close spacing, which allows one to employ the "smeared" technique. The boundary conditions can be of any transverse and in-plane variety. This includes free, s imply-supported and clamped with all possible in-plane combinations.

The loading consists of transverse (uniform lateral pressure) and eccentric in-plane loads, such as uniform axial compression and shear. Eccentric means that the line of action of these loads (applied stress resultants) is not necessarily in the plane of the reference surface.

In the derivation of the governing equations, the usual lamination theory is employed. Moreover, thin shell theory (Kirchhoff - Love hypotheses with two different approximation) and linearly elastic material behavior one assumed. The primary assumptions are listed in Appendix A. On the basis of these general assumptions two sets of field equations are derived. One, referred to as the w,F formulation, is based upon Donnell-type of kinematic

relations. For this case, the governing equations consist of the transverse equilibrium equation and the in-plane compatibility equation. These two equations and the proper boundary conditions are expressed in terms of the transverse displacement component, w, and an Airy stress resultant function, F. The second, referred to as the u, v, w - formulations, is based on Sanders' type of kinematic relations, those corresponding to small rotations about the normal and moderate rotations about in-plane axes. The governing equations, for this case, consist of the three equilibrium equations, expressed in terms of the displacement components u, v, and w. Also, the proper boundary conditions are expressed in terms of u, v, and w. In this formulation, the Donnell approximation is a special case of the more general Sanders' kinematic relations.

The solution methodology is an improvement and modification of the one employed and described in Refs. 36 and 37. For details the reader is referred to Appendix A. A brief description of the solution scheme is given below and only for the w,F - formulation.

1). First, a separated form (fourier series type) is assumed for the dependent variables , w(x,y) and F(x,y). In addition the initial geometric imperfection is also expressed in a similar form.

2). Next, these expressions are substituted into the compatibility equations. Use of trigonometric identities and use of the orthogonality of the trigonometric functions reduces this nonlinear partial differential equation (compatibility) into a system of (4k + 1) nonlinear ordinary differential equations. Furthermore, use of the Galerkin procedure in connection with the equilibrium equation (in the circumferential direction) yields (2k + 1) additional nonlinear ordinary differential equations in the (6k + 2)

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dependent (on x) functions needed to describe the response of the system. Thus, through these steps the two nonlinear partial differential equations are reduced to a set of nonlinear ordinary differential equations.

3). The nonlinear ordinary differential equations are reduced to a sequence of linear systems by employing the generalized Newton's method (Ref. 38). Iteration equation are derived, through this, based on the premise that a solution to the nonlinear set can be achieved by small corrections to an approximate solution.

4). Finally, the field equations (linearized iteration equations) and the corresponding boundary terms (linear set of equation) are cast into finite difference form by employing the usual central difference formula.

Finally, a computer program has been written (see Appendix B for Flow Charts and Program Listings) for generation of results. The solution algorithum is a modification of the one described in Ref. 43. This modification is fully described in Appendix C.

CHAPTER III

DESCRIPTION OF STRUCTURAL GEOMETRY

Three basic configurations are used in generating results. The consist of a four-ply laminated cylinder, an isotropic cylinder and an orthotropic cylinder. All configuration are geometrically imperfect but the imperfection in either symmetric or (virtually) axisymmetric.

The laminated geometries considered in the present study are variations of the one employed in (44). This reference reports experimental results for a symmetric angle-ply laminate, subjected to uniform axial compression and torsion. In addition some isotropic and orthotropic configuration are also used.

III.1 Laminated Geometry

For the laminated geometries, five different stacking combinations of the 4-ply laminate are used in the study.

First, the common geometric and structural features are: each lamina is orthotropic (Boron/Epoxy; AVCO 5505)with properties

$$E_{11} = 2.0690 \times 10^8 kN/m^2 (30 \times 10^6 psi); \nu_{12} = 0.21;$$

$$E_{22} = 0.1862 \times 10^8 kN/m^2 (2.7 \times 10^6 psi.); R = 190.5 cm. (7.5 in.);$$

$$G_{12} = 0.04482 \times 10^8 kN/m^2 (0.65 \times 10^6 psi); L = 381 cm. (15 in.);$$

$$h_{ply} = 0.013462 cm (0.0053 in.)$$

$$(h_{ply} = h_k - h_{k-1}; \text{ for } k = 1, 2, 3, 4; \text{ four plies})$$

(1)

The five different stacking combinations are denoted by I - i, i =
1, 2..5, and correspond to

$$I-1: 45°/-45°/-45°/45° \quad ; \quad I-2: 45°/-45°/45°/-45° \quad ; \quad I3 = -I2$$

$$I-4: 90°/60°/30°/0° \quad ; \quad I-5: 0°/30°/60°/90° \tag{2}$$

Where the first number denotes the orientation of the fibers of the out-
most ply with respect to x, and the last of the innermost. Geometry I-1 is
a symmetric one and it corresponds to that of (44). Geometries I-2 and I-3
denote antisymmetric regular angle-ply laminates, while geometries I-4 and
I-5 are completely asymmetric.

III.2 Isotropic Geometry

The isotropic cylinder has the following geometric and structural fea-
tures (aluminum alloy)

$$E = 7.24 \times 10^7 \, kN/m^2 \, (10.5 \times 10^6 \, psi) \, ; \quad \nu = 0.3$$

$$R = 10.16 \, Cm. \, (4in) \, ; \quad L/R = 1 \, ; \quad R/h = 1000 \tag{3}$$

III.3 Orthotropic Geometry

Finally, the properties of the orthotropic configuration are (single
$0°$ - ply shell made of the Boron/Epoxy material)

$$E_{xx} = 2.069 \times 10^8 \, kN/m^2 \, (30 \times 10^6 \, psi) \, ; \quad \nu_{xy} = 0.21$$

$$E_{yy} = 0.1862 \times 10^8 \, kN/m^2 \, (2.7 \times 10^6 \, psi)$$

$$G_{xy} = 0.04482 \times 10^8 \, kN/m^2 \, (0.65 \times 10^6 \, psi) \, ; \quad R = 190.5 \, Cm. \, (7.5in.)$$

$$L = 381.0 \, Cm. \, (15in.) \, ; \quad t = 0.05385 \, Cm. \, (0.0212in.) \tag{4}$$

III.4 Imperfection Shapes

Two imperfection shapes are used in the study, one which is symmetric, and one which is virtually axisymmetric

Symmetric : $\quad w^o(x,y) = \zeta h \sin\frac{\pi x}{L} \cos\frac{\pi y}{R}$ \qquad (5)

axisymmetric : $\quad w^o(x,y) = \zeta h (- \cos\frac{2\pi x}{L} + 0.1 \sin\frac{\pi x}{L} \cos\frac{\pi y}{R}$ \qquad (6)

where ζ is a measure of the imperfection amplitude. Note that for the symmetric imperfection, Eq. (5), $\zeta = w^o_{max}/h$, while for the (virtually) axisymmetric imperfection, Eq. (6), $\zeta = w^o_{max}/1.1h$.

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CHAPTER IV

NUMERICAL RESULTS AND DISCUSSION

Numerical results are generated, for the geometries described in the preceeding chapter, using the W-F formulation, for two load cases: (a) uniform axial compression and (b) torsion. The loads are applied individually and in combination. The results consist of finding pre- and post-limits point behavior, as well as critical, conditions for static and dynamic (sudden-some results) application of the loads.

The generated results serve a multitude of purpose. Some results serve as bench marks for the solution methodology and the computer code. These results are compared with already known and accepted numbers. Some results correspond to parametric studies, which are performed in order to enhance our understanding of the behavior of laminated shells. The effects of lamina stacking on critical conditions is studied. Furthermore, the effect of in-plane and transverse boundary conditions on critical loads is evaluated for some geometries. Moreover, the imperfection sensitivity is fully assessed for all geometries. Dynamic critical loads are obtained for very few geometries. Most of the generated results are presented in tabular and graphical form. All generated results are not presented, herein, for the sake of brevity. The conclusions, though, are based on all generated data.

IV. 1.0 Axial Compression

Several studies are performed for this load case. Each one of these studies is described and discussed separately.

IV. 1.1 Effect of Lamina Stacking (Static and Dynamic)

For this study, the load is applied through the reference surface (which is the midsurface of the laminate) and the boundary conditions are SS-3 (classical simply supported). The imperfection shape is symmetric, Eq. (5).

Table 4-1 shows critical loads, $\overline{N}^{\ell}_{xx}$ (limit point loads), for each geometry and various values of the imperfection amplitude parameter, ξ. It also presents the range of n-values used in finding critical loads, and the n-value corresponding to the critical condition. These results are also presented graphically on Fig. 4.1.

Geometry I-1 is the one reported in (44). According to this reference, the classical (linear theory) critical load is 165 lbs./in ($\overline{N}_{xx_{cl.}}$) and the experimental value is 106 lbs./in.   Note from Fig. 4.1 that through extrapolation $\overline{N}^{\ell}_{xx}$ at $\xi = 0$ is approximately equal to 148 lbs./in., which is 10% lower than the reported [44] classical value.

The results for geometries I-2 and I-3 are identical. Both geometries are antisymmetric. This is reasonable since (a) the imperfection shape is symmetric with respect to a diametral plane and (b) the axially-loaded cylinder does not distinguish between a positive 45° direction and a negative 45° direction.

Moreover, for virtually the entire range of ξ-values considered, the I-2(3) geometry seems to be the weakest configuration, while the asymmetric configuration corresponding to I-5 is the strongest. The order of going from the weakest to the strongest is I-2(3), I-1, I-4 and I-5. Note that I-5 is a geometry for which the 0° -ply is on the outside. Now since buckling occurs in an inward transverse displacement mode (w is positive), then the outside layer is in compression and it is reasonable to expect the strongest configuration to correspond to I-5, the fibers of the outer ply are in the longitudinal direction.

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Table 4.1 Critical Loads

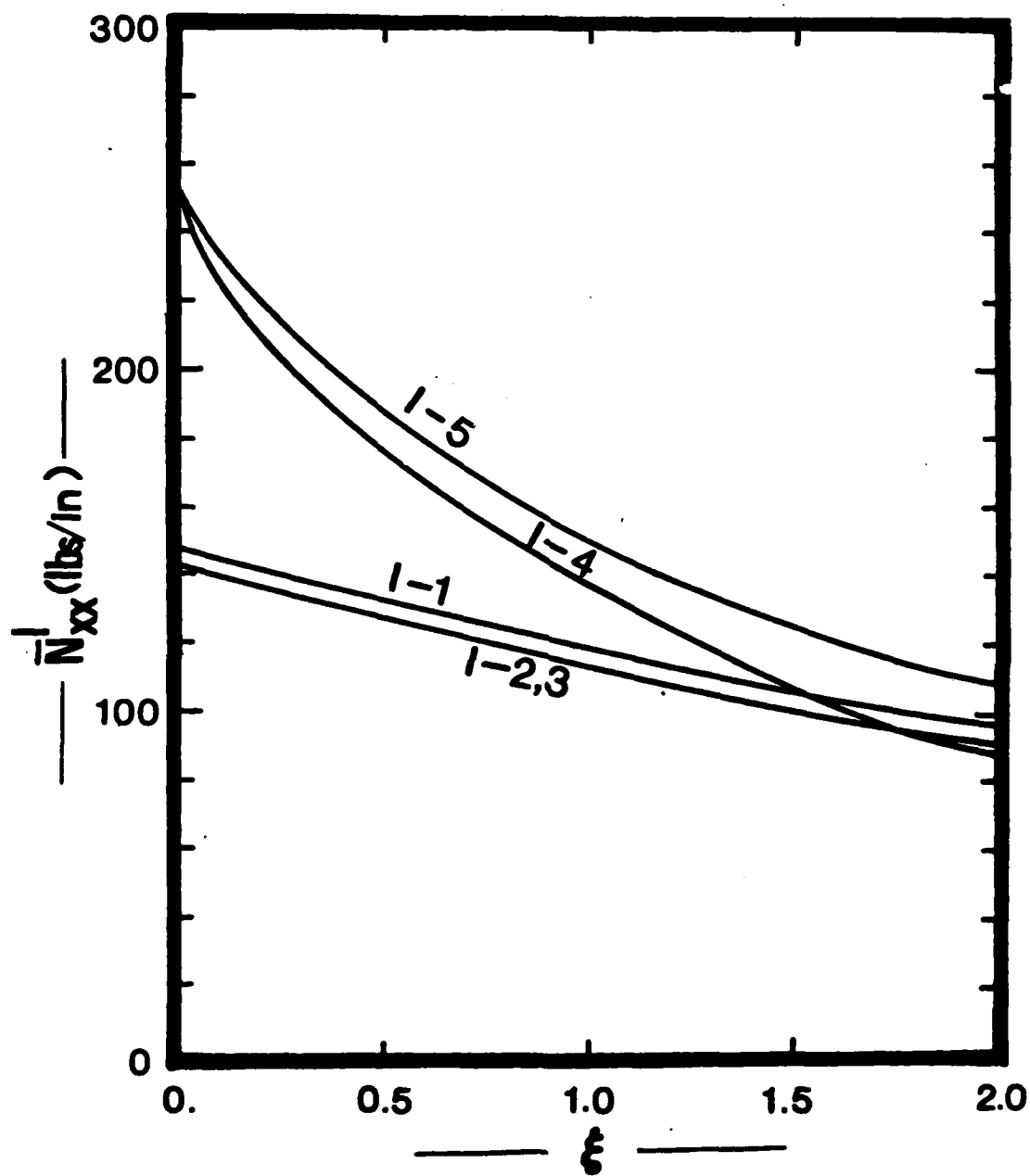| Geometry	ζ	$\bar{N}_{xx}^{\ell}$ lbs/in	n-Range	n at $\bar{N}_{xx}^{\ell}$
I-1	0.05	145.55	5-7	6
	0.50	136.0		6
	1.00	123.0		6
	2.00	98.3		6
I-2,3	0.05	138.80	5-7	6
	0.50	130.0		6
	1.00	118.7		6
	2.00	92.2		6
I-4	0.01	243.1	7-9	8
	0.05	232.03		8
	0.50	178.0		8
	1.00	137.2		8
	2.00	90.0		8
I-5	0.05	233.25	7-9	8
	0.50	191.0		8
	1.00	150.0		8
	2.00	109.5		8

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Fig. 4.1   Imperfection Sensitivity of the various Configurations

Furthermore, the difference between I-4 and I-5 geometries is the order of stacking (one is the reverse of the other). Their behavior, then, can be compared to the behavior of orthogonally stiffened metallic shells with outside and inside stiffening. Geometry I-5 is comparable to outside stiffening, while geometry I-4 to inside.

Figs. 4.2 and 4.3 present typical equilibrium paths for all geometries. Fig. 4.2 corresponds to geometry I-1, while Fig. 4.3 to geometry I-4. As seen, the response is in terms of plots of applied load $\bar{N}_{xx}$ versus average end shortening, e_{AV}. It includes, pre-limit point behavior, limit points and post-limit point behavior, for each ζ-value. The entire curves correspond to the same wave number, n. This n-value is the one that yields critical conditions (the one at the instant of buckling). If a clear picture of post-limit point behavior is desired, one should show the plots that correspond to other wave numbers. This would possibly reveal that the post-limit point curves cross each other, as in the case of isotropic shells (46).

Finally, for the two asymmetric configurations, I-4 and I-5, critical dynamic loads are calculated of the entire ζ-range (see Fig. 4.4). These are obtained by employing the criteria described in (46, 39), and they correspond to lower bounds of critical conditions when the axial compression is applied suddenly with infinite duration. According to this criterion and methodology for estimating critical dynamic conditions, when $\zeta = 0$ (perfect geometry) the static and dynamic critical loads are the same. As the imperfection amplitude increases the dynamic loads are smaller than the static loads. For these geometries, I-4 and I-5, and $0 \leq \zeta < 2.0$, the dynamic critical load, $\bar{N}_{xx}^d$ is never smaller than 60% of the corresponding static load, $\bar{N}_{xx}^\ell$.

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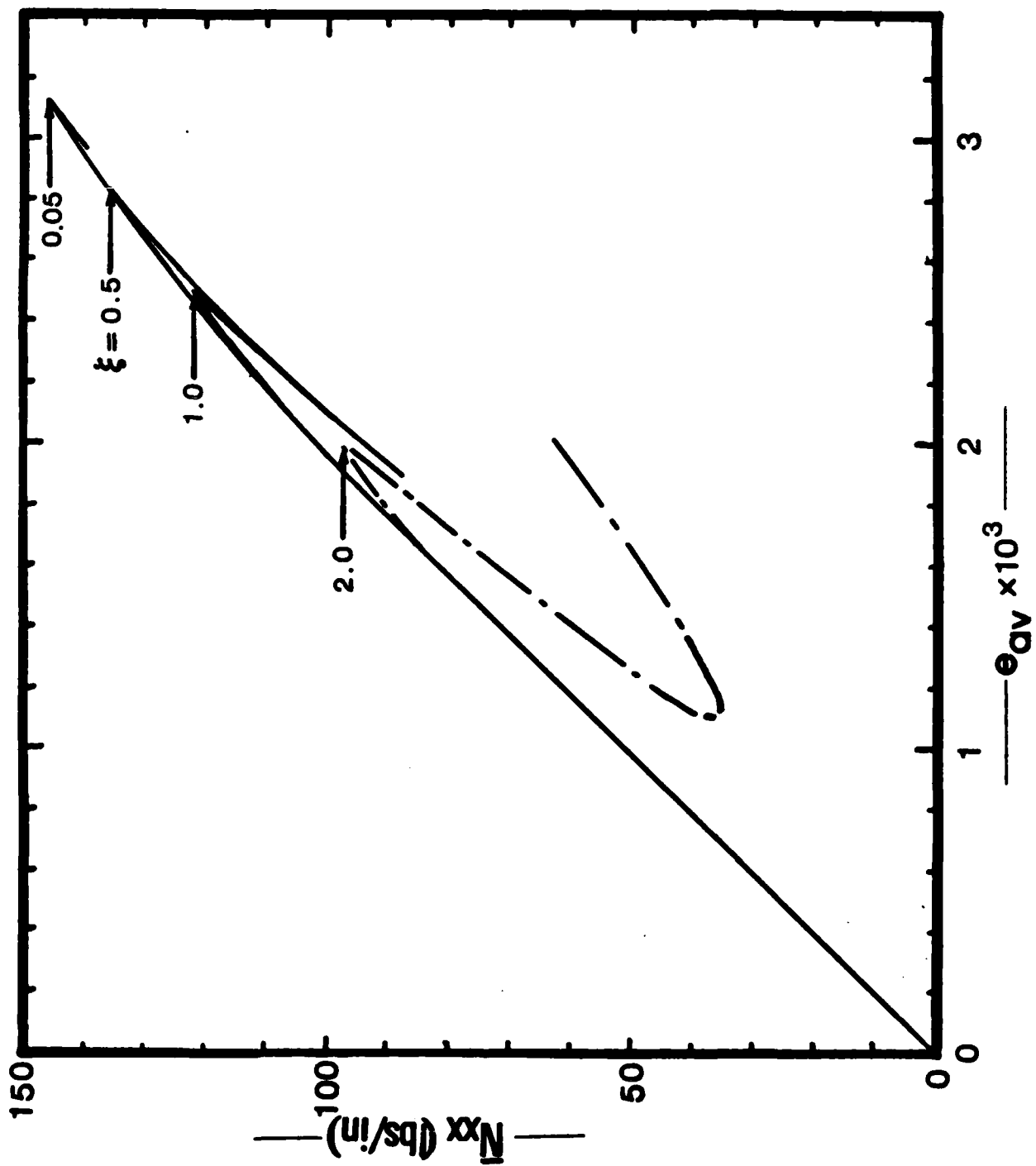
Fig. 4.2 Axial load, N_{xx}, Versus Average End Shortening, e_{av} (Conf. I-1)
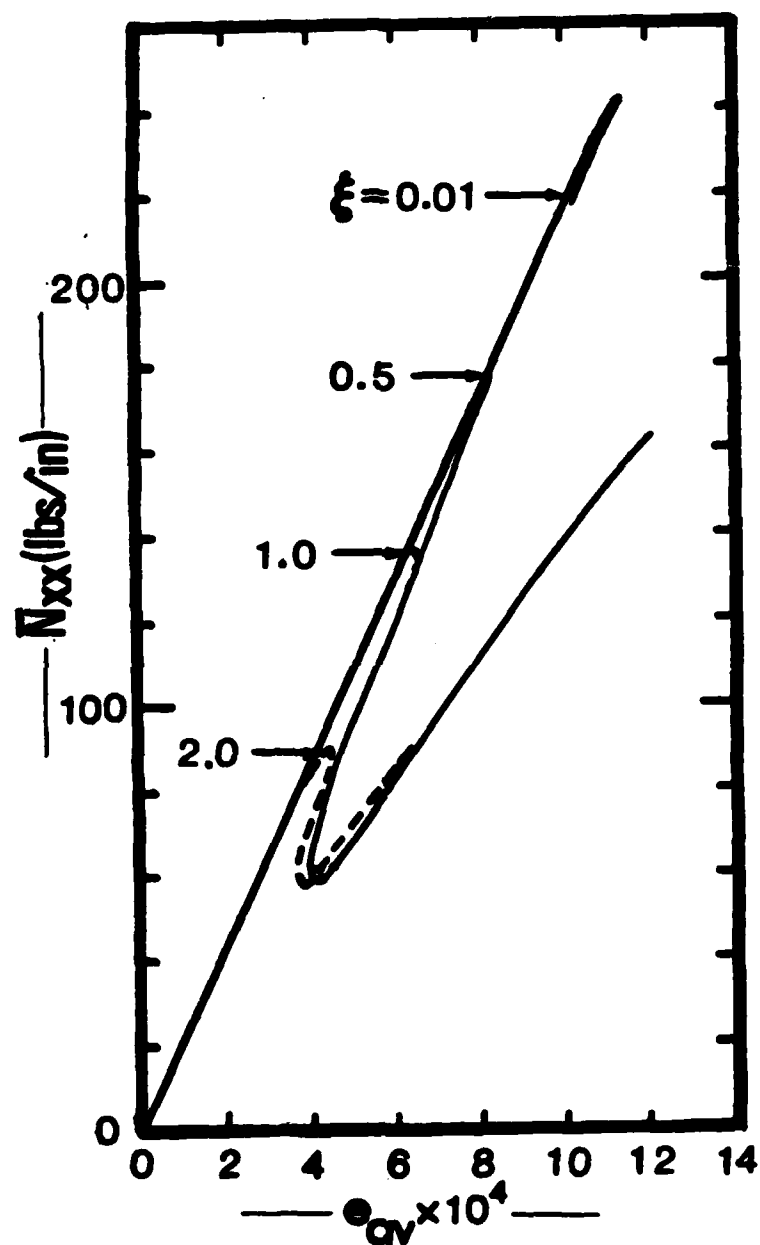
Fig. 4.3 Axial load, N_{xx}, Versis Average End shortening, e_{av} (Conf. I-4)
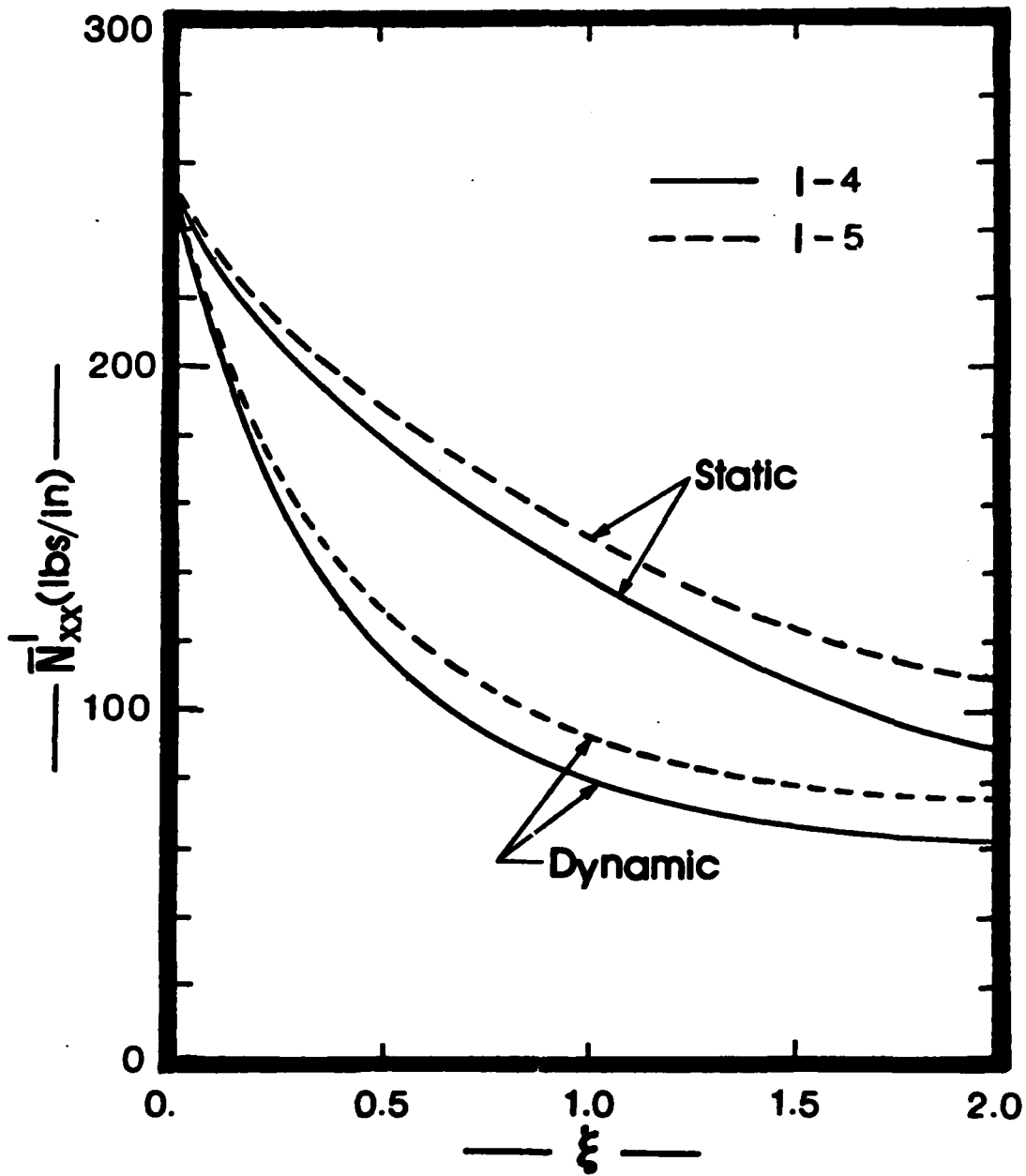
Fig. 4.4   Static and Dynamic critical loads versus Imperfection Amplitude ξ
(Conf. I-4 and I-5).

IV. 1.2 Effect of Boundary Conditions

The effect of both transverse and in-plane boundary conditions are assessed.

Results are also generated for the isotropic geometry (aluminum alloy) and various in-plane boundary conditions. These serve as bench marks for the solution scheme, and the results are presented, in part, on Table 4.2 and Fig. 4.5. For this geometry the shape of the imperfection is taken to be axisymmetric, Eq. (6). On Table 4.2, the n-value that corresponds to the critical load is given in brackets. Note that for small ζ-values (see Fig. 4.5), the trend is exactly that suggested by Hoff and Ohira, independently (see (47)), i.e., the weakest configuration is SS-1, the next one SS-2, while SS-3 and SS-4 yield the classical results. Note also that, through extrapolation, (as $\zeta \rightarrow 0$), the present results agree with those of (47). For SS-1 the ratio of critical load to classical load is 0.55, for SS-2 0.68, and for SS-3 and SS-4 0.98. Clearly here (isotropic case) the geometry for boundary conditions SS-1 and SS-2, is not very sensitive to geometric imperfection, while for SS-3 (primarily) and SS-4, it is. Note that, for small ζ-values, the v = const. in-plane boundary conditions (SS-3 and SS-4) yield a stronger configuration. For higher ζ-values the stronger configuration corresponds to u = const. in-plane boundary conditions (SS-2 and SS-4).

Table 4.2 Effect of In-Plane Boundary Condition on Critical Load (Isotropic Geometry, Simply Supported Case).

ζ	$\dfrac{\overline{\ell}}{\text{Nxx}}$, kN/m^2 (lbs/in.)			
	SS-1	SS-2	SS-3	SS-4
.10	2.52 (14.40) [n=12]	3.05 (17.40) [n=15]	3.973 (22.69) [n=13]	4.307 (24.60) [n=15]
.50	2.45 (13.98) [n=12]	2.89 (16.50) [n=15]	2.905 (16.59) [n=13]	4.027 (23.00) [n=15]
1.00	2.36 (13.50) [n=12]	2.68 (15.30) [n=15]	1.985 (11.34) [n=13]	3.192 (18.23) [n=15]

Note that, no attempt is made here to find the shape of the imperfection that yields the lowest critical load. For the case of the laminated shell, the imperfection amplitude parameter, ζ, is varied from 0.05 to two. The first number, 0.05, corresponds to a virtually perfect geometry shell, while the second number (two) denotes an amplitude in the neighborhood of two shell thicknesses (this is considered very large for thin construction).

In order to establish the imperfection sensitivity of the laminated shell and the effect of boundary conditions on the limit point load (critical load), geometry I-5 is employed, along with a symmetric type of imperfection, Eq. (5). As already established, geometry I-5 yields the strongest configuration for SS-3, by comparison to all other geometries (I-i, i = 1, 2, 3, 4).

Table 4.3 Effect of Boundary Conditions on Critical Loads. (Laminated Geometry I-5).

ζ	$\overline{N}_{xx}^{\ell}$, kN/m (lbs/in)							
	SS-1 n=7	SS-2 n=8	SS-3 n=8	SS-4 n=9	CC-1 n=8	CC-2 n=9	CC-3 n=8	CC-4 n=9
0.05	27.32 (156.0)	32.39 (185.70)	40.84 (233.25)	46.79 (267.26)	41.88 (239.20)	46.32 264.46)	41.97 (239.70)	——
0.50	26.76 (152.83)	31.78 (181.51)	33.43 (190.90)	40.15 (229.3)	37.10 (211.86)	40.75 (232.70)	37.22 (212.59)	41.44 (236.71)
1.00	25.84 (147.55)	30.04 (171.58)	26.27 (150.00)	32.92 (188.00)	29.53 (168.62)	33.62 (192.00)	29.51 (168.57)	34.63 (197.80)
2.00	20.44 (116.74)	23.21 (132.55)	18.67 (106.62)	21.20 (121.10)	19.65 (108.88)	21.27 (121.50)	19.04 (108.75)	21.95 (125.37)
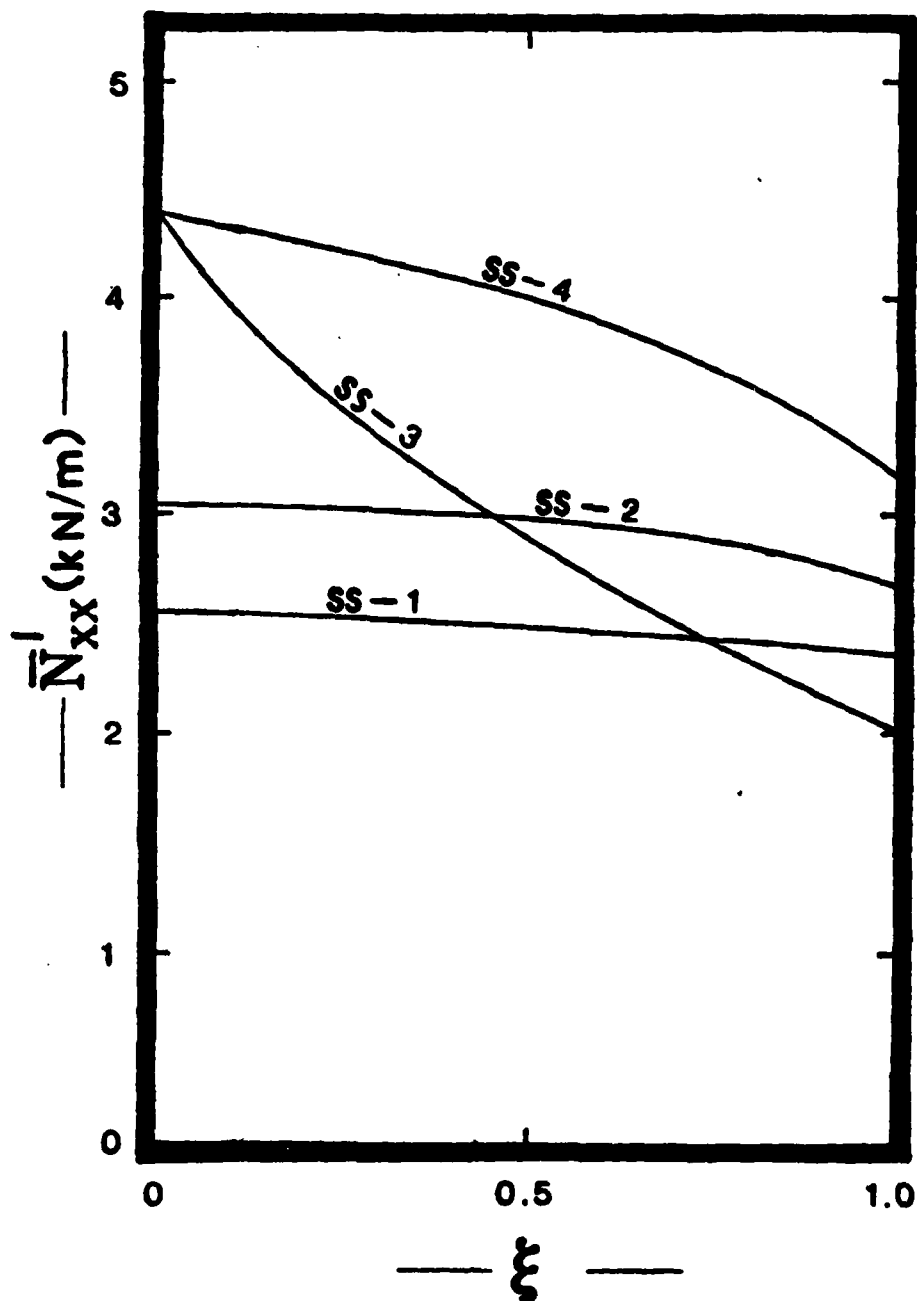
Fig. 4.5 Effect of In-plane Boundary Conditions on the Imperfection Sensitivity of Isotropic Geometry (SS-1)

Table 4.3 lists critical loads for various boundary conditions and ζ-values ($\zeta = w^o_{max}/h$; for this case). The value of n denotes the number of full waves around the circumference at the instant of buckling. These results are shown graphically on Figs. 4.6 and 4.7. A number of observations are made. First, for low ζ-values (see Fig 4.6) SS-3 and SS-4 yield stronger configurations than SS-1 and SS-2. For higher values of ζ, SS-2 and SS-4 yield stronger configurations than SS-1 and SS-3. Another way of stating the same thing is that for low ζ-values the v = const. in-plane boundary condition yields a stronger configuration, while for higher ζ-values the u = const. in-plane boundary condition yields higher critical loads. This conclusion is the same for isotropic geometries. On the other hand, for the clamped case, CC-2 and CC-4 (u = const.) yield stronger configurations than CC-1 and CC-3 for the entire ζ-range considered. Another observation is that for SS-1 and SS-2 the geometry is not as sensitive to initial geometric imperfections as it is for SS-3, SS-4, and CC-i (i = 1, 2, 3, 4) [see Figs. 4.6 and 4.7]. It is also worth mentioning that a comparison between the values at ζ = 0 between SS-1 and SS-4 is reminiscent of what happens in the isotropic case (the critical load for SS-1 is virtually half the value of that for SS-4).

IV. 1.3 Effect of In-plane Load Eccentricity

Next, the effect of load eccentricity is assessed. In all configurations for which results are generated, the shell midsurface is taken as the reference surface. Then it is assumed that the uniform axial compression is applied eccentrically, which induces a bending moment at the boundary, $\bar{M} = \bar{E} \, \bar{N}_{xx}$ [see Eqs A-35 & A-36]. Note that this load eccentricity affects only the simply supported boundary conditions.
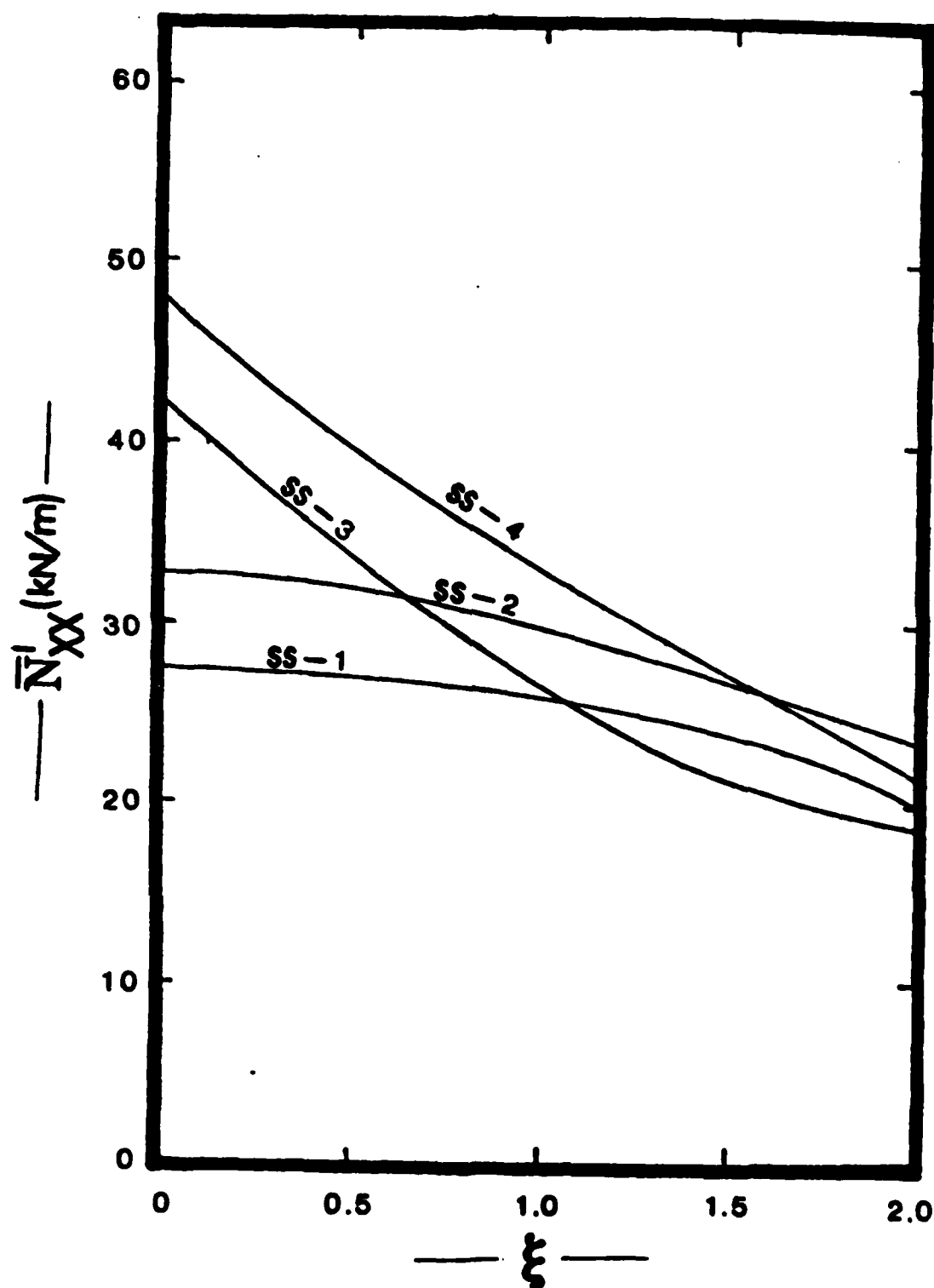
Fig. 4.6 Effect of In-plane Boundary Conditions on the imperfection Sensitivity
of Geometry I-5 (ss-1)

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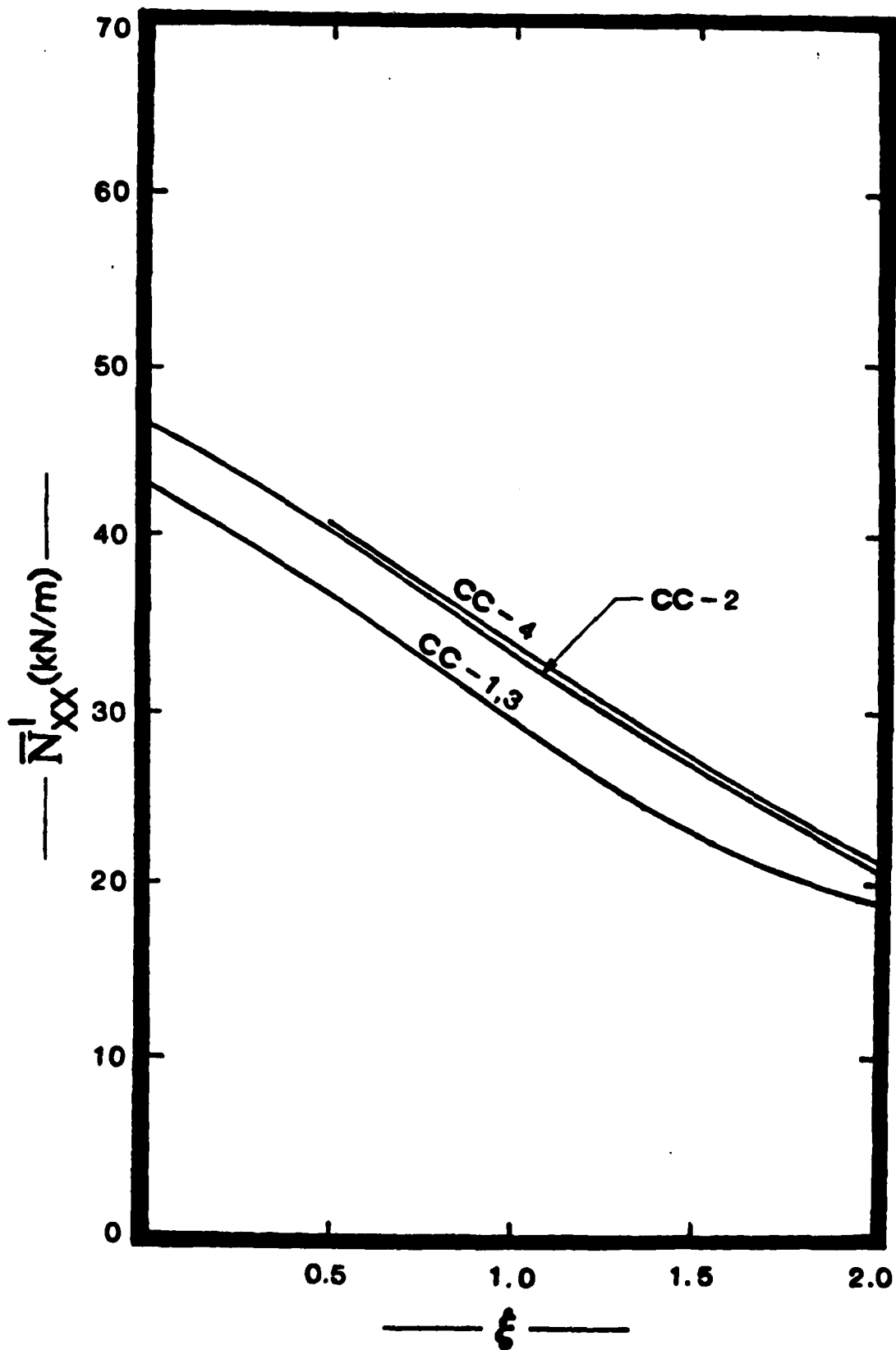
Fig. 4.7 Effect of In-plane Boundary Conditions on the Imperfection Sensitivity of Geometry (cc-1)

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Results are generated and presented for the isotropic geometry, orthotropic geometry, and laminated I-1, I-4 and I-5 geometries, using a symmetric imperfection shape Eq. (5), and classical simply supported boundary conditions SS-3.

These results are, in part, presented on Tables 4.4-4.6.

One might expect a negative edge moment (corresponding to positive load eccentricity) to have a stabilizing effect on an axially-load cylindrical shell, regardless of the construction. Contrary to this, the generated results do not support the expectation. For small eccentricities ($-0.5 < E/h < 0.5$) and isotropic geometry (see Table 4.4) the response seems to be insensitive to the eccentric application of the load. This is true for both imperfection shapes [axisymmetric and symmetric, Eq. (5) & (6)].

Table 4.4 Effect of Load Eccentricity (Isotropic & Orthotropic)

Imperf. Shape & Geometry	e/h ξ	$\overline{N}_{xx}{}^{\ell}$ in kN/m (lbs/in.)						
		12.5	2.5	0.5	0	-0.5	-2.5	-12.5
Axisym. Eq.(23)	0.5	3.08 (17.57)	2.40 (13.726)	2.84 (16.20)	2.90 (16.59)	2.92 (16.58)	2.99 (17.072)	2.47 (14.01)
Isotropic	1.0			1.98 (11.336)	1.99 (11.342)	1.98 (11.337)		
Sym, Eq. (22) Isotropic	0.5			3.026 (17.284)	3.097 (17.686)	3.100 (17.704)		
Axisym. Orthotropic	1.0			12.41 (70.89)	12.39 (70.74)	12.36 (70.57)		

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Table 4.5 Effect of Load Eccentricity (Laminated I-1 Geometry)

ζ	$\bar{N}^{\ell}_{xx_{cr}}$ in kN/m (lbs/in.)					
	$\bar{E}/h = 0.5$		$\bar{E}/h = 0$		$\bar{E}/h = -0.5$	
	Axisym. Eq. (23)	Sym. Eq. (22)	Axisym. Eq. (23)	Sym. Eq. (22)	Axisym. Eq. (23)	Sym. Eq. (22)
0.5	22.21 (126.85)	21.75 (124.2)	23.58 (134.71)	22.85 (130.49)	26.52 (151.48)	23.35 (133.34)
1.0	19.89 (113.61)	20.31 (115.98)	20.46 (116.85)	20.88 (119.25)	20.78 (118.7)	21.82 (124.6)
2.0	13.10 (74.83)	17.07 (97.46)	13.12 (74.91)	17.21 (98.30)	13.17 (75.22)	17.33 (99.00)

SS-4 boundary conditions and n = 6

Table 4.6 Effect of Load Eccentricity (Laminated I-4 and I-5 Geometries; Symmetric Imperfection; SS-3 boundary conditions).

ζ	$\bar{N}^{\ell}_{xx}$ in kN/m (lbs/in.); n = 8					
	I - 4 geometry			I - 5 geometry		
	$\bar{E}/h = 0.2569$	$\bar{E}/h = 0$	$\bar{E}/h = -0.2569$	$\bar{E}/h = 0.2569$	$\bar{E}/h = 0$	$\bar{E}/h = 0.2569$
0.5	30.61. (174.70)	30.66 (175.08)	30.67 (175.18)	33.00 (188.49)	33.44 (191.00)	36.16 (206.52)
1.0	24.07 (137.45)	24.02 (137.18)	24.08 (137.50)	28.76 (164.27)	26.27 (150.00)	29.18 (166.62)
2.0	15.78 (90.10)	15.76 (90.00)	15.75 (89.93)	18.90 (107.96)	18.67 (106.62)	18.90 (107.85)

For very large eccentricities ($|\bar{E}/h| > 12$), positive eccentricity has a stabilizing effect, while negative eccentricity has a destabilizing effect. In the intermediate range an irregularity is observed. It was suspected that one possible reason for this behavior may be attributed to the Poisson effect. As the load is applied, quasistatically, the midportion of the shell moves outward because of the Poisson effect; it reaches a maximum expansion, before the load reaches its critical value, and then an inward motion takes place, and finally at and after collapse this inward motion continues. This sequence of events and the corresponding stabilization or destabilization of the load eccentricity is heavily dependent on the value of Poisson's ratio or the A_{12} term in the extensional stiffness matrix. For instance, some data are generated, for the isotropic geometry ($\xi = 0.5$; SS-3 and axisyymmetric imperfection) but with $\nu = 0.1$. The limit point loads, N_{xx}^{ℓ}, (critical load) for three values of eccentricity ($\bar{E}/h$) are: 3.305 kN/m (18.88 lbs/in) for $\bar{E}/h = +0.5$; 2.76 kN/m (15.81 lbs/in.) for $E/h = 0$; and 2.745 kN/m (15.68 lbs/in) for $E/h = -0.5$. This clearly shows that positive eccentricity has a stabilizing effect. This observation is also true for the orthotropic geometry (see Table 4.4) for which the value of A_{12} is small by comparison to A_{11}. On the other hand, for $\nu = 0.3$ and the laminated geometries for which the values of A_{12} are of the same order of magnitude as A_{11}, it cannot be said that positive eccentricity has a stabilizing effect (see Tables 4.5 and 4.6). In reality, for these geometries no definite conclusion should be drawn regarding stabilization through load eccentricity (or applied edge moment). It is worth observing, though, that for all laminated geometries (see Tables 4.5 and 4.6), whatever the effect is, it does diminish with increasing amplitude of imperfection.

IV. 2.0 Torsion with and without Axial Compression

For this particular load case, in addition to the axisymmetric shape for the geometric imperfection, two additional shapes are employed in the studies. These additional shapes correspond to approximations of the linear theory (see Appendix D) buckling modes for positive and negative torsion for all five geometries.

In particular, Appendix C deals with solutions to the linearized buckling equations for the case of pure torsion. To this end, the Galerkin procedure is employed and the following approximation is employed for the buckling modes

$$W' = \sum_{n=1}^{N} \sum_{i=1}^{M} \left(A_{in} \cos \frac{ny}{R} + B_{in} \sin \frac{ny}{R} \right) \left[\frac{L}{i\pi} \sin \frac{i\pi x}{L} \right.$$

$$\left. - \frac{L}{(i+2)\pi} \sin \frac{(i+2)\pi x}{L} \right] \tag{7}$$

Because of orthogonality, only one n-value is needed In Appendix D a ten-term approximation (M=5) is obtained for all five geometries. By stu..ying the results, one two-term approximation for positive torsion, $w^o(+)$, and one two-term approximation for negative torsion, $w^o(-)$, for all five geometries are used in this study. The various coefficients are first normalized with respect to B_{2n}, Eq. (7), and then adjusted such that the maximum amplitude is ξh.

$$W^{\circ}(+) = \xi h \left[0.536769 \cos\frac{\pi y}{R} \left(\sin\frac{\pi x}{L} - \frac{1}{3}\sin\frac{3\pi x}{L} \right) \right.$$

$$\left. - 0.670961 \sin\frac{\pi y}{R}\left(\sin\frac{2\pi x}{L} - \frac{1}{2}\sin\frac{4\pi x}{L}\right) \right] \qquad (8)$$

$$W^{\circ}(-) = \xi h \left[0.583128 \cos\frac{\pi y}{R}\left(\sin\frac{\pi x}{L} - \frac{1}{3}\sin\frac{3\pi x}{L} \right. \right.$$

$$\left. \left. + 0.64792 \sin\frac{\pi y}{R}\left(\sin\frac{2\pi x}{L} - \frac{1}{2}\sin\frac{4\pi x}{L}\right) \right] \qquad (9) \right.$$

$$W^{\circ}_{max}/h = \xi \qquad (10)$$

The generated results for this case are presented, in part, both in tabular and graphical forms. The discussion, though, and the related conclusions are based on all data.

First, Table 4.7 shows values of critical torsion, $\bar{N}^{\ell}_{xy}$, for the two asymmetric imperfection shapes, Eqs. (8) and (9) (corresponding perfect geometry buckling modes for positive and negative torsion) and several values of the imperfection amplitude parameter. *The torsion is applied in both directions* and the critical values are recorded. The corresponding minimizing value of n (number of full waves) is shown in parenthesis.

Note that the linear theory, perfect geometry critical values (from Appendix D) for geometry I-1 are 39.9 lbs./in. for positive torsion, and -75.5 lbs./in. for negative torsion. Moreover, the experimental results obtained from (44) for this geometry (I-1) are 26.5 lbs./in. for negative torsion.

Note that the construction (orientation of the plies) is such that the configuration is much weaker when loaded in the negative direction, regardless of which of the two imperfection shapes is used. Furthermore, when $w^{\circ}(+)$ is present the configuration is somewhat sensitive for positive torsion (see second column at $\xi = 0.10$, $\bar{N}_{xy} = 35.32$ sensitive for negative torsion (see third column). On the other hand, when $w^{\circ}(-)$ the reverse is true, i.e. the

Table 4.7 Critical Shear Stress Resultant
(Geometry I-1 ; Positive & Negative Torsion)

ζ	For v^o(+) ; Eq. (8)		For v^o(-) ; Eq. (9)	
	$\bar{N}_{xy}$ lbs./in. (n)	$-\bar{N}_{xy}$ lbs./in. (n)	$\bar{N}_{xy}$ lbs./in. (n)	$-\bar{N}_{xy}$ lbs./in. (n)
0.1	35.32 (11)	-93.94 (13)	36.83 (11)	-63.44 (9)
0.5	31.57 (11)	-92.80 (13)	36.06 (10)	-57.61 (8)
1.0	28.32 (11)	-92.00 (13)	35.17 (10)	-52.11 (8)

Table 4.8 Critical Shear Stress Resultant
[for all geometries and v^o(+)]

ζ	$\bar{N}_{xy}$ in lbs./in. (n)				
	I-1	I-2	I-3	I-4	I-5
0.1	35.32 (11)	46.40 (9)	46.36 (9)	44.18 (12)	66.49 (12)
0.5	31.57 (11)	41.81 (9)	41.84 (9)	38.75 (12)	56.91 (12)
1.0	28.32 (11)	37.89 (9)	37.96 (9)	34.22 (12)	48.72 (12)

Table 4.9 Critical Axial Compression-Torsion
Interaction Data* (Geometry I-1;
Axisymmetric Imperfect)

ζ	n	6	10	10	10	11
0.1	$\bar{N}_{xx}$	146.1	135.1	95.9	40.9	0.0
	$\bar{N}_{xy}$	0	10.0	20.0	30.0	36.7
	n	6	10	11	11	11
0.5	$\bar{N}_{xx}$	140.2	128.9	81.9	28.7	0.0
	$\bar{N}_{xy}$	0.0	10.0	20.0	30.0	35.3
	n	6	6	10	10	11
1.0	$\bar{N}_{xx}$	117.7	117.2	87.3	48.4	0.0
	$\bar{N}_{xy}$	0.0	2.0	16.0	24.0	33.8
	n	6	6	10	10	11
1.5	$\bar{N}_{xx}$	93.7	93.2	73.6	37.8	0.0
	$\bar{N}_{xy}$	0.0	2.0	16.0	24.0	32.5

*The unit of the stress resultant is lbs./in.

Table 4.10 Critical Axial Compression-Torsion
Interaction Data* [Geometry I-1;
v^o(+), Eq. (8)]

ζ	n	11	12	11	11	11
0.1	$\bar{N}_{xx}$	141.5	132.1	87.5	31.0	0.0
	$\bar{N}_{xy}$	0.0	10.0	20.0	30.0	35.3
	n	11	11	11	11	11
0.5	$\bar{N}_{xx}$	137.4	123.0	87.4	43.2	0.0
	$\bar{N}_{xy}$	0.0	8.0	16.0	24.0	31.6
	n	11	12	11	11	11
1.0	$\bar{N}_{xx}$	126.8	102.9	73.1	40.4	0.0
	$\bar{N}_{xy}$	0.0	7.0	14.0	21.0	28.3
	n	11	11	11	12	11
1.5	$\bar{N}_{xx}$	105.7	80.9	63.8	26.2	0.0
	$\bar{N}_{xy}$	0	7.0	14.0	21.0	25.4

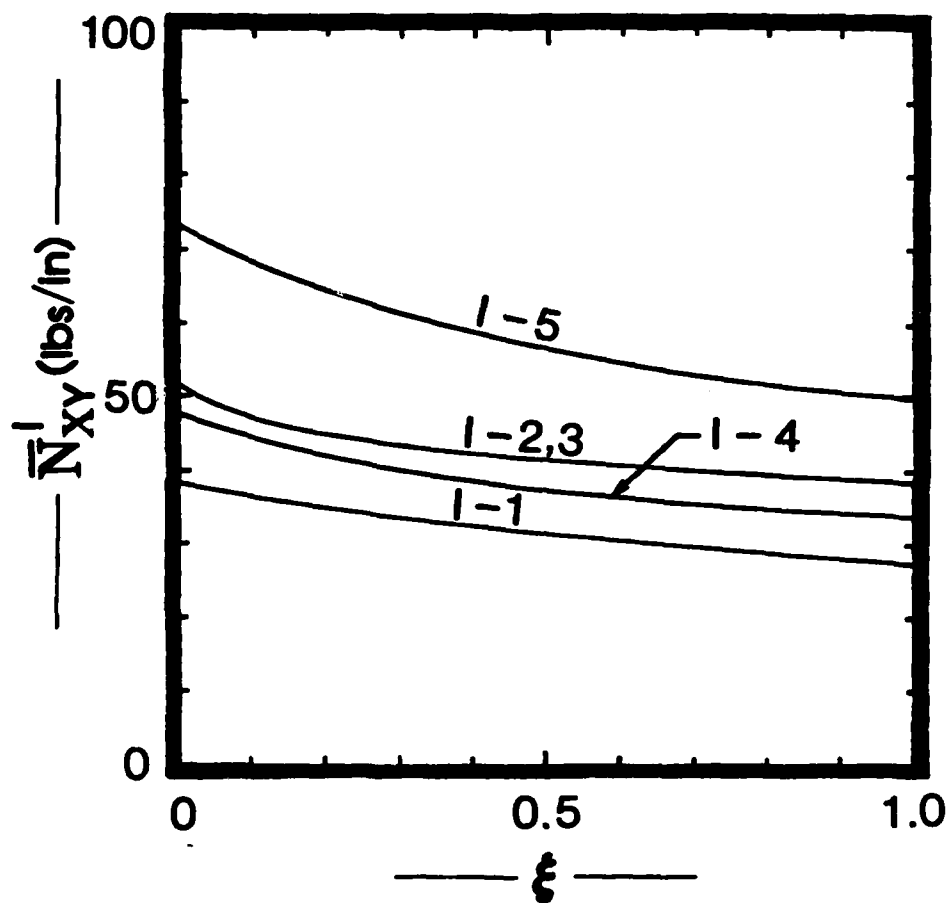*The unit of the stress resultant is lbs./in.

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Fig. 4.8 Critical shear stress vs. Imperfection Amplitude [SS-3; $w^o(+)$. Eq. A-216]
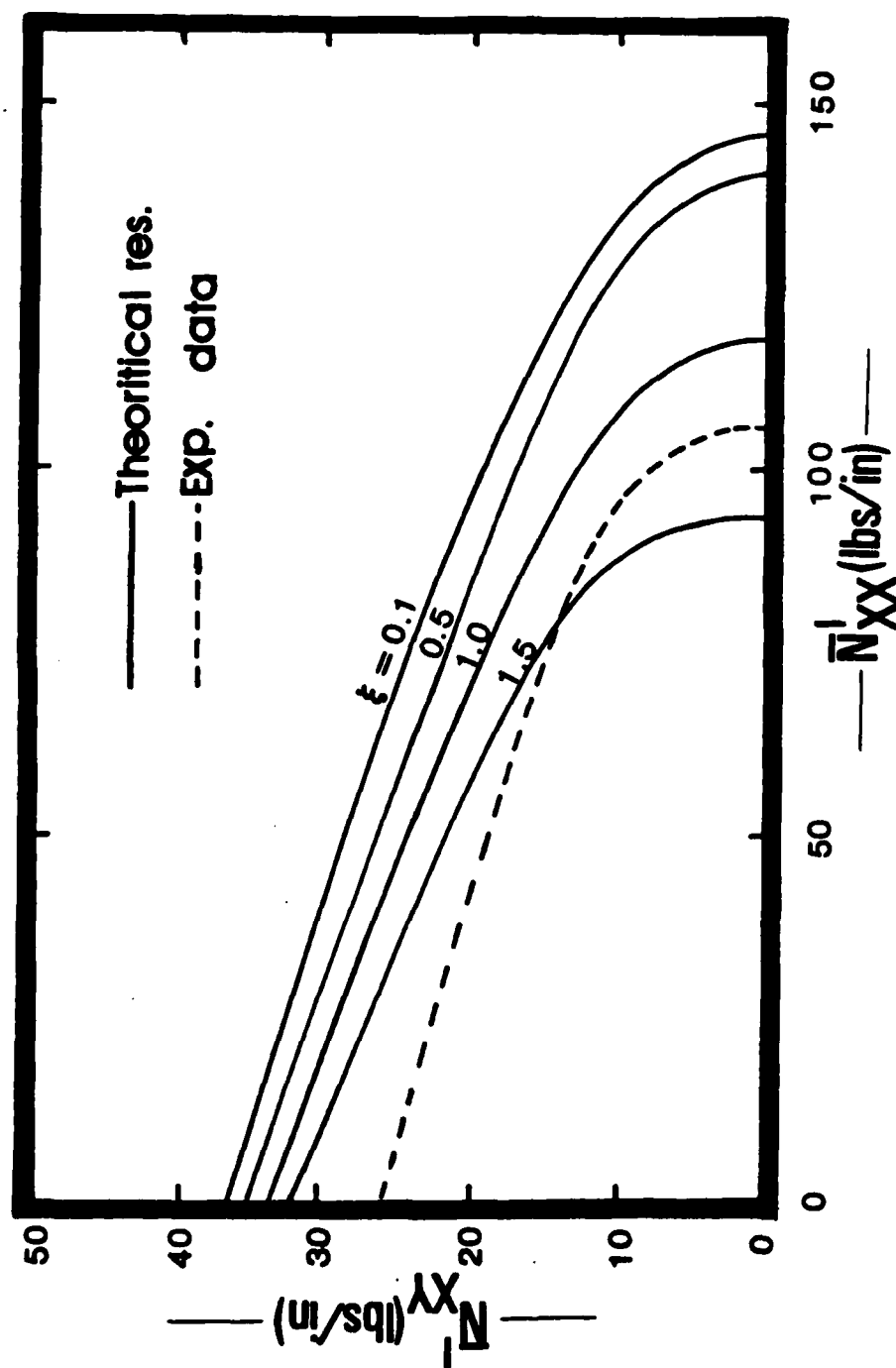
Fig. 4.9 Critical interaction curves [Geometry I-1; Axisymmetric imperfection, Eq. A-214]
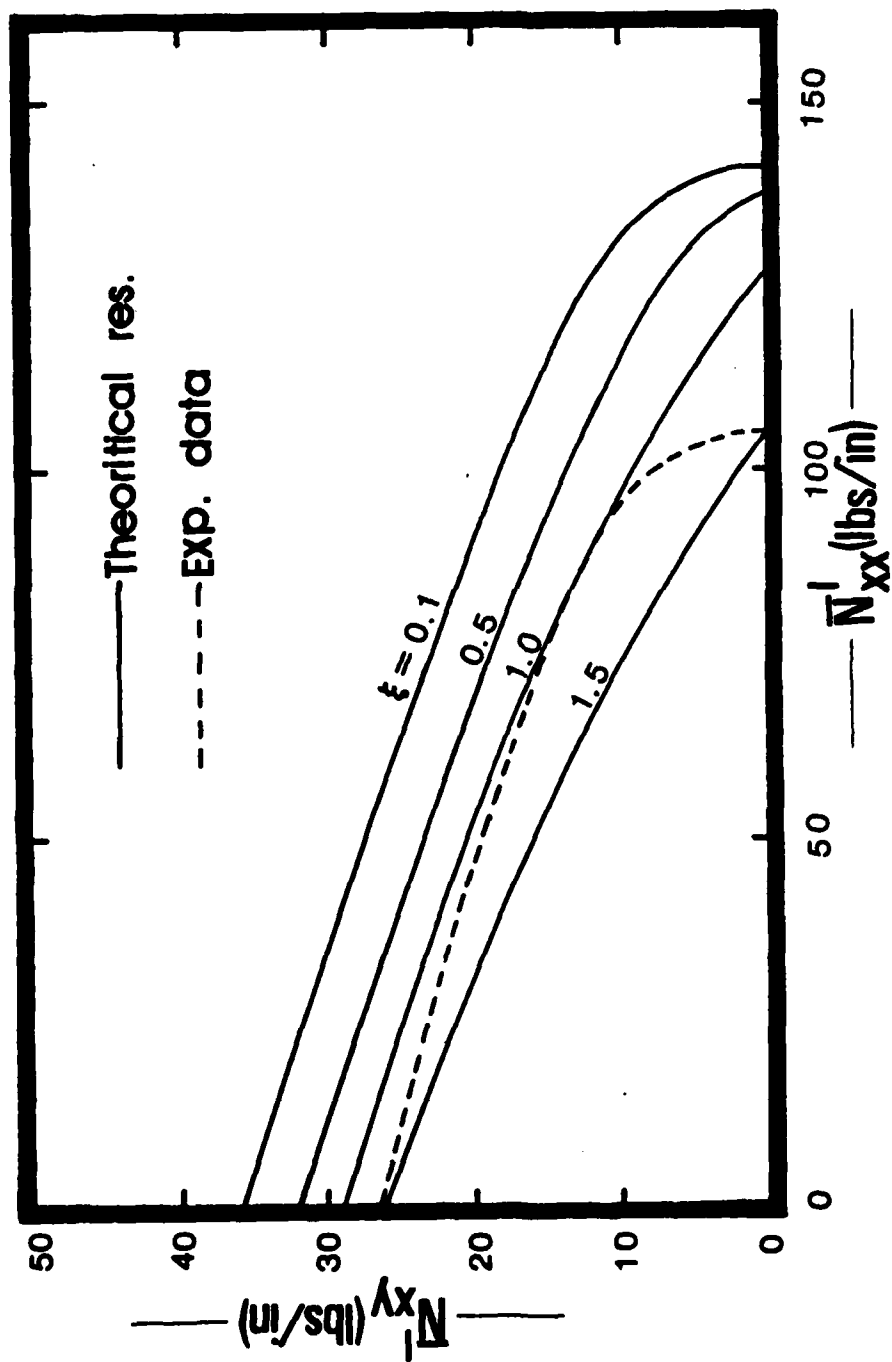
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Fig. 4.10 Critical Interaction curves [Geometry I-1; imperfection $w^o(+)$, Eq. A-216]

configuration is insensitive for positive torsion (fourth column) and rather sensitive for negative torsion (last column). Note that the experimental values (+ 26.5 lbs./in. and -65.72 lbs./in.), compare well with the theoretical values. Note that the tested specimen (44) is of unknown imperfection shape and amplitude.

Next, Table 4.8 presents critical shear stress resultants (and the minimizing n-value in parenthesis) for all five geometries and an imperfection shape similar to the positive torsion buckling mode of the perfect geometry, Eq. (8). These results are shown graphically on Fig. 4.8). Note that the strongest configuration corresponds to I-5, while the weakest to the symmetric geometry I-1. This conclusion holds true for the imperfection shape used, $w^o(+)$.

It is worth observing that the regular angle-ply antisymmetric geometries, I-2 and I-3, yield virtually the same strength for positive torsion and $w^o(+)$. Moreover, geometry I-4 is much weaker by comparison to the other asymmetric geometry (I-5) but not as weak as the symmetric geometry. These observations are reminiscent of the old external versus internal positioning of the orthogonal stiffeners controversy concerning metallic stiffened configurations. In relation to this, in the case of orthogonally stiffened complete spherical shells subjected to uniform pressure (see Ref. 48) it is observed that the weakest configuration corresponds to zero (or close to it) stiffener eccentricity, and the strength of the stiffened sphere increases as the eccentricity increases in either direction (inward or outward). Thus, one can conclude from Fig. 4.8 that all five configurations are imperfection sensitive, but not as sensitive as they are for the case of uniform axial compression (See Fig. 4.1). This conclusion is in line with the behavior of metallic cylindrical shells with or without stiffening members.

In Ref. 44, experiments are conducted for geometry I-1, to determine the interaction curve that separates the stable from the unstable region between uniform axial compression and torsion. Because of this, numerical results are obtained fro geometry I-1 and two imperfection shapes. One is virtually axi-symmetric, Eq. (6), and one similar to the (positive torsion) perfect geometry buckling mode, Eq. (8). The theoretical interaction curves are generated for several values of the imperfection amplitude parameter, ξ, by the following steps. First, the critical value for pure torsion is obtained. Then, starting with zero torsion and several values of the applied shear stress resultant, but smaller than the critical pure torsion the corresponding critical axial compression is obtained. In each combination a study of the effect of n is performed. The results are presented in tabular form on Tables 4.9 and 4.10 and graphically on Figs. 4.9 and 4.10.

The data of Table 4.9 are plotted on Fig. 4.9 and of Table 4.10 on Fig. 4.10 On both figures the experimental (44) interaction curve is shown by the dashed line. Not knowing what the imperfection shape and amplitude of the tested cylinder are, these plots may suggest a reasonable comparison between theory and test.

IV.   3.0   CONCLUSIONS

All of the conclusions are based on the generated results, which are obtained by the W, F-formulation. No results have, as yet, been generated by the u, v, w-formulation.

From all results, one may list the following as the most noteworthy conclusions.

1. Buckling, for all configurations, is of the violent type (snap through buckling through limit point instability).

2. For SS-3 boundary conditions and axial compression with zero eccentricity, the strongest configuration corresponds to the asymmetric congiguration, I-5, while the weakest configuration corresponds to the antisymmetric configurations, I-2 and I-3.

3. Again for SS-3 and axial compression, the dynamic critical loads (lower bounds, when the corresponding static loads, but their values are never smaller than 60% of the static critical loads.

4. The average end shortening (for axial compression), corresponding to the limit point for the same ξ-value, is smaller for the asymmetric geometries (I-4, I-5) than for the symmetric (I-1) and antisymmetric (I-2 and I-3) geometries by almost a factor of three.

5. For the isotropic geometry (SS-i boundary conditions)

    5a: For the perfect configuration and very small imperfections, the effect of in-plane boundary conditions is such that SS-3 and SS-4 (v = const.) yield stronger configurations than SS-1 and SS-2 ($N_{xy} = -F_{,xy} = 0$)

    5b: For higher values of the imperfection amplitude, ξ, SS-2 and SS-4 (u = const.) yield stronger configurations than SS-1 and SS-3 ($N_{xx} = F_{,yy} -\bar{N}_{xx}$)

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6. For the laminated geometry, the effect of in-plane boundary conditions for SS-1 is the same as for the isotropic geometry. For clamped boundaries, CC-2 and CC-4 (u = const.) yield stronger configurations that CC-1 and CC-3, for the entire ξ-range.

7. For both geometries, I-5 and isotropic, the sensitivity to initial geometric imperfection is dependent upon the in-plane boundary conditions for SS-i. When v = const (SS-1 and SS-2), the geometries are not very sensitive. On the other hand, when u = const the geometries are very sensitive.

8. As far as the effect of load eccentricity on critical loads is concerned, no general conclusion can be drawn. But whatever the effect is (stabilizing or destabilizing for a given geometry), it diminishes with increasing value of the imperfection amplitude parameter (ξ-values).

9. When loaded in pure torsion, the strongest configuration corresponds to geometry I-5 (asymmetric), while the weakest corresponds to the symmetric geometry I-1, for the imperfection shape corresponding to the positive torsion buckling mode, $w^o(+)$.

10. Geometry I-1 is weaker when loaded in the positive direction than when loaded in the negative direction regardless of the imperfection shape (for all that were employed).

11. When loaded in pure torsion, laminated shell configurations are sensitive to initial geometric imperfections, but not as sensitive as when loaded in axial compression.

12. Comparison between theoretical predictions (corresponding to various imperfection amplitudes and shapes) and experimental results is reasonably good.

APPENDIX A

MATHEMATICAL FORMULATION

A. 1.0 Introduction

The governing equations are derived, in this section, for the following geometry and loading. The thin, circular, cylindrical shell is assumed to be geometrically imperfect. The construction is laminated (each lamina is orthotropic) and in addition, the shell is orthogonally and eccentrically stiffened. The stiffeners are uniform and with uniform close spacing, which allows one to employ the "smeared" technique. The boundary conditions can be of any transverse and in-plane variety. This includes free, simply-supported and clamped with all possible in-plane combinations. The loading consists of transverse (uniform lateral pressure) and eccentric in-plane loads, such as uniform axial compression and shear. Eccentric means that the line of action of these loads (applied stress resultants) is not necessarily in the plane of the reference surface. In the derivation of the governing equations, the usual lamination theory is employed. Moreover, thin shell theory (Kirchhoff-Love hypotheses) and linearly elastic behavior are assumed. The primary assumptions are listed below:

(1) The shell is thin (total smeared thickness is much smaller than the initial average radius of curvature-cylinder radius).

(2) Normals remain normal and inextensional.

(3) The strains are small, the rotations about the normal are small and the rotations about in-plane axes are moderate.

(4) The imperfection shape is such that the initial curvature is small $[R|w^o,_{ii} \ll 1; i = x,y]$.

(5) The stiffness are along principal directions.

(6) The stiffener-laminate connections are monolithic.

(7) The stiffeners do not carry shear; shear is entirely trans-
mitted by the laminate .

(8) The stiffeness are torsionally weak and thus they do not con-
tribute to the shell twisting stiffness (the equations and related pro-
grams can easily be changed to accomodate the case of torsionally strong
stiffeners).

On the basis of these general assumptions, two sets of field equations
are derived. One, referred to as the w, F - formulation, is based on
Donnell-type of kinematic relations. The governing equations consist of
the transverse equilibrium equation and the in-plane compatibility equation.
These two equations and the proper boundary conditions are expressed in terms
of the transverse displacement component, w, and an Airy stress resultant
function, F. The second, referred to as the u, v, w - formulation is based
on Sanders' type of kinematic relations, those corresponding to small rotations
about the normal and moderate rotations about in-plane axes. The governing
equations for this case consist of the three equilibrium equations. These
equations are expressed in terms of the three displacement components, u, v
and w. Also, the proper boundary conditions are expressed in terms of u, v,
and w. The corresponding Donnell approximation appears as a special case of
the more general Sanders' kinematic relations. The derivation along with all
necessary relations are presented separately for each formulation.

A. 2.0 The w, F - Formulation

The geometry and sign convention for this formulation are shown on Figs.
A.1 and A.2.

The topics of kinematic relations, stress and moment resultants, governing
equations, boundary conditions and solution procedure are treated separately.

Fig. A. 1 Geometry

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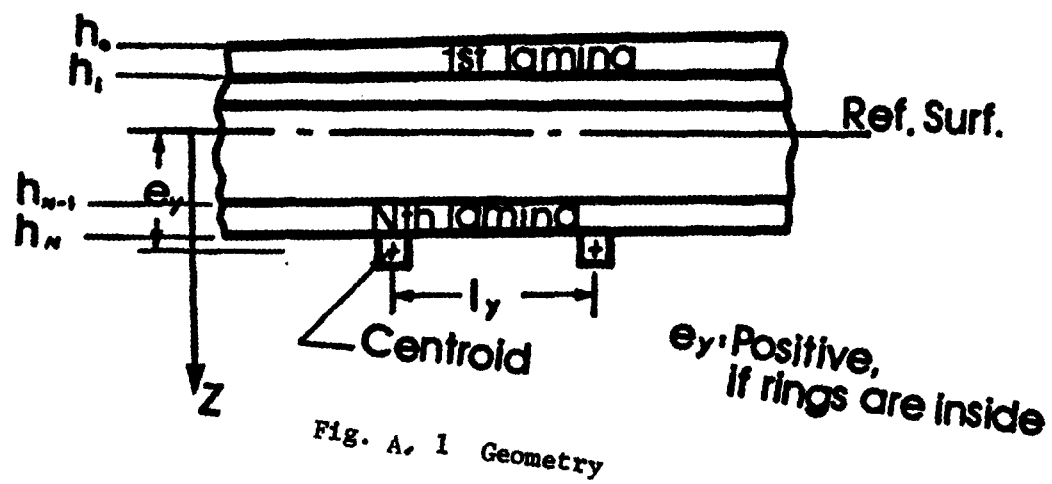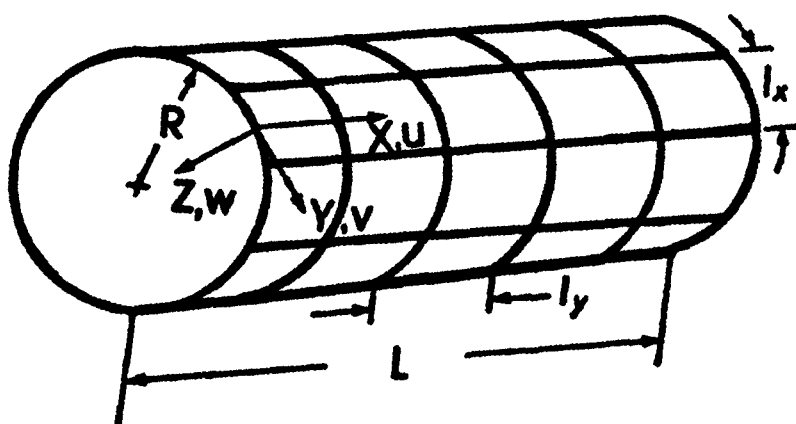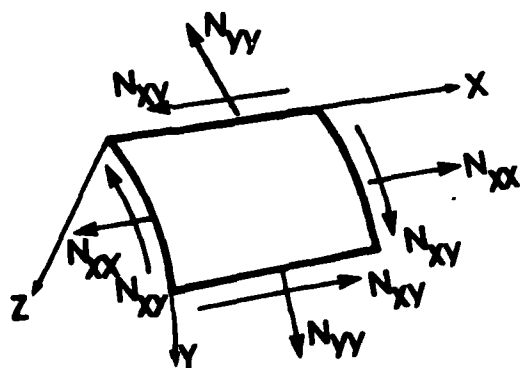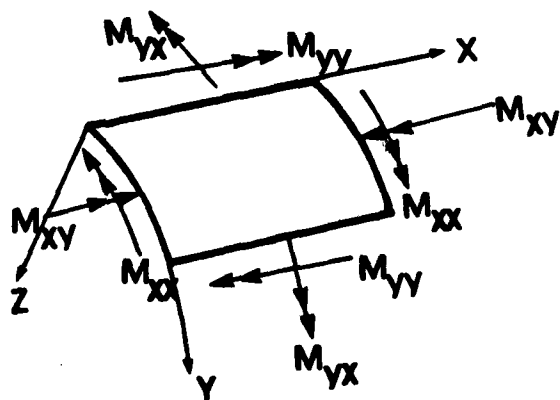
Stress Resultants

Moment Resultants

Fig. A.2 Sign Convention

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A. 2.1 Kinematic Relations

Let w^o be measured from the perfectly cylindrical surface to the refer surface of the laminated shell. Let w denote the transverse displacement component of reference surface material points and be measured from the undeformed surface. Let u and v denote the usual in-plane displacement components along the x and y directions respectively.

The Donnell-type (33) kinematic relations are given by

$$\epsilon_{xx} = \epsilon_{xx}^o - z\, k_{xx}$$

$$\epsilon_{yy} = \epsilon_{yy}^o - z k_{yy} \qquad (A-1)$$

$$\gamma_{xy} = \gamma_{xy}^o - 2z k_{xy}$$

where the superscript "o" denotes reference surface strains and the $\varkappa$'s denote the reference surface changes in curvature and torsion. Note that the positive z-direction is inward (see Fig. A.1).

According to Donnell the ϵ^o's and $\varkappa$'s are related to the displacement components by

$$\epsilon_{xx}^o = u_{,x} + \tfrac{1}{2} w_{,x}^2 + w_{,x} w_{,x}^o$$

$$\epsilon_{yy}^o = v_{,y} - \tfrac{w}{R} + \tfrac{1}{2} w_{,y}^2 + w_{,y} w_{,y}^o \qquad (A-2)$$

$$\gamma_{xy}^o = u_{,y} + v_{,x} + w_{,x} w_{,y} + w_{,x} w_{,y}^o + w_{,x}^o w_{,y}$$

$$k_{xx} = \varphi_{x,x} = (w_{,x})_{,x} = w_{,xx}$$

$$k_{yy} = \varphi_{y,y} = (w_{,y})_{,y} = w_{,yy} \qquad (A-3)$$

$$k_{xy} = \varphi_{x,y} = \varphi_{y,x} = w_{,xy}$$

A. 2.2 Stress-strain Relations

Each lamina is assumed to be orthotropic and the directions of orthotropy (1,2) make an angle θ with the in-plane axes (x,y).

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The orthotropic constitutive (it is assumed that the generalized Hooke's law holds) relations for the kth lamina are given below. Note that for an n-ply laminate k varies from <u>one</u> to <u>n</u>, and the first ply (or lamina) is on the outside, while the nth ply is on the inside (see Fig. A.1).

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}^{(k)} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}
\tag{A-4}
$$

where $2\epsilon_{12} = \gamma_{12}$ and 1, 2 are the orthotropic directions.

Since one is interested in relating the stresses to the strains in the xy frame of axes, the usual transformation relation for second order tensors are employed (see Ref. 35 for details) and the transformed constitutive equations (for the kth ply) become

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{43} & \bar{Q}_{33} \end{bmatrix}^{(k)} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}
\tag{A-5}
$$

where

$$
[\bar{Q}] = [T]^{j'}[Q][T]
\tag{A-6}
$$

and

$$
[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & -\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & \cos 2\theta \end{bmatrix}
\tag{A-7}
$$

Next, the stress-strain relations for the stiffeners are

$$
\sigma_{xx_{st}} = E_{st}\,\epsilon_{xx}
$$
$$
\sigma_{yy_{st}} = E_r\,\epsilon_{yy}
\tag{A-8}
$$

where E_{st} and E_r denote the Young's moduli for stringer and ring material respectively. Note that according to the smeared technique assumptions, stiffeners do not transmit shear.

A. 2.3 Stress and Moment Resultants

Instead of dealing with stresses, it is more convenient in thin shell and plate theory to deal with integrated stresses. This leads to the introduction and definition of stress (N_{ij}) and moment (M_{ij}) resultants. For a stiffened laminate these are

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \int_{h_o}^{h_n} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} dz + \int_{A_i} \begin{bmatrix} \frac{\sigma_{xxst}}{\ell_x} dA_x \\ \frac{\sigma_{yyr}}{\ell_y} dA_y \\ 0 \end{bmatrix} \qquad (A-9)$$

and

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \int_{h_o}^{h_n} z \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} dz + \int_{A_i} \begin{bmatrix} z \frac{\sigma_{xxst}}{\ell_x} dA_x \\ z \frac{\sigma_{yyr}}{\ell_y} dA_y \\ 0 \end{bmatrix} \qquad (A-10)$$

where

ℓ_x and ℓ_y are the stringer and ring spacings (respectively), A_i denotes the proper stiffener cross-sectional area with A_x denoting stringer area and A_y ring area, and h_o h_n denote the outer surface and inner surface coordinate of the laminate (see Fig. A.1). Note also that the above definitions lead to the sign convention shown on Fig. A.2

Substitution of Eqs. A-5 and A-8 for the stresses in Eqs. A-9 and A-10 prior substitution of Eqs. A-1 for the strains in Eqs. A-5 and A-8 and performing some minor mathematical operations lead to

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$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{N} [\bar{Q}]^{(k)} \int_{h_{k-1}}^{h_k} \left\{ \begin{bmatrix} \epsilon_{xx}^{\circ} \\ \epsilon_{yy}^{\circ} \\ \gamma_{xy}^{\circ} \end{bmatrix} - z \begin{bmatrix} K_{xx} \\ K_{yy} \\ 2K_{xy} \end{bmatrix} \right\} dz$$

$$+ \begin{bmatrix} \dfrac{E_{st}A_x}{\ell_x}\epsilon_{xx}^{\circ} \\ \dfrac{E_r A_y}{\ell_y}\epsilon_{yy}^{\circ} \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{E_{st}A_x}{\ell_x}e_x K_{xx} \\ \dfrac{E_r A_y}{\ell_y}e_y K_{yy} \\ 0 \end{bmatrix} \qquad (A-11)$$

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{N} [\bar{Q}]^{(k)} \int_{h_{k-1}}^{h_k} \left\{ z \begin{bmatrix} \epsilon_{xx}^{\circ} \\ \epsilon_{yy}^{\circ} \\ \gamma_{xy}^{\circ} \end{bmatrix} - z \begin{bmatrix} K_{xx} \\ K_{yy} \\ 2K_{xy} \end{bmatrix} \right\} dz$$

$$+ \begin{bmatrix} \dfrac{E_{st}A_x}{\ell_x}e_x \epsilon_{xx}^{\circ} \\ \dfrac{E_r A_y}{\ell_y}e_y \epsilon_{yy}^{\circ} \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{E_{st}}{\ell_x}(I_{xc}+A_x e_x^2) \\ \dfrac{E_r}{\ell_y}(I_{yc}+A_y e_y^2) \\ 0 \end{bmatrix} \qquad (A-12)$$

where $e_x \ e_y$ are the stiffener eccentricities (positive if on the side of

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positive z) and I_{x_c}, I_{y_c} are the stiffener second moment of areas about centroidal axes.

After performing the indicated operation [Eqs. A-11 and A-12], one may write

$$
\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & -\bar{B}_{11} & -\bar{B}_{12} & -\bar{B}_{13} \\ \bar{A}_{12} & \bar{A}_{22} & A_{23} & -\bar{B}_{12} & -\bar{B}_{22} & -\bar{B}_{23} \\ \bar{A}_{13} & \bar{A}_{23} & \bar{A}_{33} & -\bar{B}_{13} & -\bar{B}_{23} & -\bar{B}_{33} \\ \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{23} & -\bar{D}_{11} & -\bar{D}_{12} & -\bar{D}_{13} \\ \bar{B}_{12} & \bar{B}_{22} & \bar{B}_{23} & -\bar{D}_{12} & -\bar{D}_{22} & -\bar{D}_{23} \\ \bar{B}_{13} & \bar{B}_{23} & \bar{B}_{33} & -\bar{D}_{13} & -\bar{D}_{23} & -\bar{D}_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{xx}^{\circ} \\ \epsilon_{yy}^{\circ} \\ \gamma_{xy}^{\circ} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} \qquad (A-13)
$$

where

$$
[\bar{A}_{ij}] = [A_{ij}] + \begin{bmatrix} \dfrac{E_{st}A_x}{\ell_x} & 0 & 0 \\ 0 & \dfrac{E_r A_y}{\ell_y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (A-14a)
$$

$$
[\bar{B}_{ij}] = [B_{ij}] + \begin{bmatrix} \dfrac{E_{st}A_x}{\ell_x}\ell_x & 0 & 0 \\ 0 & \dfrac{E_r A_y}{\ell_y}\ell_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (A-14b)
$$

and

$$
[\bar{D}_{ij}] = [D_{ij}] + \begin{bmatrix} \dfrac{E_{st}}{\ell_x}(I_{x_c}+\ell_x^2 A_x) & 0 & 0 \\ 0 & \dfrac{E_r}{\ell_y}(I_{x_c}+\ell_y^2 A_y) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (A-14c)
$$

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with

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h_k - h_{k-1})$$

$$B_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h_k^2 - h_{k-1}^2) \tag{A-15}$$

$$D_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h_k^3 - h_{k-1}^3)$$

Since, in the derivation of the field equation for this formulation, the dependent variable are, w and a stress function F (through which the stress resultants are derived), then it is convenient to express the mement resultants in terms of the N_{ij}'s and the $\varkappa$'s.

Starting with Eqs. A-13, one may write

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = [\bar{A}_{ij}] \begin{bmatrix} \epsilon_{xx}^{\circ} \\ \epsilon_{yy}^{\circ} \\ \gamma_{xy}^{\circ} \end{bmatrix} - [\bar{B}_{ij}] \begin{bmatrix} \varkappa_{xx} \\ \varkappa_{yy} \\ 2\varkappa_{xy} \end{bmatrix} \tag{A-16}$$

From this, one can solve for the strain vector, or

$$\begin{bmatrix} \epsilon_{xx}^{\circ} \\ \epsilon_{yy}^{\circ} \\ \gamma_{xy}^{\circ} \end{bmatrix} = [\bar{A}_{ij}]^{-1} \begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} + [\bar{A}_{ij}]^{-1}[\bar{B}_{ij}] \begin{bmatrix} \varkappa_{xx} \\ \varkappa_{yy} \\ 2\varkappa_{xy} \end{bmatrix} \tag{A-17}$$

Another form for this equation, Eq. (17), is the following

$$\begin{bmatrix} \epsilon_{xx}^{\circ} \\ \epsilon_{yy}^{\circ} \\ \gamma_{xy}^{\circ} \end{bmatrix} = [\alpha_{ij}] \begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} + [\theta_{ij}] \begin{bmatrix} \varkappa_{xx} \\ \varkappa_{yy} \\ 2\varkappa_{xy} \end{bmatrix} \tag{A-18}$$

where

$$[\alpha_{ij}] = [\bar{A}_{ij}]^{-1} \qquad [\theta_{ij}] = [\bar{A}_{ij}]^{-1}[\bar{B}_{ij}] \tag{A-19}$$

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Next, substitution of Eqs. A-18 into the expression for the moment resultants, Eqs. A-13, yields

$$
\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = [\bar{B}_{ij}] \begin{bmatrix} \epsilon^{o}_{xx} \\ \epsilon^{o}_{yy} \\ \gamma^{o}_{xy} \end{bmatrix} - [\bar{D}_{ij}] \begin{bmatrix} K_{xy} \\ K_{yy} \\ 2K_{xy} \end{bmatrix}
$$

$$
= [\bar{B}_{ij}][a_{ij}] \begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} + [[\bar{B}_{ij}][b_{ij}] - [\bar{D}_{ij}]] \begin{bmatrix} K_{xx} \\ K_{yy} \\ 2K_{xy} \end{bmatrix}
$$

$$
= [\theta_{ij}]^{T} \begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} + [d_{ij}] \begin{bmatrix} K_{xx} \\ K_{yy} \\ 2K_{xy} \end{bmatrix} \qquad (A-20)
$$

where

$$
[d_{ij}] = [\bar{B}_{ij}][\theta_{ij}] - [\bar{D}_{ij}] \qquad (A-21)
$$

Note that $[a_{ij}]$ and $[d_{ij}]$ are symmetric three by three matrices, while $[b_{ij}]$ is a nonsymmetric three by three matrix.

A. 2.4 Equilibrium Equations

The equilibrium equations are derived by employing the principle of the stationary value of the total potential.

According to the principle, for equilibrium

$$
\delta U_{T} = 0 \qquad (A-22)
$$

where

$$
U_{T} = U_{i} + U_{P} \qquad (A-23)
$$

the sum of the strain energy and the potential of the external forces.

From Eq. A-22 one may write

$$\delta U_T = \delta U_i + \delta U_p = 0$$

$$= \int_0^{2\pi R}\int_0^L (N_{xx}\delta\epsilon_{xx}^{\circ} + N_{yy}\delta\epsilon_{yy}^{\circ} + N_{xy}\delta\gamma_{xy}^{\circ}$$

$$- M_{xx}\delta k_{xx} - M_{yy}\delta \chi_{yy} - 2 M_{xy}\delta \chi_{xy})\,dx\,dy$$

$$- \int_0^{2\pi R}\int_0^L q\,\delta w\,dx\,dy - \int_0^{2\pi R}(-\bar{N}_{xx}\delta u + \bar{N}_{xy}\delta v$$

$$+ \bar{Q}_x\delta w - \bar{M}_{xx}\delta W_{,x} - \bar{M}_{xy}\,\delta w_{,y})\Big|_0^L\,dy \qquad (A\text{-}24)$$

Where q denote the external pressure (positive in the positive z-direction) and the "bar" quantities denote external loads applied at the boundaries ($\bar{N}_{xx}$ and $\bar{N}_{xy}$ are in-plane loads, while Q_x is applied transverse shear load and $\bar{M}_{xx}$ and $\bar{M}_{xy}$ external moments). Note that $\bar{M}_{xx}$ and $\bar{M}_{xy}$ could represent moments arising from eccentrically applied $\bar{N}_{xx}$ and $\bar{N}_{xy}$.

Use of Eqs. A-2 and A-3 for expressing the variations, in the reference surface strains and changes of curvature and torsion in terms of variations in displacement components yields

$$\delta U_T = \int_0^{2\pi R}\int_0^L \Big\{ N_{xx}\left[\delta u_{,x} + w_{,x}\delta w_{,x} + w_{,x}^{\circ}\delta w_{,x}\right]$$

$$+ N_{yy}\left[\delta v_{,y} - \frac{1}{R}\delta w + w_{,y}\delta w_{,y} + w_{,y}^{\circ}\delta w_{,y}\right]$$

$$+ N_{xy}\left[\delta u_{,y} + \delta v_{,x} + w_{,x}\delta w_{,y} + w_{,y}\delta w_{,x}\right.$$

$$\left.+ w_{,x}^{\circ}\delta w_{,y} + w_{,y}^{\circ}\delta w_{,x}\right] - M_{xx}\delta w_{,xx} - M_{yy}\delta w_{,yy}$$

$$- 2 M_{xy}\delta w_{,xy}\Big\}\,dx\,dy - \int_0^{2\pi R}\int_0^L q\,\delta w\,dx\,dy$$

$$- \int_0^{2\pi R}\int_0^L[- \bar{N}_{xx}\delta u + \bar{N}_{xy}\delta v + \bar{Q}_x\delta w - \bar{M}_{xx}\delta\varphi_x$$

$$- \bar{M}_{xy}\delta\varphi_y]\Big|_0^L\,dy \qquad (A\text{-}25)$$

Re-writing the above in a convenient form in order to use Green's
theorem, one may write

$$
\begin{aligned}
\delta U_T = \int_0^{2\pi R}\int_0^L \Big\{ &\big[N_{xx}\,\delta U + N_{xx}(W_{,x}+W_{,x}^{\,o})\,\delta W \\
&+ N_{xy}(W_{,y}+W_{,y}^{\,o})\,\delta W - M_{xx}\delta W_{,x}\big]_{,x} \\
&+\big[N_{yy}\,\delta V + N_{yy}(W_{,y}+W_{,y}^{\,o})\,\delta W + N_{xy}\delta U \\
&+ N_{xy}(W_{,x}+W_{,x}^{\,o})\,\delta W - M_{yy}\delta W_{,y}\big]_{,y} \\
&-\big[N_{xx,x}\,\delta U + [N_{xx}(W_{,x}+W_{,x}^{\,o})]_{,x}\,\delta W \\
&+ N_{xy,x}\,\delta V + [N_{xy}(W_{,y}+W_{,y}^{\,o})]_{,x}\,\delta W \\
&- M_{xx,x}\delta W + N_{yy,y}\delta V + [N_{yy}(W_{,y} \\
&+ W_{,y}^{\,o})]_{,y}\,\delta W + N_{xy,y}\delta U \\
&+ [N_{xy}(W_{,x}+W_{,x}^{\,o})]_{,y}\,\delta W - M_{yy,y}\,\delta W_{,y}\big] \\
&- \frac{N_{yy}}{R}\,\delta W - 2M_{xy}\,\delta W_{,xy}\Big\}\,dx\,dy \\
-\int_0^{2\pi R}\int_0^L & \mathcal{Q}\,\delta W\,dx\,dy - \int_0^{2\pi R}[-\bar N_{xx}\delta U + \bar N_{xy}\delta V \\
&+ \bar Q_x\,\delta W - \bar M_{xx}\delta\varphi_x - \bar M_{xy}\delta\varphi_y]\big|_0^L\,dy
\end{aligned}
$$

$$= \int_0^{2\pi R} \int_0^L \left\{ \left[N_{xx} \delta \mathcal{U} + N_{xx} (W,_x + W,_x^\circ) \delta W + N_{xy} \delta \mathcal{U} + N_{xy} (W,_y \right. \right.$$

$$\left. + W,_y^\circ) \delta W - M_{xx} \delta W,_x + M_{xx,x} \delta W + 2 M_{xy,y} \delta W \right],_x$$

$$+ \left[N_{yy} \delta \mathcal{U} + N_{yy} (W,_y + W,_y^\circ) \delta W + N_{xy} \delta \mathcal{U} \right.$$

$$+ N_{xy} (W,_x + W,_x^\circ) \delta W - M_{yy} \delta W,_y + M_{yy,y} \delta W$$

$$\left. + 2 M_{xy,x} \delta W \right],_y - \left[N_{xx,x} \delta \mathcal{U} + [N_{xx} (W,_x + W,_x^\circ)],_x \delta W \right.$$

$$+ N_{xy,x} \delta \mathcal{U} + [N_{xy} (W,_y + W,_y^\circ)],_x \delta \mathcal{U}$$

$$+ M_{xx,xx} \delta W + N_{yy,y} \delta \mathcal{U} + [N_{yy} (W,_y + W,_y^\circ)],_y \delta W$$

$$+ N_{xy,y} \delta \mathcal{U} + [N_{xy} (W,_x + W,_x^\circ)],_y \delta W$$

$$\left. + M_{yy,yy} \delta W \right] - \frac{N_{yy}}{R} \delta W$$

$$- 2 M_{xy,xy} \delta W \bigg\} dx \, dy$$

$$- \int_0^{2\pi R} \int_0^L g \, \delta w \, dx \, dy - \int_0^{2\pi R} [-\bar{N}_{xx} \delta \mathcal{U} + \bar{N}_{xy} \delta \mathcal{U}$$

$$+ \bar{Q}_x \delta W - \bar{M}_{xx} \delta \varphi_x - \bar{M}_{xy} \delta \varphi_y] \bigg|_0^L dy \qquad (A\text{-}26)$$

By Green's theorem, one obtains the following equilibrium equations and associated boundary terms,

Equilibrium Equations

$$N_{xx,x} + N_{xy,y} = 0$$

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$$N_{xy,x} + N_{yy,y} = 0$$

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \frac{N_{yy}}{R} + N_{xx}(W_{,xx} + W_{,xx}^{\circ})$$

$$+ 2N_{xy}(W_{,xy} + W_{,xy}^{\circ}) + N_{yy}(W_{,yy} + W_{,yy}^{\circ}) + q = 0 \qquad (A\text{-}27)$$

Boundary Terms

either		or
$N_{xx} = -\overline{N}_{xx}$		$\delta \mathcal{U} = 0$
$N_{xy} = \overline{N}_{xy}$		$\delta \mathcal{V} = 0$
$N_{xx}(W_{,x} + W_{,x}^{\circ}) + N_{xy}(W_{,y} + W_{,y}^{\circ})$		
$+ M_{xx,x} + 2M_{xy,y} = \overline{Q}_x + \overline{M}_{xy,y}$		$\delta w = 0$
$M_{xx} = \overline{M}_{xx}$		$\delta W_{,x} = 0 \qquad (A\text{-}28)$

The first two equilibrium equations, Eqs. A-27 can be identically satisfied through the introduction of the following stress function

$$N_{xx} = F_{,yy} - \overline{N}_{xx}$$

$$N_{yy} = F_{,xx}$$

$$N_{xy} = -F_{,xy} + \overline{N}_{xy} \qquad (A\text{-}29)$$

With the introduction of the stress function, F, the third equilibrium equation becomes

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \frac{1}{R}F_{,xx} + F_{,yy}(W_{,xx} + W_{,xx}^{\circ})$$

$$+ F_{,xx}(W_{,yy} + W_{,yy}^{\circ}) - 2F_{,xy}(W_{,xy} + W_{,xy}^{\circ}) - \overline{N}_{xx}(W_{,xy} + W_{,xy}^{\circ})$$

$$+ 2\overline{N}_{xy}(W_{,xy} + W_{,xy}^{\circ}) + q = 0 \qquad (A\text{-}30)$$

A. 2.5 Compatibility Equation

Since the in-plane equilibrium equations are identically satisfied with the introduction of the Airy stress function, F, then the governing equations

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consist of the transverse equilibrium equation, Eq. A-30 and one more. This one more results from requiring compatibility of the in-plane displacement components u and v. From Eqs. A-2 one obtains

$$\epsilon^{\circ}_{xx,yy} = u_{,xyy} + \frac{1}{2} W_{,xyy}(W_{,x} + 2W^{\circ}_{,x}) + \frac{1}{2} W_{,x}(W_{,xyy} + 2W^{\circ}_{,xyy})$$

$$\epsilon^{\circ}_{yy,xx} = v_{,yxx} - \frac{W_{,xx}}{R} + \frac{1}{2}(2W_{,y}W_{,yxx} + 2W_{,yxx}W^{\circ}_{,y} + 2W_{,y}W^{\circ}_{,yxx})$$

$$\gamma^{\circ}_{xy,xy} = u_{,xyy} + v_{,xxy} + W_{,xxy}W_{,y} + W_{,x}W_{,xyy} + W_{,xyy}W^{\circ}_{,x}$$
$$+ W_{,y}W^{\circ}_{,xxy} + W_{,xxy}W^{\circ}_{,y} + W_{,x}W^{\circ}_{,xyy} \qquad (A-31)$$

Elimination of u and v leads to the following compatibility equation

$$\epsilon^{\circ}_{xx,yy} + \epsilon^{\circ}_{yy,xx} - \gamma^{\circ}_{xy,xy} = - \frac{W_{,xx}}{R} + W_{,xy}(W_{,xy} + 2W^{\circ}_{,x})$$
$$- \frac{1}{2}W_{,xx}(W_{,yy} + 2W^{\circ}_{,yy}) - \frac{1}{2}W_{,yy}(W_{,xx} + 2W^{\circ}_{,xx}) \qquad (A-32)$$

Substitution of Eqs. A-18 [Eqs. A-29 for the N's and Eqs. A-3 for the $\varkappa$'s] into the compatibility equation, Eq. A-32, yields

$$a_{11}F_{,yyyy} + a_{12}F_{,xxyy} - a_{13}F_{,xyyy} + \beta_{11}W_{,xxyy} + \beta_{12}W_{,yyyy} + 2\beta_{13}W_{,xyyy}$$

$$+ a_{12}F_{,xxyy} + a_{22}F_{,xxxx} - a_{23}F_{,xxxy} + \beta_{21}W_{,xxxx} + \beta_{22}W_{,xxyy} + 2\beta_{23}W_{,xxxy}$$

$$- a_{13}F_{,xyyy} - a_{23}F_{,xxxy} + a_{33}F_{,xxyy} - \beta_{31}W_{,xxxy} - \beta_{32}W_{,xyyy} - 2\beta_{33}W_{,xxyy}$$

$$= - \frac{W_{,xx}}{R} + W_{,xy}(W_{,xy} + 2W^{\circ}_{,xy}) - \frac{1}{2}W_{,xx}(W_{,yy} + 2W^{\circ}_{,yy}) - \frac{1}{2}W_{,yy}(W_{,xx} + 2W^{\circ}_{,xx})$$
$$(A-33)$$

Similarly, substitution of Eqs. A-19 into the transverse equilibrium equation, Eq. A-30. yields

$$\mathcal{C}_{11} F_{,xxyy} + \mathcal{C}_{21} F_{,xxxx} - \mathcal{C}_{31} F_{,xxxy} + d_{11} W_{,xxxx} + d_{12} W_{,xxy} + 2 d_{13} W_{,xxxy}$$

$$+ 2 \mathcal{C}_{13} F_{,xyyy} + 2 \mathcal{C}_{23} F_{,xxxy} - 2 \mathcal{C}_{33} F_{,xxyy} + 2 d_{31} W_{,xxxy} + 2 d_{32} W_{,xyyy} + 4 d_{33} W_{,xxyy}$$

$$+ \mathcal{C}_{12} F_{,yyyy} + \mathcal{C}_{32} F_{,xxyy} - \mathcal{C}_{32} F_{,xyyy} + d_{21} W_{,xxyy} + d_{22} W_{,yyyy} + 2 d_{23} W_{,xyyy}$$

$$+ \frac{1}{R} F_{,xx} + F_{,yy} (W_{,xx} + W_{,xx}^{\circ}) - \bar{N}_{xx} (W_{,xx} + W_{,xx}^{\circ})$$

$$+ 2 \bar{N}_{xy} (W_{,xy} + W_{,xy}^{\circ}) - 2 F_{,xy} (W_{,xy} + W_{,xy}^{\circ}) + F_{,xx} (W_{,yy} + W_{,yy}^{\circ})$$

$$+ \mathcal{G} = 0 \qquad\qquad (A-34)$$

A. 2.6 Boundary Conditions

The boundary conditions, Eqs. A-28, can be designated according to transverse one (simply supported, clamped, free) and in-plane ones. Since all of the application to be considered deal with supported boundaries, only simply supported (ss-i; i = 1, 2, 3, 4) and clamped (cc-i) boundary conditions are listed. These are (at x = 0, L):

$$SS\text{-}1: \quad W = 0 \; ; \; M_{xx} = \bar{M}_{xx} \; ; \; N_{xx} = -\bar{N}_{xx} \; ; \; N_{xy} = \bar{N}_{xy}$$

$$SS\text{-}2: \quad W = 0 \; ; \; M_{xx} = \bar{M}_{xx} \; ; \; \mathcal{U} = Const. \; ; \; N_{xy} = \bar{N}_{xy}$$

$$SS\text{-}3: \quad W = 0 \; ; \; M_{xx} = \bar{M}_{xx} \; ; \; N_{xx} = -\bar{N}_{xx} \; ; \; \mathcal{V} = Const.$$

$$SS\text{-}4: \quad W = 0 \; ; \; M_{xx} = \bar{M}_{xx} \; ; \; \mathcal{U} = Const \; ; \; \mathcal{V} = Const. \quad (A\text{-}35)$$

and

$$\text{CC-1}: \quad W = 0 \; ; \quad W_{,x} = 0 \; ; \quad N_{xx} = -\overline{N}_{xx} \; ; \quad N_{xy} = \overline{N}_{xy}$$

$$\text{CC-2}: \quad W = 0 \; ; \quad W_{,x} = 0 \; ; \quad u = \text{Const.} \; ; \quad N_{xy} = \overline{N}_{xy}$$

$$\text{CC-3}: \quad W = 0 \; ; \quad W_{,x} = 0 \; ; \quad N_{xx} = -\overline{N}_{xx} \; ; \quad v = \text{Const.}$$

$$\text{CC-4}: \quad W = 0 \; ; \quad W_{,x} = 0 \; ; \quad u = \text{Const.} \; ; \quad v = \text{Const.} \quad (A\text{-}36)$$

The above boundary conditions may be written in terms of the dependent variables F, and w. The kinematic conditions u = const and v = const are first expressed in terms of equivalent conditions. This is shown below for each of the relevant conditions separately.

Note, first that the expressions for the M_{ij}'s and N_{ij}'s are given by Eqs. A-20 and A-29.

SS-1: $\quad W = 0$

$$b_{21} F_{,xx} + d_{11} W_{,xx} + 2 d_{13} W_{,xy} = \overline{M}_{xx} + b_{11} \overline{N}_{xx} - b_{31} \overline{N}_{xy}$$

$$F_{,yy} = 0 \qquad and \qquad F_{,xy} = 0 \qquad\qquad (A\text{-}37)$$

SS-2: $\quad W = 0$

$$b_{11} F_{,yy} + b_{21} F_{,xx} + d_{11} W_{,xx} + 2 d_{13} W_{,xy} = \overline{M}_{xx} + b_{11} \overline{N}_{xx} - b_{31} \overline{N}_{xy}$$

$$F_{,yy} = 0 \qquad and \qquad F_{,xy} = 0 \qquad\qquad (A\text{-}38)$$

The u = const. condition is expressed in terms of an equivalent condition by employing the following steps.

The expressions for γ_{xy} from the kinematic relations, Eqs. A-2, and from the constitutive equations, Eqs. A-18, are first equated to each other, or

$$\gamma_{xy}^{\circ} = u_{,y} + v_{,x} + w_{,x} w_{,y} + w_{,x} w_{,y}^{\circ} + w_{,x}^{\circ} w_{,y}$$

$$= a_{13} (F_{,yy} - \overline{N}_{xx}) + a_{23} F_{,xx} + a_{33} (\overline{N}_{xy} - F_{,xy})$$

$$+ b_{31} W_{,xx} + b_{32} W_{,yy} + 2 b_{33} W_{,xy} \qquad\qquad (A\text{-}38a)$$

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One differention with respect to y and use of the conditions w = 0 and $F_{,xy}$ = 0 yields at x = 0, L

$$U_{,xy} + W_{,xy} W_{,y}^{\circ} + W_{,x} W_{,yy}^{\circ} = a_{13} F_{,yyy} + a_{23} F_{,xxy} + b_{31} W_{,xy}$$
$$+ 2 b_{33} W_{,xyy} \qquad (A-38b)$$

Similarly,

$$\mathcal{E}_{yy} = U_{,y} - \frac{W}{R} + \frac{1}{2} W_{,y}^{2} + W_{,y} W_{,y}^{\circ}$$
$$= a_{12}(F_{,yy} - \overline{N}_{xx}) + a_{22} F_{,xx} + a_{23}(\overline{N}_{xy} - F_{,xy})$$
$$+ b_{21} W_{,xx} + b_{22} W_{,yy} + 2 b_{23} W_{,xy} \qquad (A-39a)$$

from which one differentiation with respect to x yields

$$U_{,yx} - \frac{W_{,x}}{R} + W_{,xy} W_{,y}^{\circ} = a_{22} F_{,xxx} - a_{23} F_{,xxy} + 2 b_{23} W_{,xxy}$$
$$+ b_{21} W_{,xxx} + b_{22} W_{,xyy} \qquad (A-39b)$$

Elimination of $v_{,xy}$ and $v_{,yx}$ from Eqs.(A-38) and (A-39) yields the equivalent (to u = const) boundary term, which is:

$$a_{13} F_{,yyy} + 2 a_{23} F_{,xxy} - a_{22} F_{,xxx} - \frac{W_{,x}}{R} - W_{,x} W_{,yy}^{\circ} - b_{21} W_{,xxx}$$
$$+ (b_{31} - 2 b_{23}) W_{,xxy} - (b_{22} - 2 b_{33}) W_{,xyy} = 0$$

Note that because $F_{,xy}$ = 0 for this boundary condition, the term containing $F_{,xy}$ has been dropped.

Thus, for SS-2 the final form of the boundary terms becomes

$$W = 0$$
$$b_{11} F_{,yy} + b_{31} F_{,xx} + b_{11} W_{,xx} + 2 d_{13} W_{,xy} = \overline{M}_{xx} + b_{11} \overline{N}_{xx} - b_{31} \overline{N}_{xy}$$
$$F_{,xy} = 0$$
$$a_{13} F_{,yyy} + 2 a_{23} F_{,xxy} - a_{22} F_{,xxx} - \frac{W_{,x}}{R} - W_{,x} W_{,yy}^{\circ} - b_{21} W_{,xxx}$$
$$+ (b_{31} - 2 b_{23}) W_{,xxy} - (b_{22} - 2 b_{33}) W_{,xyy} = 0 \qquad (A-40)$$

SS-3

$$W = 0$$
$$b_{11} F_{,yy} + b_{21} F_{,xx} + d_{11} W_{,xx} + 2 d_{13} W_{,xy} - b_{31} F_{,xy} = \overline{M}_{xx} + b_{11} \overline{N}_{xx} - b_{31} \overline{N}_{xy}$$
$$F_{,yy} = 0 \qquad \text{and} \qquad U = Const.$$

Similarly, as in the case of SS-2 (u = const), an equivalent condition
is obtained for v = const. From Eq. A-39a , since w = 0 and $v_{,y}$ = 0, then
the equivalent condition becomes $\epsilon_{yy} = 0$ or

$$-a_{12}\bar{N}_{xx} + a_{22}F_{,xx} + a_{23}(\bar{N}_{xy} - F_{,xy}) + b_{21}W_{,xx} + 2b_{23}W_{,xy} = 0$$

Thus for SS-3 the final form of the boundary term becomes

$$W = 0$$
$$b_{21}F_{,xx} + d_{11}W_{,xx} + 2d_{13}W_{,xy} - b_{31}F_{,xy} = \bar{M}_{xx} + b_{11}\bar{N}_{xx} - b_{31}\bar{N}_{xy}$$
$$F_{,yy} = 0$$
$$a_{22}F_{,xx} - a_{23}F_{,xy} + b_{21}W_{,xx} + 2b_{23}W_{,xy} = a_{12}\bar{N}_{xx} - a_{23}\bar{N}_{xy} \quad (A-41)$$

SS-4

For this case the equivalent set of the boundary terms becomes

$$W = 0$$
$$b_{11}F_{,yy} + b_{21}F_{,xx} - b_{31}F_{,xy} + d_{11}W_{,xx} + 2d_{13}W_{,xy} = \bar{M}_{xx} + b_{11}\bar{N}_{xx} - b_{31}\bar{N}_{xy}$$
$$a_{22}F_{,xx} + a_{12}F_{,yy} - a_{23}F_{,xy} + b_{21}W_{,xx} + 2b_{23}W_{,xy} = a_{12}\bar{N}_{xx} - a_{23}\bar{N}_{xy}$$
$$a_{13}F_{,yyy} + 2a_{23}F_{,xxy} - (a_{12} + a_{33})F_{,xyy} - a_{22}F_{,xxx} - \frac{W_{,x}}{R} - W_{,x}W_{,yy}^{\circ}$$
$$+ (2b_{33} - b_{22})W_{,xyy} + (b_{31} - 2b_{23})W_{,xxy} - b_{21}W_{,xxx} = 0 \quad (A-42)$$

Following similar steps, boundary conditions CC-i, i = 1, 2, 3 and 4, are also
expressed in terms of w and F only, or

CC-1

$$W = W_{,x} = F_{,yy} = F_{,xy} = 0 \quad\quad\quad (A-43)$$

CC-2

$$W = W_{,x} = F_{,xy} = 0$$
$$a_{13}F_{,yyy} + 2a_{23}F_{,xxy} - a_{22}F_{,xxx} - b_{21}W_{,xxx} + (b_{31} - 2b_{23})W_{,xxy} = 0 \quad (A-44)$$

CC-3

$$W = W_{,x} = F_{,yy} = 0$$
$$a_{22}F_{,xx} - a_{23}F_{,xy} + b_{21}W_{,xx} = a_{12}\bar{N}_{xx} - a_{23}\bar{N}_{xy} \quad\quad (A-45)$$

CC-4

$$W = W_{,x} = 0$$
$$a_{12}F_{,yy} + a_{22}F_{,xx} - a_{23}F_{,xy} + b_{21}W_{,xx} = a_{12}\bar{N}_{xx} - a_{23}\bar{N}_{xy}$$
$$a_{13}F_{,yyy} + 2a_{23}F_{,xxy} - (a_{12} + a_{33})F_{,xyy} - a_{22}F_{,xxx} - b_{21}W_{,xxx} + (b_{31} - 2b_{23})W_{,xxy}$$
$$= 0 \quad\quad\quad\quad (A-46)$$

II. 2.7 Solution Methodology - Field Equation

The solution methodology is an improvement and modification of the one employed and outlined in Refs. 36 and 37.

The separated form, shown below, is used for the two dependent variables $w(x, y)$ and $F(x, y)$.

$$F(x,y) = C_0(x) + \sum_{i=1}^{2K} \left[C_i(x) \cos \frac{iny}{R} + D_i(x) \sin \frac{iny}{R} \right]$$

$$W(x,y) = A_0(x) + \sum_{i=1}^{K} \left[A_i(x) \cos \frac{iny}{R} + B_i(x) \sin \frac{iny}{R} \right] \qquad (A-47)$$

where n denotes the circumferential wave number.

In addition, similar expression can be employed
for the imperfection parameter $w^0(x, y)$ and the external pressure $q(x, y)$.
Note that in most applications the pressure is assumed uniform (q_0 only).

$$w^0(x,y) = A_0^0(x) + \sum_{i=1}^{K} \left[A_i^0(x) \cos \frac{iny}{R} + B_i^0(x) \sin \frac{iny}{R} \right]$$

$$q(x,y) = q_0^1(x) + \sum_{i=1}^{K} \left[q_i^1(x) \cos \frac{iny}{R} + q_i^2(x) \sin \frac{iny}{R} \right] \qquad (A-48)$$

Because of the nonlinearity of the field equations, Eqs. A-33 and A-34 substitution of Eqs. A-47, and A-48 into them yields double summations for the trigonometric functions. These double summations involve products of sine and cosine of iny/R in all four possible combinations (cosine-cosine, sine-cosine, cosine-sine, and sine-sine). Furthermore, these products have different origins. Some of them come from products of $W,_{xy}\ W,_{xy}$, others from products of $F,_{xx}\ W,_{yy}$ [see Eqs. A-33 and A-34]. In order to simplify the final expressions (and use single sums instead of double sums), and in order to cover all possible combinations of double sums, the following simplifying equations are presented. These are based on trigonometric identifies involving products.

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$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j\, b_j Cos j\theta]\, a_i\, i\, Cos i\theta = \sum_{i=0}^{K+L} A_{IJ1(K)}^{i}(b,a)\, Cos i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j\, b_j Cos j\theta]\, a_i\, i\, Sin i\theta = \sum_{i=0}^{K+L} A_{IJ2(K)}^{i}(b,a)\, Sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j\, b_j Sin j\theta]\, a_i\, i\, Cos i\theta = \sum_{i=0}^{K+L} A_{IJ3(K)}^{i}(b,a)\, Sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j\, b_j Sin j\theta]\, a_i\, i\, Sin i\theta = \sum_{i=0}^{K+L} A_{IJ4(K)}^{i}(b,a)\, Cos i\theta \qquad (A\text{-}49)$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [\, b_j Cos j\theta]\, a_i\, i^2\, Cos i\theta = \sum_{i=0}^{K+L} A_{I21(K)}^{i}(b,a)\, Cos i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [\, b_j\, Cos j\theta]\, a_i\, i^2\, Sin i\theta = \sum_{i=0}^{K+L} A_{I22(K)}^{i}(b,a)\, Sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [\, b_j\, Sin j\theta]\, a_i\, i^2\, Cos i\theta = \sum_{i=0}^{K+L} A_{I23(K)}^{i}(b,a)\, Sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [\, b_j\, Sin j\theta]\, a_i\, i^2\, Sin i\theta = \sum_{i=0}^{K+L} A_{I24(K)}^{i}(b,a)\, Cos i\theta \qquad (A\text{-}50)$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j^2 b_j Cos j\theta]\, a_i\, Cos i\theta = \sum_{i=0}^{K+L} A_{J21(K)}^{i}(b,a)\, Cos i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j^2 b_j Cos j\theta]\, a_i\, Sin i\theta = \sum_{i=0}^{K+L} A_{J22(K)}^{i}(b,a)\, Sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j^2 b_j Sin j\theta]\, a_i\, Cos i\theta = \sum_{i=0}^{K+L} A_{J23(K)}^{i}(b,a)\, Sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L} [j^2 b_j Sin j\theta]\, a_i\, Sin i\theta = \sum_{i=0}^{K+L} A_{J24(K)}^{i}(b,a)\, Cos i\theta \qquad (A\text{-}51)$$

where

$$A_{IJ1(K)}^{i}(b,a) = \frac{1}{2}\sum_{j=0}^{K} [(i+j)\, b_{i+j} + (1-\eta_{j-i}^2 + \eta_i)|i-j|\, b_{|i-j|}]\, j\, a_j$$

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$$A^i_{IJ2(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[-(i+j)b_{i+j} + (1-\eta^2_{j-i}+\eta_i)|i-j|b_{|i-j|}\right]j\,a_j$$

$$A^i_{IJ3(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)b_{i+j} + (-1+\eta_{i-j}+\eta_i)|i-j|b_{|i-j|}\right]j\,a_j$$

$$A^i_{IJ4(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)b_{i+j} + (-1-\eta_{i-j}+\eta_i)|i-j|b_{|i-j|}\right]j\,a_j \quad (A\text{-}52)$$

$$A^i_{I21(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[b_{i+j} + (1-\eta^2_{j-i}+\eta_i)b_{|i-j|}\right]j^2\,a_j$$

$$A^i_{I22(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[-b_{i+j} + (1-\eta^2_{j-i}+\eta_i)b_{|i-j|}\right]j^2\,a_j$$

$$A^i_{I23(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[b_{i+j} + (-1+\eta_{i-j}+\eta_i)b_{|i-j|}\right]j^2\,a_j$$

$$A^i_{I24(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[b_{i+j} + (-1-\eta_{i-j}+\eta_i)b_{|i-j|}\right]j^2\,a_j \quad (A\text{-}53)$$

$$A^i_{J21(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)^2 b_{i+j} + (1-\eta^2_{j-i}+\eta_i)(i-j)^2 b_{|i-j|}\right]a_j$$

$$A^i_{J22(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[-(i+j)^2 b_{i+j} + (1-\eta^2_{j-i}+\eta_i)(i-j)^2 b_{|i-j|}\right]a_j$$

$$A^i_{J23(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)^2 b_{i+j} + (-1+\eta_{i-j}+\eta_i)(i+j)^2 b_{|i-j|}\right]a_j$$

$$A^i_{J24(K)}(b,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)^2 b_{i+j} + (-1-\eta_{i-j}+\eta_i)(i+j)^2 b_{|i-j|}\right]a_j \quad (A\text{-}54)$$

and

$$K \geq L \quad ; \quad \eta_\ell = \begin{cases} 1 & \text{if} \quad \ell > 0 \\ 0 & \text{if} \quad \ell = 0 \\ 1 & \text{if} \quad \ell < 0 \end{cases}$$

Next, returning to the solution procedure, the expressions for F, w, w^o and q, Eqs A-47 and A-48 are substituted into the equilibrium and compatibility Equations, Eqs A-33 and A-34. This substitution yields the following non-linear differential equations:

Equilibrium equation

$$\sum_{i=0}^{K}\left[h_{40}\left(A_{i,xxxx}\cos\frac{iny}{R}+B_{i,xxx}\sin\frac{iny}{R}\right)+h_{31}\left(\frac{in}{R}\right)\left(-A_{i,xxx}\sin\frac{iny}{R}\right.\right.$$

$$+B_{i,xxx}\cos\frac{iny}{R}\bigg)-h_{22}\left(\frac{in}{R}\right)^{2}\left(A_{i,xx}\cos\frac{iny}{R}+B_{i,xx}\sin\frac{iny}{R}\right)$$

$$+h_{13}\left(\frac{in}{R}\right)^{3}\left(A_{i,x}\sin\frac{iny}{R}-B_{i,x}\cos\frac{iny}{R}\right)+h_{40}\left(\frac{in}{R}\right)^{4}\left(A_{i}\cos\frac{iny}{R}+B_{i}\sin\frac{iny}{R}\right)\bigg]$$

$$+\sum_{i=0}^{K}\left[g_{40}\left(C_{i,xxxx}\cos\frac{iny}{R}+D_{i,xxxx}\sin\frac{iny}{R}\right)+g_{31}\left(\frac{in}{R}\right)\left(-C_{i,xxx}\sin\frac{iny}{R}\right.\right.$$

$$+D_{i,xxx}\cos\frac{iny}{R}\bigg)-g_{22}\left(\frac{in}{R}\right)^{2}\left(C_{i,xx}\cos\frac{iny}{R}+D_{i,xx}\sin\frac{iny}{R}\right)$$

$$+g_{13}\left(\frac{in}{R}\right)^{3}\left(C_{i,x}\sin\frac{iny}{R}-D_{i,x}\cos\frac{iny}{R}\right)+g_{04}\left(\frac{in}{R}\right)^{4}\left(C_{i}\cos\frac{iny}{R}+D_{i}\sin\frac{iny}{R}\right)\bigg]$$

$$+\frac{1}{R}\sum_{i=0}^{2K}\left(C_{i,xx}\cos\frac{iny}{R}+D_{i,xx}\sin\frac{iny}{R}\right)+L(F,W+W^{o})$$

$$-\overline{N}_{xx}\sum_{i=0}^{K}\left[(A_{i,xx}+A^{o}_{i,xx})\cos\frac{iny}{R}+(B_{i,xx}+B^{o}_{i,xx})\sin\frac{iny}{R}\right]$$

$$+2\overline{N}_{xy}\sum_{i=0}^{K}\left(\frac{in}{R}\right)\left[-(A_{i,x}+A^{o}_{i,x})\sin\frac{iny}{R}+(B_{i,x}+B^{o}_{i,x})\cos\frac{iny}{R}\right]$$

$$+\sum_{i=0}^{K}\left[g^{1}_{i}\cos\frac{iny}{R}+g^{2}_{i}\sin\frac{iny}{R}\right]=0 \qquad (A\text{-}55)$$

where

$$L(F,W+W^{o})=\left[\sum_{i=0}^{2K}C_{i,xx}\cos\frac{iny}{R}+D_{i,xx}\sin\frac{iny}{R}\right]\left[\sum_{i=0}^{K}\left(\frac{in}{R}\right)^{2}\left\{-(A_{i}+A^{o}_{i})\cos\frac{iny}{R}\right.\right.$$

$$-(B_{i}+B^{o}_{i})\sin\frac{iny}{R}\bigg\}\bigg]-2\left[\sum_{i=0}^{2K}\left(\frac{in}{R}\right)\left(-C_{i,x}\sin\frac{iny}{R}+D_{i,x}\cos\frac{iny}{R}\right)\right]\left[\sum_{i=0}^{K}\right.$$

$$\left(\frac{in}{R}\right)\left\{-(A_{i,x}+A^{o}_{i,x})\sin\frac{iny}{R}+(B_{i,x}+B^{o}_{i,x})\cos\frac{iny}{R}\right\}\bigg]+\left[\sum_{i=0}^{2K}-\left(\frac{in}{R}\right)^{2}\left(C_{i}\cos\frac{iny}{R}\right.\right.$$

$$+D_{i}\sin\frac{iny}{R}\bigg)\bigg]\left[\sum_{i=0}^{K}\left\{(A_{i,xx}+A^{o}_{i,xx})\cos\frac{iny}{R}+(B_{i,xx}+B^{o}_{i,xx})\sin\frac{iny}{R}\right\}\right]=0 \ (A\text{-}56\ a)$$

or

$$L(F, W+W^\circ) = \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[-j^2(A_j+A_j^\circ)\cos\frac{iny}{R}\right] C_{i,xx}\cos\frac{iny}{R}$$

$$+ \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[-j^2(A_j+A_j^\circ)\cos\frac{iny}{R}\right] D_{i,xx}\sin\frac{iny}{R}$$

$$+ \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[-j^2(B_j+B_j^\circ)\sin\frac{iny}{R}\right] C_{i,xx}\cos\frac{iny}{R}$$

$$+ \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[-j^2(B_j+B_j^\circ)\sin\frac{iny}{R}\right] D_{i,xx}\sin\frac{iny}{R}$$

$$- \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[(A_{j,xx}+A_{j,xx}^\circ)\cos\frac{iny}{R}\right] C_i\, i^2 \cos\frac{iny}{R}$$

$$- \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[(A_{j,xx}+A_{j,xx}^\circ)\cos\frac{iny}{R}\right] D_i\, i^2 \sin\frac{iny}{R}$$

$$- \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[(B_{j,xx}+B_{j,xx}^\circ)\sin\frac{iny}{R}\right] C_i\, i^2 \cos\frac{iny}{R}$$

$$- \left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[(B_{j,xx}+B_{j,xx}^\circ)\sin\frac{iny}{R}\right] D_i\, i^2 \sin\frac{iny}{R}$$

$$+ 2\left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[-j(A_{j,x}+A_{j,x}^\circ)\sin\frac{iny}{R}\right] C_{i,x}\, i \sin\frac{iny}{R}$$

$$+ 2\left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[j(A_{j,x}+A_{j,x}^\circ)\sin\frac{iny}{R}\right] D_{i,x}\, i \cos\frac{iny}{R}$$

$$+ 2\left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[j(B_{j,x}+B_{j,x}^\circ)\cos\frac{iny}{R}\right] C_{i,x}\, i \sin\frac{iny}{R}$$

$$+ 2\left(\frac{n}{R}\right)^2 \sum_{i=0}^{2K} \sum_{j=0}^{K} \left[-j(B_{j,x}+B_{j,x}^\circ)\cos\frac{iny}{R}\right] D_{i,x}\, i \cos\frac{iny}{R} \qquad (A-56b)$$

and

$$h_{40} = d_{11} \qquad\qquad g_{40} = \theta_{21}$$

$$h_{31} = 2d_{31} + 2d_{13} \qquad\qquad g_{31} = -\theta_{31} + 2b_{23}$$

$$h_{22} = d_{12} + 4d_{33} + d_{21} \qquad\qquad g_{22} = \theta_{11} - 2\theta_{33} + \theta_{22}$$

$$h_{13} = 2d_{32} + 2d_{23} \qquad\qquad g_{13} = 2\theta_{13} - \theta_{32}$$

$$h_{04} = d_{22} \qquad\qquad g_{04} = \theta_{12}$$

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Note that the operator L(F, w + w^o), Eqs A-56, can be written in terms of a single series, which is the most appropriate form, for use in Eq. A-55. This is accomplished through the use of Eqs. A-49-A-54.

$$L(F, W+W^o) = -\left(\frac{\eta}{R}\right)^2 \sum_{i=0}^{3K} [A^i_{J21(2K)}(A+A^o, C_{,xx}) + A^i_{J24(2K)}(B+B^o, D_{,xx})$$

$$+ A^i_{I21(2K)}(A_{,xx}+A^o_{,xx}, C) + A^i_{I24(2K)}(B_{,xx}+B^o_{,xx}, D)$$

$$+2A^i_{IJ4(2K)}(A_{,x}+A^o_{,x}, C_{,x}) + 2A^i_{IJ1(2K)}(B_{,x}+B^o_{,x}, D_{,x})] \cos\frac{i\eta y}{R}$$

$$-\left(\frac{\eta}{R}\right)^2 \sum_{i=0}^{3K} [A^i_{J22(2K)}(A+A^o, D_{,xx}) + A^i_{J23(2K)}(B+B^o, C_{,xx})$$

$$+ A^i_{I22(2K)}(A_{,xx}+A^o_{,xx}, D) + A^i_{I23(2K)}(B_{,xx}+B^o_{,xx}, C)$$

$$-2A^i_{IJ3(2K)}(A_{,x}+A^o_{,x}, D_{,x}) - 2A^i_{IJ2(2K)}(B_{,x}+B^o_{,x}, C_{,x})] \sin\frac{i\eta y}{R}$$

$$(A-57)$$

Compatibility equation

$$\sum_{i=0}^{K} [g_{40}(A_{i,xxxx}\cos\frac{i\eta y}{R} + B_{i,xxxx}\sin\frac{i\eta y}{R}) + g_{31}\frac{i\eta}{R}(-A_{i,xxx}\sin\frac{i\eta y}{R} + B_{i,xxx}\cos\frac{i\eta y}{R})$$

$$+ g_{22}\left(\frac{i\eta}{R}\right)^2(-A_{i,xx}\cos\frac{i\eta y}{R} - B_{i,xx}\sin\frac{i\eta y}{R}) + g_{13}\left(\frac{i\eta}{R}\right)^3(A_{i,x}\sin\frac{i\eta y}{R} - B_{i,x}\cos\frac{i\eta y}{R})$$

$$+ g_{04}\left(\frac{i\eta}{R}\right)^4(A_i\cos\frac{i\eta y}{R} + B_i\sin\frac{i\eta y}{R})]$$

$$+ \sum_{i=0}^{K} [a_{22}(C_{i,xxxx}\cos\frac{i\eta y}{R} + D_{i,xxxx}\sin\frac{i\eta y}{R}) + 2a_{23}\left(\frac{i\eta}{R}\right)(C_{i,xxx}\sin\frac{i\eta y}{R}$$

$$- D_{i,xxx}\cos\frac{i\eta y}{R}) + (2a_{12}+a_{33})\left(\frac{i\eta}{R}\right)^2(-C_{i,xx}\sin\frac{i\eta y}{R} - D_{i,xx}\sin\frac{i\eta y}{R})$$

$$+2a_{13}\left(\frac{i\eta}{R}\right)^3\left(-C_{i,x}Sin\frac{i\eta y}{R}+D_{i,x}Cos\frac{i\eta y}{R}\right)+a_{11}\left(\frac{i\eta}{R}\right)^4\left(C_i Cos\frac{i\eta y}{R}+D_i Sin\frac{i\eta y}{R}\right)]$$

$$+\sum_{i=0}^{k}\left(\frac{A_{i,xx}}{R}Cos\frac{i\eta y}{R}+\frac{B_{i,xy}}{R}Sin\frac{i\eta y}{R}\right)-\frac{1}{2}\left(\frac{\eta}{R}\right)^2\sum_{i=0}^{2k}[A^i_{J21(k)}(A+2A^\circ,A_{,xx})$$

$$+A^i_{J24(k)}(B+2B^\circ,B_{,xx})+A^i_{I21(k)}(A_{,xx}+2A^\circ_{,xx},A)+A^i_{I24(k)}(B_{,xx}+2B^\circ_{,xx},B)$$

$$+2A^i_{IJ4(k)}(A_{,x}+2A^\circ_{,x},A_{,x})+2A^i_{IJ1(k)}(B_{,x}+2B^\circ_{,x},B_{,x})]Cos\frac{i\eta y}{R}$$

$$-\frac{1}{2}\left(\frac{\eta}{R}\right)^2\sum_{i=0}^{2k}[A^i_{J22(k)}(A+2A^\circ,B_{,xx})+A^i_{J23(k)}(B+2B^\circ,A_{,xx})$$

$$+A^i_{I22(k)}(A_{,xx}+2A^\circ_{,xx},B)+A^i_{I23(k)}(B_{,xx}+2B^\circ_{,xx},A)$$

$$-2A^i_{IJ3(k)}(A_{i,x}+2A^\circ_{,x},B_{,x})-2A^i_{IJ2(k)}(B_{,x}+2B^\circ_{,x},A_{,x})]Sin\frac{i\eta y}{R}$$

$$= 0 \tag{A-58}$$

Parenthesis

As far as the equilibrium equation is concerned, the summation starts
from zero and goes up to 3k [see Eqs A-55 and A-57] because of the nonlinea-
rity. The Galerkin procedure will be employed for this equation in the cir-
cumferential direction. This will yield (2k + 1) nonlinear ordinary differen-
tial equations [from the vanishing of (2k + 1) Galerkin integrals].

On the other hand the compatibility equation, Eq. A-58, is written in
series form, from, zero to 2k. Because of the orthogonality of the trigono-
metric functions (4k + 1) nonlinear differential equations result, which re-
late the C's and D's to the A's and B's [see Eqs A-47]. This set of ordinary
differential equations is shown in a complete form in the pages that follow.
Before showing them, though, some simplification can be made.

For the case of $i = 0$, one obtains the following equation, from the combatibility equation, Eq A-58.

$$g_{40} A_{0,xxx} + \frac{1}{R} A_{0,xx} + Q_{22} C_{0,xxxx} - \frac{1}{2} \left(\frac{\eta}{R}\right)^3 \Big[A^i_{J21(k)} (A + 2A^\circ, A_{,xx})$$

$$+ A^i_{J24(k)} (B + 2B^\circ, B_{,xx}) + A^\circ_{I21(k)} (A_{,xx} + 2A^\circ_{,xx}, A)$$

$$+ A^i_{I24(k)} (B_{,xx} + 2B^\circ_{,xx}, B) + 2 A^\circ_{IJ4(k)} (A_{,x} + 2A^\circ_{,x}, A_{,x})$$

$$+ 2 A^\circ_{IJ1(k)} (B_{,x} + 2B^\circ_{,x}, B_{,x}) \Big] = 0$$

or

$$C_{0,xxxx} = \frac{1}{Q_{22}} \Big\{ -g_{40} A_{0,xxxx} - \frac{1}{R} A_{0,xx} + \frac{1}{4}\left(\frac{\eta}{R}\right)^2 \sum_{j=1}^{K} \big[j^2 (A_j + 2A^\circ_j) A_{j,xx}$$

$$+ j^2 (B_j + 2B^\circ_j) B_{j,xx} + j^2 (A_{j,xx} + 2A^\circ_{j,xx}) A_j + j (B_{j,xx} + 2B^\circ_{j,xx}) B_j$$

$$+ 2j^2 (A_{j,x} + 2A^\circ_{j,x}) A_{j,x} + 2j^2 (B_{j,x} + 2B^\circ_{j,x}) B_{j,x} \big] \Big\} \qquad (A-59)$$

Moreover, the displacement component $v(x, y)$ is a continuous and single-valued function of y (and x), therefore

$$\int_0^{2\pi R} v_{,y}\, dy = v(x, 2\pi R) - v(x, 0) = 0 \qquad (A-60)$$

From the second of Eqs A-2 one may write

$$v_{,y} = \epsilon^\circ_{yy} + W/R - W_{,y}(W_{,y} + 2W^\circ_{,y})/2 \qquad (A-61)$$

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Furthermore, use of Eqs A-18 [relation between ϵ_{yy}^o and N_{ij}, κ_{ij}], of Eqs A-29 [definition of stress resultant function], and of Eqs A.47 and A-48 [assumed form for W, F and w^o] yields the following relation,

$$\int_0^{2\pi R} \mathcal{U}_{,y}\,dy = \int_0^{2\pi R} (-a_{12}\bar{N}_{xx} + a_{23}\bar{N}_{xy})\,dy$$

$$+ \int_0^{2\pi R} [\ a_{12}F_{,yy} + a_{22}F_{,xx} - a_{23}F_{,xy} + \theta_{23}W_{,xx}$$

$$+ \theta_{22}W_{,yy} + 2\theta_{23}W_{,xy} + \frac{W}{R} - \frac{1}{2}W_{,y}(W_{,y}$$

$$+ 2W_{,y}^{\bullet})]\,dy = 0 \qquad\qquad (A\text{-}62)$$

or

$$\int_0^{2\pi R}(-a_{12}\bar{N}_{xx} + a_{23}\bar{N}_{xy})\,dy + \int_0^{2\pi R}\Big\{a_{12}\sum_{i=0}^{2K}\left(\tfrac{in}{R}\right)^2\left(-C_i\cos\tfrac{iny}{R}\right.$$

$$\left.- D_i\sin\tfrac{iny}{R}\right] + a_{22}\sum_{i=0}^{2k}\left[C_{i,xx}\cos\tfrac{iny}{R} + D_{i,xx}\sin\tfrac{iny}{R}\right]$$

$$- a_{23}\sum_{i=0}^{2K}\left(\tfrac{in}{R}\right)\left[-C_{i,x}\sin\tfrac{iny}{R} + D_{i,x}\cos\tfrac{iny}{R}\right]$$

$$+ \theta_{21}\sum_{i=0}^{K}\left[A_{i,xx}\cos\tfrac{iny}{R} + B_{i,xx}\sin\tfrac{iny}{R}\right] + \theta_{22}\sum_{i=0}^{K}\left(\tfrac{in}{R}\right)^2\left[\right.$$

$$\left.- A_i\cos\tfrac{iny}{R} - B_i\sin\tfrac{iny}{R}\right] + 2\theta_{23}\sum_{i=0}^{K}\left(\tfrac{in}{R}\right)\left[-A_{i,x}\sin\tfrac{iny}{R} + B_{i,x}\cos\tfrac{iny}{R}\right]$$

$$+ \frac{1}{R}\sum_{i=0}^{K}\left(A_i\cos\tfrac{iny}{R} + B_i\sin\tfrac{iny}{R}\right) - \frac{1}{2}\sum_{i=0}^{K}\left(\tfrac{in}{R}\right)\left[-A_i\sin\tfrac{iny}{R}\right.$$

$$\left.+ B_i\cos\tfrac{iny}{R}\right]\cdot\sum_{j=0}^{K}\left(\tfrac{in}{R}\right)\left[-(A_j+2A_j^{\bullet})\sin\tfrac{jny}{R} + (B_j+2B_j^{\bullet})\cos\tfrac{iny}{R}\right]\Big\}\,dy$$

$$= 0 \qquad\qquad (A\text{-}63)$$

This equation, Eq. A-63, after performing the indicated operations (integration, becomes

$$\int_0^{2\pi R}\{-a_{12}\bar{N}_{xx} + a_{23}\bar{N}_{xy} + a_{22}C_{0,xx} + b_{21}A_{0,xx} + \frac{A_0}{R}$$

$$-\frac{\pi^2}{4R^2}\sum_{j=0}^{k}j^2[(A_j+2A_j^\circ)A_j + (B_j+2B_j^\circ)B_j]\}dy = 0 \qquad (A-64)$$

From which, one may write

$$C_{0,xx} = \frac{1}{a_{22}}\{-b_{21}A_{0,xx} - \frac{A_0}{R} + \frac{\pi^2}{4R^2}\sum_{j=0}^{k}j^2[(A_j+2A_j^\circ)A_j$$

$$+ (B_j+2B_j^\circ)B_j] + a_{12}\bar{N}_{xx} - a_{23}\bar{N}_{xy}\} \qquad (A-65)$$

The remaining __compatibility__ (nonlinear, ordinary differential) equations are

__For i = 1, 2, .. 2k and cosine terms__

$$a_{22}C_{i,xxxx} - 2a_{23}\left(\frac{i\pi}{R}\right)D_{i,xxx} - (2a_{12}+a_{33})\left(\frac{i\pi}{R}\right)^2 C_{i,xx} + 2a_{13}\left(\frac{i\pi}{R}\right)^3 D_{i,x}$$

$$+ a_{11}\left(\frac{i\pi}{R}\right)^4 C_i + \delta_i[g_{40}A_{i,xxxx} + g_{31}\left(\frac{i\pi}{R}\right)B_{i,xxx} - g_{22}\left(\frac{i\pi}{R}\right)^2 A_{i,xx}$$

$$- g_{13}\left(\frac{i\pi}{R}\right)^3 B_{i,x} + g_{04}\left(\frac{i\pi}{R}\right)^4 A_i + \frac{A_{i,xx}}{R} - \frac{1}{2}\left(\frac{\pi}{R}\right)^2 i^2(A_i+2A_i^\circ)A_{0,xx}]$$

$$- \left(\frac{\pi}{2R}\right)^2\sum_{j=0}^{k}\{[(i+j)^2(A_{i+j}+2A_{i+j}^\circ) + (2-\eta_{j-i}^2)(i-j)^2(A_{|i-j|}+2A_{(i-j)}^\circ)$$

$$]A_{j,xx} + [(i+j)^2(B_{i+j}+2B_{i+j}^\circ) - \eta_{i-j}(i-j)^2(B_{|i-j|}+2B_{|i-j|}^\circ)]B_{j,xx}$$

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$$+[A_{i+j,xx}+2A_{i+j,xx}^{\circ}+(2-\eta_{j-i}^{2})(A_{|i-j|,xx}+2A_{|i-j|,xx}^{\circ})]j^{2}A_{j}$$

$$+[B_{i+j,xx}+2B_{i+j,xx}^{\circ}-\eta_{i-j}(B_{|i-j|,xx}+2B_{|i-j|,xx}^{\circ})]j^{2}B_{j}$$

$$+2[(i+j)(A_{i+j,x}+2A_{i+j,x}^{\circ})-\eta_{i-j}|i-j|(A_{|i-j|,x}+2A_{|i-j|,x}^{\circ})]$$

$$jA_{j,x}+2[(i+j)(B_{i+j,x}+2B_{i+j,x}^{\circ})+(2-\eta_{j-i}^{2})|i-j|(B_{|i-j|,x}$$

$$+2B_{|i-j|,x}^{\circ})]jB_{j,x}\}=0 \qquad\qquad (A-66)$$

For i = 1, 2,.... 2k and Sine terms

$$a_{22}D_{i,xxxx}+2a_{23}\left(\frac{i\eta}{R}\right)C_{i,xxx}-(2a_{12}+a_{33})\left(\frac{i\eta}{R}\right)^{2}D_{i,xx}$$

$$-2a_{13}\left(\frac{i\eta}{R}\right)^{3}C_{i,x}+a_{11}\left(\frac{i\eta}{R}\right)^{4}D_{i}+\delta_{i}[g_{40}B_{i,xxxx}$$

$$-g_{31}\left(\frac{i\eta}{R}\right)A_{i,xxx}-g_{22}\left(\frac{i\eta}{R}\right)^{2}B_{i,xx}+g_{13}\left(\frac{i\eta}{R}\right)^{3}A_{i,x}+g_{04}\left(\frac{i\eta}{R}\right)^{4}B_{i}$$

$$+\frac{B_{i,xx}}{R}-\frac{1}{2}\left(\frac{\eta}{R}\right)^{2}i^{2}(B_{i}+B_{i}^{\circ})A_{0,xx}]-\frac{1}{4}\left(\frac{\eta}{R}\right)^{2}\sum_{j=1}^{2k}\{[-(i+j)^{2}(A_{i+j}+2A_{i+j}^{\circ})$$

$$+(2-\eta_{d-i}^{2})(i-j)^{2}(A_{|i-j|}+2A_{|i-j|}^{\circ})]B_{j,xx}+[(i+j)^{2}(B_{j+i}+2B_{i+j}^{\circ})$$

$$+\eta_{i-j}(i-j)^{2}(B_{|i-j|}+2B_{|i-j|}^{\circ})]A_{j,xx}+[-(A_{i+j,xx}+2A_{i+j,xx}^{\circ})$$

$$+(2-\eta_{j-i}^{2})(A_{|i+j|,xx}+2A_{|i+j|,xx}^{\circ})]j^{2}B_{j}+[(B_{i+j,xx}+2B_{i+j,xx}^{\circ})$$

$$+\eta_{i-j}(B_{|i-j|,xx}+2B_{|i-j|,xx}^{\circ})]j^{2}A_{j}-2[(i+j)(A_{i+j,x}+2A_{i+j,x}^{\circ})+\eta_{i-j}|i-j|$$

$$\cdot(A_{|i-j|,x}+2A_{|i-j|,x}^{\circ})]jB_{j,x}-2[-(i+j)(B_{i+j,x}+2B_{i+j,x}^{\circ})+(2-\eta_{j-i}^{2})|i-j|$$

$$\cdot(B_{|i-j|,x}+2B_{|i-j|,x}^{\circ})]jA_{j,x}\}=0 \qquad\qquad (A-67)$$

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where

$$\delta_i = \begin{cases} 0 & i > K \\ \\ 1 & i \le K \end{cases} \qquad ; \qquad \eta_i = \begin{cases} -1 & \ell < 0 \\ 0 & \ell = 0 \\ 1 & \ell > 0 \end{cases}$$

As already mentioned, the Galerkin procedure is employed in connection with the equilibrium equation, Eq. A-54, in the circumferential direction. The vanishing of the $(2k + 1)$ Galerkin integrals yields the following set of nonlinear ordinary differential equations.

For i = 0

$$h_{40} A_{0,xxxx} + g_{40} C_{0,xxxx} + \frac{1}{R} C_{0,xx} - (A_{0,xx} + A^{\circ}_{0,xx}) \bar{N}_{xx}$$

$$- \left(\frac{n}{R}\right)^2 \frac{1}{2} \sum_{j=0}^{2K} \left[j^2 (A_j + A_j^{\circ}) C_{j,xx} + j^2 (B_j + B_j^{\circ}) D_{j,xx} + j^2 (A_{j,xx} \right.$$

$$+ A^{\circ}_{j,xx}) C_j + j^2 (B_{j,xx} + B^{\circ}_{j,xx}) D_j + 2 j^2 (A_{j,x} + A^{\circ}_{j,x}) C_{j,x}$$

$$\left. + 2 j^2 (B_{j,x} + B^{\circ}_{j,x}) D_{j,x} \right] + g_0' = 0 \qquad\qquad (A\text{-}68)$$

By employing Eqs. A-59 and A-65 one obtains

$$A_{0,xxxx} \left(d_{11} - \frac{\theta_{21}^2}{a_{22}} \right) - A_{0,xx} \left(2 \frac{\theta_{21}}{R a_{22}} \right) - \bar{N}_{xx} (A_{0,xx} + A^{\circ}_{0,xx}) - \frac{A_0}{a_{22} R^2}$$

$$+ \left(\frac{n}{2R}\right)^2 \sum_{j=1}^{K} j^2 \left\{ \frac{\theta_{21}}{a_{22}} \left[(A_j + 2 A_j^{\circ}) A_{j,xx} + (A_{j,xx} + 2 A^{\circ}_{j,xx}) A_j \right. \right.$$

$$+ 2 (A_{j,x} + 2 A^{\circ}_{j,x}) A_{j,x} + (B_j + 2 B_j^{\circ}) B_{j,xx} + (B_{j,xx} + 2 B^{\circ}_{j,xx}) B_j$$

$$\left. + 2 (B_{j,x} + 2 B^{\circ}_{j,x}) B_{j,x} \right] + \frac{1}{a_{22} R} \left[(A_j + 2 A_j^{\circ}) A_j + (B_j + 2 B_j^{\circ}) B_j \right]$$

$$- 2 \left[(A_j + A_j^{\circ}) C_{j,xx} + (B_j + B_j^{\circ}) D_{j,xx} + 2 (A_{j,x} + A^{\circ}_{j,x}) C_{j,x} \right.$$

$$+ 2(B_{j,x} + B_{j,x}^{\bullet}) D_{j,x} + (A_{j,xx} + A_{j,xx}^{\circ}) C_j + (B_{j,xx} + B_{j,xx}^{\bullet}) D_j]\}$$

$$+ \frac{a_{12}}{a_{22}R} \overline{N}_{xx} - \frac{a_{23}}{a_{22}R} \overline{N}_{xy} + g_o' = 0 \qquad (A-69)$$

For i = 1, 2,... K (when the weighting function is $\cos \frac{iny}{R}$)

$$d_{11} A_{i,xxxx} + 4 d_{13} \left(\frac{in}{R}\right) B_{i,xxx} - (2 d_{12} + 4 d_{33}) \left(\frac{in}{R}\right)^2 A_{i,xx}$$

$$- 4 d_{33} \left(\frac{in}{R}\right)^3 B_{i,x} + d_{22} \left(\frac{in}{R}\right)^4 A_i + b_{21} C_{i,xxx} + (2 b_{23} - b_{31})\left(\frac{in}{R}\right) D_{i,xxx}$$

$$- (b_{11} - 2 b_{33} + b_{22}) \left(\frac{in}{R}\right)^2 C_{i,xx} - (2 b_{13} - b_{32})\left(\frac{in}{R}\right)^3 D_{i,x}$$

$$+ b_{12} \left(\frac{in}{R}\right)^4 C_i + \frac{1}{R} C_{i,xx} - \left(\frac{in}{R}\right)^2 \left(\frac{A_i + A_i^{\circ}}{a_{22}}\right)\left\{ - b_{21} A_{o,xx} - \frac{A_o}{R} \right.$$

$$+ \left(\frac{n}{2R}\right)^2 \sum_{j=1}^{K} j^2 [(A_j + 2 A_j^{\bullet}) A_j + (B_j + 2 B_j^{\bullet}) B_j]\Big\}$$

$$- \left(\frac{in}{R}\right)^2 \left(\frac{A_i + A_i^{\bullet}}{a_{22}}\right)(a_{12} \overline{N}_{xx} - a_{23} \overline{N}_{xy}) - (A_{i,xx} + A_{i,xx}^{\circ}) \overline{N}_{xx}$$

$$+ 2 \overline{N}_{xy} \left(\frac{in}{R}\right)(B_{i,x} + B_{i,x}^{\circ}) - \frac{1}{2}\left(\frac{n}{R}\right)^2 \sum_{j=1}^{2K} \Big\{ [(i+j)^2 \delta_{i+j}(A_{i+j} + A_{i+j}^{\bullet})$$

$$+ (2 - \eta_{j-i}^2)(i-j)^2 \delta_{|i-j|}(A_{|i-j|} + A_{|i+j|}^{\bullet})] C_{j,xx}$$

$$+ [(i+j)^3 \delta_{i+j}(B_{i+j} + B_{i+j}^{\bullet}) - \eta_{i-j}(i-j)^2 \delta_{|i-j|}(B_{|i-j|} + B_{|i-j|}^{\bullet})] D_{j,xx}$$

$$+ 2 [(i+j) \delta_{i+j}(A_{i+j,x} + A_{i+j,x}^{\circ}) - \eta_{i-j}|i-j| \delta_{|i-j|}(A_{|i-j|,x}$$

$$+ A_{|i-j|,x}^{\circ})] j C_{j,x} + 2 [(i+j) \delta_{i+j}(B_{i+j,x} + B_{i+j,x}^{\bullet})$$

$$+ (2 - \eta_{j-i}^2)|i-j| \delta_{|i-j|}(B_{|i-j|,x} + B_{|i-j|,x}^{\bullet})] j D_{j,x} + [\delta_{i+j}(A_{i+j,xx} + A_{i+j,xx}^{\circ})$$

$$+ (2 - \eta_{j-i}^2) \delta_{|i-j|}(A_{|i-j|,xx} + A_{|i-j|,xx}^{\circ})] j^2 C_j + [\delta_{i+j}(B_{i+j,xx} + B_{i+j,xx}^{\bullet})$$

$$- \eta_{i-j} \delta_{|i-j|}(B_{|i-j|,xx} + B_{|i-j|,xx}^{\circ})] j^2 D_j\Big\} + g_i' = 0 \qquad (A-70)$$

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For i = 1, 2...k (when the weighting function is $\sin\frac{iny}{R}$)

$$d_{11} B_{i,xxxx} - 4 d_{13}(\tfrac{i\eta}{R}) A_{i,xxx} - (2d_{12}+4d_{33})(\tfrac{i\eta}{R})^2 B_{i,xx} + 4 d_{23}(\tfrac{i\eta}{R})^3 A_{i,x}$$

$$+ d_{22}(\tfrac{i\eta}{R})^4 B_i + \beta_{21} D_{i,xxx} - (2\beta_{23} - \beta_{31})(\tfrac{i\eta}{R}) C_{i,xxx}$$

$$- (\beta_{11}-2\beta_{33}+\beta_{22})(\tfrac{i\eta}{R})^2 D_{i,xx} + (2\beta_{13}-\beta_{32})(\tfrac{i\eta}{R})^3 C_{i,x} + \beta_{12}(\tfrac{i\eta}{R})^4 D_i$$

$$+ \frac{D_{i,xx}}{R} - (\tfrac{i\eta}{R})^2 (\frac{B_i+B_i^\circ}{a_{22}})\{ -\beta_{21} A_{o,xx} - \frac{A_o}{R} + (\tfrac{\eta}{2R})^2 \sum_{j=1}^{K} j^2 [(A_j +2A_j^\circ) A_j$$

$$+ (B_j+2B_j^\circ) B_j]\} - (\tfrac{i\eta}{R})^2 (\frac{B_i+B_i^\circ}{a_{22}}) (a_{12}\overline{N}_{xx} - a_{23}\overline{N}_{xy}) - (B_{i,xx}+B_{i,xx}^\circ)\overline{N}_{xx}$$

$$- 2\overline{N}_{xy}(\tfrac{i\eta}{R})(A_{i,x} +A_{i,x}^\circ) - \tfrac{1}{2}(\tfrac{\eta}{R})^2 \sum_{j=1}^{2K} \{ [(i+j)^2 \delta_{i+j}(B_{i+j} + B_{i+j}^\circ)$$

$$+ \eta_{i-j}(i-j)^2 \delta_{|i-j|}(B_{|i-j|} + B_{|i-j|}^\circ)] C_{i,xx} + [-(i+j)^2\delta_{i+j}(A_{i+j}+A_{i+j}^\circ)$$

$$+ (2-\eta_{j-i}^2)(i-j)^2 \delta_{|i-j|}(A_{|i-j|}+A_{|i-j|}^\circ)] D_{j,xx} -2[-(i+j)\delta_{i+j}(B_{i+j,x} + B_{i+j,x}^\circ)$$

$$+ (2-\eta_{j-i}^2)|i+j| \delta_{|i-j|}(B_{|i-j|,x}+B_{|i-j|,x}^\circ)] j\, C_{j,x}$$

$$- 2[(i+j)\delta_{i+j}(A_{i+j,x}+A_{i+j,x}^\circ)+\eta_{i-j}|i-j|\delta_{|i-j|}(A_{|i-j|,x}+A_{|i-j|,x}^\circ)] j\, D_{j,x}$$

$$+ [\delta_{i+j}(B_{i+j,xx}+B_{i+j,xx}^\circ) + \eta_{i-j}\delta_{|i-j|}(B_{|i-j|,xx}+B_{|i-j|}^\circ)] j^2 C_j$$

$$+ [-\delta_{i+j}(A_{i+j,xx}+A_{i+j,xx}^\circ) + (2-\eta_{j-i}^2)\delta_{|i-j|}(A_{|i-j|,xx}+A_{|i-j|,xx}^\circ)] j^2 D_j \}$$

$$+ q_i^2 = 0 \tag{A-71}$$

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Clearly the response of the configuration is known provided that one can
solve the nonlinear ordinary differential equations. Their number is $(6k + 2)$
and the number of unknown dependent variables (functions of x) is also $(6k + 2)$
These are $(k + 1)$ A_i's, (k) B_i's, $(2k + 1)$ C_i's and $(2k)$ D_i's. Note that C_o
can and has been eliminated, through Eqs A-59 and A-65 and therefore both the
number of equations and number of unknowns is reduced by one to $(6k + 1)$. In
these equations there is one more undetermined parameter, the wave number n.
This number is determined by requiring the total potential to be a minimum
at a given level of the load. In other words the response is obtained for
various n-values and, through comparison the true response (n-value and cor-
responding values for the dependent variables) is established.

So far, the partial differential equations are reduced to a set of $(6k + 1)$
nonlinear ordinary differential equations. Next, the generalized Newton's
method (Ref. 38), applicable to differential equations is used to reduce the
nonlinear field equations and boundary conditions to a sequence of linear
systems. Iteration equations are derived by assuming that the solution to
the nonlinear set can be achieved by small corrections to an approximate
solution. The small corrections or the values of the variables at the $(m + 1)$
step in terms of the closely spaced state $\underline{m}$, can be obtained by solving the
linearized differentiate equations. Note below the way that a typical non-
linear term (product of X and Y) in the differential equation is linearized.

$$X^{m+1} Y^{m+1} = (X^m + dX^m)(Y^m + dY^m)$$

$$= X^m Y^m + X^m dY^m + Y^m dX^m + dX dY^m$$

$$\approx X^m Y^m + Y^m dX^m + X^m Y^m + X^m dY^m - X^m Y^m$$

$$= X^m (Y^m + dY^m) + Y^m (X^m + dX^m) - X^m Y^m$$

$$= X^m Y^{m+l} + Y^m X^{m+l} - X^m Y^m \qquad (A-72)$$

where X & Y can be A_i, B_i, C_i or D_i

By making use of Eqs (72), the linearized set of governing equations (iteration equation) is obtained from Eqs A-66, A-67, A-69, A-71. These are:

1. __Compatibility__ (i) [cosine terms, Eqs A-67]

For i = 1, 2,....K

$$a_{22} C^{m+l}_{i,xxxy} - 2a_{23}(\tfrac{in}{R}) D^{m+l}_{i,xxx} - (2a_{12} + a_{33})(\tfrac{in}{R})^2 C^{m+l}_{i,xx}$$

$$+ 2a_{13}(\tfrac{in}{R})^3 D^{m+l}_{i,y} + a_{11}(\tfrac{in}{R})^4 C^{m+l}_i + \delta_i \{ \ell_{21} A^{m+l}_{i,xxxy}$$

$$+ (2\ell_{23} - \ell_{31})(\tfrac{in}{R}) B^{m+l}_{i,xxx} - (\ell_{11} - 2\ell_{33} + \ell_{22})(\tfrac{in}{R})^2 A^{m+l}_{i,xx}$$

$$- (2\ell_{13} - \ell_{32})(\tfrac{in}{R})^3 B^{m+l}_{i,x} + \ell_{12}(\tfrac{in}{R})^4 A^{m+l}_i + \tfrac{1}{R} A^{m+l}_{i,xx}$$

$$- \tfrac{1}{2}(\tfrac{in}{R})^2 [A^{m+l}_i A^m_{o,xx} + (A^m_i + 2A^o_i) A^{m+l}_{o,xy} - A^m_i A^m_{o,xx}] \}$$

$$- (\tfrac{n}{2R})^2 \sum_{j=1}^{K} \{ J^{m+l}_{ij}(A+2A^o) A^m_{j,xx} + J^m_{ij}(A+2A^o) A^{m+l}_{j,xx} - J^m_{ij}(A+2A^o) A^m_{j,xx}$$

$$+ K^{m+l}_{ij}(B+2B^o) B^m_{j,xx} + K^m_{ij}(B+2B^o) B^{m+l}_{j,xx} - K^m_{ij}(B+2B^o) B^m_{j,xx}$$

$$+ 2[L^{m+l}_{ij}(A+2A^o) A^m_{j,x} + L^m_{ij}(A+2A^o) A^{m+l}_{j,x} - L^m_{ij}(A+2A^o) A^m_{j,x}]$$

$$+ 2[M^{m+l}_{ij}(B+2B^o) B^m_{j,x} + M^m_{ij}(B+2B^o) B^{m+l}_{j,x} - M^m_{ij}(B+2B^o) B^m_{j,x}]$$

$$+ N^{m+l}_{ij}(A+2A^o) A^m_j + N^m_{ij}(A+2A^o) A^{m+l}_j - N^m_{ij}(A+2A^o) A^m_j$$

$$+ O^{m+l}_{ij}(B+2B^o) B^m_j + O^m_{ij}(B+2B^o) B^{m+l}_j - O^m_{ij}(B+2B^o) B^m_j] \}$$

$$= 0 \qquad (A-73)$$

where

$$J^m_{ij}(Y) = (i+j) \delta_{i+j} Y^m_{i+j} + (2 - \eta^2_{j-i})(i-j)^2 \delta_{|i-j|} Y^m_{|i-j|}$$

$$K^m_{ij}(Y) = (i+j)^2 \delta_{i+j} Y^m_{i+j} - \eta_{i-j}(i-j)^2 \delta_{|i-j|} Y^m_{|i-j|}$$

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$$L_{ij}^m(Y) = \left[(i+j)\,\delta_{i+j}\,Y_{i+j,x}^m - \eta_{i-j}|i-j|\,\delta_{|i-j|}\,Y_{|i-j|,x}^m\right]j$$

$$M_{ij}^m(Y) = \left[(i+j)\,\delta_{i+j}\,Y_{i+j,x}^m + (2-\eta_{j-i}^2)|i-j|\,\delta_{|i-j|}\,Y_{|i-j|,x}^m\right]j$$

$$N_{ij}^m(Y) = \left[\delta_{i+j}\,Y_{i+j,x}^m + (2-\eta_{j-i}^2)\,\delta_{|i-j|}\,Y_{|i-j|,xx}^m\right]j^2$$

$$O_{ij}^m(Y) = \left[\delta_{i+j}\,Y_{i+j,xx}^m - \eta_{i-j}\,\delta_{|i-j|}\,Y_{|i-j|,xx}^m\right]j^2$$

(ii) [sine terms, Eq A-68]

For i = 1, 2,...K

$$a_{22}D_{i,xxxx}^{m+1} + 2a_{23}\left(\frac{i\eta}{R}\right)C_{i,xxx}^{m+1} - (2a_{12}+a_{33})\left(\frac{i\eta}{R}\right)^2 D_{i,xx}^{m+1}$$

$$-2a_{13}\left(\frac{i\eta}{R}\right)^3 C_{i,x}^{m+1} + a_{11}\left(\frac{i\eta}{R}\right)^4 D_i^{m+1} + \delta_i\left[\,b_{21}B_{i,xxxx}^{m+1} - (2b_{23}-b_{31})\right.$$

$$\cdot\left(\frac{i\eta}{R}\right)A_{i,xxx}^{m+1} - (b_{11}-2b_{33}+b_{22})\left(\frac{i\eta}{R}\right)^2 B_{i,xx}^{m+1} + (2b_{13}-b_{32})\left(\frac{i\eta}{R}\right)^3 A_{i,x}^{m+1}$$

$$+ b_{12}\left(\frac{i\eta}{R}\right)^4 B_i^{m+1} + \frac{B_{i,xx}^{m+1}}{R} - \frac{1}{2}\left(\frac{i\eta}{R}\right)\left[B_i^{m+1}A_{o,xx}^m + (B_i^m+2B_i^0)A_{o,xx}^{m+1}\right.$$

$$- B_i^m A_{o,xx}^m\right] - \left(\frac{\eta}{2R}\right)^2 \sum_{l=0}^{K}\left\{\,Q_{ij}^{m+1}(B+2B^0)A_{j,xx}^m + Q_{ij}^m(B+2B)A_{j,xx}^{m+1}\right.$$

$$- Q_{ij}^m(B+2B^0)A_{j,xx}^m + R_{ij}^{m+1}(A+2A^0)B_{j,xx}^m + R_{ij}^m(A+2A^0)B_{j,xx}^{m+1}$$

$$- R_{ij}^m(A+2A^0)B_{j,xx}^{m+1} - 2\left[S_{ij}^{m+1}(B+2B^0)A_{j,x}^m + S_{ij}^m(B+2B^0)A_{j,x}^{m+1}\right.$$

$$- S_{ij}^m(B+2B^0)A_{j,x}^m\right] - 2\left[T_{ij}^{m+1}(A+2A^0)B_{j,x}^m + T_{ij}^m(A+2A^0)B_{j,x}^{m+1}\right.$$

$$- T_{ij}^m(A+2A^0)B_{j,x}^m\right] + U_{ij}^{m+1}(B+2B^0)A_j^m + U_{ij}^m(B+2B^0)A_j^{m+1}$$

$$- U_{ij}^m(B+2B^0)A_j^m + V_{ij}^{m+1}(A+2A^0)B_j^m + V_{ij}^m(A+2A^0)B_j^{m+1}$$

$$- V_{ij}^m(A+2A^0)B_j^m\right\} = 0 \qquad\qquad (A-74)$$

where

$$Q_{ij}^{m}(Y) = (i+j)^2 \delta_{i+j} Y_{i+j}^{m} + \eta_{i-j}(i-j)^2 \delta_{|i-j|} Y_{|i-j|}^{m}$$

$$R_{ij}^{m}(Y) = -(i+j)^2 \delta_{i+j} Y_{i+j}^{m} + (2-\eta_{j-i}^2)(i-j)^2 \delta_{|i-j|} Y_{|i-j|}^{m}$$

$$S_{ij}^{m}(Y) = [-(i+j)\delta_{i+j} Y_{i+j,x}^{m} + (2-\eta_{j-i}^2)|i-j|\delta_{|i-j|} Y_{|i-j|,x}^{m}]j$$

$$T_{ij}^{m}(Y) = [\ (i+j)\delta_{i+j} Y_{i+j,x}^{m} + \eta_{i-j}|i-j|\delta_{|i-j|} Y_{|i-j|,x}^{m}]j$$

$$U_{ij}^{m}(Y) = [\delta_{i+j} Y_{i+j,xx}^{m} + \eta_{i-j}\delta_{|i-j|} Y_{|i-j|,xx}^{m}]j^2$$

$$V_{ij}^{m}(Y) = [-\delta_{i+j} Y_{i+j,xx}^{m} + (2-\eta_{j-i}^2)\delta_{|i-j|} Y_{|i-j|,xx}^{m}]j^2$$

(2) Equilibrium

(i) [i = 0, Eq. A-69]

$$A_{0,xxxx}^{m+1}(d_{11}-\frac{\theta_{21}^2}{a_{22}}) - A_{0,xx}^{m+1}(\frac{2\theta_{21}}{Ra_{22}}) - A_0^{m+1}(\frac{1}{R^2 a_{22}}) - \bar{N}_{xx}(A_{0,xx}^{m+1}$$

$$+ A_{0,xx}^{0}) + (\frac{n}{2R})^2 \sum_{j=1}^{K} j^2 \{\ \frac{\theta_{21}}{a_{22}} [A_j^{m+1} A_{j,xx}^{m} + (A_j^{m}+2A_j^{0}) A_{j,xx}^{m+1}$$

$$- A_j^{m} A_{j,xx}^{m} + A_{j,xx}^{m+1} A_j^{m} + (A_{j,xx}^{m}+2A_{j,xx}^{0}) A_j^{m+1} - A_{j,xx}^{m} A_j^{m} + 2A_{j,x}^{m+1} A_{j,x}^{m}$$

$$+ 2(A_{j,x}^{m}+2A_{j,x}^{0}) A_{j,x}^{m+1} - 2A_{j,x}^{m} A_{j,x}^{m} + B_j^{m+1} B_{j,xx}^{m}$$

$$+ (B_j^{m}+2B_j^{0}) B_{j,xx}^{m+1} - B_j^{m} B_{j,xx}^{m} + (B_{j,xx}^{m+1}+2B_{j,xx}^{0}) B_j^{m}$$

$$+ (B_{j,xx}^{m}+2B_{j,xx}^{0}) B_j^{m+1} - (B_{j,xx}^{m}+2B_{j,xx}^{0}) B_j^{m}$$

$$+ 2 B_{j,x}^{m+1} B_{j,x}^{m} + 2(B_{j,x}^{m}+2B_{j,x}^{0}) B_{j,x}^{m+1} - 2 B_{j,x}^{m} B_{j,x}^{m}]$$

$$+ \frac{1}{a_{22}R} [A_j^{m+1} A_j^{m} + (A_j^{m}+2A_j^{0}) A_j^{m+1} - A_j^{m} A_j^{m}$$

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$$+ B_j^{m+1} B_j^m + (B_j^m + 2 B_j^o) B_j^{m+1} - B_j^m B_j^m]\}$$

$$- \frac{1}{2}\left(\frac{\eta}{R}\right)^2 \sum_{j=1}^{K}\{ j^2 [A_j^{m+1} C_{j,xx}^m + (A_j^m + A_j^o) C_{j,xx}^{m+1} - A_j^m C_{j,xx}^m$$

$$+ B_j^{m+1} D_{j,xx}^m + (B_j^m + B_j^o) D_{j,xx}^{m+1} - B_j^m D_{j,xx}^m]$$

$$+ 2 A_{j,x}^{m+1} C_{j,x}^m + 2(A_{j,x}^m + A_{j,x}^o) C_{j,x}^{m+1} - 2 A_{j,x}^m C_{j,x}^m$$

$$+ 2 B_{j,x}^{m+1} D_{j,x}^m + (B_{j,x}^m + B_{j,x}^o) D_{j,x}^{m+1} - 2 B_{j,x}^m D_{j,x}^m$$

$$+ A_{j,xx}^{m+1} C_j^m + (A_{j,xx}^m + A_{j,xx}^o) C_j^{m+1} - A_{j,xx}^m C_j^m$$

$$+ B_{j,xx}^{m+1} D_j^m + (B_{j,xx}^m + B_{j,xx}^o) D_j^{m+1} - B_{j,xx}^m D_j^m \}$$

$$+ \frac{a_{12}}{a_{22}R} \bar{N}_{xx} - \frac{a_{23}}{a_{22}R} \bar{N}_{xy} + q_o' = 0 \qquad (A\text{-}75)$$

(ii) [i = 1, 2, ...K; weighting function is cos $\frac{iny}{R}$]

$$d_{11} A_{i,xxxx}^{m+1} + 4 d_{13}\left(\frac{i\eta}{R}\right) B_{i,xxx}^{m+1} - (2 d_{12} + 4 d_{33})\left(\frac{i\eta}{R}\right)^2 A_{i,xx}^{m+1}$$

$$- 4 d_{23}\left(\frac{i\eta}{R}\right)^3 B_{i,x}^{m+1} + d_{22}\left(\frac{i\eta}{R}\right)^4 A_i^{m+1} + \theta_{21} C_{i,xxxx}^{m+1}$$

$$+ (2\theta_{23} - \theta_{31})\left(\frac{i\eta}{R}\right) D_{i,xxx}^{m+1} - (\theta_{11} - 2\theta_{33} + \theta_{22})\left(\frac{i\eta}{R}\right)^2 C_{i,xx}^{m+1}$$

$$- (2\theta_{13} - \theta_{32})\left(\frac{i\eta}{R}\right)^3 D_{i,x}^{m+1} + \theta_{12}\left(\frac{i\eta}{R}\right)^4 C_i^{m+1} + \frac{1}{R} C_{i,xx}^{m+1}$$

$$- \left(\frac{i\eta}{R}\right)\frac{1}{a_{22}}\{- \theta_{21} A_i^{m+1} A_{o,xx}^m - \theta_{21}(A_i^m + A_i^o) A_{o,xx}^{m+1}$$

$$+ \theta_{21} A_i^m A_{o,xx}^m - \frac{1}{R} A_i^{m+1} A_o^m - \frac{1}{R}(A_i^m + A_i^o) A_o^{m+1} + \frac{1}{R} A_i^m A_o^m$$

$$+ \left(\frac{\eta}{2R}\right)^2 \sum_{j=1}^{K} j^2 [(A_j^{m+1} + A_i^o)(A_j^m + 2 A_j^o) A_j^m + (A_i^m + A_i^o)(A_j^{m+1} + 2 A_j^o) A_j^m$$

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$$+ (A_i^m + A_i^\circ)(A_j^m + 2A_j^\circ) A_j^{m+1} - 2(A_i^m + A_i^\circ)(A_j^m + 2A_j^\circ) A_j^m$$

$$+ (A_i^m + A_i^\circ)(B_j^m + 2B_j^\circ) B_j^m + (A_i^m + A_i^\circ)(B_j^{m+1} + 2B_j^\circ) B_j^m$$

$$+ (A_i^m + A_i^\circ)(B_j^m + 2B_j^\circ) B_j^{m+1} - 2(A_i^m + A_i^\circ)(B_j^m + 2B_j^\circ) B_j^m]\}$$

$$- \left(\frac{in}{R}\right)^2 \left(\frac{A_i^{m+1} + A_i^\circ}{a_{22}}\right)(a_{12} \bar{N}_{xx} - a_{23} \bar{N}_{xy})$$

$$- (A_{i,xx}^{m+1} + A_{i,xx}^\circ) \bar{N}_{xx} + 2\bar{N}_{xy}\left(\frac{in}{R}\right)(B_{i,x}^{m+1} + B_{i,x}^\circ)$$

$$- \frac{1}{2}\left(\frac{n}{R}\right)^2 \sum_{j=1}^{2K} j^2 \{[J_{ij}^{m+1}(A) C_{j,xx}^m + J_{ij}^m(A+A^\circ) C_{j,xx}^{m+1}$$

$$- J_{ij}^m(A) C_{j,xx}^m] + [K_{ij}^{m+1}(B) D_{j,xx}^m + k_{ij}^m(B+B^\circ) D_{j,xx}^{m+1}$$

$$- k_{ij}^m(B) D_{j,xx}^m] + 2[L_{ij}^{m+1}(A) C_{j,x}^m + L_{ij}^m(A+A^\circ) C_{j,x}^{m+1} - L_{ij}^m(A) C_{j,x}^m]$$

$$+ 2[M_{ij}^{m+1}(B) D_{j,x}^m + M_{ij}^m(B+B^\circ) D_{j,x}^{m+1} - M_{ij}^m(B) D_{j,x}^m]$$

$$+ N_{ij}^{m+1}(A) C_j^m + N_{ij}^m(A+A^\circ) C_j^{m+1} - N_{ij}^m(A) C_j^m$$

$$+ O_{ij}^{m+1}(B) D_j^m + O_{ij}^m(B+B^\circ) D_j^{m+1} - O_{ij}^m(B) D_j^m\} + g_i' = 0 \qquad (A\text{-}76)$$

(iii) [i = 1, 2,...K; weighting function is sin $\frac{iny}{R}$; Eq A.71]

$$d_{11} B_{i,xxxx}^{m+1} - 4d_{13}\left(\frac{in}{R}\right) A_{i,xxx}^{m+1} - (2d_{12} + 4d_{33})\left(\frac{in}{R}\right)^2 B_{i,xx}^{m+1} + 4d_{23}\left(\frac{in}{R}\right)^3 A_{i,x}^{m+1}$$

$$+ d_{22}\left(\frac{in}{R}\right)^4 B_i^{m+1} + b_{21} D_{i,xxxx}^{m+1} - (2b_{23} - b_{31})\left(\frac{in}{R}\right) C_{i,xxx}^{m+1}$$

$$- (b_{11} - 2b_{33} + b_{22})\left(\frac{in}{R}\right)^2 D_{i,xx}^{m+1} + (2b_{13} - b_{32})\left(\frac{in}{R}\right)^3 C_{i,x}^{m+1} + b_{12}\left(\frac{in}{R}\right)^4 D_i^{m+1}$$

$$+ \frac{D_{i,xx}^{m+1}}{R} - \left(\frac{in}{R}\right)^2 \frac{1}{a_{22}}\{-b_{21} B_i^{m+1} A_{0,xx}^m - b_{21}(B_i^m + B_i^\circ) A_{0,xx}^{m+1}$$

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$$+ \theta_{21} B_i^n A_{o,xx}^m - \frac{1}{R} B_i^{n+1} A_o^m - \frac{1}{R}(B_i^n + B_i^o)A_o^{m+1} + \frac{1}{R} B_i^n A_o^m$$

$$+ \left(\frac{n}{2R}\right)^2 \sum_{j=1}^{k} j^2 \left[(B_i^{m+1} + B_i^o)(A_j^m + 2A_j^o)A_j^m + (B_i^n + B_i^o)(A_j^{m+1} + 2A_j^o)A_j^m \right.$$

$$+ (B_i^n + B_i^o)(A_j^m + 2A_j^o)A_j^{m+1} - 2(B_i^n + B_i^o)(A_j^m + 2A_j^o)A_j^m$$

$$+ (B_i^{m+1} + B_i^o)(B_j^m + 2B_j^o)B_j^m + (B_i^n + B_i^o)(B_j^{m+1} + 2B_j^o)B_j^m$$

$$\left. + (B_i^n + B_i^o)(B_j^m + 2B_j^o)B_j^{m+1} - 2(B_i^n + B_i^o)(B_j^m + 2B_j^o)B_j^m \right]\}$$

$$- \left(\frac{in}{R}\right)^2 \left(\frac{B_i^{m+1} + B_i^o}{a_{22}}\right)\left(a_{12}\bar{N}_{xx} - a_{23}\bar{N}_{xy}\right) - (B_{i,xx}^{m+1} + B_{i,xx}^o)\bar{N}_{xx}$$

$$- 2\bar{N}_{xy}\left(\frac{in}{R}\right)(A_{i,x}^{m+1} + A_{i,x}^o) - \frac{1}{2}\left(\frac{n}{R}\right)^2 \sum_{j=1}^{2K} \{ Q_{ij}^{m+1}(B) C_{j,xx}^m$$

$$+ Q_{ij}^n(B+B^o) C_{j,xx}^{m+1} - Q_{ij}^n(B)C_j^m + R_{ij}^{m+1}(A) D_{j,xx}^m$$

$$+ R_{ij}^n(A+A^o) D_{j,xx}^{m+1} - R_{ij}^n(A)D_{j,xx}^m - 2(S_{ij}^{m+1}(B) C_{j,x}^m$$

$$+ S_{ij}^n(B+B^o)C_{j,x}^{m+1} - S_{ij}^n(A) C_{j,x}^m) - 2(T_{ij}^{m+1}(A) D_{j,x}^m$$

$$+ T_{ij}^n(A+A^o) D_{j,x}^{m+1} - T_{ij}^n(A) D_{j,x}^m) + U_{ij}^{m+1}(B) C_j^m + U_{ij}^n(B+B^o)C_j^{m+1}$$

$$- U_{ij}^n(B) C_j^m + V_{ij}^{m+1}(A) D_j^m + V_{ij}^n(A+A^o) D_j^m - V_{ij}^n(A)D_j^m\}$$

$$+ Z_i^2 = 0 \qquad\qquad (A-77)$$

Finally, the Boundary Conditions [SS-i, CC-i, Eqs A-37, and A-40 - A-46] are also expressed in terms of the dependent variables, through the use of Eqs A-47 They are:

$\underline{SS-1} \qquad A_o = 0$

$$A_{o,xx}\left(d_{11} - \frac{b_{21}^2}{a_{22}}\right) = \frac{b_{21}}{a_{22}}\left(-a_{12}\bar{N}_{xx} + a_{23}\bar{N}_{xy}\right) + \bar{M}_{xx} + b_{11}\bar{N}_{xx} - b_{31}\bar{N}_{xy}$$

$$A_i = B_i = 0 \quad ,$$

$$\left.\begin{array}{l} d_{11}A_{i,xx} + b_{21}C_{i,xx} + 2d_{13}\left(\frac{in}{R}\right)B_{i,x} = 0 \\[8pt] d_{11}B_{i,xx} + b_{21}D_{i,xx} - 2d_{13}\left(\frac{in}{R}\right)A_{i,x} = 0 \end{array}\right\} i = 1,2,\cdots k$$

$$C_i = D_{i,x} = D_i = C_{i,x} = 0 \; ; \; i = 1,2,\cdots 2k \qquad (A-78)$$

$\underline{SS-2} \qquad A_o = 0$

$$A_{o,xx}\left(d_{11} - \frac{b_{21}^2}{a_{22}}\right) = \frac{b_{21}}{a_{22}}\left(-a_{12}\bar{N}_{xx} + a_{33}\bar{N}_{xy}\right) + \bar{M}_{xx} + b_{11}\bar{N}_{xx} - b_{31}\bar{N}_{xy}$$

$$A_i = B_i = 0$$

$$\left.\begin{array}{l} d_{11}A_{i,xx} + b_{21}C_{i,xx} - b_{11}\left(\frac{in}{R}\right)^2 C_i + 2d_{13}\left(\frac{in}{R}\right)B_{i,x} = 0 \\[8pt] d_{11}B_{i,xx} + b_{21}D_{i,xx} - b_{11}\left(\frac{in}{R}\right)^2 D_i - 2d_{13}\left(\frac{in}{R}\right)A_{i,x} = 0 \end{array}\right\} i = 1,2,\cdots k$$

$$D_{i,x} = C_{i,x} = 0 \qquad ; \; i = 1,2,\cdots 2k$$

$$a_{22}C_{i,xxx} - 2a_{23}\left(\frac{in}{R}\right)D_{i,xx} + a_{13}\left(\frac{in}{R}\right)^3 D_i + b_{21}A_{i,xxx} + \left(2b_{33} - b_{31}\right)\frac{in}{R}B_{i,xx}$$

$$+ \left(2b_{33} - b_{22}\right)\left(\frac{in}{R}\right)^2 A_i + \frac{A_{i,x}}{R} - \frac{n^2}{2R^2}\sum_{j=0}^{K}\left\{\left[(i+j)^2 A_{i+j}^\circ\right.\right.$$

$$+ \left(1 - \eta_{j-i}^2 + \eta_i\right)(i-j)^2 A_{|i-j|}^\circ\right] A_{j,x} + \left[(i+j)^2 B_{i+j}^\circ\right.$$

$$+ \left(1 - \eta_{i-j} + \eta_i\right)(i-j)^2 B_{|i-j|}^\circ\left]B_{j,x}\right\} = 0 \; ; \; i = 1,2,\cdots 2k$$

$$a_{22}D_{i,xxx} + 2a_{23}\left(\frac{in}{R}\right)C_{i,xx} - a_{13}\left(\frac{in}{R}\right)^3 + b_{21}B_{i,xxx}$$

$$- \left(2b_{23} - b_{31}\right)\left(\frac{in}{R}\right)A_{i,xx} + \left(2b_{33} - b_{22}\right)\left(\frac{in}{R}\right)^2 B_{i,x}$$

$$+ \frac{B_{i,x}}{R} - \frac{n^2}{2R^2}\sum_{j=0}^{K}\left\{\left[-(i+j)^2 A_{i+j}^\circ\right.\right.$$

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$$+ (1 - \eta_{j-i}^2 + \eta_i)(i-j)^2 A_{|i-j|}^\circ] B_{i,xx} + [(i+j)^2 B_{i+j}^\circ$$

$$+ (-1 + \eta_{i-j} + \eta_i)(i-j)^2 B_{|i-j|}^\circ] A_{j,x}\} = 0 \quad ; i = 1,2,\cdots,2K \qquad (A-79)$$

<u>SS-3</u> $\qquad A_0 = 0$

$$A_{0,xx}(d_{11} - \frac{\theta_{21}^2}{a_{22}}) = \frac{\theta_{21}}{a_{22}}[-a_{12}\bar{N}_{xx} + a_{23}\bar{N}_{xy}] + \bar{M}_{xx} + \theta_{11}\bar{N}_{xx} - \theta_{31}\bar{N}_{yy}$$

$$A_i = B_i = 0$$

$$d_{11} B_{i,xx} + \theta_{21} D_{i,xx} + \theta_{31}(\frac{i\eta}{R})C_{i,x} - 2d_{13}(\frac{i\eta}{R})B_{i,x} = 0 \left.\right\} i = 1,2,\cdots k$$

$$d_{11} B_{i,xx} + \theta_{21} D_{i,xx} + \theta_{31}(\frac{i\eta}{R})C_{i,x} - 2d_{13}(\frac{i\eta}{R})A_{i,x} = 0 \left.\right\}$$

$$C_i = D_i = 0$$

$$a_{22} C_{i,xx} - a_{23}(\frac{i\eta}{R})D_{i,x} + \theta_{21} A_{i,xx} + 2\theta_{23}(\frac{i\eta}{R})B_{i,x} = 0 \left.\right\} i = 1,2,\cdots,2K$$

$$a_{22} D_{i,xx} + a_{23}(\frac{i\eta}{R})C_{i,x} + \theta_{21}B_{i,xx} - 2\theta_{23}A_{i,x} = 0 \left.\right\} \qquad (A-80)$$

<u>SS-4</u> $\qquad A_0 = 0$

$$A_{0,xx}(d_{11} - \frac{\theta_{21}^2}{a_{22}}) = \frac{\theta_{21}}{a_{22}}[-a_{12}\bar{N}_{xx} + a_{23}\bar{N}_{xy}] + \bar{M}_{xx} + \theta_{11}\bar{N}_{xx} - \theta_{31}\bar{N}_{xy}$$

$$A_i = B_i = 0$$

$$\theta_{21} C_{i,xx} - \theta_{11}(\frac{i\eta}{R})^2 C_i - \theta_{31}(\frac{i\eta}{R})D_{i,x} + d_{11} A_{i,xx} + 2d_{13}(\frac{i\eta}{R})B_{i,x} = 0 \left.\right\} i = 1,2,\cdots k$$

$$\theta_{21} D_{i,xx} - \theta_{11}(\frac{i\eta}{R})^2 D_i + \theta_{31}(\frac{i\eta}{R})D_{i,x} + d_{11} B_{i,xx} - 2d_{13}(\frac{i\eta}{R})B_{i,x} = 0 \left.\right\}$$

$$-a_{12}\left(\tfrac{i\eta}{R}\right)^2 C_i + a_{22} C_{i,xx} - a_{23}\left(\tfrac{i\eta}{R}\right) D_{i,x} + b_{21} A_{i,xx} + 2b_{23}\left(\tfrac{i\eta}{R}\right) B_{i,x} = 0$$

$$-a_{13}\left(\tfrac{i\eta}{R}\right)^3 D_i + 2a_{23}\left(\tfrac{i\eta}{R}\right) D_{i,xx} + (a_{33}+a_{12}) C_{i,x}\left(\tfrac{i\eta}{R}\right)^2 - a_{22} C_{i,xxx}$$

$$+ (b_{31}-2b_{23})\left(\tfrac{i\eta}{R}\right) B_{i,xx} - (2b_{33}-b_{22})\left(\tfrac{i\eta}{R}\right)^2 A_{i,x} - b_{21} A_{i,xxx} - \frac{A_{i,x}}{R}$$

$$+\frac{\eta^2}{2R^2}\sum_{j=0}^{K}\Big\{\big[(i+j)^2 A^\circ_{i+j} + (1-\gamma^2_{j-i}+\gamma_i)(i-j)^2 A^\circ_{|i-j|}\big] A_{j,x}$$

$$+\big[(i+j)^2 B_{i+j} + (-1-\gamma_{i-j}+\gamma_i)(i-j)^2 B^\circ_{|i-j|}\big] B_{j,x}\Big\} = 0$$

$$-a_{12}\left(\tfrac{i\eta}{R}\right)^2 D_i + a_{22} D_{i,xx} + a_{23}\left(\tfrac{i\eta}{R}\right) C_{i,x} + b_{21} B_{i,xx} - 2b_{23}\left(\tfrac{i\eta}{R}\right) A_{i,x} = 0$$

$$a_{13}\left(\tfrac{i\eta}{R}\right)^3 C_i - 2a_{23}\left(\tfrac{i\eta}{R}\right) C_{i,xx} + (a_{33}+a_{12}) D_{i,x}\left(\tfrac{i\eta}{R}\right)^2 - a_{22} D_{i,xxx}$$

$$- (b_{31}-2b_{33})\left(\tfrac{i\eta}{R}\right) A_{i,xx} - (2b_{33}-b_{22})\left(\tfrac{i\eta}{R}\right)^2 B_{i,x} - b_{21} B_{i,xxx} - \frac{B_{i,x}}{R}$$

$$+\frac{\eta^2}{2R^2}\sum_{j=0}^{K}\Big\{\big[-(i+j)^2 A^\circ_{i+j} + (1-\gamma^2_{j-i}+\gamma_i)(i-j)^2 A^\circ_{|i-j|}\big] B_{j,x}$$

$$+\big[(i+j)^2 B^\circ_{i+j} + (-1+\gamma_{i-j}+\gamma_i)(i-j)^2 B^\circ_{|i-j|}\big] A_{j,x}\Big\} = 0$$

$i=1,2,\cdots 2K$

$$(A-81)$$

CC-1

$$A_0 = A_{0,x} = 0$$

$$A_i = A_{i,x} = B_i = B_{i,x} = 0 \qquad ; \; i = 1,2,\cdots\cdots,K$$

$$C_i = D_{i,x} = D_i = C_{i,x} = 0 \qquad ; \; i = 1,2,\cdots\cdots 2K \qquad (A-82)$$

CC-2

$$A_0 = A_{0,x} = 0$$

$$A_i = A_{i,x} = B_i = B_{i,x} = 0 \quad ; \; i = 1, 2, \cdots, k$$

$$D_{i,x} = C_{i,x} = 0$$

$$\left.\begin{array}{l}
-a_{13}\left(\tfrac{i\eta}{R}\right)^3 D_i + 2\,a_{23}\left(\tfrac{i\eta}{R}\right) D_{i,xx} - a_{22} C_{i,xxx} - \theta_{21} A_{i,xxx} \\[6pt]
+ (\theta_{31} - 2\,\theta_{23})\left(\tfrac{i\eta}{R}\right) B_{i,xx} = 0 \\[10pt]
a_{13}\left(\tfrac{i\eta}{R}\right)^3 C_i - 2 a_{23}\left(\tfrac{i\eta}{R}\right) C_{i,xx} - a_{22} D_{i,xxx} - \theta_{21} B_{i,xxx} \\[6pt]
- (\theta_{31} - 2\,\theta_{23})\left(\tfrac{i\eta}{R}\right) A_{i,xx} = 0
\end{array}\right\} \; i = 1, 2, \cdots, 2K$$

$$(A\text{-}83)$$

CC-3

$$A_0 = A_{0,x} = 0$$

$$A_i = A_{i,x} = B_i = B_{i,x} = 0 \quad ; \; i = 1, 2, \cdots, k$$

$$\left.\begin{array}{l}
C_i = D_i = a_{22} C_{i,xx} - a_{23}\left(\tfrac{i\eta}{R}\right) D_{i,x} + \theta_{21} A_{i,xx} = 0 \\[8pt]
a_{22} D_{i,xx} + a_{23}\left(\tfrac{i\eta}{R}\right) C_{i,x} + \theta_{21} B_{i,xx} = 0
\end{array}\right\} \; i = 1, 2, \cdots, 2k$$

$$(A\text{-}84)$$

CC-4

$$A_0 = A_{0,x} = 0$$

$$A_i = A_{i,x} = B_i = B_{i,x} \quad ; \; i = 1, 2, \cdots, k$$

$$-a_{12}\left(\tfrac{i\eta}{R}\right)^2 C_i + a_{22} C_{i,xx} - a_{23}\left(\tfrac{i\eta}{R}\right) D_{i,x} + \theta_{21} A_{i,xx} = 0$$

$$-a_{13}\left(\tfrac{i\eta}{R}\right)^3 D_i + 2 a_{23}\left(\tfrac{i\eta}{R}\right) D_{i,xx} + (a_{33} + a_{12})\left(\tfrac{i\eta}{R}\right)^2 C_{i,x}$$

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$$- a_{22} C_{i,xxx} - \beta_{21} A_{i,xxx} + (\beta_{31} - 2\beta_{23})(\tfrac{i\eta}{R}) B_{i,xx} = 0$$

$$- a_{12}(\tfrac{i\eta}{R})^2 D_i + a_{22} D_{i,xx} + a_{23}(\tfrac{i\eta}{R}) C_{i,x} + \beta_{21} B_{i,xx} = 0 \left.\right\} \quad i = 1, 2, \cdots, 2K$$

$$a_{13}(\tfrac{i\eta}{R})^3 C_i - 2 a_{23}(\tfrac{i\eta}{R}) C_{i,xx} + (a_{33} + a_{12})(\tfrac{i\eta}{R})^2 D_{i,x}$$

$$- a_{22} D_{i,xxy} - \beta_{21} B_{i,xxx} - (\beta_{31} - 2\beta_{23})(\tfrac{i\eta}{R}) A_{i,xx} = 0 \left.\right. \qquad (A\text{-}85)$$

A. 2.8 Solution Methodology-Finite Difference Equations

Before casting the field equations into finite difference form, the linearized ordinary differential equations of compatibility and equilibrium, Eqs (73) - (77), can be written in matrix form.

$$[M_1]\{X_{,xxxx}\} + [M_2]\{X_{,xxx}\} + [M_3]\{X_{,xx}\}$$

$$+ [M_4]\{X_{,x}\} + [M_5]\{X\} + \{M_6\} = 0 \qquad (A\text{-}86)$$

where

$$\{X\} = \lfloor A_0^{m+1}, \cdots, A_k^{m+1}, B_1^{m+1}, \cdots, B_k^{m+1}, C_1^{m+1}, \cdots, C_{2k}^{m+1}, D_1^{m+1}, \cdots, D_{2K}^{m+1} \rfloor^T \qquad (A\text{-}87)$$

is the column matrix of the unknown function of position x, and $[M_j]$, j = 1, 2....5 are square matrices $[(6k + 1)$ by $(6k + 1)$; see Eqs A-73-A-77] with elements composed of known parameters (applied loads, geometry, and values of the unknowns evaluated at the previous step, m and therefore known). $\{M_6\}$ is a column matrix of known elements.

Next, transformation equations are introduced in order to reduce the order of the linearized differential equations. This step increases (doubles) the number of equations, but it is introduced for convenience, because it is easier to deal with low order equations when employing the finite difference scheme. These transformation equations are

$$\{\eta\} = \{X,_{xx}\}$$

and they are used in only in connection with the third and fourth derivatives.

By this transformation, Eq. A-87 Eq. A-86 becomes

$$[R]\begin{Bmatrix}\{X,_{xx}\}\\\{\eta,_{xx}\}\end{Bmatrix} + [S]\begin{Bmatrix}\{X,_{x}\}\\\{\eta,_{x}\}\end{Bmatrix} + [T]\begin{Bmatrix}\{X\}\\\{\eta\}\end{Bmatrix} = \{G\}$$

(A-88)

where

$$[R] = \begin{bmatrix}[0] & [M_1]\\[I] & [0]\end{bmatrix} \quad ; \quad [S] = \begin{bmatrix}[M_4] & [M_2]\\[0] & [0]\end{bmatrix}$$

$$[T] = \begin{bmatrix}[M_5] & [M_3]\\[0] & -[I]\end{bmatrix} \quad ; \quad \{G\} = \begin{Bmatrix}-[M_6]\\\{0\}\end{Bmatrix}$$

(A-89)

The governing equations (linearized ordinary differential equations) shown in matrix form, Eqs A-88 are next cast into finite difference form. The usual central difference formula is employed and the equation become

$$\left(\frac{1}{h^2}[R]^{(j)} + \frac{1}{2h}[S]^{(j)}\right)\begin{Bmatrix}\{X\}\\\{\eta\}\end{Bmatrix}^{(j+1)} + \left(-\frac{1}{2h}[R]^{(j)}\right.$$

$$\left. + [T]^{(j)}\right)\begin{Bmatrix}\{X\}\\\{\eta\}\end{Bmatrix}^{(j)} + \left(\frac{1}{h^2}[R]^{(j)} - \frac{1}{2h}[S]^{(j)}\right)\begin{Bmatrix}\{X\}\\\{\eta\}\end{Bmatrix}^{(j-1)} = \{G\}^{j}$$

(A-90)

where j denotes the j th node of the finite difference grid. At each end (x = 0 and L) one more fictitious point is used. This requires (12k + 2) additional equations at each end [the total number is (24k + 4)]. These needed additional equations are the boundary conditions at each end, Eqs A-78- A-79, (whichever set applies from SS-i or CC-i) and their number is (12k + 2). The boundary conditions may also be, first, expressed in matrix form and then cast into finite difference form.

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<u>at either x = 0 or L</u>

$$[N_1]\{X_{,xx}\} + [N_2]\{X_{,xx}\} + [N_3]\{X_{,x}\} + [N_4]\{X\} + \{N_5\} = 0 \quad (A-91)$$

where $\begin{bmatrix} N_j \end{bmatrix}$, j = 1, 2, 3, 4, are matrices $[(12k + 2)$ by $(6k + 1)]$ with known

element, and $\{N_s\}$ in a column matrix $[(12k + 2)$ by one$]$ with, also, known elements.

Use of the transformation equations, Eq A-87, yields

$$[BS]\begin{Bmatrix} \{X_{,x}\} \\ \{\eta_{,x}\} \end{Bmatrix} + [BT]\begin{Bmatrix} \{X\} \\ \{\eta\} \end{Bmatrix} = [BG] \quad\quad (A-92)$$

where

$$[BS] = [[N_3][N_1]]$$

$$[BT] = [[N_4][N_2]]$$

and

$$[BG] = -\{N_s\} \quad\quad (A-93)$$

Note that $[BS]$ and $[BT]$ are square matrices $[(12k + 2)$ by $(12k + 2)]$. In

finite difference form, Eq. A-92 , becomes

$$\frac{1}{2h}[BS]^j \begin{Bmatrix} \{X\} \\ \{\eta\} \end{Bmatrix}^{j+1} + [BT]^j \begin{Bmatrix} \{X\} \\ \{\eta\} \end{Bmatrix}^j - \frac{1}{2h}[BS]^j \begin{Bmatrix} \{X\} \\ \{\eta\} \end{Bmatrix}^{j-1} = \{BG\}^j \quad (A-94)$$

where j in the node number at x = 0 and x = L(1 or N)

A . 2.9 <u>End Shortening, Average Shear Strain and Total Potential</u>

Before outlining in detail the numerical scheme of the solution methodo-

logy, it is necessary to write the expressions for the average end shortening,

average shear strain and the total potential in terms of the dependent varia-

bles, A_i, B_i, C_i and D_i.

The average end shortening and shear strain are defined by

$$\varrho_{AV} = -\frac{1}{2\pi RL}\int_0^{2\pi R}\int_0^L \frac{\partial u}{\partial x}\,dx\,dy$$

$$\gamma_{AV} = \frac{1}{2\pi RL}\int_0^{2\pi R}\int_0^L \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)dx\,dy \qquad (A\text{-}95)$$

In terms of the variables $w(x, y)$ and $F(x, y)$, the above expressions become:

$$\varrho_{AV} = a_{11}\bar{N}_{xx} - a_{13}\bar{N}_{xy} - \frac{1}{2\pi RL}\int_0^{2\pi R}\int_0^L [\, a_{11}F_{,yy} + a_{12}F_{,xx} - a_{13}F_{,xy}$$

$$+\,\mathcal{B}_{11}W_{,xx} + \mathcal{B}_{12}W_{,yy} + 2\mathcal{B}_{23}W_{,xy} - \tfrac{1}{2}W_{,x}(W_{,x}+2W^\circ_{,x})]\,dx\,dy \quad (A\text{-}96)$$

$$\gamma_{AV} = -a_{13}\bar{N}_{xx} + a_{33}\bar{N}_{xy} + \frac{1}{2\pi RL}\int_0^{2\pi R}\int_0^L [\, a_{13}F_{,yy} + a_{23}F_{,xx} - a_{33}F_{,xy}$$

$$+\,\mathcal{B}_{31}W_{,xx} + \mathcal{B}_{32}W_{,yy} + 2\,\mathcal{B}_{33}W_{,xy} - \tfrac{1}{2}W_{,x}(W_{,y}+2W^\circ_{,y})$$

$$-\tfrac{1}{2}W_{,y}(W_{,x}+2W^\circ_{,x})]\,dx\,dy \qquad (A\text{-}97)$$

Finally, if the expressions for w and F are substituted into Eqs. A-96 and A-97 these equations become:

$$\varrho_{AV} = a_{11}\bar{N}_{xx} - a_{13}\bar{N}_{xy} - \tfrac{1}{2}\int_0^L \left\{\frac{a_{12}}{a_{22}}\left\{-\mathcal{B}_{21}A^{\prime\prime}_0 - A_0/R + a_{12}\bar{N}_{xx}\right.\right.$$

$$-a_{23}\bar{N}_{xy}+\left(\tfrac{n}{2R}\right)^2\sum_{j=1}^{K} j^2[(A_j+2A^\circ_j)A_j + (B_j+2B^\circ_j)B_j]\right\}$$

$$+\,\mathcal{B}_{11}A^{\prime\prime}_0 - \tfrac{1}{2}(A^\prime_0+2A^{\circ\prime}_0)A^\prime_0 - \tfrac{1}{4}[A^\prime_j(A^\prime_j+2A^{\circ\prime}_j)+B^\prime_j(B^\prime_j+2B^{\circ\prime}_j)]\right\}dx \quad (A\text{-}98)$$

$$\gamma_{AV} = -a_{13}\bar{N}_{xx} + a_{33}\bar{N}_{xy} + \tfrac{1}{2}\int_0^L \left\{\frac{a_{23}}{a_{22}}\left\{-\mathcal{B}_{21}A^{\prime\prime}_0 - A_0/R\right.\right.$$

$$+a_{12}\bar{N}_{xx} - a_{23}\bar{N}_{xy}+\left(\tfrac{n}{2R}\right)^2\sum_{j=1}^{K} j^2[(A_j+2A^\circ_j)A_j$$

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$$+ (B_j + 2B_j^\circ)\, B_j]\} + \mathcal{B}_{31}\, A_0'' - \sum_{j=1}^{K} \left(\frac{j\pi}{2R}\right)^2 [\, A_j'\,(B_j + B_j^\circ)$$

$$- B_j'\,(A_j + A_j^\circ) + A_j''\, B_j - B_j''\, A_j\,]\}\, dx \qquad\qquad (A\text{-}99)$$

Similarly, the expression for the total potential is:

$$U_T = \frac{1}{2}\int_0^{2\pi R}\!\!\int_0^L (N_{xx}\,\epsilon_{xx}^\circ + N_{yy}\,\epsilon_{yy}^\circ + N_{xy}\,\gamma_{xy}^\circ - M_{xx}\,k - M_{yy}\,k_{yy}$$

$$- 2M_{xy}\,k_{xy})\,dx\,dy - \int_0^{2\pi R}\!\!\int_0^L g\,w\,dx\,dy - \int_0^{2\pi R} [-\bar{N}_{xx}\,u$$

$$+ \bar{N}_{xy}\,v\,]\Big|_0^L dy + \int_0^{2\pi R} (\bar{M}_{xx}\,w_{,x})\Big|_0^L dy \qquad\qquad (A\text{-}100)$$

where $\bar{M}_{xx} = -\,\overline{E N}_{xx}$ and $\bar{E}$ is the load eccentricity meas ed positive in the positive z-direction and

$$u\Big|_0^L = \int_0^L \frac{\partial u}{\partial x}\,dx \qquad \& \qquad v\Big|_0^L = \int_0^L \frac{\partial v}{\partial x}\,dx$$

Thus, the contribution of the in-plane loads to the total potential becomes

$$-\int_0^{2\pi R} [-\bar{N}_{xx}\,u + \bar{N}_{xy}\,v\,]\Big|_0^L dy = -\int_0^{2\pi R} [-\bar{N}_{xx}\int_0^L \frac{\partial u}{\partial x}\,dx + \bar{N}_{xy}\int_0^L \frac{\partial v}{\partial x}\,dx\,]\,dy$$

In terms of w and F the expression for U_T becomes

$$U_T = \frac{1}{2}\int_0^{2\pi R}\!\!\int_0^L [\, a_{11}\,F_{,yy}^2 + a_{22}\,F_{,xx}^2 + a_{33}\,F_{,xy}^2 + 2a_{12}\,F_{,xx}\,F_{,yy}$$

$$- 2a_{13}\,F_{,yy}\,F_{,xy} - 2a_{23}\,F_{,xx}\,F_{,xy}\,]\,dx\,dy - \frac{1}{2}\int_0^{2\pi R}\!\!\int_0^L (\,d_{11}\,w_{,xx}^2$$

$$+ d_{22}\,w_{,yy}^2 + 4d_{33}\,w_{,xy}^2 + 2d_{12}\,w_{,xx}\,w_{,yy} + 4d_{13}\,w_{,xx}\,w_{,xy}$$

$$+ 4d_{23}\,w_{,yy}\,w_{,xy})\,dx\,dy - \bar{N}_{xx}\int_0^{2\pi R}\!\!\int_0^L (\,a_{11}\,F_{,yy} + a_{12}\,F_{,xx}$$

$$- a_{13}\,F_{,xy})\,dx\,dy + \bar{N}_{xy}\int_0^{2\pi R}\!\!\int_0^L (\,a_{23}\,F_{,xx} - a_{33}\,F_{,xy}$$

$$+ a_{13} F_{,yy}) dx dy - \int_0^{2\pi R} \int_0^L g w \, dx dy + \pi R L \, (a_{11} \bar{N}_{xx}^2$$

$$+ a_{33} \bar{N}_{xy}^2 - 2\pi R L \, (\varrho_{Av} \bar{N}_{xx} + \gamma_{Av} \bar{N}_{xy}) - 2\pi R L \, a_{13} \bar{N}_{xx} \bar{N}_{xy}$$

$$- \int_0^{2\pi R} (\bar{E} \, \bar{N}_{xx} W_{,x}) \Big|_0^L dy \qquad\qquad (A\text{-}101)$$

Finally, the expression for the total potential in terms of A_i, B_i, C_i and D_i becomes

$$U_T = \pi R \int_0^L \left\{ \frac{1}{a_{22}} \left\{ - \varrho_{21} A_o'' - A_o/R + \left(\frac{n}{2R}\right)^2 \sum_{j=1}^K j^2 \left[(A_j + 2A_j^\circ) A_j \right. \right. \right.$$

$$\left. + (B_j + 2B_j^\circ) B_j \right] + a_{12} \bar{N}_{xx} - a_{23} \bar{N}_{xy} \Big\}^2 + 2 (a_{33} \bar{N}_{xy} - a_{12} \bar{N}_{xx})$$

$$\cdot \frac{1}{a_{22}} \left\{ - \varrho_{21} A_o'' - A_o/R + \left(\frac{n}{2R}\right)^2 \sum_{j=1}^K j^2 \left[(A_j + 2A_j^\circ) A_j \right. \right.$$

$$\left. + (B_j + 2B_j^\circ) B_j \right] + a_{12} \bar{N}_{xx} - a_{23} \bar{N}_{xy} \Big\} - d_{11} (A_o'')^2$$

$$+ \frac{1}{2} \sum_{i=1}^{2K} \left\{ a_{11} \left(\frac{i n}{R}\right)^4 (C_i^2 + D_i^2) + a_{22} \left[(C_i'')^2 + (D_i'')^2 \right] \right.$$

$$+ a_{33} \left(\frac{i n}{R}\right)^2 \left[(C_i')^2 + (D_i')^2 \right] - 2 a_{12} \left(\frac{i n}{R}\right)^2 (C_i'' C_i + D_i'' D_i)$$

$$\left. - 2 a_{13} \left(\frac{i n}{R}\right)^3 (- C_i D_i' + D_i C_i') - 2 a_{23} \left(\frac{i n}{R}\right) (C_i'' D_i - D_i'' C_i) \right\}$$

$$- \frac{1}{2} \sum_{i=1}^K \left\{ d_{11} \left[(A_i'')^2 + (B_i'')^2 \right] + d_{22} \left(\frac{i n}{R}\right)^4 (A_i^2 + B_i^2) \right.$$

$$+ 4 d_{33} \left(\frac{i n}{R}\right)^2 \left[(A_i')^2 + (B_i')^2 \right] - 2 d_{12} \left(\frac{i n}{R}\right)^2 (A_i'' A_i + B_i'' B_i)$$

$$-4d_{13}\left(\tfrac{i\eta}{R}\right)\left(-A_i'' B_i' + B_i'' A_i'\right) - 4d_{33}\left(\tfrac{i\eta}{R}\right)^3\left(A_i B_i' - B_i A_i'\right)\bigg\}\bigg\} \, dx$$

$$- \pi R \int_0^l \left\{2g_o' A_o + \sum_{j=1}^{K}\left[g_j' A_j + g_j^* B_j\right]\right\} dx - 2\pi R L \left(e_{av}\bar{N}_{xx}\right)$$

$$+ \gamma_{av}\bar{N}_{xy}\right) + \pi R L \left(a_{11}\bar{N}_{xx}^2 - 2a_{13}\bar{N}_{xx}\bar{N}_{xy} + a_{33}\bar{N}_{xy}\right)$$

$$- 4\pi \bar{E}\bar{N}_{xy} R A_o' \qquad\qquad\qquad (A\text{-}102)$$

Before leaving this section, it is important to give the expression for the modified potential an expression needed in the estimation of dynamic critical loads. As explained in Ref. 39 the modification is associated with the deflectional response of the system. When an axial load is applied, an axial motion will result (with some related transverse motion). If an instability of the type described in Refs. 40-43 and 37 is to take plane, under sudden application of the axial load, it should not be expected to occur through the primary axial node, but through the existence of transverse deflectional nodes, unrelated to the axial node. Because of this and since the governing equation for dynamic buckling is (though conservation of energy)

$$U_{T_{\text{Mod.}}} + T = 0 \qquad\qquad\qquad (A\text{-}103)$$

where T is the kinetic energy (unrelated to transverse deflectional modes), then the modified potential must not contain in plane node terms, when suddenly applied in-plane loads, $\bar{N}_{xx}$ and $\bar{N}_{xy}$, are considered. In the case of lateral pressure, the modification is different, therefore the expression, given below for the modified total potential, applies only to in-plane loads. This expression is obtained by excluding strictly load-dependent terms and those terms related to F(x, y), $[C_D'']$, which correspond to in-plane motion.
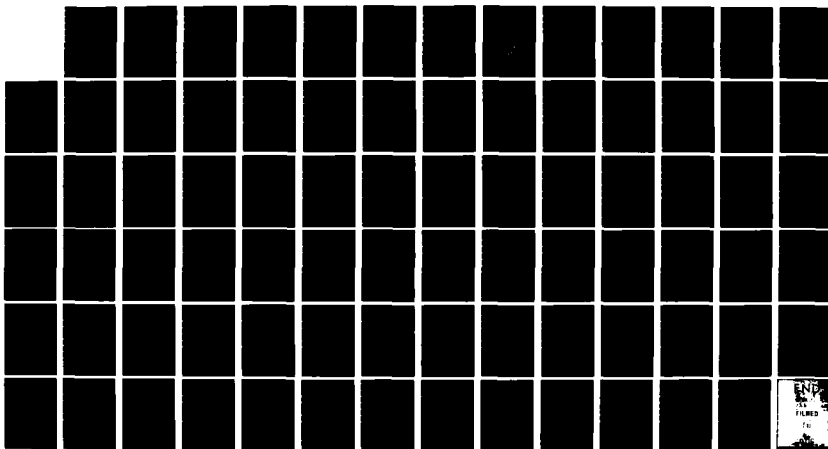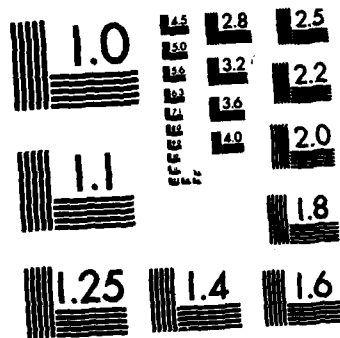
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$$U_{TMod} = U_T + \pi RL \, [\bar{N}_{xx}^2 \, (a_{11} - a_{12}^2/a_{22}) + \bar{N}_{xy} \, (a_{33}$$

$$- a_{23}^2/a_{22}) + 2\bar{N}_{xx}\bar{N}_{xy} \, (a_{12} \, a_{23}/a_{22} - a_{13})] \qquad (A\text{-}104)$$

A. 2.9 Solution Methodology - Numerical Scheme

A computer program has been written (see Appendix A for flow charts and Program Listing) for data generation. The linearized finite difference equations are solved by an algorithm which is a modification of the one described in Ref. 43. The modification, which consists of a generalization of the algorithm of Ref. 43 is fully described in Appendix B. The solution procedure used for the problem, herein, is based on the algorithm described in Appendix B.

The field equations, Eq. A-90, can be written as

$$[\bar{C}_k]\{\bar{Z}_{k-1}\} + [\bar{B}_k]\{\bar{Z}_k\} + [\bar{A}_k]\{\bar{Z}_{k+1}\} = \{G_k\} \qquad (A\text{-}105)$$

where $k = 1, 2 \ldots \ldots N$ and

$$[\bar{C}_k] = \tfrac{1}{h^2}[R]^k - \tfrac{1}{2h}[S]^k \;\; ; \;\; [\bar{B}_k] = -\tfrac{1}{2h^2}[\tilde{R}]^k + [T]^k$$

$$[\bar{A}_k] = \tfrac{1}{h^2}[R]^k + \tfrac{1}{2h}[S]^k \;\; ; \;\; \{\bar{Z}_k\} = \left\{ \begin{matrix} \{X\} \\ \{\eta\} \end{matrix} \right\}^k \qquad (A\text{-}106)$$

Note that there are $(12k + 2)$ elements in the $\{\bar{Z}_k\}$ vector.

In addition, the boundary conditions, Eqs. A-94 can be written in a similar [to Eqs A.105] form.

at $x = 0$ ($k = 1$)

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$$-\left[\bar{\bar{C}}_1\right]\left\{\bar{z}_0\right\} + \left[\bar{\bar{B}}_1\right]\left\{\bar{z}_1\right\} + \left[\bar{\bar{A}}_1\right]\left\{\bar{z}_2\right\} = \left\{BG_1\right\} \qquad (A-107)$$

and at $x = L(K = N)$

$$-\left[\bar{C}_N\right]\left\{\bar{z}_{N-1}\right\} + \left[\bar{\bar{B}}_N\right]\left\{\bar{z}_N\right\} + \left[\bar{\bar{A}}_N\right]\left\{\bar{z}_{N+1}\right\} = \left\{BG_N\right\} \qquad (A-108)$$

whefe

$$\left[\bar{\bar{C}}_i\right] = \frac{1}{2h}\left[BS\right]^i \; ; \; \left[\bar{\bar{B}}_i\right] = \left[BT\right]^i \; ; \; \left[\bar{\bar{A}}_i\right] = \frac{1}{2h}\left[BS\right]^i \qquad (A-109)$$

$$i = 1, N$$

Note that $\left\{\bar{z}_0\right\}$ and $\left\{\bar{z}_{N+1}\right\}$ denote the vectors of the unknowns at the fictitious points ($k = 0$ and $k = N + 1$).

By properly arranging Eqs. A-105, A-107 and A-108 for the entire cylinder, the following matrix representation is obtained.

Eq. A.110 can be put in the form of Fig C.1 (Appendix C) and it will be a special case of this form, by the following changes. First, there is no common unknown vector Z_1 and thus all the $\{d_i\}$ vectors are zero (tridiagonal matrix). Next,

$$[B_1] = \begin{bmatrix} [\bar{\bar{C}}_1] & [\bar{\bar{B}}_1] \\ [\bar{C}_1] & [\bar{B}_1] \end{bmatrix} \qquad (24k+4) \; by \; (24k+4)$$

$$\{Z_1\} = \begin{Bmatrix} \{\bar{\bar{Z}}_0\} \\ \{\bar{Z}_1\} \end{Bmatrix} \qquad (24k+4) \; by \; One$$

$$[A_1] = \begin{bmatrix} [\bar{\bar{A}}_1] \\ [\bar{A}_1] \end{bmatrix} \qquad (24k+4) \; by \; (12k+2)$$

$$[g_1] = \begin{Bmatrix} \{BG_1\} \\ \{G_1\} \end{Bmatrix} \qquad (24k+4) \; by \; One$$

$$[C_2] = [[0] \; [\bar{C}_1]] \qquad (12k+2) \; by \; (24k+4)$$

$$[C_j] = [\bar{C}_j] \qquad j = 3, 4, \cdots, N-1$$

$$[B_j] = [\bar{B}_j] \qquad j = 2, 3, \cdots, N-1$$

$$[A_j] = [\bar{A}_j] \qquad j = 2, 3, \cdots, N-2$$

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$$\{Z_j\} = \{\bar{Z}_j\}$$

$$j = 2, 3, \cdots, N-1$$

$$\{g_j\} = \{\bar{g}_j\}$$

$$[A_{N-1}] = [[\bar{A}_{N-1}], [0]] \qquad (12k+2) \; by \; (24k+4)$$

$$[C_N] = \begin{bmatrix} [\bar{C}_N] \\ [\bar{\bar{C}}_N] \end{bmatrix} \qquad (24k+4) \; by \; (12k+2)$$

$$[B_N] = \begin{bmatrix} [\bar{B}_N] & [\bar{A}_N] \\ [\bar{\bar{B}}_N] & [\bar{\bar{A}}_N] \end{bmatrix} \qquad (24k+4) \; by \; (24k+4)$$

$$\{Z_N\} = \begin{Bmatrix} \{\bar{Z}_N\} \\ \{\bar{\bar{Z}}_{N+1}\} \end{Bmatrix} \qquad (24k+4) \; by \; one$$

$$\{g_N\} = \begin{Bmatrix} \{G_N\} \\ \{BG_N\} \end{Bmatrix} \qquad (24k+4) \; by \; one$$

Note that $m_1 = m_N = 24k + 4$, while $m_i = 12k + 2$ for $i = 2, 3, 4, \ldots,$ N - 1.

Note also that Eqs. A-110 represents equilibrium and compatibility equations in which displacement components (A_i, B_i) and stress resultant components (C_i, D_i) [see Eq. A-86a] are the

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unknown functions, while the geometry and the loading (taken in increments) are taken on known parameters (assigned everytime the equations are solved). Thus, this special case of the algorithm, Eqs A-110, is employed for finding pre-limit point response. When approaching the critical load, the increment in the applied load parameter is kept small and the sign of the determinant of the coefficients [D in Eq.(C - 19)] must be checked. If convergence fails, the load level is over the limit point. But if convergence does not fail and the sign of the determinant changes from what it was at the previous load level, then the load level is also over the limit point. Desired accuracy can be achieved by taking smaller and smaller increments in the load parameter. It is also observed that by employing this procedure (special case of the algorithm in which the load parameter is known), no solution can be obtained past the limit point. Because of this, the more general algorithm, described in Appendix B, is employed at this point of the solution procedure. The new and more general algorithm simply changes the role of one of the displacement terms with that of the applied load parameter. By so doing the form of the equations changes and the matrix of the coefficients of the unknown ceases to be tridiagonal. Depending on the position of the particular term that replaces the load parameter [which one of the $(6k + 2)$ terms, and at which <u>node</u> (x-position)] column matrices appear all along the column corresponding to the vector $\{Z_L\}$ and the new equations assume exactly the form shown on Fig. C-1. Thus, at some level before, the limit point, the procedure is switched to the more general algorithm (Appendix C), in which one of the displacement parameters $(A_I$ or $B_I)$ at some specified node is taken as known (specified increments) and the load parameter is the unknown. This solution procedure is continued until the desired portion of the post-limit point response is obtained.

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Finally, in generating data, numerical integration is used to find the values of the total potential, the average end shortening and the average shear [see Eqs A-102, A-98, and A-99].

A.3.0 The u, v, w - Formulation

The geometry and sign convention for this formulation are shown on Figs A.3 and A.4. Note that for this case the x-axis (and therefore the transverse displacement component w) is taken as positive outward.

In this formulation two distinctly different kinematic relations (different shell theories) are employed. One is due to Sanders (Ref 34) and one due to Donnell (Ref 33). In the case of Sanders' equations, it is assumed that the reference surface strains are small, the rotation about the normal is negligibly small and the rotations about in-plane axes are moderate.

One of the reasons for expressing the governing equations in terms of u, v, and w, is that it is not possible to define a stress resultant function, in order to satisfy the in-plane equilibrium equation identically, when using the Sanders' kinematic relations. The case of using Donnell-type kinematic relations is a special case of the Sanders case.

A.3.1 Kinematic Relations

The kinematic relations derived by Sanders assume a perfect reference surface. These kinematic relations (Ref 34) are modified to include the effect of an initial geometric imperfection $w^o(x,y)$ as shown below.

$$\epsilon_{xx} = \epsilon_{xx}^o + z\, k_{xx}$$

$$\epsilon_{yy} = \epsilon_{yy}^o + z\, k_{yy}$$

$$\gamma_{xy} = \gamma_{xy}^o + 2z\, k_{xy} \qquad\qquad (A\text{-}111)$$

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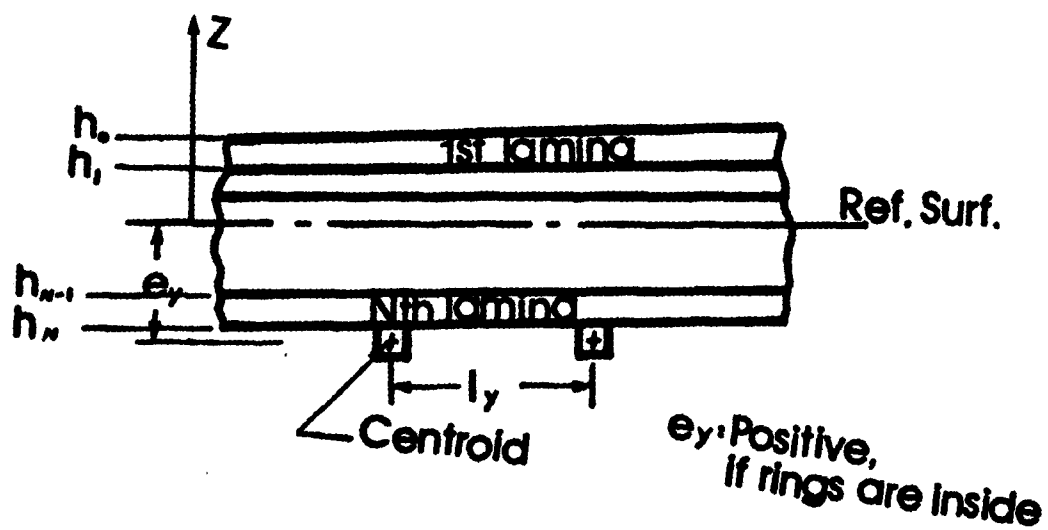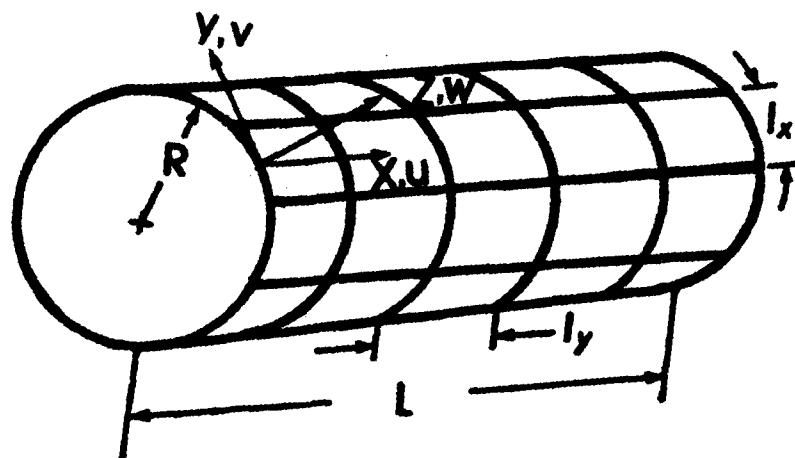
Fig. A.3 Geometry
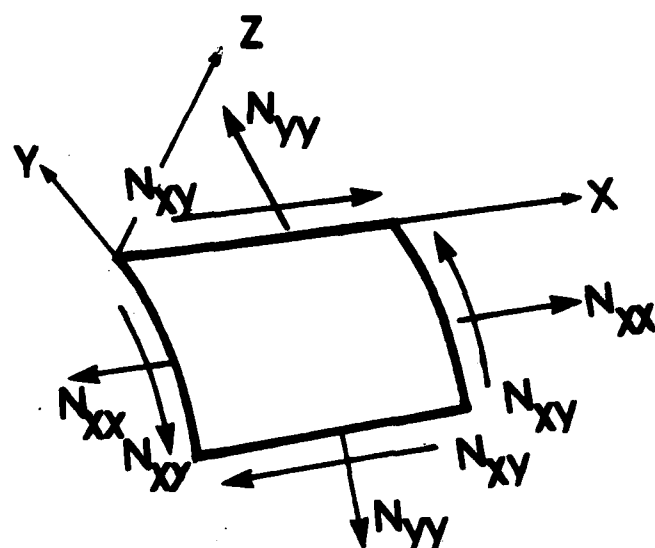
Stress Resultants
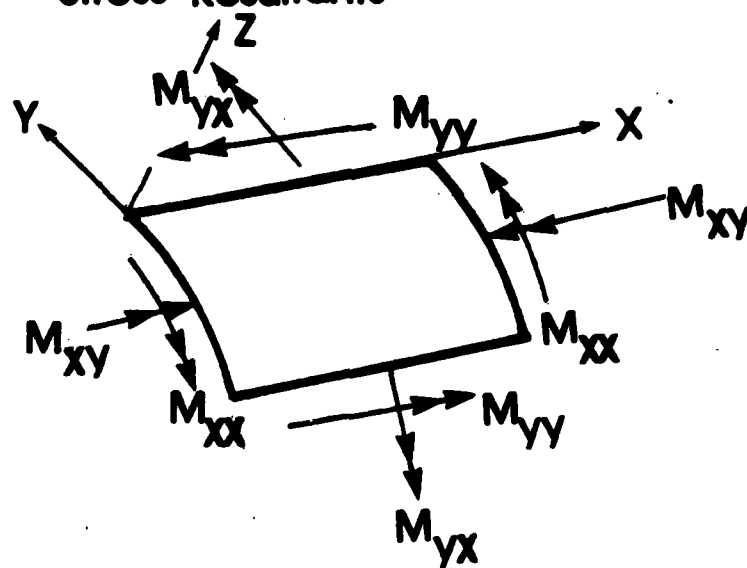
Moment Resultants

Fig. A.4 Sign Convention

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where

$$\epsilon_{xx}^{\bullet} = U_{,x} + \frac{1}{2} W_{,x}^{2} + W_{,x} W_{,x}^{\bullet}$$

$$\epsilon_{yy}^{\bullet} = U_{,y} + \frac{W}{R} + \frac{1}{2} W_{,y}^{2} + W_{,y} W_{,y}^{\bullet} + \frac{\delta_{1}}{2} \left[\frac{U^{2}}{R^{2}} - 2\frac{U}{R}(W_{,y} + W_{,y}^{\bullet}) \right]$$

$$\gamma_{xy}^{\bullet} = U_{,y} + U_{,x} + W_{,x} W_{,y} + W_{,x}^{\bullet} W_{,y} + W_{,x} W_{,y}^{\bullet} - \delta_{1} \frac{U}{R}(W_{,x} + W_{,x}^{\bullet})$$

$$\varphi_{x} = - W_{,x} \qquad ; \qquad \varphi_{y} = -W_{,y} + \delta_{1}\frac{U}{R}$$

$$K_{xx} = -W_{,xx} \qquad ; \qquad K_{yy} = -W_{,yy} + \delta_{1}\frac{U_{,y}}{R}$$

$$K_{xy} = -W_{,xy} + \frac{1}{2}\delta_{1}\frac{U_{,x}}{R} \qquad\qquad (A-113)$$

where

$$\delta_{1} = \begin{cases} 1 & \text{for Sanders' kinematic relations} \\ 0 & \text{for Donnell's kinematic relations} \end{cases} \qquad (A-114)$$

A. 3.2 Stress-Strain Relations

The constitutive equations are the same as in the w, F-formulation. Because of the different sign convention the relations between the stress and moment resultants on one hand and the reference surface strains and changes in curvature and torsions on the other, these equations are

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{13} \\ \bar{A}_{12} & \bar{A}_{22} & \bar{A}_{23} & \bar{B}_{12} & \bar{B}_{21} & \bar{B}_{23} \\ \bar{A}_{13} & \bar{A}_{23} & \bar{A}_{33} & \bar{B}_{13} & \bar{B}_{23} & \bar{B}_{33} \\ \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{13} & \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} \\ \bar{B}_{12} & \bar{B}_{32} & \bar{B}_{33} & \bar{D}_{12} & \bar{D}_{22} & \bar{D}_{23} \\ \bar{B}_{13} & \bar{B}_{23} & \bar{B}_{33} & \bar{D}_{13} & \bar{D}_{23} & \bar{D}_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{\bullet} \\ \epsilon_{yy}^{\bullet} \\ \gamma_{xy}^{\bullet} \\ K_{xx} \\ K_{yy} \\ 2K_{xy} \end{Bmatrix} \qquad (A-115)$$

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where the expressions for $\overline{A}_{1j}$, $\overline{B}_{1j}$ and $\overline{D}_{1j}$ are given by Eqs A-14 and A-15.

A. 3.3 Equilibrium Equations

Following the same procudure as the one described in section A.2.3,
the equilibrium equations and associated boundary conditions are:

Equilibrium Equations

$$N_{xx,x} + N_{xy,y} = 0$$

$$N_{xy,x} + N_{yy,y} - \delta_1 \frac{N_{yy}}{R}\left(\frac{U}{R} - (W_{,x} + W_{,x}^\circ)\right) + \delta_1 N_{xy}\frac{(W_{,x} + W_{,x}^\circ)}{R}$$
$$+ \delta_1 \frac{M_{xy,x}}{R} + \delta_1 \frac{M_{yy,y}}{R} = 0$$

$$[N_{xx}(W_{,x} + W_{,x}^\circ)]_{,x} + [N_{xy}(W_{,y} + W_{,y}^\circ)]_{,x} + [N_{xy}(W_{,x} + W_{,y}^\circ)]_{,y}$$
$$+ [N_{yy}(W_{,y} + W_{,y}^\circ)]_{,y} - \frac{N_{yy}}{R} - \frac{\delta_1}{R}\left[(N_{xy}U)_{,x} + (N_{yy}U)_{,y}\right]$$
$$+ M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + g = 0 \qquad (A-116)$$

Boundary Conditions (at x = 0, L)

Either	Or

$$N_{xx} = \overline{N}_{xx} \qquad\qquad\qquad \delta U = 0$$

$$N_{xy} + \frac{M_{xy}}{R}\delta_1 = \overline{N}_{xy} + \frac{\overline{M}_{xy}}{R}\delta_1 \qquad \delta U = 0$$

$$N_{xx}(W_{,x} + W_{,x}^\circ) + N_{xy}(W_{,y} + W_{,y}^\circ)$$

$$- \delta_1 \frac{N_{xy}}{R}U + M_{xx,x} + 2M_{xy,y} = \overline{Q}_x + \overline{M}_{xy,y} \quad \delta W = 0$$

$$M_{xx} = \overline{M}_{xx} \qquad\qquad\qquad \delta W_{,x} = 0 \qquad (A-117)$$

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Use of the first equilibrium equation in the third yields

$$(W_{,y} + W_{,y}^{\circ})(N_{xy,x} + N_{yy,y}) + N_{xx}(W_{,xx} + W_{,xx}^{\circ}) + 2N_{xy}(W_{,xy} + W_{,xy}^{\circ})$$

$$+ N_{yy}(W_{,yy} + W_{,yy}^{\circ}) - \frac{N_{yy}}{R} - \frac{\delta_{1}}{R}[U(N_{xy,x} + N_{yy,y}) + N_{xy}U_{,x} + N_{yy}U_{,y}]$$

$$+ M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + q = 0 \qquad (A\text{-}118)$$

A.3.4 Solution Methodology-Field Equations

The solution procedure for this formulation is as follows: assume a separated solution for u, v, and w; express the known (assigned) parameters w°(imperfection) and q(pressure) in a similar form; find expressions for reference surface strains, changes in curvature and torsion and stress and moment resultants; substitute these expressions into the equilibrium equations and use the G alerkin procedure in the circumferential direction (this changes the nonlinear partial differential equations to a set of nonlinear ordinary differential equations); use Newton's method, applicable to differential equations, to reduce the nonlinear field equations to a sequence of linear systems; finally cast equations into finite difference form.

All of these steps are shown herein, in detail. Then, once this step is completed, the solution scheme of Appendix B is used to solve the final set of equations.

The dependent variables are the three displacement components u(x,y), v(x,y) and w(x,y). A separated series form is assumed for each of them

$$u(x,y) = \sum_{i=0}^{K} \left[U_{1i}(x) \cos\frac{i\pi y}{R} + U_{2i}(x) \sin\frac{i\pi y}{R} \right]$$

$$v(x,y) = \sum_{i=0}^{K} \left[V_{1i}(x) \cos\frac{i\pi y}{R} + V_{2i}(x) \sin\frac{i\pi y}{R} \right]$$

$$w(x,y) = \sum_{i=0}^{K} \left[W_{1i}(x) \cos\frac{i\pi y}{R} + W_{2i}(x) \sin\frac{i\pi y}{R} \right] \qquad (A\text{-}119)$$

Thus, the number of unknown functions of x is (2k + 2) for each variable.

The total number is (6k + 6) subject to the condition that

$$\mathcal{U}_{20} = \mathcal{V}_{20} = W_{20} = 0 \qquad\qquad (A\text{-}120)$$

Note that the true number of unknown functions is (6k + 3).

Similarly the expressions for w^o and the pressure $q(x,y)$ are:

$$W^o_{(x,y)} = \sum_{i=0}^{k} [\, W^o_{1i}(x) \cos \tfrac{i\pi y}{R} + W^{\bullet}_{2i}(x) \sin \tfrac{i\pi y}{R} \,] \qquad\qquad (A\text{-}121)$$

$$q(x,y) = \sum_{i=0}^{k} [\, q_{1i} \cos \tfrac{i\pi y}{R} + q_{2i} \sin \tfrac{i\pi y}{R} \,] \qquad\qquad (A\text{-}122)$$

In this case also, the condition $w^o_{20} = q_{20} = 0$ is imposed.

In order to express the equilibrium equations in terms of the parameters of Eqs A-119 - A-122, one needs to first find the expressions for the stress resultants and therefore reference surface strains and changes in curvature and torsion.

Use of Eqs A-119 and A-120 in the expression for ϵ_{ij} and $\varkappa_{ij}$, Eqs A-112 and A-113 yields

$$\epsilon^o_{xx} = \sum_{i=0}^{2k} [\, (\delta_i \mathcal{U}_{1i,x} + t^{L}_{x1i} + t^{n}_{x1i}) \cos \tfrac{i\pi y}{R}$$

$$+ (\delta_i \mathcal{U}_{2i,x} + t^{L}_{x2i} + t^{n}_{x2i}) \sin \tfrac{i\pi y}{R} \,] \qquad\qquad (A\text{-}123)$$

where

$$t^{L}_{x1i} = A^{i}_{1(k)}(W^o_{1,x}, W_{1,x}) + A^{i}_{4(k)}(W^{\bullet}_{2,x}, W_{2,x})$$

$$t^{L}_{x2i} = A^{i}_{2(k)}(W^{\bullet}_{1,x}, W_{2,x}) + A^{i}_{3(k)}(W^{\bullet}_{2,x}, W_{1,x})$$

$$t^{n}_{x1i} = \tfrac{1}{2} \{ A^{i}_{1(k)}(W_{1,x}, W_{1,x}) + A^{i}_{4(k)}(W_{2,x}, W_{2,x}) \}$$

$$t^{n}_{x2i} = \tfrac{1}{2} \{ A^{i}_{2(k)}(W_{1,x}, W_{2,x}) + A^{i}_{3(k)}(W_{2,x}, W_{1,x}) \} \qquad\qquad (A\text{-}123a)$$

$$\epsilon_{yy}^{o} = \sum_{i=0}^{2k} \left\{ \left[\left(\frac{in}{R} \mathcal{U}_{2i} + \frac{w_{1i}}{R} \right) \delta_i + t_{y1i}^{L} + t_{y1i}^{n} \right] \cos \frac{iny}{R} \right.$$

$$\left. + \left[\left(\frac{in}{R} \mathcal{U}_{1i} + \frac{w_{2i}}{R} \right) \delta_i + t_{y2i}^{L} + t_{y2i}^{n} \right] \sin \frac{iny}{R} \right\} \qquad (A\text{-}124)$$

where

$$t_{y1i}^{L} = \left(\frac{n}{R} \right)^2 \left[A_{IJ4(k)}^{i} (W_1^o, W_1) + A_{IJ1(k)}^{i} (W_2^o, W_2) \right]$$

$$+ \frac{\delta_1 n}{R^3} \left[-A_{J1(k)}^{i} (W_2^o, \mathcal{U}_1) + A_{J4(k)}^{i} (W_1^o, \mathcal{U}_2) \right]$$

$$t_{y2i}^{L} = -\left(\frac{n}{R} \right)^2 \left[A_{IJ2(k)}^{i} (W_2^o, W_1) + A_{IJ3(k)}^{i} (W_1^o, W_2) \right]$$

$$+ \frac{\delta_1 n}{R^3} \left[A_{J3(k)}^{i} (W_1^o, \mathcal{U}_1) - A_{J2(k)}^{i} (W_2^o, \mathcal{U}_2) \right]$$

$$t_{y1i}^{n} = \frac{1}{2} \left(\frac{n}{R} \right)^2 \left(A_{IJ4(k)}^{i} (W_1, W_2) + A_{IJ1(k)}^{i} (W_2, W_2) \right)$$

$$+ \frac{\delta_1}{2R^2} \left(A_{1(k)}^{i} (\mathcal{U}_1, \mathcal{U}_1) + A_{4(k)}^{i} (\mathcal{U}_2, \mathcal{U}_2) \right)$$

$$+ \frac{\delta_1 n}{R^2} \left(A_{J4(k)}^{i} (W_1, \mathcal{U}_2) - A_{J1(k)}^{i} (W_2, \mathcal{U}_1) \right)$$

$$t_{y2i}^{n} = -\frac{1}{2} \left(\frac{n}{R} \right)^2 \left(A_{IJ2(k)}^{i} (W_2, W_1) + A_{IJ3(k)}^{i} (W_1, W_2) \right)$$

$$+ \frac{\delta_1}{2R^2} \left(A_{3(k)}^{i} (\mathcal{U}_2, \mathcal{U}_1) + A_{2(k)}^{i} (\mathcal{U}_1, \mathcal{U}_2) \right)$$

$$+ \frac{\delta_1 n}{R} \left(A_{J3(k)}^{i} (W_1, \mathcal{U}_1) - A_{J1(k)}^{i} (W_2, \mathcal{U}_2) \right) \qquad (A\text{-}124a)$$

$$\gamma_{xy}^{o} = \sum_{i=0}^{2k} \left\{ \left[\left(\frac{in}{R} \mathcal{U}_{2i} + \mathcal{U}_{1i,x} \right) \delta_i + t_{xy1i}^{L} + t_{xy1i}^{n} \right] \cos \frac{iny}{R} \right.$$

$$\left. + \left[\left(-\frac{in}{R} \mathcal{U}_{1i} + \mathcal{U}_{2i,x} \right) \delta_i + t_{xy2i}^{L} + t_{xy2i}^{n} \right] \sin \frac{iny}{R} \right. \qquad (A\text{-}125)$$

where

$$t_{xy1i}^{L} = \frac{n}{R} \left[A_{J1(k)}^{i} (W_2^o, W_{1,x}) - A_{J4(k)}^{i} (W_1^o, W_{2,x}) \right.$$

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$$- A_{14(k)}^i (W_{2,x}^{\circ}, W_1) + A_{21(k)}^i (W_{1,x}^{\circ}, W_2)]$$

$$- \frac{\delta_1}{R} \left(A_{1(k)}^i (U_1, W_{1,x}^{\circ}) + A_{4(k)}^i (U_2, W_{2,x}^{\circ}) \right)$$

$$t_{xy2i}^L = \frac{n}{R} \left[A_{J2(k)}^i (W_2^{\circ}, W_{2,x}) - A_{J3(k)}^i (W_1^{\circ}, W_{1,x}) \right.$$

$$- A_{22(k)}^i (W_{1,x}^{\circ}, W_1) + A_{23}^i (W_{2,x}^{\circ}, W_2)]$$

$$- \frac{\delta_1}{R} \left[A_{3(k)}^i (U_2, W_{1,x}^{\circ}) + A_{2(k)}^i (U_1, W_{2,x}^{\circ}) \right]$$

$$t_{xy1i}^n = \frac{n}{2R} \left[A_{J1(k)}^i (W_2, W_{1,x}) - A_{J4(k)}^i (W_1, W_{2,x}) \right.$$

$$- A_{14(k)}^i (W_{2,x}^{\circ}, W_1) + A_{21(k)}^i (W_{1,x}^{\circ}, W_2)]$$

$$- \frac{\delta_1}{R} \left(A_{1(k)}^i (U_1, W_{1,x}) + A_{4(k)}^i (U_2, W_{2,x}^{\circ}) \right)$$

$$t_{xy2i}^n = \frac{n}{2R} \left[A_{J2(k)}^i (W_2, W_{2,x}) - A_{J3(k)}^i (W_1, W_{1,x}) \right.$$

$$- A_{22(k)}^i (W_{1,x}, W_1) + A_{23(k)}^i (W_{2,x}, W_2)]$$

$$- \frac{\delta_1}{R} \left[A_{3(k)}^i (U_2, W_{1,x}) + A_{2(k)}^i (U_1, W_{2,x}) \right] \qquad (A-125a)$$

and

$$\mathcal{K}_{xx} = - \sum_{i=0}^{k} \left[W_{1i,xx} \cos \frac{iny}{R} + W_{2i,xx} \sin \frac{iny}{R} \right] \qquad (A-126)$$

$$\mathcal{K}_{yy} = \frac{1}{R^2} \sum_{i=0}^{k} \left[in (in W_{1i} + \delta_1 U_{2i}) \cos \frac{iny}{R} \right.$$

$$+ in (in W_{2i} - \delta_1 U_{1i}) \sin \frac{iny}{R}] \qquad (A-127)$$

$$\mathcal{K}_{xy} = \sum_{i=0}^{k} \left[\left(- \frac{in}{R} W_{2i,x} + \frac{\delta_1}{2R} U_{1i,x} \right) \cos \frac{iny}{R} \right.$$

$$+ \left(\frac{in}{R} W_{1i,x} + \frac{\delta_1}{2R} U_{2i,x} \right) \sin \frac{iny}{R} \qquad (A-128)$$

Note that δ_i and η_ℓ are the same as before, or

$$\delta_i = \begin{cases} 0 & i > k \\ 1 & i \leq k \end{cases} \quad ; \quad \eta_i = \begin{cases} -1 & \ell < 0 \\ 0 & \ell = 0 \\ 1 & \ell > 0 \end{cases} \quad (A\text{-}129)$$

The symbols $A^{(i)}_{j(k)}$ ($i = 1,2,3,4$), $A^{(i)}_{Jj(k)}$ ($j = 1,2,3,4$), $A_{Ij(k)}$ ($j = 1,2,3,4$), $A_{IJ_{j(k)}}$ ($j = 1,2,3,4$) and $A_{J2j(k)}$ ($j = 1,2,3,4$) result from the use of trigonometric identifies, which are employed to change double to single sums [similar to Eqs A-49 - A-51 and symbols defined by Eqs A-52 -A-54; note that some are common]. The needed trigonometric identities and definition of symbols are given below.

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [\theta_j \cos j\theta] a_i \cos i\theta = \sum_{i=0}^{K+L} A^i_{1(k)}(b,a) \cos i\theta$$

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [\theta_j \cos j\theta] a_i \sin i\theta = \sum_{i=0}^{K+L} A^i_{2(k)}(b,a) \sin i\theta$$

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [\theta_j \sin j\theta] a_i \cos i\theta = \sum_{i=0}^{K+L} A^i_{3(k)}(b,a) \sin i\theta$$

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [\theta_j \sin j\theta] a_i \sin i\theta = \sum_{i=0}^{K+L} A^i_{4(k)}(b,a) \cos i\theta \qquad (A\text{-}130)$$

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [j\theta_j \cos j\theta] a_i \cos i\theta = \sum_{i=0}^{K+L} A^i_{J1(k)}(b,a) \cos i\theta$$

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [j\theta_j \cos j\theta] a_i \sin i\theta = \sum_{i=0}^{K+L} A^i_{J2(k)}(b,a) \sin i\theta$$

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [j\theta_j \sin j\theta] a_i \cos i\theta = \sum_{i=0}^{K+L} A^i_{J3(k)}(b,a) \sin i\theta$$

$$\sum_{i=0}^{K} \sum_{j=0}^{L} [j\theta_j \sin j\theta] a_i \sin i\theta = \sum_{i=0}^{K+L} A^i_{J4(k)}(b,a) \cos i\theta \qquad (A\text{-}131)$$

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$$\sum_{i=0}^{K}\sum_{j=0}^{L}[\ \theta_j \cos j\theta]\ a_{ii}\cos i\theta = \sum_{i=0}^{K+L} A_{I1(k)}^{i}(\theta,a)\ \cos i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[\ \theta_j \cos j\theta]\ a_{ii}\sin i\theta = \sum_{i=0}^{K+L} A_{I2(k)}^{i}(\theta,a)\ \sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[\ \theta_j \sin j\theta]\ a_{ii}\cos i\theta = \sum_{i=0}^{K+L} A_{I3(k)}^{i}(\theta,a)\ \sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[\ \theta_j \sin j\theta]\ a_{ii}\sin i\theta = \sum_{i=0}^{K+L} A_{I4(k)}^{i}(\theta,a)\ \cos i\theta \qquad (A\text{-}132)$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j\ \theta_j \cos j\theta]\ a_{ii}\cos i\theta = \sum_{i=0}^{K+L} A_{IJ1(k)}^{i}(\theta,a)\ \cos i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j\ \theta_j \cos j\theta]\ a_{ii}\sin i\theta = \sum_{i=0}^{K+L} A_{IJ2(k)}^{i}(\theta,a)\ \sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j\ \theta_j \sin j\theta]\ a_{ii}\cos i\theta = \sum_{i=0}^{K+L} A_{IJ3(k)}^{i}(\theta,a)\ \sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j\ \theta_j \sin j\theta]\ a_{ii}\sin i\theta = \sum_{i=0}^{K+L} A_{IJ4(k)}^{i}(\theta,a)\ \cos i\theta \qquad (A\text{-}133)$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j^2\theta_j \cos j\theta]\ a_{i}\cos i\theta = \sum_{i=0}^{K+L} A_{J21(k)}^{i}(\theta,a)\ \cos i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j^2\theta_j \cos j\theta]\ a_{i}\sin i\theta = \sum_{i=0}^{K+L} A_{J22(k)}^{i}(\theta,a)\ \sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j^2\theta_j \sin j\theta]\ a_{i}\cos i\theta = \sum_{i=0}^{K+L} A_{J23(k)}^{i}(\theta,a)\ \sin i\theta$$

$$\sum_{i=0}^{K}\sum_{j=0}^{L}[j^2\theta_j \sin j\theta]\ a_{i}\sin i\theta = \sum_{i=0}^{K+L} A_{J24(k)}^{i}(\theta,a)\ \cos i\theta \qquad (A\text{-}134)$$

where

$$A_{1(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[\ \theta_{i+j} + (1-\eta_{j-i}^2+\eta_i)\ \theta_{|i-j|}]\ a_j$$

$$A_{2(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[-\theta_{i+j} + (1-\eta_{j-i}^2+\eta_i)\ \theta_{|i-j|}]\ a_j$$

$$A_{3(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[\ \theta_{i+j} + (-1+\eta_{i-j}+\eta_i)\ \theta_{|i-j|}]\ a_j$$

$$A_{4(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[\ \theta_{i+j} + (-1-\eta_{i-j}+\eta_i)\ \theta_{|i-j|}]\ a_j \qquad (A\text{-}135)$$

$$A_{J1(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[(i+j)\ \theta_{i+j} + (1-\eta_{j-i}^2+\eta_i)|i-j|\ \theta_{|i-j|}]\ a_j$$

$$A_{J2(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[-(i+j)\ \theta_{i+j} + (1-\eta_{j-i}^2+\eta_i)|i-j|\ \theta_{|i-j|}]\ a_j$$

$$A_{J3(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[(i+j)\ \theta_{i+j} + (-1+\eta_{i-j}+\eta_i)|i-j|\ \theta_{|i-j|}]\ a_j$$

$$A_{J4(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[(i+j)\ \theta_{i+j} + (-1-\eta_{i-j}+\eta_i)|i-j|\ \theta_{|i-j|}]\ a_j \qquad (A\text{-}136)$$

$$A_{I1(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}[\ \theta_{i+j} + (1-\eta_{j-i}^2+\eta_i)\ \theta_{|i-j|}]j\ a_j$$

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$$A_{I2(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[\,\theta_{i+j} + (1-\eta_{j-i}^{2}+\eta_{i})\,\theta_{|i-j|}\right]j\,a_{j}$$

$$A_{I3(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[\,\theta_{i+j} + (-1+\eta_{i-j}+\eta_{i})\,\theta_{|i-j|}\right]j\,a_{j}$$

$$A_{I4(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[\,\theta_{i+j} + (-1-\eta_{i-j}+\eta_{i})\,\theta_{|i-j|}\right]j\,a_{j} \qquad (A-137)$$

$$A_{IJ1(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)\,\theta_{i+j} + (1-\eta_{j-i}^{2}+\eta_{i})|i-j|\,\theta_{|i-j|}\right]j\,a_{j}$$

$$A_{IJ2(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[-(i+j)\,\theta_{i+j} + (1-\eta_{j-i}^{2}+\eta_{i})|i-j|\,\theta_{|i-j|}\right]j\,a_{j}$$

$$A_{IJ3(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)\,\theta_{i+j} + (-1+\eta_{i-j}+\eta_{i})|i-j|\,\theta_{|i-j|}\right]j\,a_{j}$$

$$A_{IJ4(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)\,\theta_{i+j} + (-1-\eta_{i-j}+\eta_{i})|i-j|\,\theta_{|i-j|}\right]j\,a_{j} \qquad (A-138)$$

$$A_{J21(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)^{2}\,\theta_{i+j} + (1-\eta_{j-i}^{2}+\eta_{i})(i-j)^{2}\,\theta_{|i-j|}\right]a_{j}$$

$$A_{J22(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[-(i+j)^{2}\,\theta_{i+j} + (1-\eta_{j-i}^{2}+\eta_{i})(i-j)^{2}\,\theta_{|i-j|}\right]a_{j}$$

$$A_{J23(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)^{2}\,\theta_{i+j} + (-1+\eta_{i-j}+\eta_{i})(i-j)^{2}\,\theta_{|i-j|}\right]a_{j}$$

$$A_{J24(k)}^{i}(\theta,a) = \frac{1}{2}\sum_{j=0}^{K}\left[(i+j)^{2}\,\theta_{i+j} + (-1-\eta_{i-j}+\eta_{i})(i-j)^{2}\,\theta_{|i-j|}\right]a_{j} \qquad (A-139)$$

In order to write the strain-displacement relations in matrix form the following definitions of column matrices (vectors) are needed.

$$\begin{Bmatrix} \epsilon_{xx}^{\bullet} \\ \epsilon_{yy}^{\bullet} \\ \gamma_{xy}^{\bullet} \\ K_{xx} \\ K_{yy} \\ 2K_{xy} \end{Bmatrix} = \sum_{i=0}^{2K}\left[\left(\{\epsilon_{1i}\}+\{t_{1i}^{i}\}+\{t_{1i}^{\eta}\}\right)\cos\frac{i\pi y}{R} + \left(\{\epsilon_{2i}\}+\{t_{2i}^{i}\}+\{t_{2i}^{\eta}\}\right)\sin\frac{i\pi y}{R}\right]$$

$$(A-140)$$

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where

$$\{\epsilon_{1i}\} = \lfloor \epsilon_{xx1}^{\circ}, \epsilon_{yy1}^{\circ}, \gamma_{xy1}^{\circ}, K_{xx1}, K_{yy1}, 2K_{xy1} \rfloor_i^T$$

$$\{\epsilon_{2i}\} = \lfloor \epsilon_{xx2}^{\circ}, \epsilon_{yy2}^{\circ}, \gamma_{xy2}^{\circ}, K_{xx2}, K_{yy2}, 2K_{xy2} \rfloor_i^T$$

$$\{t_{1i}^L\} = \lfloor t_{x1}^L, t_{y1}^L, t_{xy1}^L, 0, 0, 0 \rfloor_i^T$$

$$\{t_{2i}^L\} = \lfloor t_{x2}^L, t_{y2}^L, t_{xy2}^L, 0, 0, 0 \rfloor_i^T$$

$$\{t_{1i}^L\} = \lfloor t_{x1}^n, t_{y1}^n, t_{xy1}^n, 0, 0, 0 \rfloor_i^T$$

$$\{t_{2i}^n\} = \lfloor t_{x2}^n, t_{y2}^n, t_{xy2}^n, 0, 0, 0 \rfloor_i^T \qquad (A\text{-}141)$$

Note that t^L and t^n elements are given by Eqs A-123a, A-124a and A-125a, while the ϵ_{ij} and k_{ij} elements are:

$$\epsilon_{xx1}^{\circ} = \delta_i U_{1i,x} \qquad ; \qquad K_{xx1} = -W_{1i,xx}\delta_i$$

$$\epsilon_{xx2}^{\circ} = \delta_i U_{2i,x} \qquad ; \qquad K_{xx2} = -W_{2i,xx}\delta_i$$

$$\epsilon_{yy1}^{\circ} = (\frac{in}{R}U_{2i} + \frac{W_{1i}}{R})\delta_i \qquad ; \qquad K_{yy1} = [\frac{in}{R}(in W_{1i} + \delta_i U_{2i})]\delta_i$$

$$\epsilon_{yy2}^{\circ} = (-\frac{in}{R}U_{1i} + \frac{W_{2i}}{R})\delta_i \qquad ; \qquad K_{yy2} = [\frac{in}{R}(in W_{1i} + \delta_i U_{2i})]\delta_i$$

$$\gamma_{xy1}^{\circ} = (\frac{in}{R}U_{2i} + U_{1i,x})\delta_i \qquad ; \qquad K_{xy1} = [-\frac{in}{R}W_{2i} + \frac{\delta_i}{2R}U_{1i,x}]\delta_i$$

$$\gamma_{xy2}^{\circ} = (-\frac{in}{R}U_{1i} + U_{2i,x})\delta_i \qquad ; \qquad K_{xy2} = [\frac{in}{R}W_{1i,x} + \frac{\delta_i}{2R}U_{2i,x}]\delta_i \quad (A\text{-}142)$$

Substitution of the expressions for reference surface strains and changes in curvature and torsion into the stress-strain relations, Eqs A-115, yields

$$
\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{i=0}^{2k} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \left[(\{\epsilon_{ij}\} + \{t^{l}_{li}\} + \{t^{n}_{li}\}) \cos \frac{i\pi y}{R} \right.
$$

$$
\left. + (\{\epsilon_{li}\} + \{t^{l}_{li}\} + \{t^{n}_{li}\}) \sin \frac{i\pi y}{R} \right] \tag{A-143}
$$

or

$$
= \sum_{i=0}^{2k} \left[(\{n_j\}_i + \{n^{l}_{j}\}_i + \{n_j\}^{n}_i) \cos \frac{i\pi y}{R} \right.
$$

$$
\left. + (\{n_s\}_i + \{n^{l}_s\} + \{n^{n}_s\}) \sin \frac{i\pi y}{R} \right] \tag{A-144}
$$

where

$$
\{n_j\}_i = \begin{Bmatrix} n_{xxi} \\ n_{yyi} \\ n_{xyi} \\ m_{xxi} \\ m_{yyi} \\ m_{xyi} \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \{\epsilon_{ij}\} \tag{A-145}
$$

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$$\{n_2\}_i = \begin{Bmatrix} n_{xx2} \\ n_{yy2} \\ n_{xy2} \\ m_{xx2} \\ m_{yy2} \\ m_{xy2} \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \{\epsilon_{2i}\}$$

$$(A\text{-}146)$$

$$\{n_1^L\}_i = \begin{Bmatrix} n_{xx1}^L \\ n_{yy1}^L \\ n_{xy2}^L \\ m_{xx2}^L \\ m_{yy3}^L \\ m_{xy2}^L \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \{t_{1i}^L\}$$

$$(A\text{-}147)$$

$$\{n_2^L\}_i = \begin{Bmatrix} n_{xx2}^L \\ n_{yy2}^L \\ n_{xy2}^L \\ m_{xx2}^L \\ m_{yy2}^L \\ m_{xy2}^L \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \{t_{2i}^L\}$$

$$(A\text{-}148)$$

$$\{n_1^n\} = \begin{Bmatrix} n_{xx2}^n \\ n_{yy1}^n \\ n_{xy1}^n \\ m_{xx2}^n \\ m_{yy2}^n \\ m_{xy2}^n \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \{t_{1i}^n\}$$

$$(A\text{-}149)$$

$$\{n_2^n\} = \begin{Bmatrix} n_{xx2}^n \\ n_{yy2}^n \\ n_{xy2}^n \\ m_{xx2}^n \\ m_{yy2}^n \\ m_{xy2}^n \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \{t_{2i}^n\}$$

$$(A\text{-}150)$$

Note that the $\{s_{1i}\}$ and $\{s_{2i}\}$ vectors result from linear portion of the kinematic relations; the $\{t_{1i}^L\}$ and $\{t_{2i}^L\}$ from the coupling between the imperfection parameter, w^o, and the displacement components v and w (thus, in a sense, nonlinear relations); and the $\{t_{1i}^n\}$ and $\{t_{2i}^n\}$ vectors from the nonlinear terms of the kinematic relations (v and w coupling).

Substitution of all the derived expressions into the equilibrium equations, Eqs A-116, yields <u>in-plane equilibrium</u>

$$\sum_{i=0}^{2k} \left[\left(\mathcal{N}_{xx1i,x} + \frac{in}{R}\mathcal{N}_{xy2i} + \mathcal{N}_{xx1i,x}^L + \frac{in}{R}\mathcal{N}_{xy2i}^L + \mathcal{N}_{xx1i,x}^n + \frac{in}{R}\mathcal{N}_{xy2i}^n \right) \cos\frac{iny}{R} \right.$$

$$\left. + \left(\mathcal{N}_{xx2i,x} - \frac{in}{R}\mathcal{N}_{xy1i} + \mathcal{N}_{xx2i,x}^L - \frac{in}{R}\mathcal{N}_{xy1i}^L + \mathcal{N}_{xx2i,x}^n - \frac{in}{R}\mathcal{N}_{xy1i}^n \right) \sin\frac{iny}{R} \right]$$

$$= 0 \tag{A-151}$$

$$\sum_{i=0}^{k}\xi_{1i}'\cos\frac{iny}{R} + \sum_{i=0}^{2k}(\xi_{1iL}^2+\xi_{1in}^2)\cos\frac{iny}{R} + \sum_{i=0}^{3k}(\xi_{1iL}^3+\xi_{1in}^3)\cos\frac{iny}{R}$$

$$+ \sum_{i=0}^{k}\xi_{2i}'\sin\frac{iny}{R} + \sum_{i=0}^{2k}(\xi_{2iL}^2+\xi_{2in}^2)\cos\frac{iny}{R} + \sum_{i=0}^{3k}(\xi_{2iL}^3+\xi_{2in}^3)\sin\frac{iny}{R} = 0 \tag{A-152}$$

where

$$\xi_{1i}' = \frac{in}{R}\mathcal{N}_{yy2i} + \mathcal{N}_{xy1i,x} + \frac{\delta_1}{R}\mathcal{M}_{xy1i,x} + \frac{in\delta_1}{R}\mathcal{M}_{yy2i}$$

$$\xi_{2i}' = -\frac{in}{R}\mathcal{N}_{yy1i} + \mathcal{N}_{xy2i,x} + \frac{\delta_1}{R}\mathcal{M}_{xy2i,x} - \frac{in\delta_1}{R}\mathcal{M}_{yy1i} \tag{A-153}$$

$$\xi_{1iL}^2 = \frac{\delta_1}{R^2}\mathcal{N}\left(A_{J1(k)}^i(w_i^o,\mathcal{N}_{yy}) - A_{J4(k)}^i(w_i^o,\mathcal{N}_{yy2}) \right)$$

$$+ \frac{\delta_1}{R}\left(A_{1(k)}^i(w_{1,x}^o,\mathcal{N}_{xy1}) + A_{4(k)}^i(w_{2,x}^o,\mathcal{N}_{xy2}) \right)$$

$$+ \frac{in}{R}\mathcal{N}_{yy2i}^L + \mathcal{N}_{xy1i,x}^L + \frac{\delta_1}{R}\mathcal{M}_{xy1i,x}^L + \frac{in}{R^2}\delta_1\mathcal{M}_{yy2i}$$

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$$\xi^2_{3iL} = \frac{\delta_1}{R^3} n \left(A^i_{J2(k)}(W^\circ_2, \mathcal{N}_{yy2}) - A^i_{J3(k)}(W^\circ_1, \mathcal{N}_{yy1}) \right)$$

$$+ \frac{\delta_1}{R} \left[A^i_{3(k)}(W_{2,x}, \mathcal{N}_{xy}) + A^i_{2(k)}(W_{1,x}, \mathcal{N}_{xy2}) \right]$$

$$- \frac{in}{R} \mathcal{N}^L_{yy1i} + \mathcal{N}^L_{xy2i,x} + \frac{\delta_1}{R} \mathcal{M}^L_{xy2i,x} - \frac{in}{R^2} \delta_1 \mathcal{M}^L_{yy1i}$$

$$\xi^2_{1in} = \frac{in}{R} \mathcal{N}^n_{yy2i} + \mathcal{N}^n_{xy1i,x} + \frac{\delta_1}{R} \mathcal{M}^n_{xy1i,x} + \frac{in}{R^2} \delta_1 \mathcal{M}^n_{yy2i}$$

$$+ \frac{\delta_1}{R^2} n \left[A^i_{J1(k)}(W_2, \mathcal{N}_{yy1}) - A^i_{J4(k)}(W_1, \mathcal{N}_{yy2}) \right]$$

$$- \frac{\delta_1}{R^2} A^i_{1(k)}(\mathcal{U}_1, \mathcal{N}_{yy1}) - \frac{\delta_1}{R^2} A^i_{4(k)}(\mathcal{U}_2, \mathcal{N}_{yy2})$$

$$+ \frac{\delta_1}{R} \left[A^i_{1(k)}(W_{1,x}, \mathcal{N}_{xy1}) + A^i_{4(k)}(W_{2,x}, \mathcal{N}_{xy2}) \right]$$

$$\xi^2_{2in} = - \frac{in}{R} \mathcal{N}^n_{yy1i} + \mathcal{N}^n_{xy2i,x} + \frac{\delta_1}{R} \mathcal{M}^n_{xy2i,x} - \frac{in}{R^2} \delta_1 \mathcal{M}^n_{yy1i}$$

$$+ \frac{\delta_1}{R^2} n \left[A^i_{J2(k)}(W_2, \mathcal{N}_{yy2}) - A^i_{J3(k)}(W_1, \mathcal{N}_{yy1}) \right]$$

$$- \frac{\delta_1}{R^3} \left[A^i_{3(k)}(\mathcal{U}_2, \mathcal{N}_{yy1}) + A^i_{2(k)}(\mathcal{U}_1, \mathcal{N}_{yy2}) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_3(W_{2,x}, \mathcal{N}_{xy1}) + A^i_{2(k)}(W_{1,x}, \mathcal{N}_{xy2}) \right] \qquad (A\text{-}154)$$

$$\xi^3_{1iL} = \frac{\delta_1 n}{R^2} \left[A^i_{J1(2k)}(W^\circ_2, \mathcal{N}^L_{yy1}) - A^i_{J4(2k)}(W^\circ_1, \mathcal{N}^L_{yy1}) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{1(2k)}(W^\circ_{1,x}, \mathcal{N}^L_{xy1}) + A^i_{4(2k)}(W^\circ_{2,x}, \mathcal{N}^L_{xy2}) \right]$$

$$\xi^3_{2iL} = \frac{\delta_1 n}{R^2} \left[- A^i_{J3(2k)}(W^\circ_1, \mathcal{N}^L_{xy1}) + A^i_{J2(2k)}(W^\circ_2, \mathcal{N}^L_{yy2}) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{3(2k)}(W^\circ_{2,x}, \mathcal{N}^L_{xy1}) + A^i_{2(2k)}(W^\circ_{1,x}, \mathcal{N}^L_{xy2}) \right]$$

$$\xi^3_{1in} = \frac{\delta_1 n}{R} \left[A^i_{J1(2k)}(W_2+W^\circ_2, \mathcal{N}^n_{yy1}) - A^i_{J4(2k)}(W_1+W^\circ_1, \mathcal{N}^n_{yy2}) \right]$$

$$- \frac{\delta_1}{R^2} \left[A^i_{1(2k)}(\mathcal{U}_1, \mathcal{N}^n_{yy1}) + A^i_{4(2k)}(\mathcal{U}_2, \mathcal{N}^n_{yy2}) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{1(2k)}(W_{1,x}+W^\circ_{1,x}, \mathcal{N}^n_{xy1}) + A^i_{4(2k)}(W_{2,x}+W^\circ_{2,x}, \mathcal{N}^n_{xy2}) \right]$$

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$$+ \frac{\delta_1 n}{R^3} \left[A^i_{J1(2K)} (W_2, n^L_{yy1}) - A^i_{J4(2K)} (W_1, n^L_{yy2}) \right]$$

$$- \frac{\delta_1}{R^2} \left[A^i_{1(2K)} (U_1, n^L_{yy1}) + A^i_{4(2K)} (U_2, n^L_{yy2}) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{1(2K)} (W_{1,x}, n^L_{xy1,x}) + A^i_{4(2K)} (W_{2,x}, n^L_{xy2}) \right]$$

$$\xi^3_{2in} = \frac{\delta_1 n}{R^3} \left[- A^i_{J3(2K)} (w_1 + w^o_1, n^{\pi}_{yy1}) + A^i_{J2(2K)} (W_2 + w^o_2, n^{\pi}_{yy2}) \right]$$

$$- \frac{\delta_1}{R^2} \left[A^i_{2(2K)} (U_1, n^{\pi}_{yy2}) + A^i_{3(2K)} (U_2, n^{\pi}_{yy1}) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{3(2K)} (W_{2,x} + w^o_{2,x}, n^{\pi}_{xy1}) + A^i_{2(2K)} (W_{1,x} + w^o_{1,x}, n^{\pi}_{xy2}) \right]$$

$$+ \frac{\delta_1 n}{R^3} \left[- A^i_{J3(2K)} (w_1, n^L_{yy1}) + A^i_{J2(2K)} (W_2, n^L_{yy2}) \right]$$

$$- \frac{\delta_1}{R^2} \left[A^i_{2(2K)} (U_1, n^L_{yy2}) + A^i_{3(2K)} (U_2, n^L_{yy1}) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{3(2K)} (W_{2,x}, n^L_{xy1}) + A^i_{2(2K)} (W_{1,x}, n^L_{xy2}) \right] \qquad (A\text{-}155)$$

Transverse equilibrium

$$\sum_{i=0}^{K} \left(\eta^1_{1i} \cos\frac{i\pi y}{R} + \eta^1_{2i} \sin\frac{i\pi y}{R} \right) + \sum_{i=0}^{2K} \left[(\eta^2_{1iL} + \eta^2_{1in}) \cos\frac{i\pi y}{R} + (\eta^2_{2iL} + \eta^2_{2in}) \sin\frac{i\pi y}{R} \right]$$

$$+ \sum_{i=0}^{3K} \left[(\eta^3_{1iL} + \eta^3_{1in}) \cos\frac{i\pi y}{R} + (\eta^3_{2iL} + \eta^3_{2in}) \sin\frac{i\pi y}{R} \right] + \sum_{i=0}^{K} \left(\delta_{1i} \cos\frac{i\pi y}{R} + \delta_{2i} \sin\frac{i\pi y}{R} \right) = 0 \quad (A\text{-}156)$$

where

$$\eta^1_{1i} = m_{xx1i,xx} + 2 m_{xy2i,x} \left(\frac{i\pi}{R}\right) - \left(\frac{i\pi}{R}\right)^2 m_{yy1i} - \frac{n_{yy1i}}{R}$$

$$\eta^1_{2i} = m_{xx2i,xx} - 2 m_{xy1i,x} \left(\frac{i\pi}{R}\right) - \left(\frac{i\pi}{R}\right)^2 m_{yy2i} - \frac{n_{yy2i}}{R} \qquad (A\text{-}157)$$

$$\eta^2_{1iL} = m^L_{xx1i,xx} + 2 m^L_{xy2i,x} \left(\frac{i\pi}{R}\right) - \left(\frac{i\pi}{R}\right)^2 m^L_{yy1i} - \frac{n^L_{yy1}}{R}$$

$$+ \frac{n}{R} \left[A^i_{J1(K)} (W^o_2, n_{xy1,x}) - A^i_{J4(K)} (W^o_1, n_{xy2,x}) \right]$$

$$+ \left(\frac{n}{R}\right)^2 \left[A^i_{2J1(K)} (W^o_2, n_{yy2}) + A^i_{2J4(K)} (W^o_1, n_{yy1}) \right]$$

$$+ \left[A^i_{1(k)} (W_{1,xx}^o, \mathcal{N}_{xx1}) + A^i_{4(k)} (W_{2,xx}^\bullet, \mathcal{N}_{xx2}) \right]$$

$$+ \frac{2\eta}{R} \left[A^i_{J1(k)} (W_{2,x}^o, \mathcal{N}_{xy1}) - A^i_{J4(k)} (W_{1,x}^o, \mathcal{N}_{xy2}) \right]$$

$$+ \left(\frac{\eta}{R}\right)^2 \left[-A^i_{J21(k)} (W_1^o, \mathcal{N}_{yy1}) - A^i_{J24(k)} (W_2^\bullet, \mathcal{N}_{yy2}) \right] \qquad (A\text{-}158a)$$

$$\eta^2_{2iL} = M^L_{xx2i,xx} - 2\left(\frac{i\eta}{R}\right) M^L_{xy1L,x} - \left(\frac{i\eta}{R}\right)^2 M^L_{yy1i} - \frac{\mathcal{N}_{yy2}}{R}$$

$$+ \left(\frac{\eta}{R}\right) \left[A^i_{J2(k)} (W_2^\bullet, \mathcal{N}_{xy2,x}) - A^i_{J3(k)} (W_1^o, \mathcal{N}_{xy1,x}) \right]$$

$$+ \left(\frac{\eta}{R}\right)^2 \left[-A^i_{2J2(k)} (W_2^\bullet, \mathcal{N}_{yy1}) - A^i_{1J3(k)} (W_1^o, \mathcal{N}_{yy2}) \right]$$

$$+ A^i_{2(k)} (W_{1,xx}^o, \mathcal{N}_{xx2}) + A^i_{3(k)} (W_{2,xx}^\bullet, \mathcal{N}_{xy1})$$

$$+ \frac{2\eta}{R} \left(A^i_{J2(k)} (W_{2,x}^\bullet, \mathcal{N}_{xy2}) - A^i_{J3(k)} (W_{1,x}^o, \mathcal{N}_{xy1}) \right)$$

$$+ \left(\frac{\eta}{R}\right)^2 \left[-A^i_{J22(k)} (W_1^o, \mathcal{N}_{yy2}) - A^i_{J23} (W_2^\bullet, \mathcal{N}_{yy1}) \right] \qquad (A\text{-}158b)$$

$$\eta^2_{1iN} = M^n_{xy1i,xx} + 2 M^n_{xy2i,x} \left(\frac{i\eta}{R}\right) - \left(\frac{i\eta}{R}\right)^2 M^n_{yy1i} - \frac{\mathcal{N}_{yy1}^n}{R}$$

$$+ \frac{\eta}{R} \left[A^i_{J1(k)} (W_2, \mathcal{N}_{xy1,x}) - A^i_{J4(k)} (W_1, \mathcal{N}_{xy2,x}) \right]$$

$$+ \left(\frac{\eta}{R}\right)^2 \left[A^i_{2J1(k)} (W_2, \mathcal{N}_{yy2}) + A^i_{1J4(k)} (W_1, \mathcal{N}_{yy1}) \right]$$

$$+ \frac{\delta_i}{R} \left[-A^i_{1(k)} (\mathcal{U}_1, \mathcal{N}_{xy1,x}) - A^i_{4(k)} (\mathcal{U}_2, \mathcal{N}_{xy2,x}) \right]$$

$$+ \frac{\delta_i \eta}{R^2} \left[-A^i_{21(k)} (\mathcal{U}_1, \mathcal{N}_{yy2}) + A^i_{J4(k)} (\mathcal{U}_2, \mathcal{N}_{yy1}) \right]$$

$$+ A^i_{1(k)} (W_{1,xx}, \mathcal{N}_{x1}) + A^i_{4(k)} (W_{2,xx}, \mathcal{N}_{xy2})$$

$$+ \frac{2\eta}{R} \left[A^i_{J1(k)} (W_{2,x}, \mathcal{N}_{xy1}) - A^i_{J4(k)} (W_{1,x}, \mathcal{N}_{xy2}) \right]$$

$$- \frac{\delta_i}{R} \left[A^i_{1(k)} (\mathcal{U}_{1,x}, \mathcal{N}_{xy1}) + A^i_{4(k)} (\mathcal{U}_{2,x}, \mathcal{N}_{xy2}) \right]$$

$$+ \left(\frac{\eta}{R}\right)^2 \left[-A^2_{J21(k)} (W_1, \mathcal{N}_{yy1}) - A^i_{J24(k)} (W_2, \mathcal{N}_{yy2}) \right]$$

$$+ \frac{\delta_i \eta}{R^2} \left[-A^i_{J1(k)} (\mathcal{U}_2, \mathcal{N}_{yy1}) + A^i_{J4(k)} (\mathcal{U}_1, \mathcal{N}_{yy2}) \right] \qquad (A\text{-}158c)$$

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$$\eta^2_{2in} = \mathcal{M}^n_{xx2i,xx} - 2\mathcal{M}^n_{xy1i,x}\left(\frac{i\eta}{R}\right) - \left(\frac{i\eta}{R}\right)^2 \mathcal{M}^m_{yy2i} - \frac{\eta^n_{yy2}}{R}$$

$$+ \left(\frac{\eta}{R}\right)\left[A^i_{J2(k)}(W_2, \eta_{xy2,x}) - A^i_{J3(k)}(W_1, \eta_{xy1,x})\right]$$

$$+ \left(\frac{\eta}{R}\right)^2\left[-A^i_{IJ2(k)}(W_2, \eta_{yy1}) - A^i_{IJ3(k)}(W_1, \eta_{yy2})\right]$$

$$+ \frac{\delta_1}{R}\left[-A^i_{2(k)}(U_1, \eta_{xy2,x}) - A^i_{3(k)}(U_2, \eta_{xy1,x})\right]$$

$$+ \frac{\delta_1\eta}{R^2}\left[A^i_{I2(k)}(U_1, \eta_{yy1}) - A^i_{I3(k)}(U_2, \eta_{yy2})\right]$$

$$+ A^i_{2(k)}(W_{1,xx}, \eta_{xx2}) + A^i_{3(k)}(W_{2,xx}, \eta_{xx1})$$

$$+ \frac{2\eta}{R}\left[A^i_{J2(k)}(W_{2,x}, \eta_{xy2}) - A^i_{J3(k)}(W_{1,x}, \eta_{xy1})\right]$$

$$+ \frac{\delta_1}{R}\left[-A^i_{2(k)}(U_{1,x}, \eta_{xy2}) - A^i_{3(k)}(U_{2,x}, \eta_{xy1})\right]$$

$$+ \left(\frac{\eta}{R}\right)^2\left[-A^i_{J22(k)}(W_1, \eta_{yy2}) - A^i_{J23(k)}(W_2, \eta_{yy1})\right]$$

$$+ \frac{\delta_1\eta}{R^2}\left[-A^i_{J2(k)}(U_2, \eta_{yy2}) + A^i_{J3(k)}(U_1, \eta_{yy1})\right] \tag{A-158d}$$

$$\eta^3_{1iL} = \frac{\eta}{R}\left[A^i_{J1(2k)}(W_2^\circ, \eta^L_{xy1,x}) - A^i_{J4(2k)}(W_1^\circ, \eta^L_{xy2,x})\right.$$

$$\left. + 2A^i_{J1(2k)}(W_{2,x}^\circ, \eta^L_{xy1}) - 2A^i_{J4(2k)}(W_{1,x}^\circ, \eta^L_{xy2})\right]$$

$$+ \left(\frac{\eta}{R}\right)^2\left[A^i_{IJ1(k)}(W_2^\circ, \eta^L_{yy2}) + A^i_{IJ4(2k)}(W_1^\circ, \eta^L_{yy1})\right.$$

$$\left. - A^i_{J2k(2k)}(W_1^\circ, \eta^L_{yy1}) - A^i_{J4(2k)}(W_2^\circ, \eta^L_{yy2})\right]$$

$$+ A^i_{1(2k)}(W_{1,xx}^\circ, \eta^L_{xx1}) + A^i_{4(2k)}(W_{2,xx}^\circ, \eta^L_{xx2}) \tag{A-159a}$$

$$\eta^3_{2iL} = \frac{\eta}{R}\left[A^i_{J2(k)}(W_2^\circ, \eta^L_{xy2,x}) - A^i_{J3(2k)}(W_1^\circ, \eta^L_{xy1,x})\right.$$

$$\left. + 2A^i_{J2(2k)}(W_{2,x}^\circ, \eta^L_{xy2}) - 2A^i_{J3(2k)}(W_{1,x}^\circ, \eta^L_{xy1})\right]$$

$$+ \left(\frac{\eta}{R}\right)^2\left[-A^i_{IJ2(2k)}(W_2^\circ, \eta^L_{yy1}) - A^i_{IJ3(2k)}(W_1^\circ, \eta^L_{yy2})\right.$$

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$$- A^i_{J22(2k)}(W_1^o, \eta^L_{yy2}) - A^i_{J23(2k)}(U_2, \eta^L_{yy1})]$$

$$+ A^i_{2(2k)}(W^o_{1,xx}, \eta^L_{xx2}) + A^i_{3(2k)}(W^o_{2,xx}, \eta^L_{xx1}) \qquad (A-159b)$$

$$\eta^3_{lin} = \frac{n}{R}\left[A^i_{J1(2k)}(W_2 + W_2^o, \eta^n_{xy1,x}) - A^i_{J4(2k)}(W_1 + W_1^o, \eta^n_{xy2,x}) \right.$$

$$+ A^i_{J1(2k)}(W_2, \eta^L_{xy1,x}) - A^i_{J4(2k)}(W_1, \eta^L_{xy2,x})$$

$$+ 2A^i_{J1(2k)}(W_{2,x} + W^o_{2,x}, \eta^n_{xy1}) - 2A^i_{J4(2k)}(W_{1,x} + W^o_{1,x}, \eta^n_{xy2})$$

$$+ 2A^i_{J1(2k)}(W_{2,x}, \eta^L_{xy1}) - 2A^i_{J4(2k)}(W_{1,x}, \eta^L_{xy2})]$$

$$+ \left(\frac{n}{R}\right)^2 [A^i_{2J1(2k)}(W_2 + W_2^o, \eta^n_{yy2}) + A^i_{1J4(2k)}(W_1 + W_1^o, \eta^n_{yy1})$$

$$- A^i_{J21(2k)}(W_1 + W_1^o, \eta^n_{yy1}) - A^i_{J24(2k)}(W_2 + W_2^o, \eta^n_{yy2})$$

$$+ A^i_{1J1(2k)}(W_2, \eta^L_{yy2}) + A^i_{2J4(2k)}(W_1, \eta^L_{yy1}) - A^i_{J21(2k)}(W_1, \eta^L_{yy1})$$

$$- A^i_{J24(2k)}(W_2, \eta^L_{yy2})] + \frac{\delta_1}{R}[-A^i_{1(2k)}(U_1, \eta^n_{xy1,x}) - A^i_{4(2k)}(U_2, \eta^n_{xy2,x})$$

$$- A^i_{1(2k)}(U_{1,x}, \eta^n_{xy1}) - A^i_{4(2k)}(U_{2,x}, \eta^n_{xy2}) - A^i_{1(2k)}(U_1, \eta^L_{xy1,x})$$

$$- A^i_{4(2k)}(U_2, \eta^L_{xy2,x}) - A^i_{1(2k)}(U_{1,x}, \eta^L_{xy1}) - A^i_{4(2k)}(U_{2,x}, \eta^L_{xy2})]$$

$$+ \frac{\delta_1 n}{R^2}[-A^i_{2I1(2k)}(U_1, \eta^n_{yy2}) + A^i_{I4(2k)}(U_2, \eta^n_{yy1}) - A^i_{J1(2k)}(U_2, \eta^n_{yy1})$$

$$+ A^i_{J4(2k)}(U_1, \eta^n_{yy2}) - A^i_{2I1(2k)}(U_1, \eta^L_{yy2}) + A^i_{I4(2k)}(U_2, \eta^L_{yy1})$$

$$- A^i_{J1(2k)}(U_2, \eta^L_{yy1}) + A^i_{J4(2k)}(U_1, \eta^L_{yy2})]$$

$$+ A^i_{1(2k)}(W_{1,xx} + W^o_{1,xx}, \eta^n_{xx1}) + A^i_{4(2k)}(W_{2,xx} + W^o_{2,xx}, \eta^n_{xx2})$$

$$+ A^i_{1(2k)}(W_{1,xx}, \eta^L_{xx1}) + A^i_{4(2k)}(W_{2,xx}, \eta^L_{xx2}) \qquad (A-159C)$$

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$$\eta_{2in}^{3} = \left(\frac{n}{R}\right)\Big[A_{J2(2k)}^{i}(W_2+W_2^{\circ}, \mathcal{N}_{xy2,x}^{n}) - A_{J3(2k)}^{i}(W_1+W_1^{\circ}, \mathcal{N}_{xy1,x}^{n})$$

$$+ 2A_{J2(2k)}^{i}(W_{2,x}+W_{2,x}^{\circ}, \mathcal{N}_{xy2}^{n}) - 2A_{J3(2k)}^{i}(W_{1,x}+W_{1,x}^{\circ}, \mathcal{N}_{xy1}^{n})$$

$$+ A_{J2(2k)}^{i}(W_2, \mathcal{N}_{xy2,x}^{L}) - A_{J3(2k)}^{i}(W_1, \mathcal{N}_{xy1,x}^{L}) + 2A_{J2(2k)}^{i}(W_{2,x}, \mathcal{N}_{xy2}^{L})$$

$$- 2A_{J3(2k)}^{i}(W_{1,x}, \mathcal{N}_{xy1}^{L}) + \{ A_{IJ2(2k)}^{i}(W_2, \mathcal{N}_{yy1}^{L}) - A_{IJ3(2k)}^{i}(W_1, \mathcal{N}_{yy1}^{L})$$

$$- A_{J22(2k)}^{i}(W_1, \mathcal{N}_{yy2}^{L}) - A_{J23(2k)}^{i}(W_2, \mathcal{N}_{yy1}^{L})]\left(\frac{n}{R}\right)^{2}$$

$$+ \left(\frac{n}{R}\right)^{2}[- A_{IJ2(2k)}^{i}(W_2+W_2^{\circ}, \mathcal{N}_{yy1}^{n}) - A_{IJ3(2k)}^{i}(W_1+W_1^{\circ}, \mathcal{N}_{yy2}^{n})$$

$$- A_{J22(2k)}^{i}(W_1+W_1^{\circ}, \mathcal{N}_{yy2}^{n}) - A_{J23(2k)}^{i}(W_2+W_2^{\circ}, \mathcal{N}_{yy1}^{n})]$$

$$+ \frac{\delta_I}{R}[- A_{2(2k)}^{i}(U_1, \mathcal{N}_{xy2,x}^{n}+\mathcal{N}_{xy2,x}^{L}) - A_{3(2k)}^{i}(U_2, \mathcal{N}_{xy1,x}^{n}+\mathcal{N}_{xy1,x}^{L})$$

$$- A_{2(2k)}^{i}(U_{1,x}, \mathcal{N}_{xy2}^{n}+\mathcal{N}_{xy2}^{L}) - A_{3(2k)}^{i}(U_{2,x}, \mathcal{N}_{xy1}^{n}+\mathcal{N}_{xy1}^{L})]$$

$$+ \frac{\delta n}{R^{2}}[A_{I2(2k)}^{i}(U_1, \mathcal{N}_{yy1}^{n}+\mathcal{N}_{yy1}^{L}) - A_{I3(2k)}^{i}(U_2, \mathcal{N}_{yy2}^{n}+\mathcal{N}_{yy2}^{L})]$$

$$+ A_{2(2k)}^{i}(W_{1,xx}+W_{1,xx}^{\circ}, \mathcal{N}_{xx2}^{n}) + A_{3(2k)}^{i}(W_{2,xx}+W_{2,xx}^{\circ}, \mathcal{N}_{xx1}^{n})$$

$$+ A_{2(2k)}^{i}(W_{1,xx}, \mathcal{N}_{xx2}^{L}) + A_{3(2k)}^{i}(W_{2,xx}, \mathcal{N}_{xx1}^{L}) \qquad\qquad (A\text{-}159d)$$

According to Eqs A-119 subject to the constraint of Eq A-120, there exist 6k + 3 unknown functions of position . These are the displacement coefficients $u_{1i}(x)$, $v_{1i}(x)$, $w_{1i}(x)$ for $i = 0,1,2...k$, and $u_{2i}(x)$, $v_{2i}(x)$, $w_{2i}(x)$ for $i = 1,2,3...k$. Note that if one can solve for the displacement components the response of the system is fully characterized (deformation approach).

Next, the Galerkin procedure is employed in the circumferential direction. The vanishing of the Galerkin integrals leads to (6k + 3) nonlinear algebraic equations in the (6k + 3) unknowns. These equations are:

$$\eta_{xx1i,x} + \frac{in}{R}\eta_{xy2i} + \eta^L_{xx1i,x} + \eta^L_{xy2i}\frac{in}{R} = -\overset{n}{\eta}_{xx1i,x} - \frac{in}{R}\overset{n}{\eta}_{xy2i}$$

$$\xi'_{1i} + \xi^2_{1iL} + \xi^3_{1iL} + \xi^2_{1in} + \xi^3_{1in} = 0$$

$$\eta'_{1i,x} + \eta^2_{1iL} + \eta^3_{1iL} + \eta^2_{1in} + \eta^3_{1in} = -g_{1i}$$

$$for \quad i = 0, 1, 2, \cdots, k$$

$$\eta_{xx2i,x} - \frac{in}{R}\eta_{xy1i} + \eta^L_{xx1i,x} - \frac{in}{R}\eta^L_{xy2i} = \frac{in}{R}\overset{n}{\eta}_{xy1i} - \overset{n}{\eta}_{xx2i,x}$$

$$\xi'_{2i} + \xi^2_{2iL} + \xi^3_{2iL} + \xi^2_{2in} + \xi^3_{2in} = 0$$

$$\eta'_{2i,x} + \eta^2_{2iL} + \eta^3_{2iL} + \eta^2_{2in} + \eta^3_{2in} = -g_{2i}$$

$$for \quad i = 1, 2, \cdots, k \qquad\qquad (A\text{-}160)$$

Next, the generalized Newton's method (Ref. 38) is used to reduce the nonlinear field equations to a sequence of linear systems. This procedure is similar as the one in section II.2. Because the final set of equations, Eqs A-160, contains n's, ξ's and η's, and because these are in turn functions of other parameters, then Eq A-72 will be applied to all of the elements, needed in deriving the iteration equations. In so doing, only the nonlinear terms need be considered. Thus,

$$\langle \overset{n}{t}_{x1i}\rangle^{m+1} = A^i_{1(k)}(W^m_{1,x} \cdot W^{m+1}_{1,x}) + A^i_{4(k)}(W^m_{2,x}, W^{m+1}_{2,x})$$
$$- \frac{1}{2}\left\{ A^i_{1(k)}(W^m_{1,x}, W^m_{1,x}) + A^i_{4(k)}(W^m_{2,x}, W^m_{2,x})\right\} \qquad (A\text{-}161)$$

$$(\overset{n}{t}_{x2i})^{m+1} = A^i_{2(k)}(W^m_{1,x}, W^{m+1}_{2,x}) + A^i_{3(k)}(W^m_{2,x}, W^{m+1}_{1,x})$$
$$- \frac{1}{2}\left\{ A^i_{2(k)}(W^m_{1,x}, W^m_{2,x}) + A^i_{3(k)}(W^m_{2,x}, W^{m+1}_{1,x})\right\} \qquad (A\text{-}162)$$

$$\left(t_{yli}^{n}\right)^{m+1} = \frac{n^2}{R^3}\left[A_{IJ4(x)}^{i}(W_1^{n}, W_2^{m+1}) + A_{IJ1(x)}^{i}(W_2^{n}, W_2^{m+1})\right]$$

$$+ \frac{\delta_1}{R^3}\left[A_{1(x)}^{i}(U_1^{n}, U_1^{m+1}) + A_{4(x)}^{i}(U_2^{n}, U_2^{m+1})\right]$$

$$+ \frac{\delta_1 n}{R^2}\left[A_{J4(x)}^{i}(W_1^{n}, U_2^{m+1}) + A_{I4(x)}^{i}(U_2^{n}, W_1^{m+1})\right.$$

$$\left. - A_{J1}^{i}(W_2^{n}, U_1^{m+1}) - A_{I1(x)}^{i}(U_1^{n}, W_2^{m+1})\right]$$

$$- \frac{1}{2}\frac{n^2}{R^3}\left[A_{IJ4(x)}^{i}(W_1^{n}, W_1^{n}) + A_{IJ1(x)}^{i}(W_2^{n}, W_2^{n})\right]$$

$$- \frac{\delta_1}{2R^3}\left[A_{1(x)}^{i}(U_1^{n}, U_1^{n}) + A_{4(x)}^{i}(U_2^{n}, U_2^{n})\right]$$

$$+ \frac{\delta_1 n}{R^2}\left[A_{J4(x)}^{i}(W_1^{n}, V_2^{n}) - A_{J1}^{i}(W_2^{n}, U_1^{n})\right] \qquad (A\text{-}163)$$

$$\left(t_{yli}^{n}\right)^{m+1} = -\frac{n^2}{R^3}\left[A_{IJ2}^{i}(W_2^{n}, W_1^{m+1}) + A_{IJ3(x)}^{i}(W_1^{n}, W_2^{m+1})\right]$$

$$+ \frac{\delta_1}{R^2}\left[A_3^{i}(U_2^{n}, U_1^{m+1}) + A_{2(x)}^{i}(U_1^{n}, U_2^{m+1})\right]$$

$$+ \frac{\delta_1 n}{R^2}\left[A_{J3(x)}^{i}(W_1^{n}, U_1^{m+1}) + A_{2(x)}^{i}(U_1^{n}, W_1^{m+1})\right.$$

$$\left. - A_{J2(x)}^{i}(W_2^{n}, V_2^{m+1}) - A_{I3(x)}^{i}(U_2^{n}, W_2^{m+1})\right]$$

$$+ \frac{1}{2}\frac{n^2}{R^2}\left[A_{IJ2(x)}^{i}(W_2^{n}, W_1^{n}) + A_{IJ3(x)}^{i}(W_1^{n}, W_2^{n})\right]$$

$$- \frac{\delta_1}{2R^3}\left[A_{3(x)}^{i}(U_2^{n}, U_1^{n}) + A_{2(x)}^{i}(U_1^{n}, U_2^{n})\right]$$

$$- \frac{\delta_1 n}{R^2}\left[A_{J3(x)}^{i}(W_1^{n}, U_1^{n}) - A_{J2(x)}^{i}(W_2^{n}, U_2^{n})\right] \qquad (A\text{-}164)$$

$$\left(t_{yli}^{n}\right)^{m+1} = \frac{n}{R}\left[A_{J1(x)}^{i}(W_2^{n}, W_{1,x}^{m+1}) - A_{J4(x)}^{i}(W_1^{n}, W_{2,x}^{m+1})\right.$$

$$\left. + A_{I1(x)}^{i}(W_{1,x}^{n}, W_2^{m+1}) - A_{I4(x)}^{i}(W_{2,x}^{n}, W_1^{m+1})\right]$$

$$- \frac{\delta_1}{R}\left[A_{1(x)}^{i}(U_1^{n}, W_{1,x}^{m+1}) + A_{4(x)}^{i}(U_2^{n}, W_{2,x}^{m+1})\right.$$

$$\left. + A_{1(x)}^{i}(W_{1,x}^{n}, U_1^{m+1}) + A_{4(x)}^{i}(W_{2,x}^{n}, V_2^{m+1})\right]$$

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$$- \frac{n}{2R} \left[A^i_{J1(K)}(W^m_2, W^m_{1,x}) - A^i_{J4(K)}(W^m_1, W^m_{2,x}) \right.$$

$$\left. - A^i_{I4(K)}(W^m_{2,x}, W^m_1) + A^i_{I1(K)}(W^m_{1,x}, W^m_2) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{1(K)}(U^m_1, W^m_{1,x}) + A^i_{4(K)}(U^m_2, W^m_{2,x}) \right] \tag{A-165}$$

$$(t^n_{\eta 2i}) = \frac{n}{R} \left[A^i_{J2(K)}(W^m_2, W^m_{2,x}) - A^i_{J3(K)}(W^m_1, W^{m+1}_{1,x}) \right.$$

$$\left. - A^i_{I2(K)}(W^m_{1,x}, W^{m+1}_1) + A^i_{I3(K)}(W^m_{2,x}, W^{m+1}_2) \right]$$

$$- \frac{\delta_1}{R} \left[A^i_{3(K)}(U^m_2, W^{m+1}_{1,x}) + A^i_{2(K)}(U^m_1, W^{m+1}_{2,x}) \right.$$

$$\left. + A^i_{2(K)}(W^m_{1,x}, U^{m+1}_2) + A^i_{3(K)}(W^m_{2,x}, U^{m+1}_1) \right]$$

$$- \frac{n}{2R} \left[A^i_{J1(K)}(W^m_2, W^m_{1,x}) - A^i_{J4(K)}(W^m_1, W^m_{2,x}) \right.$$

$$\left. - A^i_{I4(K)}(W^m_{2,x}, W^m_1) + A^i_{I1(K)}(W^m_{1,x}, W^m_2) \right]$$

$$+ \frac{\delta_1}{R} \left[A^i_{1(K)}(U^m_1, W^m_{1,x}) + A^i_{4(K)}(U^m_2, W^m_{2,x}) \right] \tag{A-166}$$

$$(t^n_{x11,x})^{m+1} = A^i_{1(K)}(W^m_{1,xx}, W^{m+1}_{1,x}) + A^i_{1(K)}(W^m_{1,x}, W^{m+1}_{1,xx})$$

$$+ A^i_{4(K)}(W^m_{2,xx}, W^{m+1}_{2,x}) + A^i_{4(K)}(W^m_{2,x}, W^{m+1}_{2,xx})$$

$$- \frac{1}{2} \left[A^i_{1(K)}(W^m_{1,xx}, W^m_{1,x}) + A^i_{4(K)}(W^m_{2,xx}, W^m_{2,x}) \right.$$

$$\left. + A^i_{1(K)}(W^m_{1,x}, W^m_{1,xx}) + A^i_{4(K)}(W^m_{2,x}, W^m_{1,xx}) \right] \tag{A-167}$$

$$(t^n_{x12,x})^{m+1} = A^i_{2(K)}(W^m_{1,xx}, W^{m+1}_{2,x}) + A^i_{2(K)}(W^m_{1,x}, W^{m+1}_{2,xx})$$

$$+ A^i_{3(K)}(W^m_{2,xx}, W^{m+1}_{1,x}) + A^i_{3(K)}(W^m_{2,x}, W^{m+1}_{1,xx})$$

$$- \frac{1}{2} \left[A^i_{2(K)}(W^m_{1,xx}, W^m_{2,x}) + A^i_{2(K)}(W^m_{1,x}, W^m_{2,xx}) \right.$$

$$\left. + A^i_{3(K)}(W^m_{2,xx}, W^m_{1,x}) + A^i_{3(K)}(W^m_{2,x}, W^m_{1,xx}) \right] \tag{A-168}$$

$$\left(t_{yi,x}^{n}\right)^{m+1} = \frac{n^2}{R^2}\left[A_{IJ4(k)}^{i}(W_{1,x}^{m}, W_{1}^{m+1}) + A_{IJ4(k)}^{i}(W_{1}^{m}, W_{1,x}^{m+1})\right.$$

$$\left. + A_{IJ1(k)}^{i}(W_{2,x}^{m}, W_{2}^{m+1}) + A_{IJ1(k)}^{i}(W_{2}^{m}, W_{2,x}^{m+1})\right]$$

$$+ \frac{\delta_1}{R^2}\left[A_{1(k)}^{i}(U_{1,x}^{m}, U_{1,x}^{m+1}) + A_{4(k)}^{i}(U_{2,x}^{m}, U_{2}^{m+1})\right.$$

$$\left. + A_{1(k)}^{i}(U_{1}^{m}, U_{1,x}^{m+1}) + A_{4(k)}^{i}(U_{2}^{m}, U_{2,x}^{m+1})\right]$$

$$+ \frac{\delta_1 n}{R^3}\left[A_{J4(k)}^{i}(W_{1,x}^{m}, U_{2}^{m+1}) + A_{J4(k)}^{i}(W_{1}^{m}, U_{2,x}^{m+1})\right.$$

$$+ A_{I4(k)}^{i}(U_{2,x}^{m}, W_{1}^{m+1}) + A_{I4(k)}^{i}(U_{2}^{m}, W_{1,x}^{m+1})$$

$$- A_{J1(k)}^{i}(W_{2,x}^{m}, U_{2}^{m+1}) - A_{J1(k)}^{i}(W_{2}^{m}, U_{1,x}^{m+1})$$

$$\left. - A_{I1(k)}^{i}(U_{1,x}^{m}, W_{2}^{m+1}) - A_{I1(k)}^{i}(U_{1}^{m}, W_{2,x}^{m+1})\right]$$

$$- \frac{1}{2}\frac{n^2}{R^3}\left[A_{IJ4(k)}^{i}(W_{1,x}^{m}, W_{1}^{m}) + A_{IJ4(k)}^{i}(W_{1}^{m}, W_{1,x}^{m})\right.$$

$$\left. + A_{IJ1(k)}^{i}(W_{2,x}^{m}, W_{2}^{m}) + A_{IJ1(k)}^{i}(W_{2}^{m}, W_{2,x}^{m})\right]$$

$$- \frac{\delta_1}{2R^2}\left[A_{1(k)}^{i}(U_{1,x}^{m}, U_{1}^{m}) + A_{4(k)}^{i}(U_{2,x}^{m}, U_{2}^{m})\right.$$

$$\left. + A_{1(k)}^{i}(U_{1}^{m}, U_{1,x}^{m}) + A_{4(k)}^{i}(U_{2}^{m}, U_{2,x}^{m})\right]$$

$$- \frac{\delta_1 n}{R^3}\left[A_{J4(k)}^{i}(W_{1,x}^{m}, U_{2}^{m}) + A_{J4(k)}^{i}(W_{1}^{m}, U_{2,x}^{m})\right.$$

$$+ A_{I4(k)}^{i}(U_{2,x}^{m}, W_{1}^{m+1}) + A_{I4(k)}^{i}(U_{2}^{m}, W_{1,x}^{m+1})$$

$$- A_{J1(k)}^{i}(W_{2,x}^{m}, U_{2}^{m+1}) - A_{J1(k)}^{i}(W_{2}^{m}, U_{1,x}^{m+1})$$

$$\left. - A_{I1(k)}^{i}(U_{1,x}^{m}, W_{2}^{m}) - A_{I1(k)}^{i}(U_{1}^{m}, W_{2,x}^{m})\right] \qquad (A\text{-}169)$$

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$$\left(t_{y2ix}^n\right)^{m+1} = -\left(\frac{n}{R}\right)^2\left[A_{IJ2(k)}^i(W_{2,x}^m, W_i^{m+1}) + A_{IJ2(k)}^i(W_2^m, W_{i,x}^{m+1})\right.$$

$$\left. + A_{IJ3(k)}^i(W_{i,x}^m, W_2^{m+1}) + A_{IJ3(k)}^i(W_i^m, W_{2,x}^{m+1})\right]$$

$$+ \frac{\delta_1}{R^2}\left[A_{3(k)}^i(U_{2,x}^m, U_i^{m+1}) + A_{2(k)}^i(U_{i,x}^m, U_2^{m+1})\right.$$

$$\left. + A_{3(k)}^i(U_2^m, U_{i,x}^{m+1}) + A_{2(k)}^i(U_i^m, U_{2,x}^{m+1})\right]$$

$$+ \frac{\delta_1 n}{R^2}\left[A_{J3(k)}^i(W_{i,x}^m, U_i^{m+1}) + A_{J3(k)}^i(W_i^m, U_{i,x}^{m+1})\right.$$

$$\left. + A_{I2}^i(U_{i,x}^m, W_i^{m+1}) + A_{I2(k)}^i(U_i^m, W_{i,x}^{m+1})\right.$$

$$\left. - A_{J2(k)}^i(W_{2,x}^m, U_2^{m+1}) - A_{J2(k)}^i(W_2^m, U_{2,x}^{m+1})\right.$$

$$\left. - A_{I3(k)}^i(U_{2,x}^m, W_2^{m+1}) - A_{I3(k)}^i(U_2^m, W_{2,x}^{m+1})\right]$$

$$- \frac{\delta_1}{2R^2}\left[A_{3(k)}^i(U_{2,x}^m, U_i^m) + A_{2(k)}^i(U_{i,x}^m, U_2^m)\right.$$

$$\left. + A_{3(k)}^i(U_2^m, U_{i,x}^m) + A_{2(k)}^i(U_i^m, U_{2,x}^m)\right]$$

$$- \frac{\delta_1 n}{R^2}\left[A_{J3(k)}^i(W_{i,x}^m, U_i^m) + A_{J3(k)}^i(W_i^m, U_{i,x}^m)\right.$$

$$\left. + A_{I3(k)}^i(U_{i,x}^m, W_i^m) + A_{I2(k)}^i(U_i^m, W_{i,x}^m)\right.$$

$$\left. - A_{J2(k)}^i(W_{2,x}^m, U_2^m) - A_{J2(k)}^i(W_2^m, U_{2,x}^m)\right.$$

$$\left. - A_{I3(k)}^i(U_{2,x}^m, W_2^m) - A_{I3(k)}^i(U_2^m, W_{2,x}^m)\right]$$

$$+ \frac{1}{2}\left(\frac{n}{R}\right)^2\left[A_{IJ2(k)}^i(W_{2,x}^m, W_i^m) + A_{IJ2(k)}^i(W_2^m, W_{i,x}^m)\right.$$

$$\left. + A_{IJ3(k)}^i(W_{i,x}^m, W_2^m) + A_{IJ3(k)}^i(W_i^m, W_{2,x}^m)\right] \qquad (A\text{-}170)$$

$$(t_{xy|i,x}^n)^{m+1} = \frac{n}{R}\Big[A_{J1(k)}^i(W_{2,x}^m, W_{1,x}^{m+1}) + A_{J1(k)}^i(W_2^m, W_{1,xx}^{m+1})$$

$$- A_{J4(k)}^i(W_{1,x}^m, W_{2,x}^{m+1}) - A_{J4(k)}^i(W_1^m, W_{2,xx}^{m+1})$$

$$+ A_{I1(k)}^i(W_{1,xx}^m, W_2^{m+1}) + A_{I1(k)}^i(W_{1,x}^m, W_{2,x}^{m+1})$$

$$- A_{I4(k)}^i(W_{2,xx}^m, W_1^{m+1}) - A_{I4(k)}^i(W_{2,x}^m, W_{1,x}^{m+1})\Big]$$

$$- \frac{\delta_1}{R}\Big[A_{1(k)}^i(U_{1,x}^m, W_{1,x}^{m+1}) + A_{1(k)}^i(U_1^m, W_{1,xx}^{m+1})$$

$$+ A_{4(k)}^i(U_{2,x}^m, W_{2,x}^{m+1}) + A_{4(k)}^i(U_2^m, W_{2,xx}^{m+1})$$

$$+ A_{1(k)}^i(W_{1,xx}^m, U_2^{m+1}) + A_1^i(W_{1,x}^m, U_{1,x}^{m+1})$$

$$+ A_{4(k)}^i(W_{2,xx}^m, U_2^{m+1}) + A_{4(k)}^i(W_{2,x}^m, U_{2,x}^{m+1})\Big]$$

$$- \frac{n}{2R}\Big[A_{J1(k)}^i(W_{2,x}^m, W_{1,x}^m) + A_{J1(k)}^i(W_2^m, W_{1,xx}^m)$$

$$- A_{J4(k)}^i(W_{1,x}^m, W_{2,x}^m) - A_{J4(k)}^i(W_1^m, W_{2,xx}^m)$$

$$+ A_{I1(k)}^i(W_{1,xx}^m, W_2^m) + A_{I1(k)}^i(W_{2,x}^m, W_{1,x}^m)\Big]$$

$$- \frac{\delta_1}{R}\Big[A_{1(k)}^i(U_{1,x}^m, W_{1,x}^m) + A_{1(k)}^i(U_1^m, W_{1,xx}^m)$$

$$+ A_{4(k)}^i(U_{2,x}^m, W_{2,x}^m) + A_{4(k)}^i(U_2^m, W_{2,xx}^m)$$

$$+ A_{1(k)}^i(W_{1,xx}^m, U_1^m) + A_{1(k)}^i(W_{1,x}^m, U_{1,x}^m)$$

$$+ A_{4(k)}^i(W_{2,xx}^m, U_2^m) + A_{4(k)}^i(W_{2,x}^m, U_{2,x}^m)\Big]$$

$$+ \frac{n}{2R}\Big[A_{I4(k)}^i(W_{2,xx}^m, U_2^m) + A_{I4(k)}^i(W_{2,x}^m, U_{2,x}^m)\Big] \qquad (A-171)$$

$$\left(t_{xx2i,x}^{n}\right)^{m+1} = \frac{n}{R}\left[A_{J2(k)}^{i}(W_{2,x}^{n}, W_{2,x}^{m+1}) + A_{J2(k)}^{i}(W_{2}^{n}, W_{2,xx}^{m+1})\right.$$

$$- A_{J3(k)}^{i}(W_{1,x}^{n}, W_{1,x}^{m+1}) - A_{J3(k)}^{i}(W_{1}^{n}, W_{1,xx}^{m+1})$$

$$- A_{I2(k)}^{i}(W_{1,xx}^{n}, W_{1}^{m+1}) - A_{I2(k)}^{i}(W_{1,x}^{n}, W_{1,x}^{m+1})$$

$$\left. + A_{I3(k)}^{i}(W_{2,xx}^{n}, W_{2}^{m+1}) + A_{I3(k)}^{i}(W_{2,x}^{n}, W_{2,x}^{m+1})\right]$$

$$- \frac{\delta_1}{R}\left[A_{3(k)}^{i}(U_{2,x}^{n}, W_{1,x}^{m+1}) + A_{3(k)}^{i}(U_{2}^{n}, W_{1,xx}^{m+1})\right.$$

$$+ A_{2(k)}^{i}(U_{1,x}^{n}, W_{2,x}^{m+1}) + A_{2(k)}^{i}(U_{1}^{n}, W_{2,xx}^{m+1})$$

$$+ A_{2(k)}^{i}(W_{1,xx}^{n}, U_{2}^{m+1}) + A_{2(k)}^{i}(W_{1,x}^{n}, U_{2,x}^{m+1})$$

$$\left. + A_{3(k)}^{i}(W_{2,xx}^{n}, U_{1}^{m+1}) + A_{3(k)}^{i}(W_{2,x}^{n}, U_{1,x}^{m+1})\right]$$

$$- \frac{n}{2R}\left[A_{J2(k)}^{i}(W_{2,x}^{n}, W_{2,x}^{n}) + A_{J2(k)}^{i}(W_{2}^{n}, W_{2,xx}^{n})\right.$$

$$- A_{J3(k)}^{i}(W_{1,x}^{n}, W_{1,x}^{n}) - A_{J3(k)}^{i}(W_{1}^{n}, W_{1,x}^{n})$$

$$- A_{I2(k)}^{i}(W_{1,xx}^{n}, W_{1}^{n}) - A_{I2(k)}^{i}(W_{1,x}^{n}, W_{1,x}^{n})$$

$$\left. + A_{I3(k)}^{i}(W_{2,xx}^{n}, W_{2}^{n}) + A_{I3(k)}^{i}(W_{2,x}^{n}, W_{2,x}^{n})\right]$$

$$+ \frac{\delta_1}{R}\left[A_{3(k)}^{i}(U_{2,x}^{n}, W_{1,x}^{n}) + A_{3(k)}^{i}(U_{2}^{n}, W_{1,xx}^{n})\right.$$

$$+ A_{2(k)}^{i}(U_{1,x}^{n}, W_{2,x}^{n}) + A_{2(k)}^{i}(U_{1}^{n}, W_{2,xx}^{n})$$

$$+ A_{2(k)}^{i}(W_{1,xx}^{n}, U_{2}^{n}) + A_{2(k)}^{i}(W_{1,x}^{n}, U_{2,x}^{n})$$

$$\left. + A_{3(k)}^{i}(W_{2,xx}^{n}, U_{1}^{n}) + A_{3(k)}^{i}(W_{2,x}^{n}, U_{1,x}^{n})\right] \tag{A-172}$$

$$(t_{x1,xx}^n)^{m+1} = A_{1(k)}^i(W_{1,xxx}^m, W_{1,x}^{m+1}) + 2A_{1(k)}^i(W_{1,xx}^m, W_{1,xx}^{m+1}) + A_{1(k)}^i(W_{1,x}^m, W_{1,xxx}^{m+1})$$

$$+ A_{4(k)}^i(W_{2,xxx}^m, W_{2,x}^{m+1}) + 2A_{4(k)}^i(W_{2,xx}^m, W_{2,xx}^{m+1}) + A_{4(k)}^i(W_{2,x}^m, W_{2,xxx}^{m+1})$$

$$- \frac{1}{2}\left[A_{1(k)}^i(W_{1,xxx}^m, W_{1,x}^m) + 2A_{1(k)}^i(W_{1,xx}^m, W_{1,xx}^m) + A_{1(k)}^i(W_{1,x}^m, W_{1,xxx}^m) \right.$$

$$\left. + A_{4(k)}^i(W_{2,xxx}^m, W_{2,x}^m) + 2A_{4(k)}^i(W_{2,xx}^m, W_{2,xx}^m) + A_{4(k)}^i(W_{2,x}^m, W_{2,xxx}^m) \right] \quad (A\text{-}173)$$

$$(t_{x2,xx}^n)^{m+1} = A_{2(k)}^i(W_{1,xxx}^m, W_{2,x}^{m+1}) + 2A_{2(k)}^i(W_{1,xx}^m, W_{2,xx}^{m+1}) + A_{2(k)}^i(W_{1,x}^m, W_{2,xx}^{m+1})$$

$$+ A_{3(k)}^i(W_{2,xxx}^m, W_{1,x}^{m+1}) + 2A_{3(k)}^i(W_{2,xx}^m, W_{1,xx}^{m+1}) + A_{3(k)}^i(W_{2,x}^m, W_{1,xxx}^{m+1})$$

$$- \frac{1}{2}\left[A_{2(k)}^i(W_{1,xxx}^m, W_{2,x}^m) + 2A_{2(k)}^i(W_{1,xx}^m, W_{2,xx}^m) + A_{2(k)}^i(W_{1,xx}^m, W_{2,xxx}^m) \right.$$

$$\left. + A_{3(k)}^i(W_{2,xxx}^m, W_{1,x}^m) + 2A_{3(k)}^i(W_{2,xx}^m, W_{1,xx}^m) + A_{3(k)}^i(W_{2,x}^m, W_{1,xxx}^m) \right] \quad (A\text{-}174)$$

$$(t_{y1,xx}^n)^{m+1} = \frac{n^2}{R^2}\left[A_{IJ4}^i(W_{1,x}^m, W_1^{m+1}) + 2A_{IJ4(k)}^i(W_{1,x}^m, W_{1,x}^{m+1}) + A_{IJ4(k)}^i(W_1^m, W_{1,xx}^{m+1}) \right.$$

$$\left. + A_{IJ1(k)}^i(W_{1,xx}^m, W_2^{m+1}) + 2A_{IJ1(k)}^i(W_{1,x}^m, W_{2,x}^{m+1}) + A_{IJ1(k)}^i(W_1^m, W_{2,xx}^{m+1}) \right]$$

$$+ \frac{S_1}{R^2}\left[A_{1(k)}^i(U_{1,xx}^m, U_1^{m+1}) + 2A_{1(k)}^i(U_{1,x}^m, U_{1,x}^{m+1}) + A_{1(k)}^i(U_1^m, U_{1,xx}^{m+1}) \right.$$

$$\left. + A_{4(k)}^i(U_{2,xx}^m, U_2^{m+1}) + 2A_{4(k)}^i(U_{2,x}^m, U_{2,x}^{m+1}) + A_{4(k)}^i(U_2^m, U_{2,xx}^{m+1}) \right]$$

$$+ \frac{S_1 n}{R^2}\left[A_{J4(k)}^i(W_{1,xx}^m, U_2^{m+1}) + 2A_{J4(k)}^i(W_{1,x}^m, U_{2,x}^{m+1}) + A_{J4(k)}^i(W_1^m, U_{2,xx}^{m+1}) \right.$$

$$+ A_{I4(k)}^i(U_{2,xx}^m, W_1^{m+1}) + 2A_{I4(k)}^i(U_{2,x}^m, W_{1,x}^{m+1}) + A_{I4(k)}^i(U_2^m, W_{1,xx}^{m+1})$$

$$- A_{J1(k)}^i(W_{2,x}^m, U_1^{m+1}) - 2A_{J1(k)}^i(W_{2,x}^m, U_{1,x}^{m+1}) - A_{J1(k)}^i(W_2^m, U_{1,xx}^{m+1})$$

$$\left. - A_{I1(k)}^i(U_{1,xx}^m, W_2^{m+1}) - 2A_{I1(k)}^i(U_{1,x}^m, W_{2,x}^{m+1}) - A_{I1(k)}^i(U_1^m, W_{2,xx}^{m+1}) \right]$$

$$- \frac{n^2}{2R^2}\left[A_{IJ4(k)}^i(W_{1,xx}^m, W_1^m) + 2A_{IJ4(k)}^i(W_{1,x}^m, W_{1,x}^m) + A_{IJ4(k)}^i(W_1^m, W_{1,xx}^m) \right.$$

$$\left. + A_{IJ1(k)}^i(W_{2,xx}^m, W_2^m) + 2A_{IJ1(k)}^i(W_{2,x}^m, W_{2,x}^m) + A_{IJ1(k)}^i(W_2^m, W_{2,xx}^m) \right]$$

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$$-\frac{\delta_1}{2R^2}\left[A^i_{1(k)}(U^m_{1,xx},U^m_1)+2A^i_{1(k)}(U^m_{1,x},U^{m+1}_{1,x})+A^i_{1(k)}(U^m_1,U^{m+1}_{1,xx})\right.$$

$$\left.+A^i_{4(k)}(U^m_{2,xx},U^m_2)+2A^i_{4(k)}(U^m_{2,x},U^m_{2,x})+A^i_{4(k)}(U^m_2,U^m_{2,xx})\right]$$

$$-\frac{\delta_1\eta}{R^2}\left[A^i_{J4(k)}(W^m_{1,xx},U^m_2)+2A^i_{J4(k)}(W^m_{1,x},U^m_{2,x})+A^i_{J4(k)}(W^m_1,U_{2,xx})\right.$$

$$+A^i_{I4(k)}(U_{2,xx},W^m_1)+2A^i_{I4(k)}(U^m_{2,x},W^m_{1,x})+A^i_{I4(k)}(U^m_2,W_{1,xx})$$

$$-A^i_{J1(k)}(W_{2,xx},U^m_1)-2A^i_{J1(k)}(W^m_{2,x},U^m_{1,x})-A^i_{J1(k)}(W^m_2,U_{1,xx})$$

$$\left.-A^i_{I1(k)}(U^m_{1,xx},W^{m+1}_2)-2A^i_{I1(k)}(U^m_{1,x},W^m_{2,x})-A^i_{I1(k)}(U^m_1,W^m_{2,xx})\right] \qquad (A\text{-}175)$$

$$(t^n_{y2i,xx})^{m+1}=-\left(\frac{\eta}{R}\right)^2\left[A^i_{IJ2(k)}(W^m_{2,xx},W^{m+1}_1)+2A^i_{IJ2(k)}(W^m_{2,x},W^{m+1}_{1,x})+A^i_{IJ2(k)}(W^m_2,W^{m+1}_{1,xx})\right.$$

$$\left.+A^i_{IJ3(k)}(W^m_{1,xx},W^{m+1}_2)+2A^i_{IJ3(k)}(W^m_{1,x},W^m_{2,x})+A^i_{IJ3(k)}(W^m_1,W_{2,xx})\right]$$

$$+\frac{\delta_1}{R}\left[A^i_{3(k)}(U^m_{2,xx},U^{m+1}_1)+2A^i_{3(k)}(U^m_{2,x},U^m_{1,x})+A^i_{3(k)}(U^m_2,U_{1,xx})\right.$$

$$\left.+A^i_{2(k)}(U^m_{1,xx},U^m_2)+2A^i_{2(k)}(U^m_{1,x},U^m_{2,x})+A^i_{2(k)}(U^m_1,U^{m+1}_{2,xx})\right]$$

$$+\frac{\delta_1\eta}{R^2}\left[A^i_{J3(k)}(W^m_{1,xx},U^{m+1}_1)+2A^i_{J3(k)}(W^m_{1,x},U^m_{1,x})+A^i_{J3(k)}(W^m_1,U_{1,xx})\right.$$

$$+A^i_{I2(k)}(U^m_{1,xx},W^{m+1}_1)+2A^i_{I2(k)}(U^m_{1,x},W^{m+1}_{1,x})+A^i_{I2(k)}(U^m_1,W_{1,xx})$$

$$-A^i_{J2(k)}(W^m_{2,xx},U^{m+1}_2)-2A^i_{J2(k)}(W^m_{2,x},U^{m+1}_{2,x})-A^i_{J2(k)}(W^m_2,U_{2,xx})$$

$$\left.-A^i_{I3(k)}(U^m_{2,xx},W^{m+1}_2)-2A^i_{I3(k)}(U^m_{2,x},W_{2,x})-A^i_{I3(k)}(U^m_2,W_{2,xy})\right]$$

$$+\frac{1}{2}\left(\frac{\eta}{R}\right)^2\left[A^i_{IJ2(k)}(W^m_{1,xx},W^m_1)+2A^i_{IJ2(k)}(W^m_{2,x},W^m_{1,x})+A^i_{IJ2(k)}(W^m_2,W^m_{1,xx})\right.$$

$$\left.+A^i_{IJ3(k)}(W^m_{1,xx},W^m_2)+2A^i_{IJ3(k)}(W^m_{1,x},W^m_{2,x})+A^i_{IJ3(k)}(W^m_1,W_{2,xx})\right]$$

$$-\frac{\delta_1}{2R^2}\left[A^i_{3(k)}(U^m_{2,xy},U^m_1)+2A^i_{3(k)}(U^m_{2,x},U^m_{1,x})+A^i_{3(k)}(U^m_2,U_{1,xx})\right.$$

$$\left.+A^i_{2(k)}(U^m_{1,xx},U^m_2)+2A^i_{2(k)}(U^m_{1,x},U^m_{2,x})+A^i_{2(k)}(U^m_1,U^m_{2,xx})\right]$$

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$$-\frac{\delta_1 n}{R^3}\left[A^i_{J3(k)}(W^m_{1,xx},U^m_1) + 2A^i_{J3(k)}(W^m_{1,x},U^m_{1,x}) + A^i_{J3(k)}(W^m_1,U^m_{1,xx})\right.$$

$$+ A^i_{J2(k)}(U^m_{1,xx},W^m_1) + 2A^i_{J2(k)}(U^m_{1,x},W^m_{1,x}) + A^i_{J2(k)}(U^m_1,W^m_{1,xx})$$

$$- A^i_{J2(k)}(W^m_{2,xx},U^m_2) - 2A^i_{J2(k)}(W^m_{2,x},U^m_{2,x}) - A^i_{J2(k)}(W^m_2,U^m_{2,xx})$$

$$\left.- A^i_{J3(k)}(U^m_{2,xx},W^m_2) - 2A^i_{J3(k)}(U^m_{2,x},W^m_{2,x}) - A^i_{J3(k)}(U^m_2,W^m_{2,xx})\right] \qquad (A\text{-}176)$$

$$(t^n_{xy/i,xx})^{m+1} = \frac{n}{R}\left[A^i_{J1(k)}(W^m_{2,xx},W^m_{1,x}) + 2A^i_{J1(k)}(W^m_{2,x},W^m_{1,xx}) + A^i_{J1(k)}(W^m_2,W^m_{1,xxx})\right.$$

$$- A^i_{J4(k)}(W^m_{1,xx},W^{m+1}_{2,x}) + 2A^i_{J4(k)}(W^m_{1,x},W^{m+1}_{2,xx}) - A^i_{J4(k)}(W^m_1,W^{m+1}_{2,xxx})$$

$$+ A^i_{J1(k)}(W^m_{1,xxx},W^{m+1}_2) + 2A^i_{J1(k)}(W^m_{1,xx},W^{m+1}_{2,x}) + A^i_{J1(k)}(W^m_{1,x},W^{m+1}_{2,xx})$$

$$\left.- A^i_{J4(k)}(W^m_{2,xxx},W^{m+1}_1) - 2A^i_{J4(k)}(W^m_{2,xx},W^{m+1}_{1,x}) - A^i_{J4(k)}(W^m_{2,x},W^{m+1}_{1,xx})\right]$$

$$- \frac{\delta_1}{R}\left[A^i_{1(k)}(U^m_{1,xx},W^{m+1}_{1,x}) + 2A^i_{1(k)}(U^m_{1,x},W^{m+1}_{1,xx}) + A^i_{1(k)}(U^m_1,W^{m+1}_{1,xxx})\right.$$

$$+ A^i_{4(k)}(U^m_{2,xx},W^{m+1}_{2,x}) + 2A^i_{4(k)}(U^m_{2,x},W^{m+1}_{2,xx}) + A^i_{4(k)}(U^m_2,W^{m+1}_{2,xxx})$$

$$+ A^i_{1(k)}(W^m_{1,xx},U^{m+1}_1) + 2A^i_{1(k)}(W^m_{1,xx},U^{m+1}_{1,x}) + A^i_{1(k)}(W^m_{1,x},U^{m+1}_{1,xx})$$

$$\left.+ A^i_{4(k)}(W^m_{2,xxx},U^{m+1}_2) + 2A^i_{4(k)}(W^m_{2,xx},U^{m+1}_{2,x}) + A^i_{4(k)}(W^m_{2,x},U^{m+1}_{2,xxx})\right]$$

$$- \frac{n}{2R}\left[A^i_{J1(k)}(W^m_{2,xx},W^m_{1,x}) + 2A^i_{J1(k)}(W^m_{2,x},W^m_{1,xx}) + A^i_{J1}(W^m_2,W^m_{1,xxx})\right.$$

$$- A^i_{J4(k)}(W^m_{1,xx},W^m_{2,x}) - 2A^i_{J4(k)}(W^m_{1,x},W^m_{2,xx}) + A^i_{J4(k)}(W^m_1,W^m_{2,xxx})$$

$$+ A^i_{1(k)}(W^m_{1,xxx},W^m_2) + 2A^i_{1(k)}(W^m_{1,xx},W^m_{2,x}) + A^i_{1(k)}(W^m_{1,x},W^m_{2,xx})$$

$$- A^i_{I4(k)}(W^m_{2,xxx}, W^m_1) - 2A^i_{I4(k)}(W^m_{2,xx}, W^m_{1,x}) - A^i_{I4(k)}(W^m_{2,x}, W^m_{1,xx})]$$

$$+ \frac{\delta_1}{R}[A^i_{1(k)}(U^m_{1,xx}, W^m_{1,x}) + 2A^i_{1(k)}(U^m_{1,x}, W^m_{1,xx}) + A^i_{1(k)}(U^m_1, W^m_{1,xxx})$$

$$+ A^i_{4(k)}(U^m_{2,xx}, W^m_{2,x}) + 2A^i_{4(k)}(U^m_{2,x}, W^m_{2,xx}) + A^i_{4(k)}(U^m_2, W^m_{2,xxx})$$

$$+ A^i_{1(k)}(W^m_{1,xxx}, U^m_1) + 2A^i_{1(k)}(W^m_{1,xx}, U^m_{1,x}) + A^i_{1(k)}(W^m_{1,x}, U^m_{1,xx})$$

$$+ A^i_{4(k)}(W^m_{2,xxx}, U^m_2) + 2A^i_{4(k)}(W^m_{2,xx}, U^m_{2,x}) + A^i_{4(k)}(W^m_{2,x}, U^m_{2,xx})] \qquad (A\text{-}177)$$

$$(t^n_{xy2i,xx})^{m+1} = \frac{n}{R}[A^i_{J2(k)}(W^m_{2,xx}, W^{m+1}_{2,x}) + 2A^i_{J2(k)}(W^m_{2,x}, W^{m+1}_{2,xx}) + A^i_{J2(k)}(W^m_2, W^{m+1}_{2,xxx})$$

$$- A^i_{J3(k)}(W^m_{1,xx}, W^{m+1}_{1,x}) - 2A^i_{J3(k)}(W^m_{1,x}, W^{m+1}_{1,xx}) - A^i_{J3(k)}(W^m_1, W^{m+1}_{1,xxx})$$

$$- A^i_{I2(k)}(W^m_{1,xxx}, W^{m+1}_1) - 2A^i_{I2(k)}(W^m_{1,xx}, W^{m+1}_{1,x}) - A^i_{I2(k)}(W^m_{1,x}, W^{m+1}_{1,xx})$$

$$+ A^i_{I3(k)}(W^m_{2,xxx}, W^{m+1}_2) + 2A^i_{I3(k)}(W^m_{2,xx}, W^{m+1}_{2,x}) + A^i_{I3(k)}(W^m_{2,x}, W^{m+1}_{2,xx})]$$

$$- \frac{\delta_1}{R}[A^i_{3(k)}(U^m_{2,xx}, W^{m+1}_{2,x}) + 2A^i_{3(k)}(U^m_{2,x}, W^{m+1}_{2,xx}) + A^i_{3(k)}(U^m_2, W^{m+1}_{2,xxx})$$

$$+ A^i_{2(k)}(U^m_{1,xx}, W^{m+1}_{2,x}) + 2A^i_{2(k)}(U^m_{1,x}, W^{m+1}_{2,xx}) + A^i_{2(k)}(U^m_1, W^{m+1}_{2,xxx})$$

$$+ A^i_{2(k)}(W^m_{1,xxx}, U^{m+1}_2) + 2A^i_{2(k)}(W^m_{1,xxx}, U^{m+1}_{2,x}) + A^i_{2(k)}(W^m_{1,x}, U^{m+1}_{2,xx})$$

$$+ A^i_{3(k)}(W^m_{2,xxx}, U^{m+1}_1) + 2A^i_{3(k)}(W^m_{2,xx}, U^{m+1}_{1,x}) + A^i_{3(k)}(W^m_{2,x}, U^{m+1}_{1,xx})]$$

$$- \frac{n}{2R}[A^i_{J2(k)}(W^m_{2,xx}, W^m_{2,x}) + 2A^i_{J2(k)}(W^m_{2,x}, W^m_{1,xx}) + A^i_{J2(k)}(W^m_2, W^m_{1,xxx})$$

$$- A^i_{J3(k)}(W^m_{1,xx}, W^m_{1,x}) - 2A^i_{J3(k)}(W^m_{1,x}, W^m_{1,xx}) - A^i_{J3(k)}(W^m_1, W^m_{1,xxx})$$

$$- A_{I2(k)}^i (W_{1,xxx}^m, W_1^m) - 2 A_{I2(k)}^i (W_{1,xx}^m, W_{1,x}^m) - A_{I2(k)}^i (W_{1,x}^m, W_{1,xx}^m)$$

$$+ A_{I3(k)}^i (W_{2,xxx}^m, W_2^m) + 2 A_{I3(k)}^i (W_{2,xx}^m, W_{2,x}^m) + A_{I3(k)}^i (W_{2,x}^m, W_{2,xx}^m)$$

$$+ \frac{\delta_i}{R} [A_{3(k)}^i (U_{2,xx}^m, W_{1,x}^m) + 2 A_{3(k)}^i (U_{2,x}^m, W_{1,xx}^m) + A_{3(k)}^i (U_2^m, W_{1,xxx}^m)$$

$$+ A_{2(k)}^i (U_{1,xx}^m, W_{2,x}^m) + 2 A_{2(k)}^i (U_{1,x}^m, W_{2,xx}^m) + A_{2(k)}^i (U_1^m, W_{2,xxx}^m)$$

$$+ A_{2(k)}^i (W_{1,xxx}^m, U_2^m) + 2 A_{2(k)}^i (W_{1,xx}^m, U_{2,x}^m) + A_{2(k)}^i (W_{1,x}^m, U_{2,xx}^m)$$

$$+ A_{3(k)}^i (W_{2,xxx}^m, U_1^m) + 2 A_{3(k)}^i (W_{2,xx}^m, U_{1,x}^m) + A_{3(k)}^i (W_{2,x}^m, U_{1,xx}^m)] \qquad (A-178)$$

where m is the number of the iteration step.

Substitution of Eqs A-161- A-178 into Eqs A-149 and A-150 one may obtain the iteration equations for the nonlinear part of the stress and moment resultant vectors $\left(\{ n_1^n \}_i \text{ and } \{ n_2^n \}_i \right)$. In so doing, new symbols are introduced and defined. The part of the t's or n^n's that is linearized (linear) with respect to the iteration parameters (containing u^{m+1}, v^{m+1} & w^{m+1}) is denoted by superscript L next to n, i.e. $\{ t_{1L}^{nL} \}$. The part that only depends on the value of the parameters at the previous step (u^m, v^m, w^m), is denoted by superscript n next to n, i.e. $\{ t_{1L}^{nn} \}$.

$$\{ n_1^n \}_i^{m+1} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} (\{ t_{1i}^{nL} \}^{m+1} + \{ t_{1i}^{nn} \}^m)$$

$$= \{ n_1^{nL} \}_i^{m+1} + \{ n_1^{nn} \}_i^m \qquad (A-179)$$

$$\{ n_2^n \} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} (\{ t_{2i}^{nL} \}^{m+1} + \{ t_{2i}^{nn} \}^n)$$

$$= \{ n_1^{nL} \}_i + \{ n_1^{nn} \}_i \qquad (A-180)$$

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$$\left\{ n_{i,x}^{n} \right\}_{i}^{m+1} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \left(\left\{ t_{1i,x}^{nL} \right\}^{m+1} + \left\{ t_{1i,x}^{nn} \right\}^{m} \right)$$

$$= \left\{ n_{1,x}^{nL} \right\}_{i}^{m+1} + \left\{ n_{1,x}^{nn} \right\}_{i}^{m} \qquad (A\text{-}181)$$

$$\left\{ n_{2,x}^{n} \right\}_{i}^{m+1} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \left(\left\{ t_{2i,x}^{nL} \right\}^{m+1} + \left\{ t_{2i,x}^{nn} \right\}^{m} \right)$$

$$= \left\{ n_{2,x}^{nL} \right\}_{i}^{m+1} + \left\{ n_{2,x}^{nn} \right\}_{i}^{m} \qquad (A\text{-}182)$$

$$\left\{ n_{1,xx}^{n} \right\}^{m+1} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \left(\left\{ t_{1i,xx}^{nL} \right\}^{m+1} + \left\{ t_{1i,xx}^{nn} \right\}^{m} \right)$$

$$= \left\{ n_{1,xx}^{nL} \right\}^{m+1} + \left\{ n_{1,xx}^{nn} \right\}^{m} \qquad (A\text{-}183)$$

$$\left\{ n_{2,xx}^{n} \right\}^{m+1} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \left(\left\{ t_{2i,xx}^{nL} \right\}^{m+1} + \left\{ t_{2i,xx}^{nn} \right\}^{m} \right)$$

$$= \left\{ n_{2,xx}^{nL} \right\}^{m+1} + \left\{ n_{2i,xx}^{nn} \right\}^{m} \qquad (A\text{-}184)$$

In a very similar manner, the nonlinear terms of the equilibrium equations are also linearized by Newton's method:

$$\xi_{1in}^{2} = -\frac{in}{R}\left(n_{yy1i}^{nL} + n_{yy1i}^{nn} \right) + n_{xy2i,x}^{nL} + n_{xy2i,x}^{nn} + \frac{\delta_{1}}{R}\left(m_{xy2i,x}^{nL} + m_{xy2i,x}^{nn} \right)$$

$$- \frac{in}{R^{2}}\delta_{1}\left(m_{yy1i}^{nL} + m_{yy1i}^{nn} \right) + \frac{\delta_{1}}{R^{2}}n\left(A_{J1(k)}^{i}(W_{2}^{m}, n_{yy1}^{m+1}) + A_{J1(k)}^{i}(W_{2}^{m+1}, n_{yy1}^{m}) \right.$$

$$\left. - A_{J1(k)}^{i}(W_{2}^{m}, n_{yy1}^{m}) - A_{J4(k)}^{i}(W_{1}^{m+1}, n_{yy2}^{m}) - A_{J4(k)}^{i}(W_{1}^{m}, n_{yy2}^{m+1}) + A_{J4(k)}^{i}(W_{1}^{m}, n_{yy2}^{m}) \right)$$

$$- \frac{\delta_{1}}{R^{2}}\left(A_{1(k)}^{i}(U_{1}^{m+1}, n_{yy1}^{m}) + A_{1(k)}^{i}(U_{1}^{m}, n_{yy1}^{m+1}) + A_{1(k)}^{i}(U_{1}^{m}, n_{yy1}^{m}) \right)$$

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$$+ A_{4(k)}^i(U_2^{m+1}, \mathcal{N}_{yy2}^m) + A_{4(k)}^i(U_2^m, \mathcal{N}_{yy2}^{m+1}) - A_{4(k)}^i(U_2^m, \mathcal{N}_{yy2}^m)]$$

$$+ \frac{\delta_1}{R}[\, A_{1(k)}^i(W_{1,x}^{m+1}, \mathcal{N}_{xy1}^m) + A_{1(k)}^i(W_{1,x}^m, \mathcal{N}_{xy1}^{m+1}) - A_{1(k)}^i(W_{1,x}^m, \mathcal{N}_{yy1}^m)$$

$$+ A_{4(k)}^i(W_{2,x}^{m+1}, \mathcal{N}_{xy2}^m) + A_{4(k)}^i(W_{2,x}^m, \mathcal{N}_{xy2}^{m+1}) - A_{4(k)}^i(W_{2,x}^m, \mathcal{N}_{xy2}^m)] \qquad (A-185)$$

$$\xi_{2in}^2 = -\frac{in}{R}(\mathcal{N}_{yy1i}^{nL} + \mathcal{N}_{yy1i}^{nn}) + \mathcal{N}_{xy2i,x}^{nL} + \mathcal{N}_{xy2i,x}^{nn} + \frac{\delta_1}{R}(\mathcal{M}_{xy2i,x}^{nL} + \mathcal{M}_{xy2i,x}^{nn})$$

$$- \frac{in}{R}\delta_1(\mathcal{M}_{yy1i}^{nL} + \mathcal{M}_{yy1i}^{nn}) + \frac{\delta_1 n}{R^2}[\, A_{J2(k)}^i(W_2^{m+1}, \mathcal{N}_{yy2}^m) + A_{J2(k)}^i(W_2^m, \mathcal{N}_{yy2}^{mH})$$

$$- A_{J2(k)}^i(W_2^m, \mathcal{N}_{yy2}^m) + A_{J3(k)}^i(W_1^{m+1}, \mathcal{N}_{yy1}^m) + A_{J3(k)}^i(W_1^m, \mathcal{N}_{yy1}^{m+1})$$

$$- A_{J3(k)}^i(W_1^m, \mathcal{N}_{yy1}^m)] - \frac{\delta_1}{R^2}[\, A_{3(k)}^i(U_2^{m+1}, \mathcal{N}_{yy1}^m) + A_{3(k)}^i(U_2^m, \mathcal{N}_{yy1}^{m+1})$$

$$- A_{3(k)}^i(U_2^m, \mathcal{N}_{yy1}^m) + A_{2(k)}^i(U_1^{m+1}, \mathcal{N}_{yy2}^m) + A_{2(k)}^i(U_1^m, \mathcal{N}_{yy2}^{m+1}) - A_{2(k)}^i(U_1^m, \mathcal{N}_{yy2}^m)] \quad (A-186)$$

$$\xi_{iin}^3 = \frac{\delta_1 n}{R^2}[\, A_{J1(2k)}^i(W_2^0, \mathcal{N}_{yy1}^{nL}) + A_{J1(2k)}^i(W_2^*, \mathcal{N}_{yy1}^{nn}) + A_{J1(2k)}^i(W_2^n, \mathcal{N}_{yy1}^{nn})$$

$$- A_{J4(2k)}^i(W_1^n, \mathcal{N}_{yy2}^{nn}) - A_{J4(2k)}^i(W_1^0, \mathcal{N}_{yy2}^{nL}) - A_{J4(2k)}^i(W_1^0, \mathcal{N}_{yy2}^{nn})$$

$$- \frac{\delta_1}{R^2}[\, A_{1(2k)}^i(U_1^m, \mathcal{N}_{yy1}^{nn}) + A_{4(2k)}^i(U_2, \mathcal{N}_{yy2}^{nn})]$$

$$+ \frac{\delta_1}{R}[\, A_{1(2k)}^i(W_{1,x}^0, \mathcal{N}_{xy1}^{nn}) + A_{1(2k)}^i(W_{1,x}^0, \mathcal{N}_{xy1}^{nn}) + A_{1(2k)}^i(W_{1,x}^n, \mathcal{N}_{xy1}^{nn})$$

$$+ A_{4(2k)}^i(W_{2,x}^0, \mathcal{N}_{xy2}^{nn}) + A_{4(2k)}^i(W_{1,x}^0, \mathcal{N}_{xy1}^{nn}) + A_{4(2k)}^i(W_{2,x}^n, \mathcal{N}_{xy2}^{nn})]$$

$$+ \frac{\delta_1 n}{R^2}[\, A_{J1(2k)}^i(W_2^{mH}, \mathcal{N}_{yy1}^{Lm}) + A_{J1(2k)}^i(W_2^m, \mathcal{N}_{yy1}^{Lm+1}) - A_{J1(2k)}^i(W_2^m, \mathcal{N}_{yy1}^{Lm})$$

$$- A_{J4(2k)}^i(W_1^m, \mathcal{N}_{yy2}^{Lm+1}) - A_{J4(2k)}^i(W_1^{mH}, \mathcal{N}_{yy2}^{Lm}) + A_{J4(2k)}^i(W_1^m, \mathcal{N}_{yy2}^{Lm})]$$

$$- \frac{\delta_1}{R^2}[\, A_{1(2k)}^i(U_1^{m+1}, \mathcal{N}_{yy1}^{Lm}) + A_{1(2k)}^i(U_1^m, \mathcal{N}_{yy1}^{Lm+1}) - A_{1(2k)}^i(U_1^m, \mathcal{N}_{yy1}^{Lm})$$

$$+ A_{4(2k)}^i(U_2^{m+1}, \mathcal{N}_{yy2}^{Lm}) + A_{4(2k)}^i(U_2^m, \mathcal{N}_{yy2}^{Lm+1}) - A_{4(2k)}^i(U_2^m, \mathcal{N}_{yy2}^{Lm})]$$

$$+ \frac{\delta_1}{R}[\, A_{1(2k)}^i(W_{1,x}^{m+1}, \mathcal{N}_{xy1,x}^{Lm}) + A_{1(2k)}^i(W_{1,x}^m, \mathcal{N}_{xy1,x}^{Lm+1}) - A_{1(2k)}^i(W_{1,x}^m, \mathcal{N}_{xy1,x}^{Lm})$$

$$+ A^i_{4(2k)}(W^{m+1}_{2,x}, \mathcal{N}^{Lm}_{xy2}) + A^i_{4(2k)}(W^m_{2,x}, \mathcal{N}^{Lm+1}_{xy2}) - A^i_{4(2k)}(W^m_{2,x}, \mathcal{N}^{Lm}_{xy2})] \qquad (A-187)$$

$$\xi^3_{2in} = \frac{\delta_1 n}{R^2}[-A^i_{J3(2k)}(W^m_1, \mathcal{N}^{nm}_{yy1}) - A^i_{J3(2k)}(W^0_1, \mathcal{N}^{nL}_{yy1}) - A^i_{J3(2k)}(W^0_1, \mathcal{N}^{nn}_{yy1})$$

$$+ A^i_{J2(2k)}(W^m_2, \mathcal{N}^{nm}_{yy2}) + A^i_{J2(2k)}(W^0_2, \mathcal{N}^{nL}_{yy2}) + A^i_{J2(2k)}(W^0_2, \mathcal{N}^{nn}_{yy2})]$$

$$- \frac{\delta_1}{R^2}[A^i_{2(2k)}(\mathcal{U}^m_1, \mathcal{N}^{nm}_{yy2}) + A^i_{3(2k)}(\mathcal{U}^m_2, \mathcal{N}^{nm}_{yy1})]$$

$$+ \frac{\delta_1}{R}[A^i_{3(2k)}(W^m_{2,x}, \mathcal{N}^{nm}_{xy1}) + A^i_{3(2k)}(W^0_{2,x}, \mathcal{N}^{nL}_{xy1}) + A^i_{3(2k)}(W^0_{2,x}, \mathcal{N}^{nn}_{xy1})$$

$$+ A^i_{2(2k)}(W^m_{1,x}, \mathcal{N}^{nm}_{xy2}) + A^i_{2(2k)}(W^0_{1,x}, \mathcal{N}^{nL}_{xy2}) + A^i_{2(2k)}(W^0_{1,x}, \mathcal{N}^{nn}_{xy2})]$$

$$+ \frac{\delta_1 n}{R^2}[-A^i_{J3(2k)}(W^{m+1}_1, \mathcal{N}^{Lm}_{yy1}) - A^i_{J3(2k)}(W^m_1, \mathcal{N}^{Lm+1}_{yy1}) + A^i_{J3(2k)}(W^m_1, \mathcal{N}^{Lm}_{yy1})$$

$$+ A^i_{J2(2k)}(W^{m+1}_2, \mathcal{N}^{Lm}_{yy2}) + A^i_{J2(2k)}(W^m_2, \mathcal{N}^{Lm+1}_{yy2}) - A^i_{J2(2k)}(W^m_2, \mathcal{N}^{Lm}_{yy2})]$$

$$- \frac{\delta_1}{R^2}[A^i_{2(2k)}(\mathcal{U}^{m+1}_1, \mathcal{N}^{Lm}_{yy2}) + A^i_{2(2k)}(\mathcal{U}^m_1, \mathcal{N}^{Lm+1}_{yy2}) - A^i_{2(2k)}(\mathcal{U}^m_1, \mathcal{N}^m_{yy2})$$

$$+ A^i_{3(2k)}(\mathcal{U}^{m+1}_2, \mathcal{N}^{Lm}_{yy1}) + A^i_{3(2k)}(\mathcal{U}^m_2, \mathcal{N}^{Lm+1}_{yy1}) - A^i_{3(2k)}(\mathcal{U}^m_2, \mathcal{N}^{Lm}_{yy1})]$$

$$+ \frac{\delta_1}{R}[A^i_{3(2k)}(W^{m+1}_{2,x}, \mathcal{N}^{Lm}_{xy1}) + A^i_{3(2k)}(W^m_{2,x}, \mathcal{N}^{Lm+1}_{xy1}) - A^i_{3(2k)}(W^m_{2,x}, \mathcal{N}^{Lm}_{xy1})$$

$$+ A^i_{2(2k)}(W^{m+1}_{1,x}, \mathcal{N}^{Lm}_{xy2}) + A^i_{2(2k)}(W^m_{1,x}, \mathcal{N}^{Lm+1}_{xy2}) - A^i_{2(2k)}(W^m_{1,x}, \mathcal{N}^{Lm}_{xy2})] \qquad (A-188)$$

$$\eta^2_{1in} = \mathcal{M}^{nL}_{xx1i,xx} + \mathcal{M}^{nn}_{xx1i,xx} + 2(\frac{in}{R})(\mathcal{M}^{nL}_{xy2i,x} + \mathcal{M}^{nn}_{xy2i,x}) - (\frac{in}{R})^2(\mathcal{N}^{nL}_{yy1i} + \mathcal{M}^{nn}_{yy1i})$$

$$- \frac{1}{R}(\mathcal{N}^{nL}_{yy1i} + \mathcal{N}^{nn}_{yy1i}) + \frac{n}{R}[A^i_{J1(k)}(W^{m+1}_2, \mathcal{N}^m_{xy1,x}) + A^i_{J1(k)}(W^m_2, \mathcal{N}^{m+1}_{xy1,x})$$

$$- A^i_{J1(k)}(W^m_2, \mathcal{N}^m_{xy,x}) - A^i_{J4(k)}(W^{m+1}_1, \mathcal{N}^{m+1}_{xy1,x}) - A^i_{J4(k)}(W^m_1, \mathcal{N}^{m+1}_{xy2,x}) + A^i_{J4(k)}(W^m_1, \mathcal{N}^m_{xy2,x})]$$

$$+ (\frac{n}{R})^2[A^i_{IJ1(k)}(W^m_2, \mathcal{N}^m_{yy2}) + A^i_{IJ4(k)}(W^{m+1}_1, \mathcal{N}^m_{yy1}) - A^i_{IJ4(k)}(W^m_1, \mathcal{N}^m_{yy1})]$$

$$+ A^i_{IJ4(k)}(W^{m+1}_1, \mathcal{N}^m_{yy1}) + A^i_{IJ4(k)}(W^m_1, \mathcal{N}^{m+1}_{yy1}) - A^i_{IJ4(k)}(W^m_1, \mathcal{N}^m_{yy1})]$$

$$+ \frac{\delta_1}{R}[-A^i_{1(k)}(\mathcal{U}^{m+1}_1, \mathcal{N}^m_{xy1,x}) - A^i_{1(k)}(\mathcal{U}^m_1, \mathcal{N}^{m+1}_{xy1,x}) + A^i_{1(k)}(\mathcal{U}^m_1, \mathcal{N}^m_{xy1,x})$$

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$$- A_{4(k)}^{i} (\mathcal{U}_2^{m+1}, \mathcal{N}_{xy2,x}^{m}) - A_{4(k)}^{i} (\mathcal{U}_2^{m}, \mathcal{N}_{xy2,x}^{m+1}) + A_{4(k)}^{i} (\mathcal{U}_2^{m}, \mathcal{N}_{xy2,x}^{m})]$$

$$+ \frac{\delta_i \eta}{R^2} [-A_{I1(k)}^{i} (\mathcal{U}_1^{m+1}, \mathcal{N}_{yy2}^{m}) - A_{I1(k)}^{i} (\mathcal{U}_1^{m}, \mathcal{N}_{yy2}^{m+1}) + A_{I1(k)}^{i} (\mathcal{U}_1^{m}, \mathcal{N}_{yy2}^{m})$$

$$+ A_{I4(k)}^{i} (\mathcal{U}_2^{m+1}, \mathcal{N}_{yy1}^{m}) + A_{I4(k)}^{i} (\mathcal{U}_2^{m}, \mathcal{N}_{yy1}^{m+1}) - A_{I4(k)}^{i} (\mathcal{U}_2^{m}, \mathcal{N}_{yy1}^{m})$$

$$+ A_{1(k)}^{i} (W_{1,xx}^{m+1}, \mathcal{N}_{xx1}^{m}) + A_{1(k)}^{i} (W_{1,xx}^{m}, \mathcal{N}_{xx1}^{m+1}) - A_{1(k)}^{i} (W_{1,xx}^{m}, \mathcal{N}_{xx1}^{m})$$

$$+ A_{4(k)}^{i} (W_{2,xx}^{m+1}, \mathcal{N}_{xx2}^{m}) + A_{4(k)}^{i} (W_{2,xx}^{m}, \mathcal{N}_{xx2}^{m+1}) - A_{4(k)}^{i} (W_{2,xx}^{m}, \mathcal{N}_{xx2}^{m})]$$

$$+ \frac{2\eta}{R} [A_{J1(k)}^{i} (W_{2,x}^{m+1}, \mathcal{N}_{xy1}^{m}) + A_{J1(k)}^{i} (W_{2,x}^{m}, \mathcal{N}_{xy1}^{m+1}) - A_{J1(k)}^{i} (W_{2,x}^{m}, \mathcal{N}_{xy1}^{m})$$

$$- A_{J4(k)}^{i} (W_{1,x}^{m+1}, \mathcal{N}_{xy2}^{m}) - A_{J4(k)}^{i} (W_{1,x}^{m}, \mathcal{N}_{xy2}^{m+1}) + A_{J4(k)}^{i} (W_{1,x}^{m}, \mathcal{N}_{xy2}^{m})]$$

$$+ (\frac{\eta}{R})^2 [- A_{J21(k)}^{i} (W_{1}^{m+1}, \mathcal{N}_{yy1}^{m}) - A_{J21(k)}^{i} (W_{1}^{m+1}, \mathcal{N}_{yy1}^{m}) + A_{J21(k)}^{i} (W_{1}^{m}, \mathcal{N}_{yy1}^{m})$$

$$- A_{J24(k)}^{i} (W_{2}^{m+1}, \mathcal{N}_{yy2}^{m}) - A_{J24(k)}^{i} (W_{2}^{m}, \mathcal{N}_{yy2}^{m+1}) + A_{J24(k)}^{i} (W_{2}^{m}, \mathcal{N}_{yy2}^{m})]$$

$$+ \frac{\delta_i \eta}{R^2} [-A_{J1(k)}^{i} (\mathcal{U}_2^{m+1}, \mathcal{N}_{yy1}^{m}) - A_{J1(k)}^{i} (\mathcal{U}_2^{m}, \mathcal{N}_{yy1}^{m+1}) + A_{J1(k)}^{i} (\mathcal{U}_2^{m}, \mathcal{N}_{yy1}^{m})$$

$$+ A_{J4(k)}^{i} (\mathcal{U}_1^{m+1}, \mathcal{N}_{yy2}^{m}) + A_{J4(k)}^{i} (\mathcal{U}_1^{m}, \mathcal{N}_{yy2}^{m+1}) - A_{J4(k)}^{i} (\mathcal{U}_1^{m}, \mathcal{N}_{yy2}^{m})] \qquad (A-189)$$

$$\eta_{2i\eta}^{2} = m_{xx2i,xx}^{nL} + m_{xx2i,xx}^{nn} - 2(\frac{i\eta}{R})(m_{xy1i,x}^{nL} + m_{xy2i,x}^{nn})$$

$$- (\frac{i\eta}{R})^2 (m_{yy2i}^{nL} + m_{yy2i}^{nn}) - \frac{1}{R} (\mathcal{N}_{yy2}^{nL} + \mathcal{N}_{yy2}^{nn})$$

$$+ (\frac{\eta}{R})[A_{J2(k)}^{i} (W_{2}^{m+1}, \mathcal{N}_{xy2,x}^{m}) + A_{J2(k)}^{i} (W_{2}^{m}, \mathcal{N}_{xy2,x}^{m+1}) - A_{J2(k)}^{i} (W_{2}^{m}, \mathcal{N}_{xy2,x}^{m})$$

$$- A_{J3(k)}^{i} (W_{1}^{m+1}, \mathcal{N}_{xy2,x}^{m}) - A_{J3(k)}^{i} (W_{1}^{m}, \mathcal{N}_{xy2,x}^{m+1}) + A_{J3(k)}^{i} (W_{1}^{m}, \mathcal{N}_{xy2,x}^{m})]$$

$$+ (\frac{\eta}{R})^2 [- A_{IJ2(k)}^{i} (W_{2}^{m+1}, \mathcal{N}_{yy1}^{m}) - A_{IJ2(k)}^{i} (W_{2}^{m}, \mathcal{N}_{yy1}^{m+1}) + A_{IJ2(k)}^{i} (W_{2}^{m}, \mathcal{N}_{yy1}^{m})$$

$$- A_{IJ3(k)}^{i} (W_{1}^{m+1}, \mathcal{N}_{yy2}^{m}) - A_{IJ3(k)}^{i} (W_{1}^{m}, \mathcal{N}_{yy2}^{m+1}) + A_{IJ3(k)}^{i} (W_{1}^{m}, \mathcal{N}_{yy2}^{m})]$$

$$+ \frac{\delta_i}{R} [-A_{2(k)}^{i} (\mathcal{U}_1^{m+1}, \mathcal{N}_{xy2,x}^{m}) - A_{2(k)}^{i} (\mathcal{U}_1^{m}, \mathcal{N}_{xy2,x}^{m+1}) + A_{2(k)}^{i} (\mathcal{U}_1^{m}, \mathcal{N}_{xy2,x}^{m})$$

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$$- A_{3(k)}^i (U_2^{m+1}, \mathcal{N}_{xy1,x}^m) - A_{3(k)}^i (U_2^m, \mathcal{N}_{xy1,x}^{m+1}) + A_{3(k)}^i (U_2^m, \mathcal{N}_{xy1,x}^m)]$$

$$+ \frac{\delta_1 \eta}{R} [A_{I2(k)}^i (U_1^{m+1}, \mathcal{N}_{yy1}^m) + A_{I2(k)}^i (U_1^m, \mathcal{N}_{yy1}^{m+1}) - A_{I2(k)}^i (U_1^m, \mathcal{N}_{yy1}^m)$$

$$+ A_{I3(k)}^i (U_2^{m+1}, \mathcal{N}_{yy2}^m) + A_{I3(k)}^i (U_2^m, \mathcal{N}_{yy2}^{m+1}) - A_{I3(k)}^i (U_2^m, \mathcal{N}_{yy2}^m)]$$

$$+ A_{2(k)}^i (W_{1,xx}^{m+1}, \mathcal{N}_{xx2}^m) + A_{2(k)}^i (W_{1,xx}^m, \mathcal{N}_{xx2}^{m+1}) - A_{2(k)}^i (W_{1,xx}^m, \mathcal{N}_{xx2}^m)$$

$$+ A_{3(k)}^i (W_{2,xx}^{m+1}, \mathcal{N}_{xx1}^m) + A_{3(k)}^i (W_{2,xx}^m, \mathcal{N}_{xx1}^{m+1}) - A_{3(k)}^i (W_{2,xx}^m, \mathcal{N}_{xx1}^m)$$

$$+ \frac{2\eta}{R} [A_{J2(k)}^i (W_{2,x}^{m+1}, \mathcal{N}_{xy1}^m) + A_{J2(k)}^i (W_{2,x}^m, \mathcal{N}_{xy1}^{m+1}) - A_{J2(k)}^i (W_{2,x}^m, \mathcal{N}_{xy}^m)$$

$$- A_{J3(k)}^i (W_{1,x}^{m+1}, \mathcal{N}_{xy1}^m) - A_{J3(k)}^i (W_{1,x}^m, \mathcal{N}_{xy1}^{m+1}) + A_{J3(k)}^i (W_{1,x}^m, \mathcal{N}_{xy1}^m)]$$

$$+ \frac{\delta_1}{R} [- A_{2(k)}^i (U_{1,x}^{m+1}, \mathcal{N}_{xy2}^m) - A_{2(k)}^i (U_{1,x}^m, \mathcal{N}_{xy2}^{m+1}) + A_{2(k)}^i (U_{1,x}^m, \mathcal{N}_{xy2}^m)$$

$$- A_{3(k)}^i (U_{2,x}^{m+1}, \mathcal{N}_{xy1}^m) - A_{3(k)}^i (U_{2,x}^m, \mathcal{N}_{xy1}^{m+1}) + A_{3(k)}^i (U_{2,x}^m, \mathcal{N}_{xy1}^m)]$$

$$+ (\frac{\eta}{R})^2 [- A_{J22(k)}^i (W_1^{m+1}, \mathcal{N}_{yy2}^m) - A_{J22(k)}^i (W_1^m, \mathcal{N}_{yy2}^{m+1}) + A_{J22(k)}^i (W_1^m, \mathcal{N}_{yy2}^m)$$

$$- A_{J23(k)}^i (W_2^{m+1}, \mathcal{N}_{yy1}^m) - A_{J23(k)}^i (W_2^m, \mathcal{N}_{yy1}^{m+1}) + A_{J23(k)}^i (W_2^m, \mathcal{N}_{yy1}^m)]$$

$$+ \frac{\delta_1 \eta}{R} [- A_{J2(k)}^i (U_2^{m+1}, \mathcal{N}_{yy2}^m) - A_{J2(k)}^i (U_2^m, \mathcal{N}_{yy2}^{m+1}) + A_{J2(k)}^i (U_2^m, \mathcal{N}_{yy2}^m)$$

$$+ A_{J3(k)}^i (U_1^{m+1}, \mathcal{N}_{yy1}^m) + A_{J3(k)}^i (U_1^m, \mathcal{N}_{yy1}^{m+1}) - A_{J3(k)}^i (U_1^m, \mathcal{N}_{yy1}^m)] \quad (A\text{-}190)$$

$$\eta_{1,x}^3 = \frac{\eta}{R} [A_{J1(2k)}^i (W_2^m, \mathcal{N}_{xy1,x}^{nm}) + A_{J1(2k)}^i (W_2^o, \mathcal{N}_{xy1,x}^{nL}) + A_{J1(2k)}^i (W_2^o, \mathcal{N}_{xy1,x}^{nn})$$

$$- A_{J4(2k)}^i (W_1^m, \mathcal{N}_{xy2,x}^{nm}) - A_{J4(2k)}^i (W_1^o, \mathcal{N}_{xy2,x}^i) - A_{J4(2k)}^i (W_1^o, \mathcal{N}_{xy2,x}^{nn})$$

$$- 2 A_{J4(2k)}^i (W_{1,x}^m, \mathcal{N}_{xy2}^{nm}) - 2 A_{J4(2k)}^i (W_{1,x}^o, \mathcal{N}_{xy2}^{nL}) - 2 A_{J4(2k)}^i (W_{1,x}^o, \mathcal{N}_{xy2}^{nn})$$

$$+ A_{J1(k)}^i (W_2^{m+1}, \mathcal{N}_{xy1,x}^{Lm}) + A_{J1(k)}^i (W_2^m, \mathcal{N}_{xy1,x}^{Lm+1}) - A_{J1(k)}^i (W_2^m, \mathcal{N}_{xy1,x}^{Lm})$$

$$- A_{J4(k)}^i (W_1^{m+1}, \mathcal{N}_{xy2,x}^{Lm}) - A_{J4(k)}^i (W_1^m, \mathcal{N}_{xy2,x}^{Lm+1}) + A_{J4(k)}^i (W_1^m, \mathcal{N}_{xy2,x}^{Lm})$$

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$$+ 2 A^i_{J1(K)} (W^{m+1}_{2,x}, n^{LM}_{xy1}) + 2 A^i_{J1(K)} (W^m_{2,x}, n^{LM+1}_{xy1}) - 2 A^i_{J1(K)} (W^m_{2,x}, n^{LM}_{xy1})$$

$$- 2 A^i_{J4(2K)} (W^{m+1}_{1,x}, n^{LM}_{xy2}) - 2 A^i_{J4(2K)} (W^m_{1,x}, n^{LM+1}_{xy2}) + 2 A^i_{J4(2K)} (W^m_{1,x}, n^{LM}_{xy2})]$$

$$+ (\tfrac{n}{R})^2 [A^i_{IJ1(2K)} (W^m_2, n^{nm}_{yy2}) + A^i_{IJ1(2K)} (W^0_2, n^{nL}_{yy2}) + A^i_{IJ1(2K)} (W^0_2, n^{nn}_{yy2})$$

$$+ A^i_{IJ4(2K)} (W^m_1, n^{nm}_{yy1}) + A^i_{IJ4(2K)} (W^0_1, n^{nL}_{yy1}) + A^i_{IJ4(2K)} (W^0_1, n^{nn}_{yy1})$$

$$- A^i_{J21(2K)} (W^m_1, n^{nm}_{yy1}) - A^i_{J21(2K)} (W^0_1, n^{nL}_{yy1}) - A^i_{J21(2K)} (W^0_1, n^{nn}_{yy1})$$

$$- A^i_{J24(2K)} (W^m_2, n^{nm}_{yy2}) - A^i_{J24(2K)} (W^0_2, n^{nL}_{yy2}) - A^i_{J24(2K)} (W^0_2, n^{nn}_{yy2})$$

$$+ A^i_{IJ1(2K)} (W^{m+1}_2, n^{LM}_{yy2}) + A^i_{IJ1(2K)} (W^m_2, n^{LM+1}_{yy2}) - A^i_{IJ1(2K)} (W^m_2, n^{LM}_{yy2})$$

$$+ A^i_{IJ4(2K)} (W^{m+1}_1, n^{LM}_{yy1}) + A^i_{IJ4(2K)} (W^m_1, n^{LM+1}_{yy1}) - A^i_{IJ4(2K)} (W^m_1, n^{LM}_{yy1})$$

$$- A^i_{J21(2K)} (W^{m+1}_1, n^{LM}_{yy1}) - A^i_{J21(2K)} (W^m_1, n^{LM+1}_{yy1}) + A^i_{J21(2K)} (W^m_1, n^{LM}_{yy1})$$

$$+ A^i_{J24(2K)} (W^{m+1}_0, n^{LM}_{yy}) - A^i_{J24(2K)} (W^m_2, n^{LM+1}_{yy}) + A^i_{J24(2K)} (W^m_2, n^{LM}_{yy})]$$

$$+ \tfrac{\delta_1}{R} [- A^i_{1(2K)} (U^m_1, n^{nm}_{xy1,x}) - A^i_{4(2K)} (U^m_2, n^{nm}_{xy2,x}) - A^i_{1(2K)} (U_{1,x}, n^n_{xy1})$$

$$- A^i_{4(2K)} (U^m_{2,x}, n^{nm}_{xy2}) - A^i_{1(2K)} (U^{m+1}_1, n^{LM}_{xy1,x}) - A^i_{1(2K)} (U^m_1, n^{LM+1}_{xy1,x})$$

$$+ A^i_{1(2K)} (U^m_1, n^{LM}_{xy1,x}) - A^i_{4(2K)} (U^{m+1}_{2,x}, n^{LM}_{xy2}) - A^i_{4(2K)} (U^m_{2,x}, n^{LM}_{xy2})$$

$$+ A^i_{4(2K)} (U^m_2, n^{LM}_{xy2}) - A^i_{1(2K)} (U^{m+1}_{1,x}, n^{LM}_{xy1}) - A^i_{1(2K)} (U^m_{1,x}, n^{LM+1}_{xy1})$$

$$+ A^i_{1(2K)} (U^m_{1,x}, n^{LM}_{xy1}) - A^i_{4(2K)} (U^{m+1}_{2,x}, n^{LM}_{xy2}) - A^i_{4(2K)} (U^m_{2,x}, n^{LM+1}_{xy2})$$

$$+ A^i_{4(2K)} (U^m_{2,x}, n^{LM}_{xy2})]$$

$$+ \tfrac{\delta_1 n}{R^3} [- A^i_{I1(2K)} (U^m_1, n^{nm}_{yy2}) + A^i_{I4(2K)} (U^m_2, n^{nm}_{yy1}) - A^i_{J1(2K)} (U^m_2, n^{nm}_{yy1})$$

$$+ A^i_{J4(2K)} (U^m_1, n^{nm}_{yy2}) - A^i_{I1(2K)} (U^{m+1}_1, n^{LM}_{yy2}) - A^i_{I1(2K)} (U^m_1, n^{LM+1}_{yy2})$$

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$$+ A^i_{I1(2k)}(U^m_1, \eta^{Lm}_{yy2}) + A^i_{I4(2k)}(U^{m+1}_2, \eta^{Lm}_{yy1}) + A^i_{I4(2k)}(U^m_2, \eta^{Lm+1}_{yy1})$$

$$+ A^i_{I4(2k)}(U^m_2, \eta^{Lm}_{yy1}) - A^i_{J1(2k)}(U^{m+1}_2, \eta^{Lm}_{yy1}) - A^i_{J1(2k)}(U^m_2, \eta^{Lm+1}_{yy1})$$

$$+ A^i_{J1(2k)}(U^m_2, \eta^{Lm}_{yy1}) + A^i_{J4(2k)}(U^{m+1}_1, \eta^{Lm}_{yy2}) + A^i_{J4(2k)}(U^m_1, \eta^{Lm+1}_{yy2})$$

$$+ A^i_{J4(2k)}(U^m_1, \eta^{Lm}_{yy2})]$$

$$+ A^i_{1(2k)}(W^m_{1,xx}, \eta^{nm}_{xx1}) + A^i_{1(2k)}(W^0_{1,xx}, \eta^{nL}_{xx1}) + A^i_{1(2k)}(W^0_{1,xx}, \eta^{nn}_{xx1})$$

$$+ A^i_{4(2k)}(W^m_{2,xx}, \eta^{nm}_{xx2}) + A^i_{4(2k)}(W^0_{2,xx}, \eta^{nL}_{xx2}) + A^i_{4(2k)}(W^0_{2,xx}, \eta^{nn}_{xx2})$$

$$+ A^i_{1(2k)}(W^{m+1}_{1,xy}, \eta^{Lm}_{xx1}) + A^i_{1(2k)}(W^m_{1,xx}, \eta^{Lm+1}_{xx1}) - A^i_{1(2k)}(W^m_{1,xx}, \eta^{Lm}_{xx1})$$

$$+ A^i_{4(2k)}(W^{m+1}_{2,xx}, \eta^{Lm}_{xx2}) + A^i_{4(2k)}(W^m_{2,xx}, \eta^{Lm+1}_{xx2}) - A^i_{4(2k)}(W^m_{2,xx}, \eta^{Lm}_{xx2}) \qquad (A\text{-}191)$$

$$\eta^3_{2in} = \frac{\eta}{R}\Big[A^i_{J2(2k)}(W^m_2, \eta^{nm}_{xy2,x}) + A^i_{J2(2k)}(W^0_2, \eta^{nL}_{xy2,x}) + A^i_{J2(2k)}(W^0_2, \eta^{nn}_{xy2,x})$$

$$- A^i_{J3(2k)}(W^m_1, \eta^{nm}_{xy1,x}) - A^i_{J3(2k)}(W^0_1, \eta^{nL}_{xy1,x}) - A^i_{J3(2k)}(W^0_1, \eta^{nn}_{xy1,x})$$

$$+ 2 A^i_{J2(2k)}(W^m_{2,x}, \eta^{nm}_{xy2}) + 2 A^i_{J2(2k)}(W^0_{2,x}, \eta^{nL}_{xy2}) + 2 A^i_{J2(2k)}(W^0_{2,x}, \eta^{nn}_{xy2})$$

$$- 2 A^i_{J3(2k)}(W^m_{1,x}, \eta^{nm}_{xy1}) - 2 A^i_{J3(2k)}(W^0_{1,x}, \eta^{nL}_{xy1}) - 2 A^i_{J3(2k)}(W^0_{1,x}, \eta^{nn}_{xy1})$$

$$+ A^i_{J2(2k)}(W^{m+1}_2, \eta^{Lm}_{xy2,x}) + A^i_{J2(2k)}(W^m_2, \eta^{Lm+1}_{xy2,x}) - A^i_{J2(2k)}(W^m_2, \eta^{Lm}_{xy2,x})$$

$$- A^i_{J3(k)}(N^{m+1}_1, \eta^{Lm}_{xy1,x}) - A^i_{J3(2k)}(W^m_1, \eta^{Lm+1}_{xy1,x}) + A^i_{J3(2k)}(W^m_1, \eta^{Lm}_{xy2,x})$$

$$+ 2 A^i_{J2(2k)}(W^{m+1}_{2,x}, \eta^{Lm}_{xy2}) + 2 A^i_{J2(2k)}(W^m_{2,x}, \eta^{Lm+1}_{xy2}) - 2 A^i_{J2(2k)}(W^m_{2,x}, \eta^{Lm}_{xy2})$$

$$- 2 A^i_{J3(k)}(W^{m+1}_{1,x}, \eta^{Lm}_{xy1}) - 2 A^i_{J3(k)}(W^m_{1,x}, \eta^{Lm+1}_{xy1}) + 2 A^i_{J3(k)}(W^m_{1,x}, \eta^{Lm}_{xy1})\Big]$$

$$+ \Big(\frac{\eta}{R}\Big)^2 \Big[- A^i_{IJ2(2k)}(W^m_2, \eta^{nm}_{yy1}) - A^i_{IJ2(2k)}(W^0_2, \eta^{nL}_{yy1}) - A^i_{IJ2(2k)}(W^0_2, \eta^{nn}_{yy})$$

$$- A^i_{IJ3(2k)}(W^m_1, \eta^{nm}_{yy2}) - A^i_{IJ3(2k)}(W^0_1, \eta^{nL}_{yy2}) - A^i_{IJ3(2k)}(W^0_1, \eta^{nn}_{yy2})$$

$$-A^i_{J22(2k)}(W_1^m, \eta^{nm}_{yy2}) - A^i_{J22(2k)}(W_1^0, \eta^{nL}_{yy2}) - A^i_{J22(2k)}(W_1^0, \eta^{nn}_{yy2})$$

$$-A^i_{J23(2k)}(W_2^m, \eta^{nm}_{yy1}) - A^i_{J23(2k)}(W_2^0, \eta^{nL}_{yy1}) - A^i_{J23(2k)}(W_2^0, \eta^{nn}_{yy1})$$

$$-A^i_{IJ2(2k)}(W_2^{m+l}, \eta^{Lm}_{yy1}) - A^i_{IJ3(2k)}(W_2^m, \eta^{Lm+l}_{yy1}) - A^i_{IJ2(2k)}(W_2^m, \eta^{Lm}_{yy1})$$

$$-A^i_{IJ3(2k)}(W_1^{m+l}, \eta^{Lm}_{yy2}) - A^i_{IJ3(2k)}(W_1^m, \eta^{Lm+l}_{yy2}) + A^i_{IJ3(2k)}(W_1^m, \eta^{Lm}_{yy2})$$

$$-A^i_{J22(2k)}(W_1^{m+l}, \eta^{Lm}_{yy2}) - A^i_{J22(2k)}(W_1^m, \eta^{Lm+l}_{yy2}) + A^i_{J22(2k)}(W_1^m, \eta^{Lm}_{yy2})$$

$$-A^i_{J23(2k)}(W_2^{m+l}, \eta^{Lm}_{yy1}) - A^i_{J23(2k)}(W_2^m, \eta^{Lm+l}_{yy1}) + A^i_{J23(2k)}(W_2^m, \eta^{Lm}_{yy1})]$$

$$+\frac{\delta_1}{R}[-A^i_{2(2k)}(\upsilon_1^m, \eta^{nm}_{xy2,x}) - A^i_{2(2k)}(\upsilon_1^{m+l}, \eta^{Lm}_{xy2,x}) - A^i_{2(2k)}(\upsilon_1^m, \eta^{Lm+l}_{xy2,x})$$

$$+A^i_{2(2k)}(\upsilon_1^m, \eta^{Lm}_{xy2,x}) - A^i_{3(2k)}(\upsilon_2^m, \eta^{nm}_{xy1,x}) - A^i_{3(2k)}(\upsilon_2^{m+l}, \eta^{Lm}_{xy1,x})$$

$$-A^i_{3(2k)}(\upsilon_2^m, \eta^{Lm+l}_{xy1,x}) + A^i_{3(2k)}(\upsilon_2^m, \eta^{Lm}_{xy1,x})$$

$$-A^i_{2(2k)}(\upsilon_{1,x}^m, \eta^{nm}_{xy2}) - A^i_{2(2k)}(\upsilon_{1,x}^{m+l}, \eta^{Lm}_{xy2}) - A^i_{2(2k)}(\upsilon_{1,x}^m, \eta^{Lm+l}_{xy2})$$

$$+A^i_{2(2k)}(\upsilon_{1,x}^m, \eta^{nm}_{xy2}) - A^i_{3(2k)}(\upsilon_{2,x}^m, \eta^{nm}_{xy1}) - A^i_{3(2k)}(\upsilon_{2,x}^{m+l}, \eta^{Lm}_{xy1})$$

$$-A^i_{3(2k)}(\upsilon_{2,x}^m, \eta^{Lm+l}_{xy1}) + A^i_{3(2k)}(\upsilon_{2,x}^m, \eta^{Lm}_{xy1})]$$

$$+\frac{\delta_1 \eta}{R}[A^i_{I2(2k)}(\upsilon_1^m, \eta^{nm}_{yy1}) + A^i_{I2(2k)}(\upsilon_1^{m+l}, \eta^{Lm}_{yy1}) + A^i_{I2(2k)}(\upsilon_1^m, \eta^{Lm+l}_{yy1})$$

$$-A^i_{I2(2k)}(\upsilon_1^m, \eta^{Lm}_{yy1}) - A^i_{I3(2k)}(\upsilon_2^m, \eta^{nm}_{yy2}) - A^i_{I3(2k)}(\upsilon_2^{m+l}, \eta^{Lm}_{yy2})$$

$$-A^i_{I3(2k)}(\upsilon_2^m, \eta^{Lm+l}_{yy2}) + A^i_{I3(2k)}(\upsilon_2^m, \eta^{Lm}_{yy2})]$$

$$+A^i_{2(2k)}(W_{1,xx}^m, \eta^{nm}_{xx2}) + A^i_{2(2k)}(W_{1,xx}^0, \eta^{nL}_{xx2}) + A^i_{2(2k)}(W_{1,xx}^0, \eta^{nn}_{xx2})$$

$$+A^i_{3(2k)}(W_{2,xx}^m, \eta^{nm}_{xx1}) + A^i_{3(2k)}(W_{2,xx}^0, \eta^{nL}_{xx1}) + A^i_{3(2k)}(W_{2,xx}^0, \eta^{nn}_{xx1})$$

$$+A^i_{2(2k)}(W_{1,xx}^m, \eta^{Lm}_{xx2}) + A^i_{2(2k)}(W_{1,xx}^m, \eta^{Lm+l}_{xx2}) - A^i_{2(2k)}(W_{1,xx}^m, \eta^{Lm}_{xx2})$$

$$+A^i_{3(2k)}(W_{2,xx}^{m+l}, \eta^{Lm}_{xx1}) + A^i_{3(2k)}(W_{2,xx}^m, \eta^{Lm+l}_{xx1}) - A^i_{3(2k)}(W_{2,xx}^m, \eta^{Lm}_{xx1}) \quad (A-192)$$

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After linearization the equilibrium equations, Eqs A-160, can be written in matrix form

$$[C_{12}]\{n_{1,xx}\} + [C_{11}]\{n_{1,x}\} + [C_{10}]\{n_1\} + [E_{12}]\{n_{1,xx}^L\} + [E_{11}]\{n_{1,x}^L\}$$

$$+ [E_{10}]\{n_{1,x}^L\} + [B_{12}]\{n_{1,xx}^{nL}\} + [B_{11}]\{n_{1,x}^{nL}\} + [B_{10}]\{n_1^{nL}\} + [C_{22}]\{n_{2,xx}\}$$

$$+ [C_{21}]\{n_{2,x}\} + [C_{20}]\{n_2\} + [E_{22}]\{n_{2,xx}^L\} + [E_{21}]\{n_{2,x}^L\} + [E_{10}]\{n_{1,x}^L\}$$

$$+ [B_{22}]\{n_{1,xx}^{nL}\} + [B_{21}]\{n_{1,x}^{nL}\} + [B_{20}]\{n_1^{nL}\} + [A_{12}]\{X_{,xx}\} + [A_{11}]\{X_{,x}\}$$

$$+ [A_{10}]\{X\} = \{g\} \tag{A-193}$$

where

$$\{n_1\}^T = \lfloor n_{xx1i}, n_{yy1i}, n_{xy1i}, m_{xx1i}, m_{yy1i}, m_{xy1i}\rfloor^{m+1}$$

$$\{n_2\}^T = \lfloor n_{xx2i}, n_{yy2i}, n_{xy2i}, m_{xx2i}, m_{yy2i}, m_{xy2i}\rfloor^{m+1}$$

$$\{X\}^T = \lfloor u_{1i}, v_{1i}, w_{1i}, u_{2i}, v_{2i}, w_{2i}\rfloor^{m+1}$$

$$\{n_1^L\}^T = \lfloor n_{xx1i}^L, n_{yy1i}^L, n_{xy1i}^L, m_{xx1i}^L, m_{yy1i}^L, m_{xy1i}^L\rfloor^{m+1}$$

$$\{n_2^L\}^T = \lfloor n_{xx2i}^L, n_{yy2i}^L, n_{xy2i}^L, m_{xx2i}^L, m_{yy2i}^L, m_{xy2i}^L\rfloor^{m+1}$$

$$\{n_1^{nL}\}^T = \lfloor n_{xx1i}^{nL}, n_{yy1i}^{nL}, n_{xy1i}^{nL}, m_{xx1i}^{nL}, m_{yy1i}^{nL}, m_{xy1i}^{nL}\rfloor^{m+1}$$

$$\{n_2^{nL}\}^T = \lfloor n_{xx2i}^{nL}, n_{yy2i}^{nL}, n_{xy2i}^{nL}, m_{xx2i}^{nL}, m_{yy2i}^{nL}, m_{xy2i}^{nL}\rfloor^{m+1}$$

In Eqs A-145 A-150 $\{\epsilon\}$, $\{t^1\}$ and $\{t^n\}$ can be written as:

$$\{\epsilon_{1i}\} = [k1_1]\{X_{,x}\} + [k1_0]\{X\}$$

$$\{\epsilon_{2i}\} = [k2_1]\{X_{,x}\} + [k2_0]\{X\}$$

$$\{t_1^L\} = [k1_1^L]\{X_{,x}\} + [k1_0^L]\{X\}$$

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$$\{t_2^L\} = [K2_1^L]\{X_{,x}\} + [K2_o^L]\{X\}$$

$$\{t_1^n\} = [K1_1^n]\{X_{,x}\} + [K2_o^n]\{X\}$$

$$\{t_{1,x}^n\} = [K1_{2x}^n]\{X_{,xx}\} + [K1_{1x}^n]\{X_{,x}\} + [K1_{ox}^n]\{X\}$$

$$\{t_{2,x}^n\} = [K2_{2x}^n]\{X_{,xx}\} + [K2_{1x}^n]\{X_{,x}\} + [K2_{ox}^n]\{X\}$$

$$\{t_{1,xx}^n\} = [K1_{3xx}^n]\{X_{,xxx}\} + [K1_{2xx}^n]\{X_{,xx}\} + [K1_{1xx}^n]\{X_{,x}\} + [K1_{oxx}^n]\{X\}$$

$$\{t_{2,xx}^n\} = [K2_{3xx}^n]\{X_{,xxx}\} + [K2_{2xx}^n]\{X_{,xx}\} + [K2_{1xx}^n]\{X_{,x}\} + [K1_{oxx}^n]\{X\} \quad (A\text{-}194)$$

Substitute of Eqs A-145--A-150, and A-194 into Eq A-193 yields a matrix equation which only contain the vector of unknown, $\{x\}$

$$[R4]\{X_{,xxx}\} + [R3]\{X_{,xxx}\} + [R2]\{X_{,xx}\} + [R1]\{X_{,x}\} + [R0]\{X\} = \{g\} \quad (A\text{-}195)$$

As in the case of W-F formulation transformation equation are introduced in order to reduce the order of the linear equations.

$$\{\eta\} = \{X_{,xx}\}$$

By this transformation, Eq A-195 can be written in the following form:

$$[R]\begin{Bmatrix} X_{,xx} \\ \eta_{,xx} \end{Bmatrix} + [S]\begin{Bmatrix} X_{,x} \\ \eta_{,x} \end{Bmatrix} + [T]\begin{Bmatrix} X \\ \eta \end{Bmatrix} = \{G\} \quad (A\text{-}196)$$

A.3.5 Boundary Condition

Boundary condition A-117 can be presented in the following form

Either   Or

$$N_{xx} = \bar{N}_{xx} \qquad\qquad U = Const.$$

$$N_{xy}^* = \bar{N}_{xy}^* \qquad\qquad V = Const.$$

$$Q^* = \bar{Q} + \bar{M}_{xy,y} \qquad\qquad W = 0$$

$$M_{xx} = \bar{M}_{xx} \qquad\qquad W_{,x} = 0 \qquad (A\text{-}117a)$$

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where

$$N_{xy}^* = N_{xy} + \delta \frac{M_{xy}}{R}$$

$$Q^* = N_{xx}(W_{,x} + W_{,x}^\circ) + N_{xy}(W_{,y} + W_{,y}^\circ) - \frac{N_{xy}}{R} U \delta_1$$

$$+ M_{xx,x} + 2M_{xy,y} \tag{A-197}$$

Obviously, the boundary condition can be written in matrix form (at x = 0, L)

$$[\Omega1]\begin{Bmatrix} N_{xy} \\ N_{xy}^* \\ Q^* \\ M_{xx} \end{Bmatrix} + [\lambda1]\begin{Bmatrix} U \\ V \\ W \\ W_{,x} \end{Bmatrix} = \{Bg\} \tag{A-198}$$

where the form of $[\Omega1]$ and $[\lambda1]$ depends on the type of boundary conditions.
The stress and moment results, and the displacements are represented in series form.

$$[\Omega1]\begin{Bmatrix} \sum_{i=0}^{2K}[\bar{n}_{xx}^1 \cos\frac{iny}{R} + \bar{n}_{xx}^2 \sin\frac{iny}{R}] \\ \sum_{i=0}^{2K}[\bar{n}_{xy}^{*1} \cos\frac{iny}{R} + \bar{n}_{xy}^{*2} \sin\frac{iny}{R}] \\ \sum_{i=0}^{2K}[Q^{*1} \cos\frac{iny}{R} + Q^{*2} \sin\frac{iny}{R}] \\ \sum_{i=0}^{2K}[\bar{m}_{xx}^1 \cos\frac{iny}{R} + \bar{m}_{xx}^2 \sin\frac{iny}{R}] \end{Bmatrix} + [\lambda1]\begin{Bmatrix} \sum_{i=0}^{K}[U_{1i}\cos\frac{iny}{R} + U_{2i}\sin\frac{iny}{R}] \\ \sum_{i=0}^{K}[V_{1i}\cos\frac{iny}{R} + V_{2i}\sin\frac{iny}{R}] \\ \sum_{i=0}^{K}[W_{1i}\cos\frac{iny}{R} + W_{2i}\sin\frac{iny}{R}] \\ \sum_{i=0}^{K}[W_{1i,x}\cos\frac{iny}{R} + W_{2i,x}\sin\frac{iny}{R}] \end{Bmatrix}$$

$$= \{Bg\} \tag{A-199}$$

After applying the Galerkin Procedure, the boundary conditions can be written as:

$$
[\Omega]
\begin{Bmatrix}
\bar{n}_{xxi}^{1} \\
\bar{n}_{xyi}^{*1} \\
\bar{Q}_{i}^{*1} \\
\bar{m}_{xxi}^{1} \\
\bar{n}_{xxi}^{2} \\
\bar{n}_{xyi}^{*2} \\
\bar{Q}^{*2} \\
\bar{m}_{xxi}^{1}
\end{Bmatrix}
+ [\lambda]
\begin{Bmatrix}
U_{1i} \\
V_{1i} \\
W_{1i} \\
W_{1i,x} \\
U_{2i} \\
V_{2i} \\
W_{2i} \\
W_{2i,x}
\end{Bmatrix}
= \{e\}
\qquad i = 0, \cdots , k
\tag{A-200}
$$

where

$$\bar{n}_{xxi}^{1} = n_{xxii} + n_{xxii}^{L} + n_{xxii}^{n}$$

$$\bar{n}_{xxi}^{2} = n_{xx2i} + n_{xx2i}^{L} + n_{xx2i}^{n}$$

$$\bar{n}_{xyi}^{*1} = n_{xy1i} + n_{xy1i}^{L} + n_{xy1i}^{n} + \frac{\delta_1}{R}\left(m_{xy1i} + m_{xy1i}^{L} + m_{xy1i}^{n}\right)$$

$$\bar{n}_{xyi}^{*2} = n_{xy2i} + n_{xy2i}^{L} + n_{xy2i}^{n} + \frac{\delta_1}{R}\left(m_{xy2i} + m_{xy2i}^{L} + m_{xy2i}^{n}\right)$$

$$\bar{Q}_{i}^{*1} = \bar{m}_{xx1,x}^{1} + 2\frac{in}{R}\bar{m}_{xy}^{2} + A_{1(2k)}^{i}(W_{1,x}+W_{1,x}^{o},\bar{n}_{xx}^{1}) + A_{4(2k)}^{i}(W_{2,x}+W_{2,x}^{o},\bar{n}_{xx}^{2})$$
$$+ \frac{n}{R}\left[A_{J1(2k)}^{i}(W_1+W_1^{o},\bar{n}_{xy}^{1}) - A_{J4(2k)}^{i}(W_1+W_1^{o},\bar{n}_{xy}^{2})\right]$$
$$- \frac{\delta_1}{R}\left[A_{1(2k)}^{i}(U_1,\bar{n}_{xy}^{1}) + A_{4(2k)}^{i}(U_2,\bar{n}_{xy}^{2})\right]$$

$$\bar{Q}_{i}^{*2} = \bar{m}_{xx2,x}^{2} - 2\frac{in}{R}\bar{m}_{xy}^{1} + A_{2(2k)}^{i}(W_{1,x}+W_{1,x}^{o},\bar{n}_{xx}^{2}) + A_{3(2k)}^{i}(W_{2,x}+W_{2,x}^{o},\bar{n}_{xx}^{1})$$
$$+ \frac{n}{R}\left[A_{J2(2k)}^{i}(W_2+W_2^{o}),\bar{n}_{xy}^{2}) - A_{J3(2k)}^{i}(W_1+W_1^{o},\bar{n}_{xy}^{1})\right]$$
$$- \frac{\delta_1}{R}\left[A_{2(2k)}^{i}(U_1,\bar{n}_{xy}^{2}) + A_{3(2k)}^{i}(U_2,\bar{n}_{xy}^{1})\right]$$

$$\bar{m}_{xxi}^{1} = m_{xx1i} + m_{xx1i}^{L} + m_{xx1i}^{n}$$

$$\bar{m}_{xxi}^{2} = m_{xx2i} + m_{xx2i}^{L} + m_{xx2i}^{n}
\tag{A-201}$$

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Using the similar procedure as used in sectionII, Eqs A-200 can be linearized
and written in matrix form:

$$[\Omega]\{N^\beta\} + [\lambda]\{X^\beta\} \doteq [\Omega](\{N_\delta\} + \{N_\delta^L\} + \{N_\delta^{NL}\} + \{N_\delta^{NN}\}) + [\lambda]\{X^\beta\}$$

$$= \{e\}$$

or
$$[\Omega](\{N_\delta\} + \{N_\delta^L\} + \{N_\delta^{NL}\}) + [\lambda]\{X^\beta\} = \{e\} - [\Omega]\{N_\delta^{NN}\} \qquad (A\text{-}202)$$

where

$$\{N_\delta\} = \begin{Bmatrix} \bar{n}_{xx}^{'1} \\ \bar{n}_{xy}^{'1} \\ Q_{xy}^{*1} \\ \bar{m}_{xy}^{1} \\ \bar{n}_{xx}^{'2} \\ \bar{n}_{xy}^{'2} \\ Q_{xx}^{*2} \\ \bar{m}_{xx}^{2} \end{Bmatrix}$$

Substituting of Eqs A-145-A-150, A-194 into Eqs A-202 yields the following form
for the boundary conditions

$$[DB]\begin{Bmatrix} \{X_\eta\} \\ \{\eta_\omega\} \end{Bmatrix} + [DC]\begin{Bmatrix} \{X\} \\ \{\eta\} \end{Bmatrix} = \begin{Bmatrix} \{e\} - [\Omega]\{N_\delta^{NN}\} \\ 0 \end{Bmatrix}$$

$$= \{BG\} \qquad (A\text{-}203)$$

A.3.6 Solution Methodology - Finite Difference Equations

The linearized iteration equations (equilibrium) assume the form

$$[R] \left\{ \begin{matrix} \{X_{,xx}\} \\ \{\eta_{,xx}\} \end{matrix} \right\} + [S] \left\{ \begin{matrix} \{X_{,x}\} \\ \{\eta_{,x}\} \end{matrix} \right\} + [T] \left\{ \begin{matrix} \{X\} \\ \{\eta_{,xx}\} \end{matrix} \right\} = \{G\} \qquad (A\text{-}196)$$

Note that the true number of unknown is $(6k + 3)$. These are u_{1i}, v_{1i}, w_{1i} $(i = 1,2...k)$ and u_{2i}, v_{2i}, w_{2i} $(i = 1,2...k)$ [see Eqs (119)]. For convenience though the number of unknown is treated as $(6k + 6)$ with u_{20}, v_{20} & w_{20} existing for the count, but subject to the constraint $u_{20} = v_{20} = w_{20} = 0$. Thus with the transformation, $\{\eta\} = \{X_{,xx}\}$, the number of unknowns is $(12k + 12)$.

The equilibrium equation, Eqs A-196, are next cast into finite difference form, by employing the usual central difference formula. Thus at each node point j, the equations become (in matrix form)

$$\left(\frac{1}{h^2}[R]^{(j)} + \frac{1}{2h}[S]^{(j)} \right) \left\{ \begin{matrix} \{X\} \\ \{\eta\} \end{matrix} \right\}^{(j+1)} + \left(-\frac{1}{2h^2}[R]^{(j)} + [T]^{(j)} \right) \left\{ \begin{matrix} \{X\} \\ \{\eta\} \end{matrix} \right\}^{(j)}$$

$$+ \left(\frac{1}{h^2}[R]^{(j)} - \frac{1}{2h}[S]^{(j)} \right) \left\{ \begin{matrix} \{X\} \\ \{\eta\} \end{matrix} \right\}^{(j-1)} = \{G\}^{(j)} \qquad (A\text{-}204)$$

At each end one fictitious point is used. This requires $(12k + 12)$ additional equations at each end $(j = 1$ and N; the fictitious points are denoted by $j = 0$ and $j = N + 1)$. These additional equations come from the boundary conditions.

Paradoxically, the number of boundary equations is $(8k + 8)$ at each end. Note that these are either natural $(8k + 8$ through the Galerkin (procedure) or kinematic $(8k + 8,$ $u_{1i} = u_{2i} = 0,$ $v_{1i} = v_{2i} = 0,$ $w_{1i} = w_{2i} = 0$ & $w_{1i,x} = w_{2i,x} = 0$ for $i = 0,1,2...k)$. This necessitates the requirement of $(4k + 4)$ additional conditions at each boundary.

The additional boundary terms are given below and they only involve $u_{1i,xx}$, $u_{2i,xx}$, $v_{1i,xx}$, $v_{21,xx}$ at each boundary. Their existence derivatives with respect to x of the displacement components u and v in the equilibrium equations. On the other hand, regardless of whether or not the boundary conditions are natural or kinematic, they do not contain second derivatives of u and v with respect to x.

$$\bar{\eta}_1 \, \mathcal{U}_{1i,xx}\binom{j=0 \atop or}{j=N+1} + \bar{\eta}_2 \, \mathcal{U}_{1i,xx}\binom{j=1 \atop or}{j=N} + \bar{\eta}_3 \, \mathcal{U}_{1i,xx}\binom{j=2 \atop or}{j=N-1} = 0$$

$$\bar{\eta}_1 \, \mathcal{V}_{1i,xx}\binom{j=0 \atop or}{j=N+1} + \bar{\eta}_2 \, \mathcal{V}_{1i,xx}\binom{j=1 \atop or}{j=N} + \bar{\eta}_3 \, \mathcal{V}_{1i,xx}\binom{j=2 \atop or}{j=N-1} = 0$$

$$\bar{\eta}_1 \, \mathcal{U}_{2i,xx}\binom{j=0 \atop or}{j=N+1} + \bar{\eta}_2 \, \mathcal{U}_{2i,xx}\binom{j=1 \atop or}{j=N} + \bar{\eta}_3 \, \mathcal{U}_{2i,xx}\binom{j=2 \atop or}{j=N-1} = 0$$

$$\bar{\eta}_1 \, \mathcal{V}_{2i,xx}\binom{j=0 \atop or}{j=N+1} + \bar{\eta}_2 \, \mathcal{V}_{2i,xx}\binom{j=1 \atop or}{j=N} + \bar{\eta}_3 \, \mathcal{V}_{2i,xx}\binom{j=2 \atop or}{j=N-1} = 0 \qquad (A-205)$$

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Where the constant $\bar{\eta}_1$, $\bar{\eta}_2$ and $\bar{\eta}_3$ are assigned to achieve certain goals (in generating some results $\bar{\eta}_1 = 1$, $\eta_2 = -2$ and $\bar{\eta}_3 = 1$ are used, which implies that a derivative at a boundary is obtained in a forward manner).

Note that Eqs A-205 are the additional (4k + 4) boundary terms and that these equations are incorporated in the matrix form shown in Eqs A-203. This means that [DB] and [DC] are square matrices, [(12k + 12) by (12k + 12)]. These boundary equations, Eqs A-203, are also cast into finite difference form.

$$\frac{1}{2h} [DB]^j \left\{ \begin{matrix} \{X\} \\ \{\eta\} \end{matrix} \right\}^{j+1} + [DC]^j \left\{ \begin{matrix} \{X\} \\ \{\eta\} \end{matrix} \right\}^{j} - \frac{1}{2h} [DB]^j \left\{ \begin{matrix} \{X\} \\ \{\eta\} \end{matrix} \right\}^{j-1} = \{BG\}^j \qquad (A\text{-}206)$$

where j = 1 or N.

A.3.7 Total Potential & End Shortening

The expression for the total potential for a supported (ss-i, cc-i) cylindrical shell is given by

$$U_T = \frac{1}{2} \int_0^{2\pi R} \int_0^L [N_{xx} \epsilon_{xx}^0 + N_{yy} \epsilon_{yy}^0 + N_{xy} \gamma_{xy}^0$$

$$+ M_{xx} k_{xx} + M_{yy} k_{yy} + 2 M_{xy} k_{xy}] \, dx \, dy$$

$$+ \int_0^{2\pi R} [-\bar{N}_{xx} U - \bar{N}_{xy} \upsilon + \bar{M}_{xx} W_{,x}] \Big|_0^L \, dy$$

$$- \int_0^{2\pi R} \int_0^L \bar{g} w \, dx \, dy \qquad\qquad (A\text{-}207)$$

or

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$$U_T = \frac{1}{2}\int_0^L \mathcal{N}'_{xx_0}\,\epsilon'_{xx_0} + \mathcal{N}'_{yy_0}\,\epsilon'_{yy_0} + \mathcal{N}'_{xy_0}\,\gamma'_{xy_0} + \mathcal{M}'_{xx_0}\,K'_{xx_0}$$

$$+ \mathcal{M}'_{yy_0}\,K'_{yy_0} + 2\mathcal{M}'_{xy_0}\,K'_{xy_0} + \frac{1}{2}\sum_{i=1}^{2K}\left(\mathcal{N}'_{xx_i}\,\epsilon'_{xx_i} + \mathcal{N}'_{yy_i}\,\epsilon'_{yy_i}\right.$$

$$+ \mathcal{N}'_{xy_i}\,\gamma'_{xy_i} + \mathcal{N}^2_{xx_i}\,\epsilon^2_{xx_i} + \mathcal{N}^2_{yy_i}\,\epsilon^2_{yy_i} + \mathcal{N}^2_{xy_i}\,\gamma^2_{xy_i}$$

$$+ \mathcal{M}'_{xx_i}\,K'_{xx_i} + \mathcal{M}'_{yy_i}\,K'_{yy_i} + 2\mathcal{M}'_{xy_i}\,K'_{xy_i} + \mathcal{M}^2_{xx_i}\,K^2_{xx_i}$$

$$+ \mathcal{M}^2_{yy_i}\,K^2_{yy_i} + 2\mathcal{M}^2_{xy_i}\,K^2_{xy_i}\Big)\Big]\,dX$$

$$+ 2\pi R\left(-\mathcal{N}'^{\ell}_{xx_0}\,U_0'^{\ell} + \mathcal{N}'^0_{xx_0}\,U_0'^0 - \mathcal{N}'^{\ell}_{xy}\,V_0'^{\ell} + \mathcal{N}'^{\ell}_{xy_0}\,V_0'^0\right.$$

$$+ \mathcal{M}'^{\ell}_{xx_0}\,W'^{\ell} - \mathcal{M}'^0_{xx_0}\,W_{0,x}'^0\Big) \qquad\qquad (A\text{-}208)$$

where

$$\begin{Bmatrix}\mathcal{N}'_{xx_i}\\ \mathcal{N}'_{yy_i}\\ \mathcal{N}'_{xy_i}\\ \mathcal{M}'_{xx_i}\\ \mathcal{M}'_{yy_i}\\ \mathcal{M}'_{xy_i}\end{Bmatrix} = \{\mathcal{N}_1\}_i + \{\mathcal{N}_1^L\}_i + \{\mathcal{N}_1^\pi\}_i$$

$$\begin{Bmatrix}\mathcal{N}^2_{xx_i}\\ \mathcal{N}^2_{yy_i}\\ \mathcal{N}^2_{xy_i}\\ \mathcal{M}^2_{xx_i}\\ \mathcal{M}^2_{yy_i}\\ \mathcal{M}^2_{xy_i}\end{Bmatrix} = \{\mathcal{N}_2\}_i + \{\mathcal{N}_2^L\}_i + \{\mathcal{N}_2^\pi\}_i$$

$$
\begin{Bmatrix}
\epsilon'_{xxi} \\
\epsilon'_{yyi} \\
\gamma'_{xyi} \\
K'_{xxi} \\
K'_{yyi} \\
2K'_{xyi}
\end{Bmatrix} = \{\epsilon_{1i}\} + \{t^{l}_{1i}\} + \{t^{\eta}_{1i}\}
$$

$$
\begin{Bmatrix}
\epsilon'_{xxi} \\
\epsilon^{2}_{yyi} \\
\gamma^{2}_{xyi} \\
K^{2}_{xxi} \\
K^{2}_{yyi} \\
2K^{2}_{xyi}
\end{Bmatrix} = \{\epsilon_{2i}\} + \{t^{l}_{2i}\} + \{t^{\eta}_{2i}\}
$$

and n^{1l}_{xxi}, $n^{1^{\ell}}_{xxi}$, m^{1l}_{xxi}, $u^{1^{\ell}}_{i}$, v^{1l}_{i}, w^{1l}_{1} are the values at $x = \ell$, n^{10}_{xxi}, n^{10}_{xxi}, m^{10}_{xxi}, u^{10}_{1}, v^{10}_{1}, w^{10}_{1} are the values at $x = 0$

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APPENDIX B

COMPUTER PROGRAM

B.1   w, F-Formulation

B.2   u, v, w-Formulation

Flow charts and program listing, for both formulations, will be made available upon request. (Write to Professor G. J. Simitses).

APPENDIX C

MODIFICATION AND GENERALIZATION
OF POTTER'S METHOD.

The behavior of several structural configurations is often fully described by a set of linear algebraic equations. In general, when these linear equations are put in matrix form, they can be partitioned as shown in Fig. B-1.

The blank spaces in the coefficient matrix are zeroes and $[C_i]$, $[B_i]$ and $[A_i]$ are matrices of orders m_i by m_{i-1}, m_i by m_i and m_i by m_{i+1} respectively Z_i is the vector of unknowns, each of order m_i by one and there are N such vectors. Let Z_L be the common unknown vector. Moreover, g_i is also a vector of order m_i by one and d_i is a vector or order m_i by one, which includes the coefficients of the common unknown.

Note that the presence of vectors d_i make the whole coefficient matrix nonbanding and irregular. If, on the other hand, the d_i-vectors do not exist then the coefficient matrix is identical to that of Ref C-1. In this case, the matrix is a banded tridiagonal matrix with zeroes everywhere and with, at most, three submatrices banded along the diagonal as shown on Fig. C-1. Therefore, the present case is a bit more general than that of Ref C-1. The solution procedure, though, is basically the same on that of Ref. C-1.

C.1 Description of the Algorithm

The explicit form of the system of linear equations of Fig. C.1 is given by

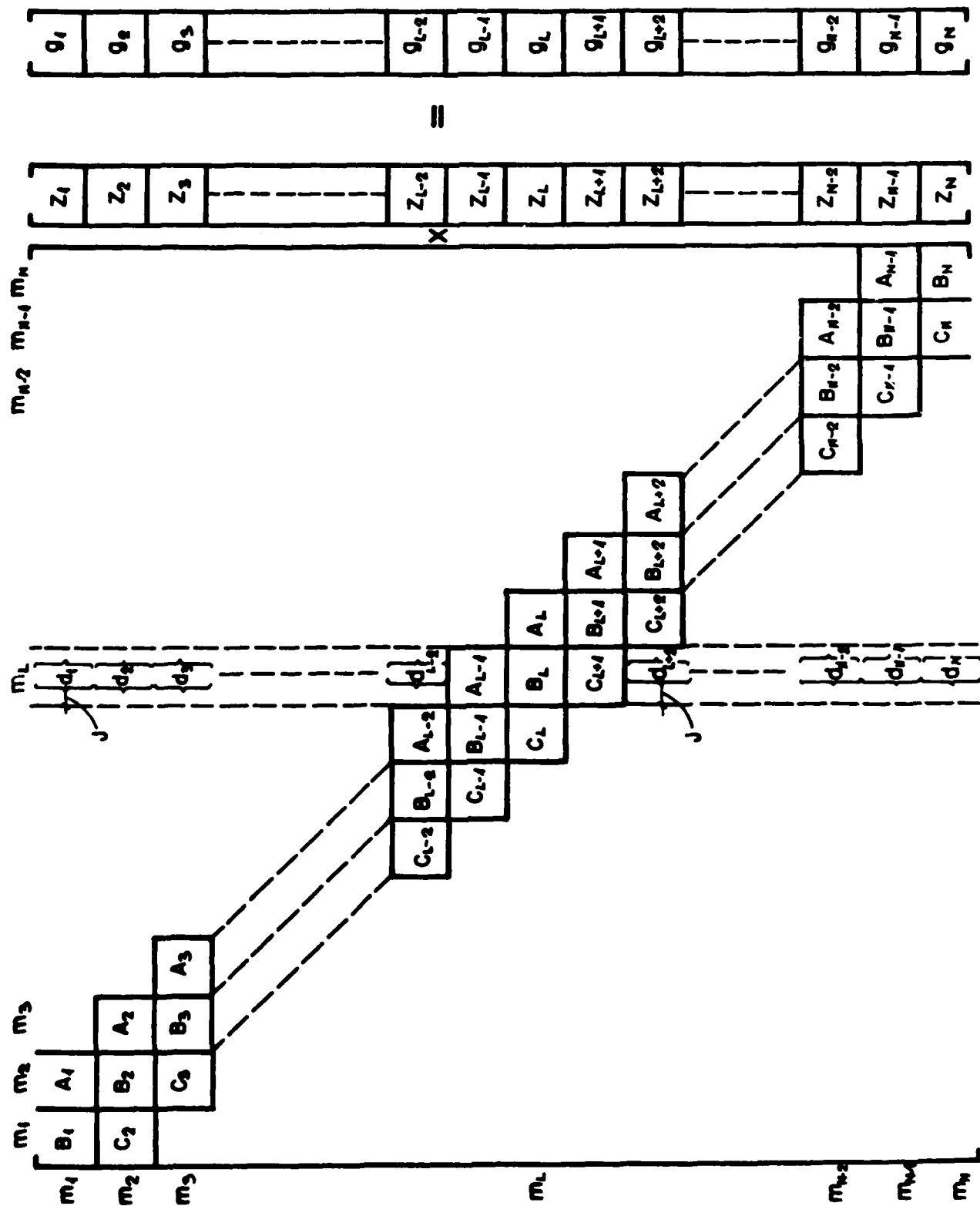$$[B_i]\{Z_i\} + [A_i]\{Z_j\} + \{d_i\} \, Z_L(j) = \{g_i\}$$

Fig. C.1 Matrix Equation

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$$[C_i]\{Z_{i-1}\} + [B_i]\{Z_i\} + [A_i]\{Z_{i+1}\} + \{d_i\}\, Z_L(j) = \{g_i\}$$

$$\text{with} \quad i = 2, 3, \cdots\cdots, N-1$$

$$[C_i]\{Z_{i-1}\} + [B_i]\{Z_i\} + [A_i]\{Z_{i+1}\} = \{g_i\}$$

$$\text{with} \quad i = L-1, L, L+1$$

$$[C_N]\{Z_{N-1}\} + [B_N]\{Z_N\} + \{d_N\}\, Z_L(j) = \{g_N\} \qquad (C\text{-}1)$$

Note that $Z_L(j)$ is one element of the common unknown vector Z_L (see Fig C.1).

A short description of the solution procedure is next outlined.

By using Gaussian elimination for the first (L-2) matrix equations, one may find the equivalent set of equations, which is

$$\{Z_i\} + [P_i]\{Z_{i+1}\} + \{E_i\}\, Z_L(j) = \{X_i\}$$
$$i = 1, 2, \cdots\cdots, L-2 \qquad (C\text{-}2)$$

where

$$[P_1] = [B_1]^{-1}[A_1] \quad ; \quad \{E_1\} = [B_1]^{-1}\{d_1\}$$

$$\{X_1\} = [B_1]^{-1}\{g_1\} \qquad (C\text{-}3)$$

and

$$[P_i] = \big[[B_i] - [C_i][P_{i-1}]\big]^{-1}[A_i]$$

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$$\{E_i\} = [[B_i] - [C_i][P_{i-1}]]^{-1}\{\{d_i\} - [C_i]\{E_{i-1}\}\}$$

$$\{X_i\} = [[B_i] - [C_i][P_{i-1}]]^{-1}\{\{g_i\} - [C_i]\{X_{i-1}\}\} \qquad (C-4)$$
$$\text{for } i = 2, 3, \cdots, L-2$$

Note that the order of the various matrices is as follows:

$$[C_i] \qquad\qquad m_i \text{ by } m_{i-1}$$

$$[B_i] \qquad\qquad m_i \text{ by } m_i$$

$$[A_i] \qquad\qquad m_i \text{ by } m_{i+1}$$

$$[P_i] \qquad\qquad m_i \text{ by } m_{i+1}$$

$\{z_i\}$, $\{g_i\}$, $\{d_i\}$, $\{X_i\}$ and $\{E_i\}$ are all m_i by 1

Next, for i = L-1, L, and L+1 the equivalent equations are:

$$\{Z_{i-1}\} + [P_{i-1}]\{Z_i\} = \{X_{i-1}\}$$

$$\text{for } i = L-1, L, L+1 \qquad (C-5)$$

where, for i = L-1

$$[P_i] = [[B_i] - [C_i][P_{i-1}]]^{-1}[[A_i] - [C_i][\bar{E}_{i-1}]]$$

$$\{X_i\} = [[B_i] - [C_i][P_{i-1}]]^{-1}(\{g_i\} - [C_i]\{X_{i-1}\}) \qquad (C-6)$$

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with

$$[\bar{E}_{i\text{-}1}] = [\; 0 \;\; \{E_{i\text{-}1}\} \;\; 0 \;] \qquad\qquad (C\text{-}7)$$

Note that $\lfloor \bar{E}_{1\text{-}1} \rfloor$ is an m_{i-1} by m_{i+1} matrix (defined, as shown, for convenience). and for $i = L, L + 1$

$$[P_i] = [[B_i] - [C_i][P_{i\text{-}1}]]^{-1}[A_i]$$

$$\{X_i\} = [[B_i] - [C_i][P_{i\text{-}1}]]^{-1}(\{g_i\} - [C_i]\{X_{i\text{-}1}\}) \qquad\qquad (C\text{-}8)$$

Finally, for $i = L + 2, L + 3, \ldots.N$, before writing the equivalent equations, $\{d_L\}$ is eliminated from each matrix equation. The elimination is accomplished by multiplying $\{d_L\}$ with the appropriate terms of matrix $\lfloor P_L \rfloor$. This leads to a matrix with only one nonzero column (vector), as shown below

$$\left[\{d_i\} \oplus [P_L] \right] = \left[\; 0 \quad \begin{Bmatrix} d_i(1)\, P_L(L,1) \\ d_i(2)\, P_L(L,2) \\ \vdots \\ d_i(m_i)\, P_L(L,m_i) \end{Bmatrix} \quad 0 \; \right] \qquad (C\text{-}9)$$

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Note that the symbol $\oplus$ is introduced to define the operation that leads to the matrix of Eq (C-9).

Similarly, the symbol $\odot$ is introduced to define an operation that leads to a column matrix.

$$\{V_i\} \odot \{V_2\} = \begin{Bmatrix} V_1(1) \; V_2(1) \\ V_1(2) \; V_2(2) \\ \vdots \\ \vdots \\ V_1(m_i) V_2(m_i) \end{Bmatrix} \qquad (C-10)$$

With these definitions one may now write the equivalent equations for $i = L+2,\; L+3,\; \ldots .N-1$. These are

$$\{Z_i\} + [P_i]\{Z_{i+1}\} = \{X_i\} \qquad (C-11)$$

where

$$[P_i] = [[B_i] - [\bar{C}_i][P_{i-1}]]^{-1} [A_i] \qquad (C-12)$$

$$\{X_i\} = [[B_i] - [\bar{C}_i][P_{i-1}]]^{-1} (\{\bar{g}_i\} - [\bar{C}_i]\{X_{i-1}\}) \qquad (C-13)$$

with

$$[\bar{C}_i] = [C_i] - (-1)^{i-L-2}\{d_i\} \oplus [P_L][P_{L+1}] \cdots [P_{i-2}] \qquad (C-14)$$

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and

$$\{\hat{g}_i\} = \{g_i\} - \{d_i\} \odot (\{X_i\} - [P_i]\{X_{i+1}\} + [P_i][P_{i+1}]\{X_{i+2}\}$$

$$- [P_i][P_{i+1}][P_{i+2}]\{X_{i+3}\} + \cdots + (-1)^{i-L-2}[P_i]\cdots[P_{i-3}]\{X_{i-2}\}) \qquad (C\text{-}15)$$

Finally, for i = N

$$\{Z_N\} = \{X_N\} \qquad\qquad (C\text{-}16)$$

where $\{X_N\}$ is given by Eq (C-13) with i=N. The recurrence formulae for backward substitution, in order to calculate Z_{N-1}, Z_{N-2}, $\cdots$ Z_2, and Z_1 are

$$\{Z_N\} = \{X_N\}$$

$$\{Z_i\} = \{X_i\} - [P_i]\{Z_{i+1}\} \;;\; i = N-1, N-2, \cdots, L-1$$

$$\{Z_i\} = \{X_i\} - [P_i]\{Z_{i+1}\} - \{E_i\} Z_L(i) \;;\; i = L-2, L-3, \cdots, 2, 1 \qquad (C\text{-}17)$$

C 2 Determininant Calculation

In each step of the inversion process, one must calculate the corresponding determinant e_i, namely

$$\varrho_1 = det [B_1]$$

$$e_i = det [[B_i] - [C_i][P_{i-1}]] \;;\; i = 2, 3, \cdots, L+2$$

$$e_i = det [[B_i] - [\bar{C}_i][P_{i-1}]] \;;\; i = L+2, L+3, \cdots N \qquad (C\text{-}18)$$

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Thus the determinant, D, of the entire coefficient matrix of the system can easily be computed by

$$D = \prod_{i=1}^{N} e_i \qquad\qquad (C\text{-}19)$$

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Appendix D

INSTABILITY OF LAMINATED CYLINDERS IN TORSION

by

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Introduction

A Galerkin-type solution, for the buckling analysis of a perfect
geometry, laminated, circular, cylindrical thin shell subjected to pure
torsion, is presented. The torsion is applied through the reference
surface, which is the midsurface of the laminate and the boundaries
are classical simple supports (SS-3). The analysis is based on Donnell-
type nonlinear kinematic relations and linearly elastic material behavior.
It is assumed that a primary state exists and that it is axisymmetric.
This primary state can be obtained by solving the field equations. Through
perturbation of the governing field equation a set of (linearized) buckling
equations is obtained, along with the related boundary conditions. A
Galerkin procedure is employed for solving the buckling equations. Thus,
the problem is reduced to an eigen-boundary-value problem. Critical
torsional loads are obtained for several Boron/Epoxy configurations of
symmetric, antisymmetric and asymmetric stacking. In addition, approxi-
mate buckling modes are established for both positive and negative torsion.
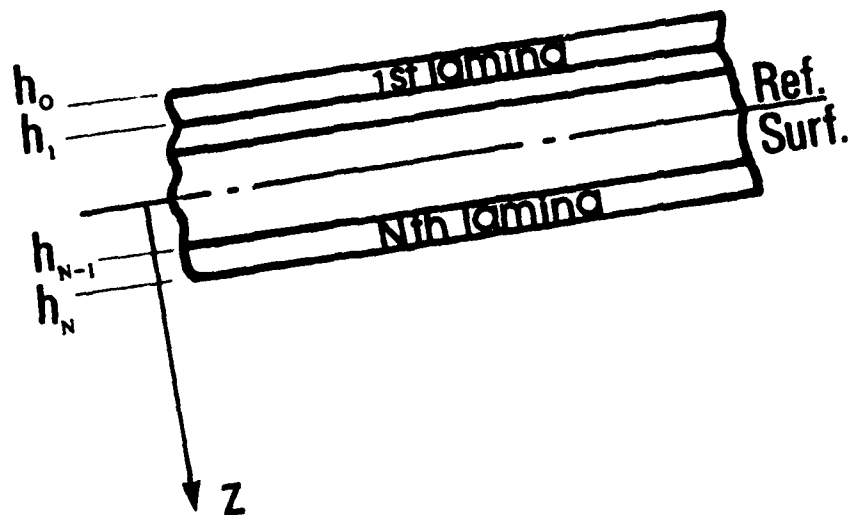

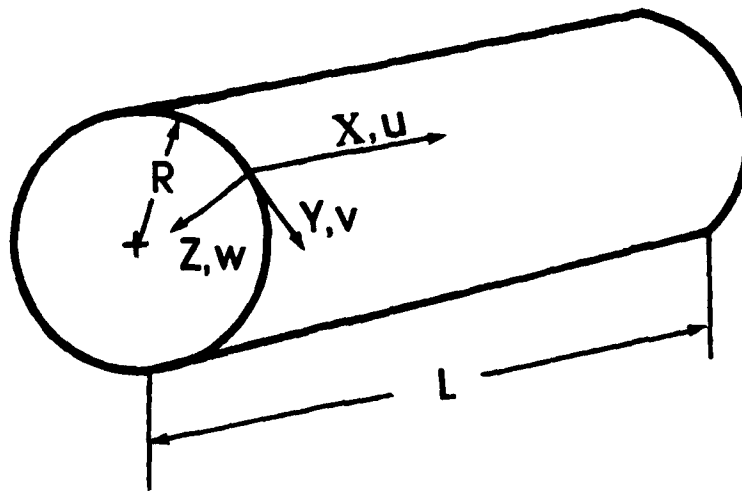[†]Graduate Research Assistant
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Fig.D.1 Geometry and Sign Convention

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Governing Equations and Solution Procedure

The geometry and sign convention are shown on Fig. 1. The torsion is positive if applied clockwise at the right end (x = L) and counter-clockwise at the left end (x = 0). The governing equations for a general laminated circular cylindrical shell, with or without orthogonal stiffeners, without geometric imperfections, and subjected to a pure torsion, consist of two coupled partial nonlinear differential equations in the transverse displacement component w(x,y) and an Airy stress (resultant) function, F(x,y). One of the equations characterizes transverse equilibrium and the other in-plane compatibility. These equations are taken from [D.1] by setting $\bar{N}_{xx} = q = w^o(x,y) = 0$, where $\bar{N}_{xx}$ denotes the uniform axial compression, q lateral pressure and $w^o(x,y)$ an initial geometric imperfection. The two equations are

Equilibrium:

$$b_{11}F,_{yyxx} + b_{21}F,_{xxxx} - b_{31}F,_{xxxy} + d_{11}w,_{xxxx} + d_{12}w,_{xxzz} + 2d_{13}w,_{xxxy}$$

$$+ 2b_{13}F,_{xyyy} + 2b_{23}F,_{xxyy} + 2\,b_{33}\,F,_{xxyy} + 2\,d_{31}\,w,_{xxxy} + 2\,d_{32}\,w,_{xyyy} + 4\,d_{33}\,w,_{xxyy}$$

$$+ b_{12}F,_{yyyy} + b_{22}F,_{xxyy} - b_{32}F,_{xyyy} + d_{21}\,w,_{xxyy} + d_{22}\,w,_{yyyy} + 2\,d_{23}\,w,_{xyyy}$$

$$+ \frac{1}{R}\,F,_{xx} + F,_{yy}w,_{xx} + 2\bar{N}_{xy}w,_{xy} - 2F,_{xy}w,_{xy} + F,_{xx}w,_{yy} = 0 \qquad \text{(D-1)}$$

Compatibility:

$$a_{11}F,_{yyyy} + a_{12}F,_{xxyy} - a_{13}F,_{xyyy} + b_{11}w,_{xxyy} + b_{12}w,_{yyyy} + 2b_{13}w,_{xyyy}$$

$$+ a_{12}F,_{xxyy} + a_{22}F,_{xxxx} - a_{23}F,_{xxxy} + b_{21}w,_{xxxx} + b_{22}w,_{xxyy} + 2\,b_{23}\,w,_{xxxy}$$

$$- a_{13}F,_{xyyy} - a_{23}F,_{xxxy} + a_{33}F,_{xxyy} - b_{31}w,_{xxxy} - b_{32}w,_{xyyy} - 2\,b_{33}\,w,_{xxyy} =$$

$$- \frac{w,_{xx}}{R} + w,_{xy}\,w,_{xy} - w,_{xx}\,w,_{yy} \qquad \text{(D-2)}$$

where

$$[a_{ij}] = [A_{ij}]^{-1} \quad ; \quad [b_{ij}] = [A_{ij}]^{-1} [B_{ij}]$$

$$[d_{ij}] = [B_{ij}] [b_{ij}] - [D_{ij}] \tag{D-3}$$

and $[A_{ij}]$, $[B_{ij}]$ and $[D_{ij}]$ are the extensional, coupling and flexural stiffnesses appearing in the usual lamination theory.

The expressions for the simply supported boundary conditions (SS - 3) are given below in terms of w and F (at x = 0, L).

$$w = 0 \quad ; \quad F_{,yy} = 0 \quad ;$$

$$b_{21}F_{,xx} + d_{11} w_{,xx} + 2 d_{13}w_{,xy} - b_{31}F_{,xy} = - b_{31}\overline{N}_{xy} \quad ;$$

$$a_{22}F_{,xx} - a_{23}F_{,xy} + b_{21}w_{,xx} + 2 b_{23}w_{,xy} = - a_{23}\overline{N}_{xy} \quad . \tag{D-4}$$

where $\overline{N}_{xy}$ is the applied torsional stress resultant. For more details see [c.1].

It is assumed that, under the action of pure torsion, a primary state exists, which is axisymmetric (all three reference surface displacment components, u, v and w, are independent of the circumferential coordinate y). Note that for symmetric construction (regular angle-ply or cross-ply with odd number of plies, for example) a membrane state exists and, therefore, the above is not an assumption. How reasonable this assumption is depends on the nature and magnitude of the coupling stiffnesses $[B_{ij}]$. Primary state quantities are denoted by tilda. With this assumption, the field equation becomes

$$b_{21}\hat{F}_{,xxxx} + d_{11}\hat{w}_{,xxxx} + \hat{F}_{,xx}/R = 0 \tag{D-5}$$

$$a_{22}\hat{F}_{,xxxx} + b_{21}\hat{w}_{,xxxx} + \hat{w}_{,xx}/R = 0 \tag{D-6}$$

Moreover, the expression for the reference surface hoop strain ϵ^o_{yy} is given by

$$\epsilon^o_{yy} = -\hat{w}/R$$

$$= a_{22}\hat{F},_{xx} + a_{23}\bar{N}_{xy} + b_{21}\hat{w},_{xx} \left.\right\} \tag{D-7}$$

These three equations, Eqs. D-5, D-5 and D-7, are employed to eliminate $\hat{F}$ and thus there is only a single field equation. This resulting equation is:

$$\left(d_{11} - \frac{b^2_{21}}{a_{22}}\right)\hat{w},_{xxxx} + 2\frac{b_{21}}{a_{22}R}\hat{w},_{xx} - \frac{\hat{w}}{a_{22}R^2} = \frac{a_{23}}{a_{22}R}\bar{N}_{xy} \tag{D-8}$$

The general solutions for $\hat{w}$ and consequently [from Eq. D-7] for $\hat{F},_{xx}$ become

$$\hat{w} = B_1 \sinh\lambda_1(x - \tfrac{L}{2})\sin\lambda_2(x - \tfrac{L}{2})$$

$$+ B_2 \cosh\lambda_1(x - \tfrac{L}{2})\cos\lambda_2(x - \tfrac{L}{2}) - R\,a_{23}N_{xy} \tag{D-9}$$

$$\hat{F},_{xx} = \frac{-1}{a_{22}}\left(b_{21}B_2(\lambda^2_1 - \lambda^2_2) + 2\,b_{21}B_1\lambda_1\lambda_2 + \right.$$

$$+ \frac{B_2}{R}\right)\cosh\lambda_1(x - \tfrac{L}{2})\cos\lambda_2(x - \tfrac{L}{2})$$

$$- \frac{1}{a_{22}}\left(b_{21}B_1(\lambda^2_1 - \lambda^2_2) - 2\,b_{21}B_2\lambda_1\lambda_2 + \right.$$

$$+ \frac{B_1}{R}\right)\sinh\lambda_1(x - \tfrac{1}{2})\sin\lambda_2(x - \tfrac{L}{2}) \tag{D-10}$$

$$\lambda_1 = \left\{ \frac{1}{2} \left[a_{22} R^2 \left(\frac{b_{21}^2}{a_{22}} - d_{11} \right) \right]^{-\frac{1}{2}} + \frac{1}{2} \left(\frac{b_{21}}{a_{22}R} \right) \left(d_{11} - \frac{b_{21}^2}{a_{22}} \right)^{-1} \right\}^{\frac{1}{2}}$$

$$\lambda_2 = \left\{ \frac{1}{2} \left[a_{22} R^2 \left(\frac{b_{21}^2}{a_{22}} - d_{11} \right) \right]^{\frac{1}{2}} - \frac{1}{2} \left(\frac{b_{21}}{a_{22}R} \right) \left(d_{11} - \frac{b_{21}}{a_{22}} \right)^{-1} \right\}^{\frac{1}{2}} \qquad \text{(D-11)}$$

The constants B_1 and B_2 can be obtained by making use of the boundary conditions, Eqs. D.4.

Next, the buckling equations are obtained through a perturbation of the nonlinear governing equations. The dependent variables, w and F, are replaced by the sum of the primary state parameters, $\hat{w}$ and $\hat{F}$, and small additional quantities, w^1 and F^1, necessary to represent the buckled state. Moreover, the related boundary conditions for the buckling equations are also obtained in the same manner. Note that since the additional quantities can be made small as one wishes, only the linear terms in w^1 and F^1 are retained.

The buckling equations and related boundary conditions are:

$$b_{21} F^1_{,xxxx} + (2 b_{23} - b_{31}) F^1_{,xxxy} + (b_{11} - 2 b_{33} + b_{22}) F^1_{,xxyy} + (2 b_{13} - b_{32}) F^1_{,xyyy}$$

$$+ b_{12} F^1_{,yyyy} + d_{11} w^1_{,xxxx} + (2 d_{31} + 2 d_{13}) w^1_{,xxxy} + (d_{12} + 4 d_{33} + d_{21}) w^1_{,xxyy}$$

$$+ (2 d_{32} + 2 d_{23}) w^1_{,xyyy} + d_{22} w^1_{,yyyy} + \frac{F^1_{,xx}}{R} + \hat{F}_{,xx} w^1_{,yy}$$

$$+ \hat{w}_{,xx} F^1_{,yy} + 2 \bar{N}_{xy} w^1_{,xy} = 0 \qquad \text{(D-12)}$$

$$a_{22} F^1_{,xxxx} - 2 a_{23} F^1_{,xxxy} + (2 a_{12} + a_{33}) F^1_{,xxyy} - 2 a_{13} F^1_{,xyyy} + a_{11} F^1_{,yyyy}$$

$$+ b_{21} w^1_{,xxxx} + (2 b_{23} - b_{31}) w^1_{,xxxy} + (b_{11} - 2 b_{33} + b_{22}) w^1_{,xxyy}$$

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$$+ (2 b_{13} - b_{32})w^1_{,xyyy} + b_{12}w^1_{,yyyy} + \frac{w^1_{,xx}}{R} + \hat{w}_{,xx} \, w^1_{,yy} = 0 \qquad (D\text{-}13)$$

at
x = 0,L
$$w = 0 \; ; \quad b_{21}F^1_{,xx} + b_{31}F^1_{,xy} + d_{11}w^1_{,xx} + 2d_{13}w^1_{,xy} = 0 \; ;$$

$$F_{,yy} \; ; \quad a_{22}F^1_{,xx} + a_{23}F^1_{,xy} + b_{12}w^1_{,xx} + 2b_{23}w^1_{,xy} = 0 \; . \qquad (D\text{-}14)$$

The Galerkin procedure is employed for both equations. The following
approximate series is used for generating the Galerkin integrals. Note
that the boundary conditions are satisfied by each term in the series.

$$w^1 = \sum_{n=1}^{N} \sum_{i=1}^{M} (A_{in}\cos\frac{ny}{R} + B_{in}\sin\frac{ny}{R}) \left[\frac{L}{i\pi} \sin\frac{i\pi x}{L} - \frac{L}{(i+2)\pi} \sin\frac{(i+2)\pi x}{L} \right]$$

$$F^1 = \sum_{n=1}^{N} \sum_{i=1}^{M} (C_{in}\cos\frac{ny}{R} + D_{in}\sin\frac{ny}{R}) \left[\frac{L}{i\pi} \sin\frac{i\pi x}{L} - \frac{L}{(i+2)\pi} \sin\frac{(i+2)\pi x}{L} \right] \quad (D\text{-}15)$$

Substitution of the above expressions, Eqs. D.15, into the buckling
equations results into a set of systems of linear homogeneous algebraic
equations in A_{in}, B_{in}, C_{in} and D_{in} for each n (decoupled with respect to n).
Assuming that the lowest eigenvalue corresponds to the critical load,
$\bar{N}_{xy_{cr}}$, a computer program has been written to this effect. The Georgia
Tech high speed digital computer CDC - CYBER - 170/760 is used for
generating data. Note that a minimization with respect to n is per-
formed in order to find the lowest eigenvalue.

Numerical Results and Conclusions

The geometries considered in the investigation represent variations
of the one report in D.2. Each lamina is orthotropic (Boron/Epoxy;
AVCO 5505) with the following properties:

$$E_{11} = 2.0690 \times 10^8 \text{ kN/m}^2 \ (30 \times 10^6 \text{psi.}) \ ; \quad \mu = 0.21 \ ;$$

$$E_{22} = 0.1862 \times 10^8 \text{ kN/m}^2 \ (2.7 \times 10^6 \text{psi.}); \quad R = 190.5 \text{ cm} (7.5 \text{in.});$$

$$G_{12} = 0.04482 \times 10^8 \text{ kN/m}^2 \ (0.65 \times 10^6 \text{ psi}) \ ; \quad L = 381 \text{ cm} \ (15 \text{ in.});$$

$$h_{ply} = 0.013462 \text{ cm.} \ (0.0053 \text{ in.}) \tag{D.16}$$

$(h_{ply} = h_k - h_{k-1}$ for $k = 1,2,3,4$; four plies)

Five different stacking combinations of the four-ply laminate comprise the various geometries, $I - i$, $i = 1,2, -- 5$. These are

$I - 1 \ : \ 45^o / -45^o / -45^o / 45^o$

$I - 2 \ : \ 45^o / -45^o / 45^o / -45^o$

$I - 3 \ : \ -45^o / 45^o / -45^o / 45^o$ \hfill (D-17)

$I - 4 \ : \ 90^o / 60^o / 30^o / 0^o$

$I - 5 \ : \ 0^o / 30^o / 60^o / 90^o$

where the first number denotes the orientation of the fibers of the outermost ply with respect to x, and the last of the innermost. A pure torsion is applied through the midsurface of the four-ply laminate.

Some of the generated results are shown on Table D.1. For each geometry, the critical torsion (for both positive and negative application; clockwise and counterclockwise at the end $x = L$), the minimizing value of n (full number of circumferential waves), and the values of the coefficients A_{in} and B_{in} (normalized with respect to B_{2n}) are shown. Note that the A_{in} and B_{in} when substituted into the first of Eqs. D.15, yields the buckling mode. It was concluded that $M = 5$ suffices for determining critical loads.

Table D-1. Numerical Results

Geo.	Minimizing n	$\bar{N}_{xy_{cr}}$ in N/m (lbs./in.)	A_2	B_1	A_2	B_2	A_3	B_3	A_4	B_4	A_5	B_5
I-1	12	6987 (39.90)	-0.3353	0.	0.	1.0	0.7520	0.	0.	0.2038	0.3439	0.
	9	-13220 (-75.50)	0.7627	-0.1954	0.2561	1.0	0.0980	-0.0251	0.1185	0.4626	0.0225	0.005
I-2	10	9534 (54.45)	-0.5830	0.	0.	1.0	0.1696	0.	0.	0.4230	0.1023	0.
	10	-9454 (-53.99)	0.5804	0.	0.	1.0	-0.1753	0.	0.	0.4218	-0.1063	0.
I-3	10	9454 (53.99)	-0.5830	0.	0.	1.0	-0.1753	0.	0.	0.4218	-0.1063	0.
	10	-9534 (54.45)	0.5804	0.	0.	1.0	0.1696	0.	0.	0.4230	0.1023	0.
I-4	13	8597 (49.01)	-0.3290	0.0226	0.0685	1.0	0.7107	0.0487	0.0189	0.2759	0.4182	-0.028
	12	-7790 (-44.59)	0.3431	0.	0.	1.0	0.7355	0.	0.	0.10374	-0.2708	0.
I-5	13	13082 (74.71)	-0.4035	-0.0062	0.0153	1.0	0.5681	0.0087	0.0056	0.3666	0.3994	0.006
	12	-7846 (-44.81)	0.3413	0.0183	0.0536	1.0	-0.8158	-0.0437	0.0010	0.0184	-0.263	-0.014

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Note that Geometry I - 1 is symmetric (with respect to the midsurface), Geometries I-2 and I-3 antisymmetric, and Geometries I-4 and I-5 asymmetric. For the symmetric geometry (I-1), the positive direction critical torsion is 6987 N/m (39.9 lbs./in.), while the negative critical torsion is 13,220 N/m (75.5 lbs./in.). The respective reported D-2 experimental values are 4640 N/m (26.5 lbs./in.) for the positive direction and 11,508 N/m (65.72 lbs./in.) for the negative. This suggest that the geometric imperfection in the tested cylinder D-2 is such that the configuration is more sensitive to it, when loaded in the positive direction, than in the negative (the ratio of the experimental to theoretical value is 0.664 for the former and 0.87 for the latter). The difference in response is understandable, because of the anisotropy. The antisymmetric geometries, I-2 and I-3, yield the same response when loaded opposite to each other. Note that the positive direction critical load for I-2 is the same as the negative direction critical load for I-3 (the same is true for the buckling mode). Also, observe that the two ($\pm$ direction) critical loads are very close (9534 N/m. and 9454 N/m.). This is due to the fact that the extensional, $[A_{ij}]$, and flexural, $[D_{ij}]$, stiffness have the same form as if the shell were isotropic. The difference from isotropy is the existence of some small (in value) terms in the coupling, $[B_{ij}]$, stiffnesses.

Finally, for the asymmetric configurations, I-4 and I-5 the response is completely different when each geometry is loaded in the positive and in the negative direction. Although the $[A_{ij}]$ and $[D_{ij}]$ stiffnesses, for the two configurations, are the same and only the signs are different in the $[B_{ij}]$ stiffness, the geometries behave (radically) differently. The only similarity is that the number of full waves, n, is approximately the same (12 and 13).

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