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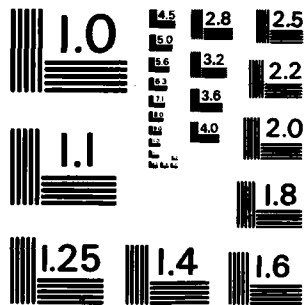
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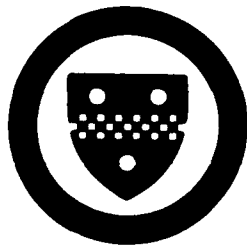
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LIKELIHOOD RATIO TESTS FOR RELATIONSHIPS
BETWEEN TWO COVARIANCE MATRICES

C. Radhakrishna Rao
University of Pittsburgh

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Center for Multivariate Analysis
University of Pittsburgh
Ninth Floor, Schenley Hall
Pittsburgh, PA 15260

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C. Radhakrishna Rao
University of Pittsburgh

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ABSTRACT

Likelihood ratio tests for hypotheses on relationships between two population covariance matrices Σ_1^T and Σ_2^T are derived on the basis of the sample covariance matrices having Wishart distributions. The specific hypotheses considered are (i) $\Sigma_2^T = \sigma^2 \Sigma_1^T$, (ii) $\Sigma_2^T = \Gamma + \sigma^2 \Sigma_1^T$, (iii) $\Sigma_2^T = \Gamma + \Sigma_1^T$ where Γ may be n.n.d. or arbitrary and the rank of Γ is less than that of Σ_1^T . Some applications of these tests are given.

AMS Subject Classification: 62H15, 62H25

KEY WORDS: Analysis of dispersion, familial correlations, likelihood ratio tests, MANOVA, Principal components, Wilks A.

LIKELIHOOD RATIO TESTS FOR RELATIONSHIPS
BETWEEN TWO COVARIANCE MATRICES

C. Radhakrishna Rao

1. INTRODUCTION

Let $S_1: p \times p$ and $S_2: p \times p$ be two random symmetric matrices having Wishart distributions $W_p(n_1, \Sigma_1)$ and $W_p(n_2, \Sigma_2)$ respectively, where n_1 and n_2 are degrees of freedom, and Σ_1 and Σ_2 are population covariance matrices. In this paper the likelihood ratio tests are derived for the following hypotheses on Σ_1 and Σ_2 :

$$H_1: \Sigma_2 = \sigma^2 \Sigma_1, \sigma^2 \text{ unknown,}$$

$$H_2: \Sigma_2 = \Gamma + \sigma^2 \Sigma_1, \Gamma \text{ is n.n.d. and } \rho(\Gamma) = k < p, \sigma^2 \text{ unknown,}$$

$$H_3: \Sigma_2 = \Gamma + \Sigma_1, \Gamma \text{ is n.n.d. and } \rho(\Gamma) = k < p,$$

$$H_4: \Sigma_2 = \Gamma + \Sigma_1, \rho(\Gamma) = k < p,$$

where $\rho(A)$ = the rank of the matrix A and n.n.d. stands for non-negative definiteness.

Applications of the above tests to problems of inference on "familial correlations" introduced by the author (see Rao, 1945 and the follow up in Rao, 1953) are discussed.

The following well known results and notations are used.

(i) If Σ_1 and Σ_2 are nonsingular, then S_1 and S_2 are nonsingular with probability 1

(ii) If S_1 is nonsingular, then there exist matrices P and $T = (P')^{-1}$ such that

$$\underline{S}_1 = \underline{T}\underline{T}', \quad \underline{S}_2 = \underline{T}\underline{\Lambda}\underline{T}' \quad (1.1)$$

$$\underline{P}'\underline{S}_1\underline{P} = \underline{I}, \quad \underline{P}'\underline{S}_2\underline{P} = \underline{\Lambda} \quad (1.2)$$

where $\underline{\Lambda}$ is the diagonal matrix with the roots ℓ_1, \dots, ℓ_p of

$$|\underline{S}_2 - \lambda \underline{S}_1| = 0 \text{ as the diagonal elements (see Rao, 1973, p. 41).}$$

(iii) Let \underline{A} : $p \times p$ be a real symmetric matrix and \underline{T}_i : $p \times k_i$, $i = 1, \dots, r$ be such that

$$\sum k_i = p, \quad \rho(\underline{T}_1 : \dots : \underline{T}_r) = p, \quad (1.3)$$

$$\underline{T}_i' \underline{A} \underline{T}_j = 0, \quad \underline{T}_i' \underline{T}_j = 0, \quad i \neq j. \quad (1.4)$$

Then there exists a choice of eigen vectors $\underline{R}_1, \dots, \underline{R}_p$, of

\underline{A} such that the columns of each \underline{T}_i depend on an exclusive subset of $\underline{R}_1, \dots, \underline{R}_p$.

(iv) If \underline{X} : $p \times p$, then the matrix derivative with respect to $\underline{X} = (X_{ij})$ of a scalar function $f(\underline{X})$ is defined by

$$\frac{\partial f}{\partial \underline{X}} = \left(\frac{\partial f}{\partial X_{ij}} \right): p \times p. \quad (1.5)$$

For particular choices of f , we have (Rao, 1973, p.72),

$$\frac{\partial |\underline{X}|}{\partial \underline{X}} = |\underline{X}| (\underline{X}^{-1})' \quad (1.6)$$

$$\frac{\partial \text{tr } \underline{M} \underline{X}}{\partial \underline{X}} = \underline{M}' \quad (1.7)$$

$$\frac{\partial \text{tr } \underline{X}^{-1} \underline{M}}{\partial \underline{X}} = -(\underline{X}^{-1} \underline{M} \underline{X}^{-1})' \quad (1.8)$$

(v) The log likelihood of $\underline{\Sigma}_1, \underline{\Sigma}_2$ given $\underline{S}_1, \underline{S}_2$ (considering only the terms depending on $\underline{\Sigma}_1, \underline{\Sigma}_2$) multiplied by 2 is

$$L(\underline{\Sigma}_1, \underline{\Sigma}_2 | \underline{S}_1, \underline{S}_2) \\ = -n_1 \log |\underline{\Sigma}_1| - \text{tr } \underline{\Sigma}_1^{-1} \underline{S}_1 - n_2 \log |\underline{\Sigma}_2| - \text{tr } \underline{\Sigma}_2^{-1} \underline{S}_2 \quad (1.9)$$

so that

$$\left[\frac{\partial L}{\partial \underline{\Sigma}_1} \right]' = -n_1 \underline{\Sigma}_1^{-1} + \underline{\Sigma}_1^{-1} \underline{S}_1 \underline{\Sigma}_1^{-1}, \quad \left[\frac{\partial L}{\partial \underline{\Sigma}_2} \right]' = -n_2 \underline{\Sigma}_2^{-1} + \underline{S}_2 \underline{\Sigma}_2^{-1}. \quad (1.10)$$

[Note that in taking the derivatives we do not consider $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ as symmetric matrices. This does not matter so long as the optimum solutions for $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ turn out to be symmetric.]

2. TEST FOR $\underline{\Sigma}_2 = \sigma^2 \underline{\Sigma}_1$ (σ^2 unknown)

Substituting $\sigma^2 \underline{\Sigma}_1$ for $\underline{\Sigma}_2$ in $L(\underline{\Sigma}_1, \underline{\Sigma}_2 | \underline{S}_1, \underline{S}_2)$ of (1.9), and taking derivatives with respect of $\underline{\Sigma}_1$ and σ^2 using the formulae (1.6)-(1.8), we have

$$\left[\frac{\partial L}{\partial \underline{\Sigma}_1} \right]' = -(n_1 + n_2) \underline{\Sigma}_1^{-1} + \underline{\Sigma}_1^{-1} (\underline{S}_1 + \sigma^{-2} \underline{S}_2) \underline{\Sigma}_1^{-1} = 0 \quad (2.1)$$

$$\frac{\partial L}{\partial \sigma^2} = -pn_2 + \sigma^{-2} \text{tr } \underline{\Sigma}_1^{-1} \underline{S}_2 = 0. \quad (2.2)$$

From (2.1),

$$(n_1 + n_2) \underline{\Sigma}_1 = \underline{S}_1 + \sigma^{-2} \underline{S}_2 = \underline{T} (\underline{I} + \sigma^{-2} \underline{\Lambda}) \underline{T}' \\ (n_1 + n_2)^{-1} \underline{\Sigma}_1^{-1} = \underline{P} (\underline{I} + \sigma^{-2} \underline{\Lambda})^{-1} \underline{P}' \quad (2.3)$$

where \underline{T} and \underline{P} are as defined in (1.1) and (1.2). Eliminating $\underline{\Sigma}_1^{-1}$ from (2.2) using (2.3), we obtain the equation for estimating the unknown σ^2 as

$$\frac{pn_2}{n_1 + n_2} = \sum_{i=1}^p \frac{\ell_i}{\ell_i + \sigma^2} = \sum_{i=1}^p \frac{n_2 m_i}{n_2 m_i + n_1 \sigma^2} \quad (2.4)$$

where ℓ_1, \dots, ℓ_p are the roots of $|S_2 - \lambda S_1| = 0$ and $m_i = n_1 \ell_i / n_2$. The equation (2.4) has only one non-negative solution which we represent by $\hat{\sigma}^2$. Then the estimate of Σ_1 is

$$\hat{\Sigma}_1 = \frac{S_1 + \hat{\sigma}^{-2} S_2}{n_1 + n_2}. \quad (2.5)$$

The likelihood ratio test (LRT) for testing $H_1: \Sigma_2 = \sigma^2 \Sigma_1$ is based on the difference

$$\begin{aligned} & \sup_{\Sigma_1, \Sigma_2} L(\Sigma_1, \Sigma_2 | S_1, S_2) - \sup_{\Sigma_1, \sigma^2} L(\Sigma_1, \sigma^2 \Sigma_1 | S_1, S_2) \\ &= \log \pi \prod_{i=1}^p \left[\frac{1}{n_2 \hat{\sigma}^{2n_1}} \left(\frac{n_2 m_i + n_1 \hat{\sigma}^2}{n_2 + n_1} \right)^{n_1 + n_2} \right]. \end{aligned} \quad (2.6)$$

The statistic (2.6) has an asymptotic χ^2 distribution on $[p^2 + p - 2]/2$ degrees of freedom (d.f.) when n_1 and n_2 tend to infinity.

It may be recalled that the LR test for $\Sigma_2 = \Sigma_1$ (Kshirsagar, 1978, p.404) is

$$\log \pi \left[\left(\frac{n_1 + n_2 m_1}{n_1 + n_2} \right)^{n_1 + n_2} \frac{1}{m_1^{n_2}} \right] \quad (2.7)$$

which can be written as the sum of (2.6) and

$$\log \pi \left[\left(\frac{n_1 + n_2 m_1}{n_1 \hat{\sigma}^2 + n_2 m_1} \right)^{n_1 + n_2} \hat{\sigma}^{2n_1} \right]. \quad (2.8)$$

The statistic (2.8) has χ^2 distribution on 1 d.f. when $\Sigma_2 = \sigma^2 \Sigma_1$ and n_1, n_2 are large, and can be used to test the hypothesis $\sigma^2 = 1$.

Further, if a confidence interval for σ^2 is needed, we can use the statistic

$$\log \pi \left[\left(\frac{n_1 \sigma^2 + n_2 m_1}{n_1 \hat{\sigma}^2 + n_2 m_1} \right)^{n_1 + n_2} \left(\frac{\hat{\sigma}}{\sigma} \right)^{2n_1} \right] \quad (2.9)$$

as χ^2 on 1 d.f.

A hypothesis of the type $\Sigma_2 = \sigma^2 \Sigma_1$ occurs in examining whether two response vectors differ by a scalar multiplier. For example, in the evaluation of drugs, two drugs will be considered equivalent if their response vectors, $\underline{x}, \underline{y}$, differ by a scalar multiplier, since by a change of dosage the effects may be made equal. Such a hypothesis specifies that $E(\underline{x}) = c E(\underline{y})$ and $D(\underline{x}) = c^2 D(\underline{y})$. We have considered only the hypothesis $D(\underline{x}) = c^2 D(\underline{y})$. Testing of the hypothesis, $E(\underline{x}) = c E(\underline{y})$, under the condition $D(\underline{x}) = D(\underline{y})$ was considered by Cochran (1943) and Kraft, Olkin and van Eeden (1972).

3. TEST FOR $\Sigma_2 = \Gamma + \sigma^2 \Sigma_1$

Under the hypothesis $\Sigma_2 = \Gamma + \sigma^2 \Sigma_1$ with Γ as an n.n.d. matrix of rank $k < p$ and σ^2 unknown, we can write

$$\Sigma_1 = R_1 R_1' + \dots + R_p R_p' = R R' \quad (3.1)$$

$$\Sigma_2 = \lambda_1 R_1 R_1' + \dots + \lambda_k R_k R_k' + \sigma^2 (R_{k+1} R_{k+1}' + \dots + R_p R_p') = R \Delta R' \quad (3.2)$$

where Δ is a diagonal matrix with $\lambda_1 > \dots > \lambda_k > \sigma^2, \dots, \sigma^2$ as diagonal elements.

We shall maximize

$$L(\Sigma_1, \Sigma_2 | S_1, S_2) + \text{tr } M_1 (\Sigma_1 - R R') + \text{tr } M_2 (\Sigma_2 - R \Delta R')$$

where M_1 and M_2 are matrices of Lagrangian multipliers. The optimizing equations are

$$-n_1 \Sigma_1^{-1} + \Sigma_1^{-1} S_1 \Sigma_1^{-1} + M_1 = 0, \quad -n_2 \Sigma_2^{-1} + \Sigma_2^{-1} S_2 \Sigma_2^{-1} + M_2 = 0 \quad (3.3)$$

$$M_2 R \Delta + M_1 R = 0 \quad (3.4)$$

$$\underline{R}'_i \underline{M}_i \underline{R}_i = 0, i=1, \dots, k, \sum_{i=k+1}^p \underline{R}'_i \underline{M}_i \underline{R}_i = 0 \quad (3.5)$$

With $\underline{U}' = \underline{R}^{-1}$, we have from (3.3)

$$n_1 \underline{I} = \underline{U}' \underline{S}_1 \underline{U} + \underline{R}' \underline{M}_1 \underline{R}, \quad (3.6)$$

$$n_2 \underline{I} = \underline{\Delta}^{-1} \underline{U}' \underline{S}_2 \underline{U} + \underline{R}' \underline{M}_2 \underline{R} \underline{\Delta}. \quad (3.7)$$

Adding (3.6) and (3.7) and using (3.4), we have

$$(n_1 + n_2) \underline{I} = \underline{U}' \underline{S}_1 \underline{U} + \underline{\Delta}^{-1} \underline{U}' \underline{S}_2 \underline{U}. \quad (3.8)$$

The equations (3.6), (3.7) and (3.8) \Rightarrow

$$\underline{U}'_i \underline{S}_1 \underline{U}_i = n_1, \underline{U}'_i \underline{S}_2 \underline{U}_i = n_2 \lambda_i, \underline{U}'_i \underline{S}_1 \underline{U}_j = 0 = \underline{U}'_i \underline{S}_2 \underline{U}_j, i \neq j \quad (3.9)$$

$$i, j = 1, \dots, k$$

$$\underline{U}'_i \underline{S}_1 \underline{V} = 0, \underline{U}'_i \underline{S}_2 \underline{V} = 0, i = 1, \dots, k, \underline{V} = (\underline{U}_{k+1} : \dots : \underline{U}_p) \quad (3.10)$$

The results (3.9) and (3.10) show, by using the results (1.3) and (1.4),

that the estimates of λ_i , \underline{U}_i and \underline{V} are

$$\hat{\lambda}_i = m_i, \hat{\underline{U}}_i = n_1^{1/2} \underline{P}_i, i = 1, \dots, k, \hat{\underline{V}} = (\underline{P}_{k+1} : \dots : \underline{P}_p) \underline{G} \quad (3.11)$$

where \underline{G} : $(p-k) \times (p-k)$ is any matrix such that

$$\underline{G}' (\underline{I} + \sigma^{-2} \underline{K}) \underline{G} = (n_1 + n_2) \underline{I} \quad (3.12)$$

In (3.12), \underline{K} is the diagonal matrix with $\ell_{k+1}, \dots, \ell_p$, the last $(p-k)$ eigen values of $|\underline{S}_2 - \lambda \underline{S}_1| = 0$ as the diagonal elements. The equation (3.12) together with (3.5)-(3.7) provide the estimate $\hat{\sigma}^2$ of σ^2 as the nonnegative root of the equation

$$\frac{(p-k)n_2}{n_1+n_2} = \prod_{i=k+1}^p \frac{n_2 m_i}{n_2 m_i + n_1 \sigma^2} \quad (3.13)$$

The LRT of the hypothesis $H_2: \Sigma_2 = \Gamma + \sigma^2 \Sigma_1$ is based on the difference

$$\begin{aligned} & \sup_{\Sigma_1, \Sigma_2} L(\Sigma_1, \Sigma_2 | S_1, S_2) - \sup_{H_2} L(\Sigma_1, \Sigma_2 | S_1, S_2) \\ &= \log \prod_{i=k+1}^p \left[\frac{n_2 m_i + n_1 \sigma^2}{n_1 + n_2} \right]^{n_1 + n_2} \frac{1}{n_2 \hat{\sigma}^{2n_1}} \quad (3.14) \end{aligned}$$

which has a χ^2 distribution, asymptotically as n_1 and $n_2 \rightarrow \infty$, on $[(p-k)(p-k+1)-2]/2$ d.f.

Muirhead (1978) obtained a representation of the conditional asymptotic distribution of $\ell_{k+1}, \dots, \ell_p$ given ℓ_1, \dots, ℓ_k and the last $p-k$ roots are equal, and observed that by neglecting a linkage factor, this distribution is the same as that of the roots of $V_2 V_1^{-1}$ where V_1 and V_2 have Wishart distributions $W_{p-k}(n_1, B)$ and $W_{p-k}(n_2 - k, \sigma^2 B)$ respectively. In such a case we may expect a better approximation to the χ^2 distribution by considering the statistic (3.14) with n_2 replaced by $n_2 - k$.

4. TEST FOR $\Sigma_2 = \Gamma + \Sigma_1$ (Γ , n.n.d.)

Under the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ with Γ as an n.n.d. matrix of rank k , we can write

$$\Sigma_1 = R_1 R_1' + \dots + R_p R_p' \quad (4.1)$$

$$\Sigma_2 = \lambda_1 R_1 R_1' + \dots + \lambda_k R_k R_k' + R_{k+1} R_{k+1}' + \dots + R_p R_p' \quad (4.2)$$

where

$$\lambda_1 > \dots > \lambda_k > 1. \quad (4.3)$$

In order to compute the likelihood ratio test, it is necessary to obtain the estimates of $\lambda_1, \dots, \lambda_k$ subject to the condition (4.3) which is somewhat difficult.

However, we can approach the problem of testing the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ by breaking up into two parts. One is for testing the hypothesis, $\Sigma_2 = \Gamma + \sigma^2 \Sigma_1$ (i.e., the last $p-k$ eigen values of Σ_2 with respect to Σ_1 are equal), and another for testing the hypothesis, $\sigma^2 = 1$ given that the first hypothesis holds.

The appropriate test statistic for the hypothesis, $\Sigma_2 = \Gamma + \sigma^2 \Sigma_1$, is given in (3.14). If this hypothesis is not disproved, we proceed to test the hypothesis $\sigma^2 = 1$ by using the statistic

$$z = \left[\frac{n_1 n_2 (p-k)}{2(n_1 + n_2)} \right]^{1/2} (\hat{\sigma}^2 - 1) \quad (4.4)$$

which is asymptotically distributed as a normal deviate. To deduce the result (4.5), observe that $\hat{\sigma}^2$ is a root of the equation

$$\frac{(p-k)n_2}{n_1 + n_2} = \sum_{i=k+1}^p \frac{n_2 m_i}{n_2 m_i + n_1 \sigma^2} \quad (4.5)$$

so that by the δ -method

$$\delta \hat{\sigma}^2 = \frac{1}{p-k} \sum \delta m_i \quad (4.6)$$

under the assumption that the true values of m_{k+1}, \dots, m_p are all equal to σ^2 .

Thus the asymptotic distribution of $\hat{\sigma}^2$ is the same as that of the average $(m_{k+1} + \dots + m_p)/(p-k)$. Then, using the results on the asymptotic distribution of the functions of the roots m_{k+1}, \dots, m_p (see Fang and Krishnaiah, 1982 and Muirhead, 1978), the asymptotic distribution of (4.4) is seen to be normal.

As observed earlier, it is difficult to derive the LR test of the hypothesis that the last $(p-k)$ roots of $\Sigma_2 \Sigma_1^{-1}$ are equal with common value unity. Even if the exact LR test is obtained, it may not have an asymptotic χ^2 distribution. [It may be noted that in a similar context, of the principal component analysis, the statistic for testing that the last $(p-k)$ roots of a covariance matrix are all equal to a given value, given by Anderson (1963) and quoted by Kshirsagar (1978, p. 448) is not an LRT.]

However, substituting $\hat{\sigma}^2 = 1$ in (3.14) we obtain the statistic

$$\log \prod_{i=k+1}^p \left[\left(\frac{n_1 + n_2 m_i}{n_1 + n_2} \right)^{n_1 + n_2} \frac{1}{m_i^{n_2}} \right] \quad (4.7)$$

which provides an overall measure of the difference between the vectors (m_{k+1}, \dots, m_p) and $(1, \dots, 1)$. The statistic (4.7) can be written as the sum of (3.14) and

$$\log \prod_{i=k+1}^p \left[\left(\frac{n_1 + n_2 m_i}{n_1 \hat{\sigma}^2 + n_2 m_i} \right)^{n_1 + n_2} \hat{\sigma}^{2n_1} \right]. \quad (4.8)$$

If the true values of m_{k+1}, \dots, m_p are equal, then the statistic (4.8) is asymptotically equivalent to

$$\frac{n_1 n_2 (p-k)}{2(n_1 + n_2)} (\hat{\sigma}^2 - 1)^2 \quad (4.9)$$

which is the square of the statistic (4.4), and hence is asymptotically distributed as χ^2 on 1 d.f. Thus (4.8) is an alternative statistic to (4.4)

to test the hypothesis that the common value of the last $(p-k)$ roots is unity.

In view of the remark made at the end of Section 3 based on Muirhead's observation, the statistic (4.7) is asymptotically distributed as χ^2 on $(p-k)(p-k+1)/2$ d.f. if the last $(p-k)$ roots of $\Sigma_2 \Sigma_1^{-1}$ are equal with the common value unity. But as observed earlier, it is more meaningful to use the statistics (3.14) and (4.8) [or (4.4)] by breaking the hypothesis into two parts, one specifying the equality of the roots and other specifying the common value.

If the common value specified is c , instead of unity, the statistic (4.8) is changed to

$$\log \prod_{i=k+1}^p \left[\frac{n_1 c + n_2 m_i}{n_1 \hat{\sigma}^2 + n_2 m_i} \right]^{n_1 + n_2} \left(\frac{\hat{\sigma}^2}{c} \right)^{n_1}. \quad (4.10)$$

Also, there may be some theoretical advantage in replacing n_2 by $(n_2 - k)$ in defining the statistics (3.14), (4.8) and (4.10).

5. TEST FOR $\Sigma_2 = \Gamma + \Sigma_1$

In Section 4, we considered the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ where Γ is n.n.d. and $\rho(\Gamma) = k$. If the n.n.d. condition is not imposed, then the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ implies that some $(p-k)$ eigen values of Σ_2 with respect to Σ_1 are equal to unity. In such a case Σ_1 and Σ_2 can be written as

$$\Sigma_1 = R_1 R_1' + \dots + R_p R_p' \quad (5.1)$$

$$\Sigma_2 = \lambda_1 R_1 R_1' + \dots + \lambda_k R_k R_k' + R_{k+1} R_{k+1}' + \dots + R_p R_p' \quad (5.2)$$

where λ_i need not be greater than unity.

The equations for estimating the unknown parameters under the hypothesis

$\Sigma_2 = \Gamma + \Sigma_1$ are

$$-n_1 \Sigma_1^{-1} + \Sigma_1^{-1} S_1 \Sigma_1^{-1} + M_1 = 0, \quad -n_2 \Sigma_2^{-1} + \Sigma_2^{-1} S_2 \Sigma_2^{-1} + M_2 = 0 \quad (5.3)$$

$$M_1 R \Delta + M_1 R = 0, \quad R_1' M_1 R_1 = 0, \quad i=1, \dots, k \quad (5.4)$$

where Δ is a diagonal matrix with $\lambda_1, \dots, \lambda_k, 1, \dots, 1$ as diagonal elements.

The equations (5.3) and (5.4) are the same as those in (3.3)-(3.5) except

for the equation $\sum_{i=k+1}^p R_1' M_1 R_1 = 0$ corresponding to σ^2 .

Proceeding as in Section 3, the LRT for the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ is seen to be

$$\inf_j \log \prod_{i=j+1}^{j+(p-k)} \left[\left(\frac{n_2 m_i + n_1}{n_2 + n_1} \right)^{n_1 + n_2} \frac{1}{m_i^{n_2}} \right] \quad (5.5)$$

which is asymptotically distributed as χ^2 on $(p-k)(p-k+1)/2$ d.f. The statistic (5.5) is different from (4.7).

6. FAMILIAL CORRELATIONS

In an early paper (Rao, 1945), the author introduced the concept of "familial correlations" as a generalization of the intraclass correlation. They arose in a natural way in defining a single measure of correlation between members (such as brothers) of a family with respect to a number of measurements. Typically we have a $b \times p$ matrix variable

$$X = \begin{bmatrix} x_{11} & \dots & x_{p1} \\ \cdot & \dots & \cdot \\ x_{1b} & \dots & x_{pb} \end{bmatrix} = \begin{bmatrix} x_1' \\ \vdots \\ x_b' \end{bmatrix} \quad (6.1)$$

where the i -th row vector x_i' corresponds to the measurements of p characteristics on the i -th member of a family. For instance, if we are considering brothers in a family, the row may correspond to the parity of a brother. In such a case, a natural model for the means and variances and covariances of the variables in (6.1) is

$$E(\underline{x}_i) = \underline{\mu}_i, \quad i = 1, \dots, k \quad (6.2)$$

$$D(X) = \begin{pmatrix} A & B & \dots & B \\ B & A & \dots & B \\ \vdots & \vdots & \dots & \vdots \\ B & B & \dots & A \end{pmatrix}. \quad (6.3)$$

Krishnaiah and Lee (1974) and Olkin (1973) considered the problem of testing the structure of the dispersion matrix of X as given in (6.3).

In earlier papers of the author (Rao, 1945 and the follow up in Rao, 1953), the mean vectors $\underline{\mu}_i$ were taken to be the same (which is valid when the members of a family are not distinguishable) and $D(X)$ is as in (6.3). Under this model, familial correlations were defined as intraclass correlations for suitably chosen linear functions of the p measurements.

If we have observations on X from N families, then we can write down the Analysis of Dispersion (MANOVA) for a two way classification (families \times parity) in the usual way.

Table 1. Analysis of dispersion for two way classification by family and parity

Due to	D.F.	Sums of squares and products (SSP)	Mean squares and products (MSP)	E(MSP)
Families	$N-1$	$(F_{ij}) = F$	(f_{ij})	$bB + A - B = \Gamma + \Sigma_1$
Parity	$b-1$	$(P_{ij}) = P$	(p_{ij})	$\phi + A - B = \phi + \Sigma_1$
Interaction	$(N-1)(b-1)$	$(W_{ij}) = W$	(w_{ij})	$A - B = \Sigma_1$

In Table 1, ϕ represents the non-centrality parameter which becomes a null matrix if $\underline{\mu}_i$ are all equal, and

$$\underline{F} \sim W_p(N-1, \underline{\Gamma} + \underline{\Sigma}_1), \underline{W} \sim W_p((N-1)(k-1), \underline{\Sigma}_1) \quad (6.4)$$

$$\underline{P} \sim W_p(N-1, \underline{\phi}, \underline{\Sigma}_1) \quad (6.5)$$

are all independently distributed. The joint distribution of the familial correlations can be obtained from that of the roots of the equation $|\underline{F} - \lambda \underline{W}| = 0$, derived by Roy (1939).

Two hypotheses of interest in such studies are

$$H_{01}: \mu_1 = \dots = \mu_b \quad \text{or} \quad \underline{\phi} = 0 \quad (6.6)$$

$$H_{02}: \rho(\underline{B}) = \rho(\underline{\Gamma}) = k. \quad (6.7)$$

The hypothesis H_{01} can be tested by Wilks $\Lambda = |\underline{W}|/|\underline{P} + \underline{W}|$, and the hypothesis H_{02} can be tested by using the statistic (5.5), writing \underline{F} for \underline{S}_2 and \underline{W} for \underline{S}_1 .

The familial correlations, ρ_1, ρ_2, \dots , defined by Rao (1945, 1953) are the roots of the equation $|\underline{B} - \rho \underline{A}| = 0$. The maximum root is the maximum correlation between any two members of a family with respect to a linear combination of the measurements. The number of nonzero familial correlations is equal to the rank of \underline{B} , and the hypothesis (6.7) is therefore relevant in drawing inferences on familial correlations.

It is seen that if ℓ_1, ℓ_2, \dots are the roots of the equation $|\underline{F} - \lambda \underline{W}| = 0$, then the estimate r_i of ρ_i is obtained from the relationship

$$\frac{(1 + \overline{b-1} r_i)}{(b-1)(1-r_i)} = \ell_i, \quad i = 1, 2, \dots \quad (6.8)$$

If $\mu_1 = \dots = \mu_b$, then the estimates of ρ_1, ρ_2, \dots are obtained from the roots ℓ'_1, ℓ'_2, \dots of $|\underline{F} - \lambda(\underline{W} + \underline{P})| = 0$ by the formula

$$\frac{(n-1)}{n(b-1)} \frac{1 + \overline{b-1} r_i}{1-r_i} = \ell'_i \quad (6.9)$$

which shows the relevance of the hypothesis (6.6).

In the statement of the hypothesis (6.7), no further condition was imposed on \underline{B} . If \underline{B} is not n.n.d., then some of the familial correlations will be negative.

Let the matrix variable \underline{X} : $b \times p$ have a structure of the type

$$X_{ij} = \mu_{ij} + \gamma_j + \epsilon_{ij} \quad (6.10)$$

where μ_{ij} are constants, and γ_j and ϵ_{ij} are stochastic variables representing family effects (common to all members of a family) and random effects respectively, such that

$$\begin{aligned} \text{Cov}(\gamma_j, \epsilon_{ij}) &= 0, \text{Cov}(\epsilon_{ij}, \epsilon_{km}) = 0, i \neq k \\ D(\gamma_1, \dots, \gamma_p) &= \underline{B}, D(\epsilon_{i1}, \dots, \epsilon_{ip}) = \underline{C}, i = 1, \dots, b. \end{aligned} \quad (6.11)$$

Then

$$D(\underline{X}) = \begin{pmatrix} \underline{A} & \underline{B} & \dots & \underline{B} \\ \dots & \dots & \dots & \dots \\ \underline{B} & \underline{A} & \dots & \underline{B} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \underline{B} & \underline{B} & \dots & \underline{A} \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad (6.12)$$

where $\underline{A} = \underline{B} + \underline{C}$ and \underline{B} is n.n.d. The rank of \underline{B} is equal to the number of linearly independent variables among $\gamma_1, \dots, \gamma_p$ (family effects specific to the p measurements). Thus, if the covariance matrix of \underline{X} is specified to be of the form (6.12), then the hypothesis (6.10) on the structure of the random variable \underline{X} with k linearly independent γ_1 is equivalent to

$$H_{03}: \underline{B} \text{ is n.n.d. and } \rho(\underline{B}) = k.$$

Such a hypothesis can be tested by using the statistics (3.14) and (4.5), with $\underline{F} = \underline{S}_2$, $\underline{W} = \underline{S}_1$ and the corresponding changes in the degrees of freedom.

There is some similarity between the hypotheses considered in the present paper with those of Fisher (1939) and Anderson (1951). Let $\gamma_1, \dots, \gamma_N$ be

the unknown vectors of family effects in N observed families. The problems considered by Fisher and Anderson relate to hypotheses on $\gamma_1, \dots, \gamma_N$ considered as fixed parameters. In the present paper, we consider γ_1 as stochastic and test hypotheses concerning the common covariance matrix of γ_1 .

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7. REFERENCES

- [1] Anderson, T.W. (1951). Estimating linear restrictions on regression coefficients for multivariate normal distributions. Ann. Math. Statist., 22, 327-351.
- [2] Anderson, T.W. (1963). Asymptotic theory for principal component analysis. Ann. Math. Statist., 34, 122-
- [3] Cochran, W.G. (1943). The comparisons of different scales of measures for experimental results. Ann. Math. Statist. 14, 205-216.
- [4] Fang, C. and Krishnaiah, P.R. (1982). Asymptotic distributions of functions of the eigen values of the doubly noncentral F matrix. (Sankhya, in press).
- [5] Fisher, R.A. (1939). The sampling distribution of some statistics obtained from non-linear equations. Ann. Eugen., 9, 238-249.
- [6] Kraft, C.H., Olkin, I. and van Eeden (1972). Estimation and testing for differences in magnitude or displacement in the mean vectors of two multivariate normal population. Ann. Math. Statist., 43, 455-467.
- [7] Krishnaiah, P.R. and Lee, J.C. (1974). On covariance structures. Sankhya A, 44, 357-371.
- [8] Kshirsagar, A.M. (1978). Multivariate Analysis, Marcel Dekkar, Inc. New York.
- [9] Muirhead, R.J. (1978). Latent roots and matrix variates: A review of some asymptotic results. Ann. Statist., 6, 5-33.
- [10] Olkin, I. (1973). Testing and estimation for structures which are circularly symmetric in blocks, in Multivariate Statistical Inference, p.183-195, North Holland (Eds. D.G. Kabe and R.P. Gupta).
- [11] Rao, C.R. (1945). Familial correlations or the multivariate generalization of the intraclass correlation. Current Science 14, 66-67.

- [12] Rao, C.R. (1953). Discriminant functions for genetic differentiation and selection. Sankhya, 12, 229-246.
- [13] Rao, C.R. (1973). Linear Statistical Inference and its Applications, Second edition. John Wiley, New York.
- [14] Roy, S.N. (1939). p-statistics or some generalizations of analysis of variance appropriate to multivariate problems. Sankhya, 4, 381-396.

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