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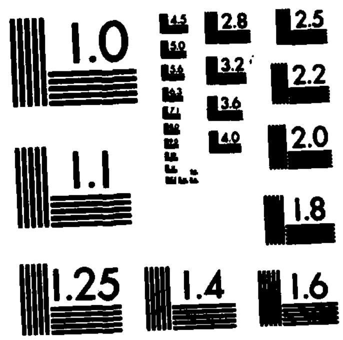

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Period: 30 September 1981 through 29 September 1982

Title of Research: Eigenvalues, Numerical Ranges, Stability Analysis, and Applications of Number Theory to Computing

Principal Investigators: M. Marcus
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Abstract

Work in the period under review centered principally on the topics: (1) numerical methods for systems of linear equations, (2) the numerical range, including computer routines for its evaluation, (3) inequalities involving matrices, (4) properties of eigenvalues, singular values, and invariant factors, (5) properties of norms and condition numbers on matrices.

Under (1), p-adic arithmetic (from pure number theory) has now been found to be well suited for computing exact solutions of linear systems that may be ill conditioned, and work is under way to develop computer routines to implement this new procedure. For (2), a blend of analytical and computer techniques, supplemented by geometrical considerations, continues to yield new insights concerning the various numerical ranges and numerical radii on matrices. This research is motivated by applications to partial differential equations. (3) Insightful understanding of numerical aspects of linear algebra requires insightful understanding of the numerous inequalities that pervade the subject. Continued progress in this direction of research is reported, there being inequalities involving permanents, nonnegative matrices, submatrices of Hermitian matrices, diagonal elements of normal matrices, matrix valued inequalities, and others. (4) Eigenvalues, singular values,

invariant factors, important in every application of linear algebra, continued under investigation, with new results reported for matrices of algebraic integers, and for normal matrices. (5) Norms and condition numbers, central tools for the numerical analyst, remain under active investigation, with new results connecting von Neumann norms to condition numbers, and also new results for the very intransigent multiplicativity factors that make norms submultiplicative.

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Abstract

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ABSTRACT

MARVIN MARCUS: MATRIX NORMS, CONDITION NUMBERS, AND THE NUMERICAL SOLUTION OF LARGE, SPARSE, LINEAR SYSTEMS

The author's research during the period October 1, 1981 through September 30, 1982 was primarily concerned with the following topics: (1) inequalities for unitarily invariant norms and condition numbers; (2) convexity properties of certain eigenvalue inclusion regions arising from quadratic and bilinear forms; (3) inequalities for principal submatrices of positive definite hermitian matrices; (4) generalized matrix functions and the index of a symmetry class of tensors. Activities included the publication of three papers in these areas and the presentation of some of these results at the April 1982 conference on Applied Linear Algebra held by the Society for Industrial and Applied Mathematics (SIAM).

The present annual report summarizes this research of M. Marcus sponsored by the Air Force Office of Scientific Research.

1. Inequalities for Unitarily Invariant Matrix Norms and Condition Numbers.

It is pointed out in the LINPACK User's Guide [3] that obtaining estimates of norms and condition numbers is decisive in dealing with two general problems that often arise in the numerical solution of large linear systems.

> If A is an n-square complex matrix, how close is A to being singular?
>
> How much does the solution change when A is replaced by A+E, where E is a perterbation matrix representing the total error arising from initial data errors, round- off errors, etc.?

A well known theorem [12] states that the relative error in A+E may be magnified by as much as the condition number of A, c(A). Recall that if $\|A\|$ is a matrix norm then the condition number is defined by

$$c(A) = \|A\|\|A^{-1}\|$$

In order to obtain computable lower bounds on c(A), the investigator recently proved that for the Hadamard product

$$A \cdot B = [a_{ij}b_{ij}]$$

the following inequality holds

$$\| A \cdot B \| \leq \| A \| \, \| B \|$$

where the indicated norm is the usual matrix norm subordinate to the Euclidean vector norm. We thus have that

$$\| A \cdot A^{-1} \| \leq c(A). \qquad (1)$$

We have extended this result to a large class of von Neumann norms [10]. For example, if A is an n-square matrix with singular values

$$a_1 \geq \cdots \geq a_n \geq 0$$

and if

$$\| A \| = \| A \|_k^p = \{ \sum_{m=1}^{k} a_m^p \}^{1/p}$$

then (1) holds. Research is still underway to determine whether or not this result holds for all unitarily invariant norms and in what cases equality can hold for (1).

2. Convexity Properties of Eigenvalue Inclusion Regions Arising from Quadratic and Bilinear forms.

Let A be an n-square complex matrix. The numerical range of A, W(A), is defined as the image of the unit sphere under the quadratic mapping

$$x \longrightarrow x^*Ax.$$

A classical result due originally to F. Hausdorff and O. Toeplitz [4,16] states that the numerical range of A is always a convex region in the plane which contains the spectrum of A. Thus W(A) can be used to yield estimates of the eigenvalues of a complex matrix and hence it is of interest to examine the discrepancies between W(A) and the convex hull of the spectrum of A. Towards this end, the investigator and two graduate students have developed FORTRAN programs for plotting W(A) and some of the higher numerical ranges

$$W(A,c) = \{ \sum_{m=1}^{n} c_m x_m^* A x_m \mid x_1,\ldots,x_n \text{ orthonormal} \}$$

where $c_1,\ldots,c_n$ are complex numbers. Currently we are examining PASCAL implementations of these programs which might be easily translated into the Department of Defense's forthcoming language ADA.

An interesting result concerning W(A,c) which has been established under the current grant characterizes the class of essentially hermitian matrices whenever $c_1 + \cdots + c_n$ is nonzero and at least two of the c's are distinct. This extends a result originally due to O. Taussky [13].

Another topic of research in this area concerns the geometry of the G-bilinear range. If $1 \leq r < n$ and if $G = [G_1 : G_2 : G_3]$ is an r by 3r complex matrix then W(A,G) is defined to be the totality of sums

$$y_1^* Ax + \cdots + y_r^* Ax,$$

where $x_1, \ldots, x_r, y_1, \ldots, y_r$ run over all n-tuples of complex numbers satisfying

$$[x_j^* x_i] = G_1, \ [y_j^* y_i] = G_2, \ [y_j^* x_i] = G_3.$$

In addition to the results concerning the convexity of W(A,G) reported in [9], the authors have obtained bounds on the maximum modulus of any element in W(A,G). These results have since appeared in [8].

Results have also been obtained concerning the special case $r=1$, $G=[1,1,q]$, $|q| \leq 1$. Several papers concerned with this case have appeared [1,5,8,15]. Computer programs to plot approximations to W(A,G) in this case have been developed and are currently being expanded.

All of the above results were surveyed at the SIAM conference on Applied Linear Algebra which was held in April of 1982.

3. Inequalities for Principal Submatrices of Positive Definite Hermitian Matrices.

In [6], a simple proof of a result due to J. Chollet [2] is given. Specifically, let A be an n-square positive definite hermitian matrix. Then for every integer $1 \le r \le n$ and every integer sequence $1 \le w_1 \le \cdots \le w_r \le n$ the matrix

$$A^{-1}[w] - (A[w])^{-1}$$

is positive semidefinite hermitian. Here $A[w]$ denotes the principal submatrix of A lying in the rows and columns labeled by w. This extends a result originally due to I. Schur [11].

4. Generalized Matrix Functions and The Index of a Symmetry Class of Tensors.

This past period of research has also seen the publication of some joint work with J. Chollet [7] in which we show that the m^{th} Exterior product of an n $(\ge m-1)$ dimensional vector space is the only symmetry class of tensors with the property that a decomposable tensor $x_1 * \cdots * x_m$ is zero when the dimension of the linear span of $x_1, \ldots, x_m$ is m-1. This property of the exterior product is then abstracted to obtain a definition of the index for a symmetry class of tensors. The proof of this result is based on several interesting characterizations of the determinant and permanent functions amoung the class of generalized matrix funtions induced by a group and a character.

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[15] N. Tsing & Y. Au Yeung, A conjecture of Marcus on the Generalized Numerical Range, in press, Linear and Multilinear Algebra.

PUBLICATIONS

1. Three proofs of the Goldberg-Straus Theorem (with M. Sandy), Linear and Multilinear Algebra 11 (1982), 243-252.

2. The G-bilinear range (with M. Sandy), Linear and Multilinear Algebra 11 (1982), 317-322.

3. Solution to Problem 1122 in Mathematics Magaxine (with M. Newman), to appear.

4. The index of a symmetry class of tensors (with J. Chollet), Linear and Multilinear Algebra 11 (1982), 277-281.

5. A remark on the preceding paper ("On principal submatrices" by Chollet), Linear and Multilinear Algebra 11 (1982), 287.

6. Interior points of the generalized numerical range (with M. Sandy), to be submitted.

7. Bilinear ranges, convexity and elementary doubly stochastic matrices (with M. Sandy and K. Kidman), in progress.

8. Unitarily invariant norms, condition numbers and the Hadamard product (with M. Sandy and K. Kidman), in progress.

ABSTRACT

R.C. THOMPSON: NORMAL MATRICES, SINGULAR VALUES, SMITH INVARIANTS, HANKEL AND TOEPLITZ MATRICES

The author's research during the period October 1, 1981 - September 30, 1982 was concerned with: (1) eigenvalues and diagonal elements of normal matrices, (2) Smith invariants of matrices, (3) matrix valued inequalities, and (4) Hankel and Toeplitz matrices. There was a one hour invited address on Core Linear Algebra at the SIAM meeting on Applied Linear Algebra at Raleigh, April 1982, and a one half hour invited address on Smith invariants at the Linear Algebra meeting immediately preceding the American Math. Society meeting in Toronto, August 1982.

1. Diagonal elements of normal matrices

The relationship between the diagonal elements and the eigenvalues of a normal matrix N is a long standing open question, with applications in elementary particle physics. Let $d_1,\dots,d_n$ be the diagonal elements and $\lambda_1,\dots,\lambda_n$ the eigenvalues. The following inequality was proved by R.C. Thompson some years ago. For every complex number z,

$$\sum_{i=1}^{n} |z-d_i| - 2 \min_{1\le i\le n} |z-d_i| \le \sum_{i=1}^{n} |z-\lambda_i| - 2 \min_{1\le i\le n} |z-\lambda_i|. \quad (1)$$

This is an infinite class of conditions relating diagonal elements and eigenvalues. It is also clear that

$$\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = S \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \quad (2)$$

where S is a doubly stochastic matrix constructed from the eigenvectors of N. It was conjectured in the proposal, and in research problem that appeared in Linear and Multilinear Algebra, that (1) and (2) are not adequate to characterize the diagonal elements of a normal matrix. This has now been established, in a paper accepted by the journal Linear Algebra and Its Applications.

2. Smith Invariants of matrix sums

A paper, Author vs Referee, somewhat expository, discussing the problem of how invariant factors behave when matrices of integers add, has been submitted to a journal.

3. Matrices of algebraic integers

Jointly with M. Newman, a manuscript has been prepared discussing the Smith invariants and the eigenvalues of matrices of algebraic integers. This sort of research is significant since it can shed light on the behaviour of singular values. The manuscript is still undergoing revision.

4. L. Puctyl´nikov published certain assertions, in two papers, concerning the structure of the spaces of Hankel and Toeplitz matrices, without proof. These matrices have many applications in signal processing and in control theory. Proofs have been constructed for Puctyl´nikov's assertions from his first paper. Work is underway concerning the material in the second paper. Along the same lines, a manuscript of 150 typed pages has been prepared translating and improving a Russian language book "Hankel and Toeplitz matrices - Algebraic theory" published by I.S. Iohvidov in 1976.

5. Matrix valued inequalities

Matrices with quaternion elements occur naturally in representation theory. For these matrices a triangle inequality has been found: Given a matrix A of quaternion entries, let $|A| = (AA^*)^{1/2}$ where $*$ denotes conjugate transpose. Then: If A, B have quaternion entries, unitary matrices U, V always exist such that

$$|A + B| \leq U|A|U^* + V|B|V^*.$$

This will appear in Linear and Multilinear Algebra.

6. Invited talks

An invited one hour lecture on "Core linear algebra" was given at the Applied Linear Algebra meeting held by SIAM (= Society for Industrial and Applied Mathematics) in Raleigh, NC., in April 1982. A one half hour invited talk on "Smith invariants of matrix sums" was given at the Linear Algebra meeting held in Toronto, August 1982. A paper based on the first talk is being prepared, and one based on the second has been submitted to a journal.

Papers published

1. Cyclic relations and the Goldberg coefficients in the Campbell-Baker-Hausdorff formula, Proceedings Amer. Math. Soc., 86, 12-14, 1982.

2. The Jacobi-Gundelfinger-Frobenius-Iohvidov rule and the Hasse symbol, Reports in Linear Algebra, 44, 189-196, 1982.

3. The Smith form, the inversion rule for 2 × 2 matrics, and the uniqueness of the invariant factors for finitely generated modules. Reports in Linear Algebra, 44, 197-201, 1982.

4. An inequality for invariant factors, Proceedings Amer. Math. Soc., 86, 9-11, 1982.

Papers in press (accepted)

5. Doubly stochastic, unitary, unimodular, and complex orthogonal power embeddings. Acta Scientiarium Mathematicarcem (Szeged).

6. A converse to the Ostrowski-Taussky determinantal inequality, Linear Algebra and Applications.

7. A counter example to a linear algebra conjecture, Linear Algebra and Applications.

8. The true growth rate and the inflation balancing principle, Amer. Math. Monthly.

9. The matrix valued triangle inequality: quaternion version, Linear and Multilinear Algebra.

Papers submitted

1. Author vs Referee, a case history for middle class mathematicians, submitted.

Manuscripts still in preparation

1. p-adic matrix values inequalities.

2. Sums of integral matrices.

3. Hankel and Toeplitz matrices - algebraic theory, 150 typed pages.

4. Matrices of algebraic integers (jointly with M. Newman).

ABSTRACT

M. NEWMAN: EXACT SOLUTIONS OF LINEAR SYSTEMS

The author prepared a program using p-adic arithmetic to find the exact solution of an integral linear system of equations. The program compares favorably in execution time with a standard elimination method program using real arithmetic.

Detailed Report:

The work accomplished this period was the preparation (for a serial machine) of a program to find the exact solution of a linear system by p-adic arithmetic, rather than residue arithmetic. The advantage is that only one prime modulus is necessary, and the numerous local solutions modulo differing primes are replaced by simple matrix by vector multiplications modulo the single prime. In addition the Chinese remainder theorem is <u>not</u> required; only some rather simple multi-precision multiplications, divisions, and additions need be performed. The time required to find the exact solution in this way compares very favorably with the time required to find an ordinary solution. Some disadvantages are that the determinant is not readily computed this way, and that no advantage is gained over ordinary residue arithmetic in finding the inverse. The method appears to have great potential, and was first suggested by J. Dixon.

Morris Newman

PUBLICATIONS

October 1, 1981, to date.

1. A radical diophantine equation, J. Number Theory 13 (1981), 495-498.

2. Cyclotomic units and Hilbert's Satz 90, to appear in Acta Arith.

3. Pseudo-similarity and partial unit regularity (with F.J. Hall, R.E. Hartwig, and I.J. Katz), to appear.

4. Similarity over SL(n,F), to appear in Linear and Multilinear Algebra.

5. On a problem of H.J. Ryser, to appear in Linear and Multilinear Algebra.

6. Lyapunov revisited, to appear in Linear Algebra and its Applications.

ABSTRACT

M. GOLDBERG: PROBLEMS IN STABILITY ANALYSIS OF FINITE DIFFERENCE SCHEMES FOR HYPERBOLIC SYSTEMS AND RELATED TOPICS

The research completed by M. Goldberg under AFOSR Grant 79-0127 during the period October 1981 - September 1982, consists of the following topics: (1) Convenient stability criteria for finite difference approximations to hyperbolic initial-boundary value problems. (2) Matrix norms: submultiplicativity of C-numerical radii and p-norms. (3) Generalizations of the Perron-Frobenius Theorem and localization of the eigenvalues with maximal absolute value.

The purpose of this interim report is to summarize my Air Force sponsored research in stability analysis of finite difference approximations for hyperbolic initial-boundary value problems and related topics, during the period October 1, 1981 - September 30, 1982.

1. Convenient Stability Criteria for Difference Approximations of Hyperbolic Initial-Boundary Value Problems.

In the past few years, E. Tadmor and I [12,13,15] have succeeded in obtaining convenient stability criteria for difference approximations of initial-boundary value problems asociated with the hyperbolic differential system

$$\partial u(x,t)/\partial t = A\partial u(x,t)/\partial x + Bu(x,t) + f(x,t), \quad x \geq 0, \ t \geq 0.$$

Here, $u(x,t)$ is the unknown vector, A a Hermitian matrix, B an arbitrary matrix, and $f(x,t)$ a given vector.

Our difference approximation consists of a well-behaved arbitrary basic scheme (explicit or implicit, dissipative or unitary, two-level or multi-level), and boundary conditions of a rather general type. Although the basic scheme is assumed to be stable for the pure Cauchy problem, it is well known that the introduction of boundary conditions may destroy stability. Thus, the question is whether overall stability of the entire approximation for the initial-boundary value problem is retained.

Our main stability criteria in [12,13] for the above difference approximations were given in terms of the boundary conditions and were essentially independent of the basic scheme. Such scheme-independent criteria eliminate the need to analyze the intricate and often complicated interaction between the basic scheme and the boundary conditions; hence providing a convenient alternative to the well known stability criterion of Gustafsson, Kreiss, and Sundstrom [17] which is usually very hard to check.

In the past two summers, Tadmor and I [15] have extended our stability criteria in [12,13] to include a wider range of examples. As in [12,13], our stability analysis is restricted to the case where the outflow boundary conditions are translatory, i.e., determined at all boundary points by the same coefficients. Such boundary conditions are commonly used in practice and, in particular, when the numerical boundary consists of a single point, the boundary conditions are translatory by definition.

Our new criteria in [15] depend both on the basic scheme and on the boundary conditions; yet they are as convenient as our scheme-independent results in [12,13]. In fact, in many cases our new results imply stability almost automatically as stated, for example, by the following theorem:

Let the basic scheme be stable for the pure initial value problem, and let the boundary conditions be explicit and satisfy the von Neumann condition. If either the basic scheme or the boundary conditions are dissipative, and if either the basic scheme or the boundary conditions are two-level, then the entire difference approximation is stable.

It can be easily verified that the new stability criteria in [15] yield all the examples in our previous papers [12,13] and much more. For instance, we can show that if the basic scheme is dissipative and if the outflow boundary conditions are generated by oblique extrapolation, by the Box-scheme, or by the right-sided weighted Euler scheme, then overall stability of the approximation is assured. For general basic schemes (dissipative or unitary) we can show that overall stability holds if the outflow boundary conditions are determined by the right-sided explicit or implicit Euler schemes. We also showed, for example, that if the basic scheme is two-level and the outflow boundary conditions are generated by horizontal extrapolation, then overall stability holds -- a result which may fail for multi-level basic schemes. These examples and others incorporate most special cases discussed in recent literature [2,3,12,13,16,17, 19-21,27,28,30].

2. <u>Matrix Norms</u>: <u>Sub-Multiplicativity of C-Numerical Radii and p-Norms</u>.

In the past few years, E.G. Straus and I [5-10] have continued to study sub-multiplicativity properties of various norms on $C_{n\times n}$, the algebra of n × n complex matrices. This topic, which is strongly related to most fields of numerical analysis, is particularly useful in stability analysis of difference schemes for partial differential equations and error analysis for numerical solutions of linear systems.

In our work we considered arbitrary norms on $C_{n\times n}$ which may or may not be (sub-) multiplicative. Given such a norm N and a fixed constant $m \geq 0$, then obviously $N_m \equiv mN$ is a norm too. If N_m is multiplicative, i.e.,

$$N_m(AB) \leq N_m(A)N_m(B), \quad A,B \in C_{n\times n},$$

then m is called a multiplicativity factor for N. With this definition our question was whether multiplicativity factors exist; and if so, how to find them in order to obtain multiplicativity by very simple means. A formal answer to this question was given by us in [7]:

(i) All norms on $C_{n\times n}$ have multiplicativity factors.

(ii) If N is a norm on $C_{n\times n}$ then m is a multiplicativity factor for N if and only if

$$m \geq m_N \equiv \max\{N(AB):\ N(A)=N(B)=1\}.$$

The original reason for introducing the idea of multiplicativity factors was to investigate the multiplicativity properties of one of the best known norms on $C_{n\times n}$, namely the numerical radius [1,4,14,18,26],

$$r(A) = \max\{|x^*Ax|: x^*x = 1\}.$$

We found [5,7] that m is a multiplicativity factor for r if and only if $m \geq 4$, independently of dimension. This unexpected result is of interest since r plays an important role in stability analysis of difference schemes for multi-space-dimensional hyperbolic initial value problems [14,22,23,29].

Our next step was to investigate C-numerical radii which constitute a generalization of the classical radius r, defined by us in [5] as follows: For given matrices $A,C \in C_{n\times n}$, the C-numerical radius of A is

$$r_C(A) = \max\{|\mathrm{tr}(CU^*AU)| : U \ n\times n \text{ unitary}\}.$$

Evidently, for C=diag(1,0,...,0), r_C reduces to r; thus r_C is indeed a generalization of the classical radius.

We have shown [5] (compare [24]) that r_C is a norm on $C_{n\times n}$ -- and so has multiplicativity factors -- if and only if C is not a scalar matrix and tr $C \neq 0$. Multiplicativity factors for the above r_C were found in [5-8] except for the case where C has equal eigenvalues. The project was completed last September when multiplicativity factors for all relevant C-radii were established in [9].

Our most recent effort in this area was to obtain multiplicativity factors for the well known ℓ_p-norms $(1 \le p \le \infty)$:

$$|A|_p = \{ \sum_{i,j} |a_{i,j}|^p \}^{1/p}, \; A = (a_{i,j}) \in C_{n\times n}.$$

It was shown by Ostrowski [25] that these norms are multiplicative if and only if $1 \le p \le 2$. For $p \ge 2$ we have shown [10] that m is a multiplicativity factor for $|A|_p$ if and only if $m \ge n^{1-2/p}$; thus, in particular, obtaining the useful result that $n^{1-2/p}|A|_p$ is a multiplicative norm on $C_{n\times n}$.

3. <u>Generalizations of the Perron-Frobenius Theorem and Localization of Eigenvalues with Maximal Absolute Value</u>.

In many instances one is interested in localizing an eigenvalue of maximal absolute value for a given matrix. The most famous result in this vein is the Perron-Frobenius Theorem which states that a matrix with nonnegative elements has at least one nonnegative eigenvalue of maximal absolute value.

Last summer, E.G. Straus and I were looking for generalizations of this celebrated theorem that locate an eigenvalue of maximal absolute value within a certain angle of the complex plane depending on the angle which contains the elements of the matrix. More precisely, let $A_n(\alpha)$ denote the family of all $n \times n$ complex matrices whose entries are contained in a sector

$$S(\alpha) = \{z: |\arg z| \le \alpha; \ 0 \le \alpha \le \pi \ \text{ fixed}\}.$$

For each $A \in A_n(\alpha)$ let $\beta(A)$ denote the minimal (nonnegative) angle for which the sector $S(\beta)$ contains an eigenvalue of A with maximal absolute value. Thus, defining

$$\beta_n(\alpha) = \sup \{\beta(A): A \in A_n(\alpha)\}$$

we posed the problem of finding $\beta_n(\alpha)$ as a function of α and n.

Since the Perron-Frobenius Theorem states that $\beta_n(0)=0$, a wishful generalization would read $\beta_n(\alpha)=\alpha$. This unfortunately is not so, as shown by us in [11] where we give a complete description of the 2×2 case as well as partial results for $n \ge 3$.

For n=2, for example, we have

$$\beta_2(\alpha) = \begin{cases} \alpha & \text{for } \alpha \le \pi/4 \\ \alpha + \pi/2 & \text{for } \pi/4 < \alpha \le \pi/2 \\ \pi & \text{for } \alpha > \pi/2, \end{cases}$$

where the discontinuity in $\beta_2(\alpha)$ is typical for all n. More expected properties of $\beta_n(\alpha)$ are:

(i) $$\beta_n(\alpha) \geq \alpha$$

(ii) $\beta_n(\alpha)$ is a nondecreasing function of α and n.

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1. Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems. II, (with E. Tadmor), Math. Comp. 36 (1981), 603-626.

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8. Multiplicativity of ℓ_p norms for matrices, (with E.G. Straus), in preparation.

9. Convenient stability criteria for difference approximations of hyperbolic initial-boundary value problems (with E. Tadmor), in preparation.

ABSTRACT

HENRYK MINC: PERMANENTS, NONNEGATIVE MATRICES, AND HADAMARD MATRICES

The author's research was mainly concerned with techniques developed by Egoryčev and Falikman in their solutions of the van der Waerden permanent conjecture. Their proofs were studied and the methods were applied to extend the result to the problem of determining the minimum permanent on a face of the polyhedron of doubly stochastic $n \times n$ matrices.

The author also studied Fréchet's problem of finding a simple analytic (or at least asymptotic) expression for the minimum determinant of $n \times n$ (1,-1)-matrices.

(I) VAN DER WAERDEN PERMANENT CONJECTURE

Research in the theory of permanents and its applications in the last two years has been strongly influenced by the recent solutions of the van der Waerden permanent conjecture by Egoryčev and by Falikman. They both proved that

$$\mathrm{per}(S) \geq n!/n^n \tag{1}$$

for any $n \times n$ doubly stochastic matrix. In addition, Egoryčev showed that equality can hold in (1) if and only if S is the $n \times n$ matrix all of whose entries are $1/n$. In paper [4] a version of Egoryčev's proof is given. In [3] two alternative variants of his proof are presented. In [5] versions of Egoryčev's proof and of the more difficult Falikman's proof are given in detail.

(II) MINIMUM OF THE PERMANENT OF A DOUBLY STOCHASTIC MATRIX WITH PRESCRIBED ZERO ENTRIES

Paper [7] is a study of properties of matrices with minimum permanent in a face of Ω_n, the polyhedron of doubly stochastic $n \times n$ matrices, i.e., for doubly stochastic matrices with zero entries in prescribed fixed positions. Egoryčev proved that all permanental cofactors of a matrix with minimal permanent in Ω_n are equal. This implies that in such matrix any pair of rows (columns) can be replaced by their mean without change in permanent. This averaging process leads to a proof of the van der Waerden conjecture. Unfortunately, in the case of doubly stochastic matrices with prescribed zero entries such averaging method has only a restricted application (viz., to rows (columns) with same prescribed zero pattern). In [7] we show that permanental cofactors of a matrix with minimal permanent in a face of Ω_n cannot exceed the permanent of the matrix, and the permanental cofactors of entries which are not prescribed are actually all equal to the permanent of the matrix. This result is then used to obtain minimum permanents in faces of Ω_n in which all prescribed zeros are restricted to two rows or columns, or in which the prescribed zeros form a submatrix.

In some cases in which prescribed zeros are located in many rows (columns), Falikman's method seems to be more appropriate than that of Egoryčev. Some partial results, as yet unpublished, have been obtained.

(III) HADAMARD MATRICES

Let $\Delta(n)$ be the maximum of the determinant of $n \times n$ $(1,-1)$-matrices. Fréchet asked if there exists a simple analytic expression for $\Delta(n)$ as a function of n, and he proposed the problem of determining an analytic asymptotic expression for $\Delta(n)$. It is known that $\Delta(n) \leq n^{n/2}$, and that equality, for $n > 2$, can hold only if $n \equiv 0 \bmod 4$. In [6] relevant known results for all n are surveyed, and it is concluded that Fréchet's question should be answered in the negative, although

$$\log \Delta(n) \sim \log n^{n/2}.$$

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