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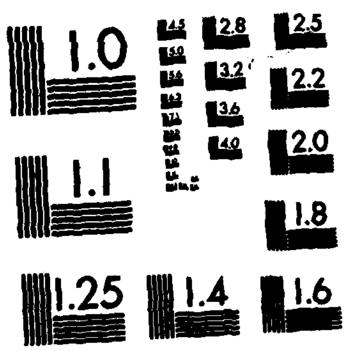

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The Shape of a Liquid Drop in the Flow of a Perfect Fluid

by Clyde A. Morrison

Abstract

A method of analysis previously used to determine the shape of a liquid drop in an electric field has been applied to determine the shape of a drop in the flow of a perfect fluid. The surface of the drop is expanded in a series of Legendre polynomials with arbitrary coefficients. Minimization of the total energy with respect to the coefficients then gives equations for their determination. Due to the similarity of the equations for determining the coefficients to previous results, it is expected that the shape of the drops will deviate considerably from the usually assumed oblate spheroids.

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For further copies of this technical letter, contact the author at the following address:

Commander
Harry Diamond Laboratories
ATTN DELHD-RT-CA
2800 Powder Mill Road
Adelphi, MD 20783

1. INTRODUCTION

In this report we determine the shape of a liquid drop in the flow of an ideal fluid (no viscosity). The drop is assumed to be held together by the interface surface tension between the liquid inside the drop and the external fluid. In previous analyses of this problem, the shapes of the drops have been assumed to be oblate spheroids with a minor axis parallel to the direction of flow of the external fluid.[1,2] In our earlier analysis of the shape of liquid drops in an electric field, the commonly accepted assumption that the drops formed prolate spheroids with the major axis parallel to the electric field was shown to be generally invalid.[3,4] The spheroidal approximation for drops was only valid for small drops or low electric fields.

Since the spheroidal approximation was invalid for drops in an electric field, we suspected that perhaps the same approximation would be invalid for falling liquid drops in air or other fluids. We have made the assumption that the external fluid has no viscosity; otherwise, the problem is extremely complicated. Here we shall use the technique used in our previous analysis.[4] That is, we express the total energy of the system, the kinetic energy of the external fluid, and the potential energy due to surface tension, in terms of the coefficients, a_n, in the expansion

$$r(\theta) = a_o + \sum_{n=1}^{\infty} a_n P_n(\cos\theta) \quad , \tag{1}$$

which relates an arbitrary point on the surface of the drop to the angle θ measured from the direction of the fluid flow (z axis). In equation (1) the $P_n(\cos\theta)$ are Legendre polynomials. Minimizing the total energy with respect to the a_n's gives a set of equations for these coefficients. The solutions of this set of equations then are used in equation (1) to determine the shape of the drop.

2. THEORY

In equation (1), we have made the physically realistic assumption that the drop is symmetric about the direction of the external fluid flow. In standard texts on fluid mechanics,[5,6] it is shown that the velocity in a perfect incompressible fluid can be derived from a velocity potential, ϕ, which is a solution of Laplace's equation. That is,

$$\nabla^2\phi = 0 \tag{2}$$

and

$$v = -\nabla\phi \quad . \tag{3}$$

The solution to equation (2) appropriate for the region external to the drop is

$$\phi = \sum_n \frac{A_n P_n}{r^{n+1}} - VrP_1 \quad , \qquad (4)$$

where we have dropped the arguments of the Legendre polynomials $[P_n = P_n(\cos\theta)]$ and assumed that far from the drop the velocity of the external fluid is given by

$$\mathbf{v} = \hat{e}_z V \quad , \qquad (5)$$

and $\hat{e}_z$ is a unit vector in the z direction.

As in our previous analyses[3,4] we shall rewrite equation (1) as

$$r(\theta) = a_o + \delta \sum a_n P_n \quad , \qquad (6)$$

where δ is an order parameter. We then expand the A_n in equation (4) in powers of δ as

$$A_n = A_n^{(0)} + \delta A_n^{(1)} + \delta^2 A_n^{(2)} + \ldots , \qquad (7)$$

and retain the terms through δ^2.

The A_n of equation (4) are determined by the boundary condition, which, for a perfect incompressible fluid, is that the velocity normal to the surface, v_n, must vanish. The velocity normal to the surface is given by equation (3.13) of Morrison et al[4] or

$$v_n = -\left[\frac{\partial}{\partial r} - \frac{(1-\mu^2)}{r^2}\frac{dr}{d\mu}\frac{\partial}{\partial \mu}\right]\phi \quad , \tag{8}$$

where $\mu = \cos\theta$ and r is given by equation (6). Using equations (6) and (7) in equation (9) and letting $v_n = 0$ we obtain

$$A_n^{(0)} = \frac{Va_o^3}{2}\delta_{n1}$$

$$A_n^{(1)} = -\frac{3Va_o^{n+1}}{4(n+1)}\sum_k a_k C_{1n,k}\langle 1k|n\rangle^2 \tag{9}$$

$$A_n^{(2)} = -\frac{3Va_o^n}{8(n+1)}\sum_{k\ell m} a_k a_m \langle 1k|\ell\rangle^2 \langle \ell m|n\rangle^2 C_{1\ell,k}(2\ell - C_{\ell m,n}) \quad ,$$

where $\langle \ell k|n\rangle$ are Clebsch-Gordan coefficients.[7] (In Rose,[7] the $C(\ell kn;00) = \langle \ell k|n\rangle$), the $C_{ab,c}$, are given by

$$C_{ab,c} = a(a+1) + b(b+1) - c(c+1) \quad ,$$

and δ_{n1} is the Kronecker delta function.

3. ENERGY

For the change in the kinetic energy of the external fluid, we use

$$T = \int \frac{1}{2} \rho V^2 \, d\tau - \int \frac{1}{2} \rho V^2 d\tau \quad , \tag{10}$$

where the first integral covers the region $r(\theta) < r < R_o$, and the second integral covers the region $0 < r < R_o$, and in the end result we let R_o become infinite. When equation (4) is used in equation (10), we obtain ($R_o \to \infty$)

$$T = \frac{2\pi\rho V}{3} A_1 \quad . \tag{11}$$

Thus, we need only A_1 through second order to obtain the change in kinetic energy. All the Clebsch-Gordan coefficients appearing in equation (9) are simple for n = 1, and algebraic expressions for them can be found in Rose.[7] Thus,

$$A_1 = -\frac{Va^3}{2} - \frac{3Va}{2} \sum_n \frac{h_n a_n^2}{2n+1} + \frac{9Va}{2} \sum_n g_{n+1} a_n a_{n+2} \quad , \tag{12}$$

where

$$h_n = \frac{(n - 1)\ 6n^3 + 7n^2 - 2n + 3)}{(2n - 1)(2n + 1)^2(2n + 3)} ,$$

$$g_n = \frac{n^2(n + 1)}{(2n - 1)(2n + 1)(2n + 3)} .$$

To obtain the result given in equation (12), we have used the constraint of constant volume as given in Page.[5],

$$a_o = a - \frac{1}{a} \sum_n \frac{a_n^2}{2n + 1} . \tag{13}$$

The surface energy of the drop is simply the product of the surface tension, γ, and the surface area of the drop and is given by

$$U_s = 2\pi\gamma \sum_n \frac{(n - 1)(n + 2)a_n^2}{2n + 1} , \tag{14}$$

and the constraint given in equation (13) was used in Page[5] to obtain this result. We now have the total energy of the system expressed in terms of the a_n.

4. THE EQUATIONS FOR a_n

The condition for minimum energy can be written

$$\frac{\partial}{\partial a_n}(T + U_s) = 0 \quad . \qquad (15)$$

The derivative of U_s is simple; from equation (14), this is

$$\frac{\partial U_s}{\partial a_n} = \frac{4\pi\gamma(n-1)(n+2)a_n}{(2n+1)} \quad . \qquad (16)$$

Using equation (12) in equation (11), we have the kinetic energy; from this result, we obtain

$$\frac{\partial T}{\partial a_n} = \frac{\pi\rho a^2 V^2}{5}\,\delta_{n2} - 2\pi\rho a V^2 \left[h_n a_n - \frac{3}{2}\left(g_{n+1}a_{n+2} + g_{n-1}a_{n-2}\right)\right] \quad . \qquad (17)$$

If the results given in equations (16) and (17) are substituted into equation (15), we get

$$\frac{\left[(n-1)(n+2) - y\,h_n\right]}{2n+1}\,x_n + \frac{3}{2}\,y\,\left[g_{n+1}x_{n+2} + g_{n-1}x_{n-2}\right] = -\frac{y}{2}\,\delta_{n,2} \quad , \qquad (18)$$

where $x_n = \frac{a_n}{a}$ and $y = \frac{\rho V^2}{2\gamma}$.

The result given in equation (18) can be used to determine the x_n in the same manner as was used in Page.[5] The resulting x_n can then be used in equation

(1) ($a_n = ax_n$) to determine the shape of the drop as a function of fluid velocity, V. Because of the similarity of equation (18) to the corresponding equation in Page,[5] it is expected that the drop will deviate considerably from an oblate spheroid for moderate fluid velocities or for large drop radii a.

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