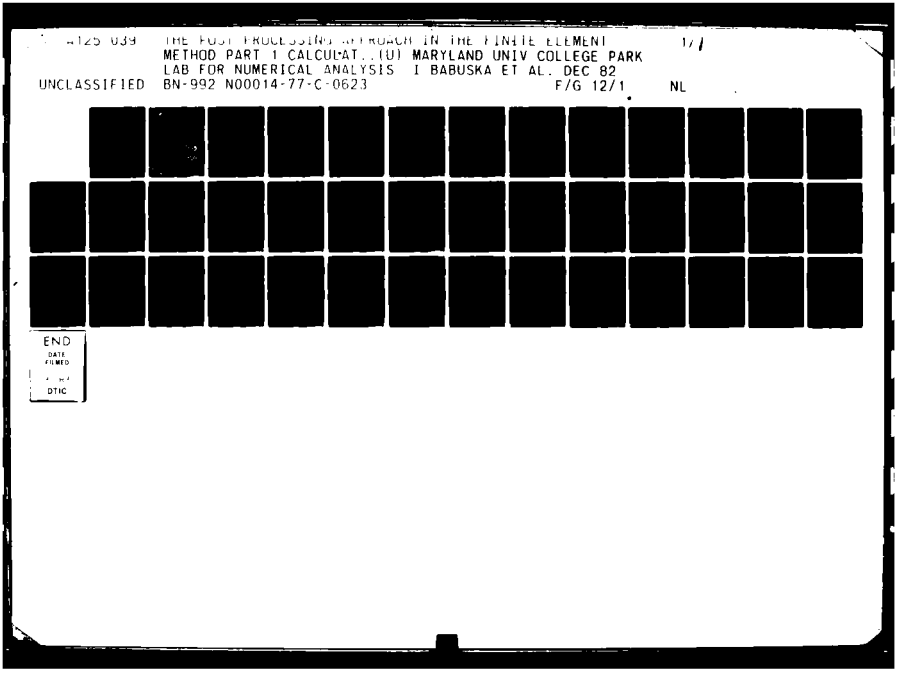
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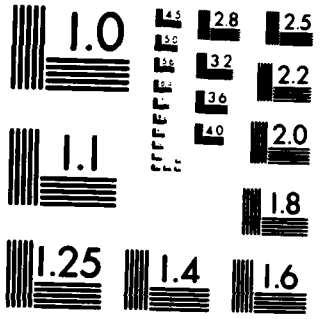

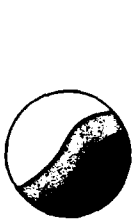

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

Laboratory for Numerical Analysis

Technical Note BN-992

THE POST-PROCESSING APPROACH IN THE FINITE ELEMENT METHOD

by

I. Babuška

and

T. Miller

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This is the first in a series of three papers in which we discuss a method for "post-processing" a finite element solution to obtain high accuracy approximations for displacements, stresses, stress intensity factors etc. Rather than take the values of these quantities "directly" from the finite element solution, we evaluate certain weighted averages of the solution over the entire region. These yield approximations that are of the same order of accuracy as the strain energy. We obtain error estimates, and also present some numerical examples to illustrate the practical effectivity of the technique. In the third paper

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of this series we address the matters of adaptive mesh selection and a'posteriori error estimation.

THE POST-PROCESSING APPROACH IN THE FINITE ELEMENT METHOD

PART 1: Calculation of Displacements, Stresses, and other Higher Derivatives of the Displacements

by

I. Babuška and A. Miller

Technical Note BN-992

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Abstract

This is the first in a series of three papers in which the author discuss a method for "post-processing" a finite element solution to obtain high accuracy approximations for displacements, stresses, stress intensity factors etc. Rather than take the values of these quantities "directly" from the finite element solution, we evaluate certain weighted averages of the solution over the entire region. These yield approximations that are of the same order of accuracy as the strain energy. We obtain error estimates, and also present some numerical examples to illustrate the practical effectivity of the technique. In the third paper of this series we address the matters of adaptive mesh selection and a'posteriori error estimation.

§1 Introduction

§1.1 The role of post-processing.

In many instances the primary aim of a finite element analysis is to obtain the values of a few important quantities with a rather high accuracy. For instance, in structural mechanics, the values of displacements, stresses or stress intensity factors at a small number of critical sites in the structure are important design criteria. Decisions on whether the structure meets design specifications, or whether it is safe are made on the basis of these few quantities. The bulk of the remaining numerical output of the finite element analysis is generally not scrutinized so closely, but rather is looked at from a more qualitative viewpoint. It may, for instance, be used to identify the critical points in the construction, to obtain a graphical display of the structure's deformation or to examine the solution's plausibility with a view to replacing, if need be, the particular mathematical model or constitutive law employed.

These considerations suggest that some thought should be paid to how the solution should be "post-processed" to obtain values for these quantities. Since only a few quantities ever need to be calculated, we should be willing, if necessary, to expend a modest amount of computational effort on any post-processing calculation. A straightforward approach to post-processing is to take the displacements or stresses directly as they are output from the finite element computations. "Curve fitting" methods for stress intensity factor calculations also fall into this "direct" category. There are, however, more sophisticated techniques which are currently implemented in many commercial codes. We mention, for instance, the calculation of stresses at the Gaussian points of certain element types, and the use of the "stiffness derivative" method of Parks [1] or the "J-integral" method of Rice [2] for the deter-

mination of stress intensity factors.

We shall show in this and two succeeding papers that there are post-processing procedures available which give an accuracy for many physically important quantities of the same order as the error in the energy of the finite element approximation. For example, in solving the plate problem (that is, the two dimensional biharmonic equation) with conforming elements of degree $p \geq 5$ it is possible (with an appropriate finite element mesh) to determine at any point the displacement, rotation, moment and shear force, all with an order of accuracy $O(N^{-(p-1)})$, where N is the number of degrees-of-freedom of the finite element model. Compare this with the "direct" approach which gives displacements to $O(N^{-(p+1)/2})$, rotation to $O(N^{-(p/2)})$ and moments to $O(N^{-(p-1)/2})$. The fact that the "direct" approach results in higher accuracy for displacements than moments is widely known. We remark also that the post processing approaches that we shall outline can be well understood from within the standard "energy" theory of finite elements. This contrasts with the complex mathematical theory which is needed to analyze the direct approach to the determination of pointwise quantities.

§1.2 A general form for post-processing calculations

Let us denote by Φ the quantity (e.g. stress at a point) which we wish to determine. We shall write $\Phi(w)$ for Φ to indicate that Φ relates to a problem whose exact solution is w. Now, let us suppose that we know a function ζ so that

$$\Phi = \Phi(w) = \int_{\Omega} w\zeta \, d\Omega + R \tag{1.1}$$

where Ω is the region on which our problem is posed, and R is an integral which may be computed using only the problem's input data (applied tractions

and forces etc.). As a simple illustration of what (1.1) may describe, consider the case where Φ is the value of w at some given point. Were the influence function (Green's function) known for this point, then we could take $\zeta=0$ and Φ would be expressible in terms of the input data alone. So (1.1) would take the form $\Phi=R$. Of course, the influence function is not in general available. At the other extreme, if we take ζ to be the Dirac delta function at the point under consideration then we could set $R=0$ and $\Phi = \int_\Omega w\zeta \, d\Omega$. This is simply a formal way of saying, evaluate w directly at the point. However, as we shall see later, there are many choices for ζ between these two extremes. Let us also mention that (1.1) may also be written in other equivalent forms (e.g. after an integration by parts we are able to obtain an integral in terms of the derivatives of w instead of w itself). The well known J-integral of Rice [2] could be thought of as arising from such a modified version of (1.1). The path independence property of the J-integral then corresponds to different choices for ζ.

Having (1.1) suggests an obvious method of approximation for Φ. If $\tilde{w}$ is a finite element approximation to w, then we could try to approximate $\Phi = \Phi(w)$ by

$$\tilde{\Phi} = \tilde{\Phi}(\tilde{w}) = \int_\Omega \tilde{w}\zeta \, d\Omega + R \, . \tag{1.2}$$

The difference between Φ and $\tilde{\Phi}$ is given by

$$e = \Phi - \tilde{\Phi} = \int_\Omega (w-\tilde{w})\zeta \, d\Omega \, , \tag{1.3}$$

and we see clearly that the choice of ζ affects the magnitude of this difference. If ζ is the Dirac delta function, then $\tilde{\Phi}$ is the point value of $\tilde{w}$ and e is the pointwise difference between w and $\tilde{w}$.

If however ζ is not concentrated at one point, then e becomes some weighted average of $w-\tilde{w}$. It is well known that the finite element method appears more reliable when its accuracy is measured in an average, rather than a pointwise sense. Spurious oscillation which may cause serious loss of accuracy in pointwise values of the approximate solution (especially of its higher derivatives) are filtered out by averaging. This is especially the case with the p-version of the finite element method. So, even at this early stage, we see that choices of ζ's which have large support are likely to give superior approximations $\tilde{\Phi}$. A good choice for ζ is an important feature of any successful implementation of (1.2). Numerical experience however has shown that, provided ζ meets a few simple criteria, $\tilde{\Phi}$ is quite insensitive to the choice of ζ.

Another implementation issue that we shall address is the optimal choice of mesh (and consequently of $\tilde{w}$) for use in (1.2) This is especially significant for adaptive finite element codes, where some adaptive criteria must be set. Obviously, if a good approximation to Φ is our ultimate goal, this adaptive criteria should be directed towards producing a $\tilde{w}$ that performs well in (1.2).

From a computational point of view, evaluation of $\tilde{\Phi}$ needs at most $O(N)$ operations, while the solution of the finite element problem itself usually needs about $O(N^2)$ operations in two dimensions and $O(N^{7/3})$ in three dimensions. Since only a few evaluations are ever needed, the computational effort entailed is relatively insignificant.

§1.3 Outline of the paper

This is the first of a series of three papers which shall deal with post-processing in the finite element method. In this paper we detail some particular applications of the general theory outlined in §1.2. In §2

we treat the example of a one-dimensional, elastic supported beam. Here we shall be interested in post-processed values for the displacement, rotation, moment and shear force at certain points, along with the average displacement over a subsection of the beam. §3 deals with the simple two dimensional problem of a membrane on an elastic support. For this problem we shall be interested in the displacements and stresses at certain points. Finally in §4 we discuss a numerical example related to the problem treated in §3.

The second paper of the series will be devoted to applications in linear facture mechanics. In particular, we shall be concerned with the computation of stress intensity factors (including both k_1 and k_2 for mixed mode fracture). In the third paper we address the issues of a'posteriori error estimates for $\tilde{\phi}$ and adaptive mesh selection for $\tilde{w}$.

§2. A One-dimensional Example

§2.1 Formulation of the example

By way of a one-dimensional application of the techniques outlined in §1.2 we shall consider the problem of a clamped beam (of unit length) on an elastic support. The governing differential equation is

$$(2.1) \qquad (aw'')'' - (bw')' + cw = f \qquad \text{on } (0,1)$$

with the boundary conditions

$$(2.2) \qquad w(0) = w'(0) = 0 \quad \text{and} \quad w(1) = w'(1) = 0 \ .$$

We shall assume that a, b, c and f are smooth functions which satisfy

$$(2.3) \qquad a(t) \geq \alpha > 0 \qquad (0 < t < 1)$$
$$b(t),\ c(t) \geq 0 \ .$$

The coefficient a is the rigidity of the beam while b and c relate to the elastic properties of the support. For this problem we shall be interested in the evaluation of the following five important mechanical quantities:

(I) The displacement of the beam $\Phi_1 = \Phi_1(w) = w$ at $0 < \bar{t} < 1$.

(II) The rotation of the beam $\Phi_2 = \Phi_2(w) = w'$ at $0 < \bar{t} < 1$.

(III) The bending moment $\Phi_3 = \Phi_3(w) = aw''$ at $0 \leq \bar{t} \leq 1$.

(IV) The shear force $\Phi_4 = \Phi_4(w) = (aw'')' - bw'$ at $0 \leq \bar{t} \leq 1$.

(V) The average displacement of the subsection $t_1 \leq t \leq t_2$ of the beam,

$$\Phi_5 = \Phi_5(w) = (t_2 - t_1)^{-1} \int_{t_1}^{t_2} w \, dt \ .$$

§2.2 Expressions for Φ_i (i=1, ..,4)

For the moment, let ϕ be any function defined on (0,1) which satisfies the boundary conditions (2.2). Suppose also that ϕ is sufficiently smooth to allow any operations that we carry out. Now, multiply (2.1) by ϕ and integrate by parts four times over the entire interval,

$$\int_0^1 f\,\phi\,dt = \int_0^1 ((aw'')''\phi - (bw')'\phi + cw\phi)\,dt$$

$$= [(aw'')'\phi - aw''\phi' + aw'\phi'' - w(a\phi'')']_{\bar{t}+0}^{\bar{t}-0}$$

$$-[bw'\phi - bw\phi']_{\bar{t}+0}^{\bar{t}-0} + \left(\int_0^{\bar{t}} + \int_{\bar{t}}^1\right) L[\phi]w\,dt$$

$$= [w(-(a\phi'')' + b\phi') + w'(a\phi'') + aw''(-\phi') +$$

$$(2.4)\qquad ((aw'')' - bw')\phi]_{\bar{t}+0}^{\bar{t}-0} + \left(\int_0^{\bar{t}} + \int_{\bar{t}}^1\right) L[\phi]w\,dt$$

where

$$L[\phi] = (a\phi'')'' - (b\phi')' + c\phi.$$

Let us now be more specific about the behavior of ϕ near $\bar{t}$. Depending upon which derivatives of ϕ are continuous at $\bar{t}$, (2.4) will form the basis of our expressions for Φ_i (i=1, ..,4).

Case (I): Suppose that

$$(2.5a)\qquad [\,\phi^{(i)}\,]_{\bar{t}+0}^{\bar{t}-0} = 0 \quad (i=0,1,2) \qquad \text{while}$$

$$[\,\phi^{(3)}\,]_{\bar{t}+0}^{\bar{t}-0} = -a(\bar{t})^{-1},$$

then (2.4) gives

$$\phi_1(w) = w(\bar{t}) = -\left(\int_0^{\bar{t}} + \int_{\bar{t}}^1 \right) L[\phi]w \, dt + \int_0^1 f\phi \, dt \, .$$

This is exactly in the form (1.1) with $R = \int_0^1 f\phi \, dt$ and $\zeta = -L[\phi]$. Notice that $L[\phi]$ will in general be discontinuous at $\bar{t}$. We shall see later that from a numerical point of view it is important that ζ be smooth on $(0,1)$. So let us append to (2.5a) the condition

$$(2.5b) \qquad [\, L[\phi]^{(j)} \,]_{\bar{t}+0}^{\bar{t}-0} = 0 \qquad j=0,\ldots,n$$

where n is some integer, which, for the moment, will remain arbitrary.

If we select for ϕ the influence function (Green's function), then (2.5a) and (2.5b) are satisfied. Indeed, $L[\phi] = 0$ on $(0,\bar{t})$ and $(\bar{t},1)$, and we have $\phi_1(w) = \int_0^1 f\phi \, dt$. Of course, in general, we cannot find the influence function. Nevertheless, functions ϕ which satisfy (2.5a) and (2.5b) are readily constructed, as the following example shows:

Example 1: We shall construct a function ϕ which fulfills all the necessary conditions, with $n=1$ in (2.5b). The construction is done in a number of steps. First, define

$$\phi_0(t) = \begin{cases} 0 & 0 < t \leq \bar{t} \\ a(\bar{t})^{-1} & \bar{t} < t < 1 \end{cases} ,$$

and then set

$$\phi_1(t) = \int_0^t \int_0^y \int_0^x \phi_0(s) ds \, dx \, dy \, .$$

So $\phi_1(t)$ satisfies (2.5a). To meet the requirement (2.5b) with $n=1$ we

define

$$\phi_2(t) = \begin{cases} \phi_1(t) + \alpha (t-\bar{t})^4 + \beta (t-\bar{t})^5 & 0 < t \leq \bar{t} \\ \phi_1(t) & \bar{t} < t < 1 \end{cases}$$

where the coefficients α and β are to be chosen so that (2.5b) holds. It is easy to see that this needs

$$\alpha = \frac{-1}{4!a(\bar{t})} \quad L[\phi_1] \Big|_{\bar{t}+0} \qquad \text{and}$$

$$\beta = \frac{-1}{5!a(\bar{t})} (L[\phi_1]^{(1)} \Big|_{\bar{t}+0} + 3\, a'(t) 4! \alpha)$$

(This step can be extended in an obvious fashion to handle $n > 1$.)

We now have a function that satisfies (2.5), however it may not yet satisfy the boundary conditions (2.2). There are many ways this can be remedied. Two possibilities are:

(i) Let χ be a smooth function which vanishes, along with its first derivative, at $t=0$ and $t=1$, but has a value of 1 in an interval about $\bar{t}$. Then $\phi = \chi\phi_2$ meets all the requirements. We shall refer to χ as a cut off function.

(ii) Let ϕ_3 be a smooth function such that $\phi_3^{(i)}(0) = \phi_2^{(i)}(0)$ and $\phi_3^{(i)}(1) = \phi_2^{(i)}(1)(i=0,1)$. Then we may set $\phi = \phi_2 - \phi_3$. We shall refer to ϕ_3 as a blending function. Various mixtures of these two techniques are possible; for example, using a cut off function to impose the boundary condition at one end, and a blending function approach to handle the other endpoint.□

The above example typifies a general method of construction that we shall employ, either explicitly or implicitly, throughout this series of papers: Firstly, a relatively simple function is constructed that behaves in some prescribed "singular" manner near a given point. This function must then be modified to ensure that it satisfies a set of boundary conditions on the entire boundary of the region of interest. Either cut off function, blending function or a combination of these techniques may be used to achieve this. Let us note at this point, that blending function techniques will usually lead to a smoother modified function than will the use of a cutoff function. For this reason, in a numerical setting, the blending function approach is to be preferred.

Case II: Suppose that

$$[\,\phi^{(i)}\,]_{\bar{t}+0}^{\bar{t}-0} = 0 \qquad (i=0,1)\ ,$$

$$(2.6a)\quad [\,\phi''\,]_{\bar{t}+0}^{\bar{t}-0} = a(\bar{t})^{-1}\ , \qquad \text{and}$$

$$[\,(a\phi'')'\,]_{\bar{t}+0}^{\bar{t}-0} = 0$$

then (2.4) gives

$$\phi_2(w) = w'(\bar{t}) = -\left(\int_0^{\bar{t}} + \int_{\bar{t}}^1\right) L[\phi]\, w\, dt + \int_0^1 f\,\phi\, dt\ .$$

As for Case I, it will turn out to be important to add the condition

$$(2.6b) \qquad \left[L[\phi]^{(j)} \right]_{\bar{t}+0}^{\bar{t}-0} = 0 \qquad j=0,\ldots,n \ .$$

Example 2: We shall construct a function ϕ that meets our requirements. We proceed much as in Example 1. First we define ϕ_0 as in that example, but now set

$$\phi_1(t) = -\int_0^t \int_0^x \phi_0(s)\, ds\, dx \ .$$

To satisfy (2.6b) (with n=1, say) and the third part of (2.6a) we may define

$$\phi_2(t) = \begin{cases} \phi_1(t) + \alpha(t-\bar{t})^3 + \beta(t-\bar{t})^4 + \gamma(t-\bar{t})^5 & 0 < t \leq \bar{t} \\ \phi_1(t) & \bar{t} < t < 1 \end{cases}$$

where α,β and γ are to be chosen so that (2.6b) and the last part of (2.6a) hold. (That this may always be done follows since $a(\bar{t})\neq 0$.) We then proceed to impose the boundary conditions (2.2) by the same methods described in Example 1.□

Cases III and IV: If $\bar{t}$ is internal to the interval then selecting

$$(2.7) \qquad \begin{aligned} &\left[\phi \right]_{\bar{t}+0}^{\bar{t}-0} = 0 \ , \\ &\left[\phi' \right]_{\bar{t}+0}^{\bar{t}-0} = -1 \ , \\ &\left[\phi'' \right]_{\bar{t}+0}^{\bar{t}-0} = 0 \ , \\ &\left[-(a\phi'')' + b\phi' \right]_{\bar{t}+0}^{\bar{t}-0} = 0 \ , \text{ and} \\ &\left[L[\phi]^{(j)} \right]_{\bar{t}+0}^{\bar{t}-0} = 0 \qquad j=0,\ldots,n \ , \end{aligned}$$

gives

$$\phi_3(w) = aw''(\bar{t}) = -\left(\int_0^{\bar{t}} + \int_{\bar{t}}^1\right) L[\phi]w\, dt + \int_0^1 f\phi\, dt$$

with smooth $L[\phi]$. While choosing ϕ to satisfy

$$[\,\phi\,]_{\bar{t}+0}^{\bar{t}-0} = 1 \quad ,$$

$$[\,\phi''\,]_{\bar{t}+0}^{\bar{t}-0} = 0 \quad ,$$

(2.8)

$$[\,a\phi''\,]_{\bar{t}+0}^{\bar{t}-0} = 0 \quad ,$$

$$[\,-(a\phi'')' + b\phi'\,]_{\bar{t}+0}^{\bar{t}-0} = 0 \quad , \text{ and}$$

$$[\,L[\phi]^{(j)}\,]_{\bar{t}+0}^{\bar{t}-0} = 0 \quad (j=0,\ldots,n)$$

leads to the expression

$$\phi_4(w) = ((aw'')' - bw')(\bar{t}) = -\left(\int_0^{\bar{t}} + \int_{\bar{t}}^1\right) L[\phi]w\, dt + \int_0^1 f\phi\, dt$$

for $\phi_4(w)$.

To treat these two cases when $\bar{t}$ is one of the endpoints we need a slightly different argument. For definiteness, suppose that $\bar{t}$ is the right hand endpoint $\bar{t}=1$. We now let ϕ be a smooth function on $(0,1)$ which satisfies the boundary condition $\phi(0) = \phi'(0) = 0$. Analogous to (2.4) we now have

$$\int_0^1 f\phi\, dt = \int_0^1 ((aw'')''\phi - (bw')'\phi + c\, w\phi)\, dt$$

$$= ((aw'')' \phi - aw'' \phi')\Big|_{t=1} + \int_0^1 L[\phi]w\,dt \ . \tag{2.9}$$

If we further set

$$\phi(1) = 0 \quad \text{and} \quad \phi'(1) = -1 \tag{2.10}$$

we obtain after rearranging (2.9)

$$\Phi_3(w) = aw''(1) = -\int_0^1 L[\phi]\ w\,dt + \int_0^1 f\phi\ dt \ ,$$

while, using in place of (2.10)

$$\phi(1) = 1 \quad \text{and} \quad \phi'(1) = 0 \ , \tag{2.11}$$

leads to the expression

$$\Phi_4(w) = (aw'')'(1) = -\int_0^1 L[\phi]\ w\,dt + \int_0^1 f\phi\ dt \ .$$

Functions ϕ satisfying (2.7), (2.8), (2.10) or (2.11) and the appropriate boundary conditions are readily constructed using techniques similar to those described in Examples 1 and 2.

§2.3. An expression for Φ_5 .

The definition of Φ_5 in §2.1 is already in the form (1.1), for we may certainly write

$$\Phi_5 = \Phi_5(w) = \int_0^1 w\ \zeta_o dt \ , \tag{2.12}$$

where $\zeta_o = \begin{cases} (t_2 - t_1)^{-1} & t_1 \le t \le t_2 \\ 0 & \text{otherwise} \end{cases}$.

However, from our point of view, this is an unsuitable expression for ϕ_5 since ζ_o is not smooth. To overcome this failing we may proceed as follows: Define

$$\phi_1(t) = \int_0^t \int_0^{s_4} \frac{1}{a(s_3)} \int_0^{s_3} \int_0^{s_2} \zeta_0(s_1) ds_1\, ds_2\, ds_3\, ds_4 \quad .$$

Let ϕ_2 be a smooth blending function that satisfies the boundary conditions

$$\phi_2^{(i)}(0) = \phi_1^{(i)}(0) \qquad \text{and} \qquad \phi_2^{(i)}(1) = \phi_1^{(i)}(1) \qquad (i=0,1)$$

and set $\phi = \phi_1 - \phi_2$. Multiply (2.1) by ϕ and integrate by parts four times over the entire interval,

$$\begin{aligned} \int_0^1 f\phi &= \int_0^1 (aw'')''\phi - (bw')'\phi + c\phi w \, dt \\ &= \int_0^1 L[\phi]w \, dt \\ &= \int_0^1 (L[\phi_1]w - L[\phi_2]w)dt \\ &= \int_0^1 (\zeta_o w - (b\phi_1')w + c\phi_1 w - L[\phi_2]w)dt \quad . \end{aligned}$$

Upon rearrangement then

$$\phi_5 = \phi_5(w) = \int_0^1 ((b\phi_1')' - c\phi_1 + L[\phi_2])w dt + \int_0^1 f\,\phi \, dt$$

which is of the form discussed in §1.2 with

$$\zeta = (b\phi_1')' - c\phi_1 + L[\phi_2] \qquad \text{and} \qquad R = \int_0^1 f\,\phi \, dt \quad .$$

In contrast to ζ_o, notice that ζ has a continuous first derivative (though, in general, a discontinuous second derivative at t_1 and t_2). A ζ with more smoothness can be constructed by iterating the above process.

§2.4 The accuracy of the approximations $\tilde{\phi}_i (i=1,..,5)$

In §2.2 and §2.3 we derived some integral expressions for the ϕ_i. These fitted into the general pattern discussed in §1.2. We shall now address the important question of the accuracy of the approximations $\tilde{\phi}_i = \phi_i(\tilde{w})$ which arise when the finite element solution $\tilde{w}$ is used in place of w in these expressions.

To be definite, suppose we have set up a finite element model of (2.1)/(2.2) using C^1 polynomial elements of degree $p(\geq 3)$. Write $I_1,\ldots,I_N$ for the intervals which comprise the finite element mesh. Denote by S the set of all admissible finite element functions. Let $h = \max_k$ (length I_k). In the case of the problem (2.1)/(2.2) the fundamental orthogonality property of the finite element error $w-\tilde{w}$ takes the specific form

$$\int_0^1 (a(w-\tilde{w})''v'' + b(w-\tilde{w})'v' + c(w-\tilde{w})v)\, dt = 0 \tag{2.13}$$

for all v in S. Denoting by $E(\cdot)$ the energy expression

$$E(\cdot) = \int_0^1 (a((\cdot)'')^2 + b((\cdot)')^2 + c(\cdot)^2)\, dt \quad ,$$

the standard finite element error estimate may be written as

$$E(w-\tilde{w}) \leq \min_{v^* \in S} E(w-v^*) \tag{2.14}$$

where the minimum is taken over all v^* from S.

For the purposes of the analysis, we need to introduce the auxilliary function ψ which satisfies

$$L[\psi] = \zeta \quad \text{on } (0,1) \quad \text{and} \tag{2.15.a}$$

$$\psi(0) = \psi'(0) = 0 = \psi(1) = \psi'(1) \ ,$$

or what, after integration by parts, is the same thing

$$(2.15b) \qquad \int_0^1 (a\psi''u'' + b\psi'u' + c\psi u)\,dt = \int_0^1 \zeta u\,dt$$

for all u with boundary values as in (2.2). Now, recalling (1.3), we have the following estimate for the error $\Phi - \tilde{\Phi}$

$$e = \Phi - \tilde{\Phi} = \int_0^1 \zeta(w-\tilde{w})\,dt$$

$$= \int_0^1 (a(w-\tilde{w})''\psi'' + b(w-\tilde{w})'\psi' + c(w-\tilde{w})\psi)\,dt$$

(putting $u = w-\tilde{w}$ in (2.15b)

$$= \int_0^1 (a(w-\tilde{w})''(\psi-v)'' + b(w-\tilde{w})'(\psi-v)' + c(w-\tilde{w})(\psi-v))\,dt$$

for any v from S (by(2.13)). So

$$|\Phi-\tilde{\Phi}| \le \min_{v\in S} \left(\int_0^1 (a(w-\tilde{w})''(\psi-v)'' + b(w-\tilde{w})'(\psi-v)' + c(w-\tilde{w})(\psi-v))\,dt\right)$$

$$\le \min_{v\in S} (E(w-\tilde{w})^{\frac{1}{2}} E(\psi-v)^{\frac{1}{2}})$$

$$(2.16) \qquad \le \min_{v^*\in S}(E(w-v^*)^{\frac{1}{2}}) \min_{v\in S}(E(\psi-v)^{\frac{1}{2}})$$

using (2.14). In words then, the error in Φ is bounded by the product of the energy norm difference between w and its best approximation from S, and the energy norm difference between ψ and its best approximation from S. The importance of the smoothness of ζ can now be appreciated. Smooth functions ζ will give smooth auxillary functions ψ, and these will be approximated well by the functions in S.

Let us now try to obtain an asymtotic rate of convergence for $\tilde{\Phi}$ which will be applicable to both the h and p-versions of the finite element method. First, recall the approximation result (see [3]): If z is a

function defined on (0,1) and if the smoothness measuring quantity

$$||z||_s = \left(\sum_{\ell=0}^{s} \sum_{k=1}^{N} \int_{I_k} |z^{(\ell)}|^2 \, dt \right)^{\frac{1}{2}}$$

is finite for some integer $s \geq 2$, then

$$\min_{v \in S} E(z-v) \leq C_1 \frac{h^{2m}}{p^{2(s-2)}} ||z||_s^2 \tag{2.17}$$

where C_1 is a constant which does not depend on the function z, the finite element mesh or the order of elements used; and $m=\min(p-1,s-2)$. If the load f in (2.1) and the function ζ are smooth enough to guarantee that $||f||_{s_1} < \infty$ and $||\zeta||_{s_2} < \infty$, then, provided the coefficients a, b and c are sufficiently smooth, it may be shown that

$$\begin{aligned} ||w||_{s_1+4} &\leq C_2 ||f||_{s_1} \quad \text{and} \\ ||\psi||_{s_2+4} &\leq C_3 ||\zeta||_{3_2} \end{aligned} \tag{2.18}$$

where C_2 and C_3 do not depend on f and ζ respectively. Having these smoothness properties of w and ψ, we may make use of the approximation bounds (2.17) in (2.16) to obtain

$$\begin{aligned} |\phi-\tilde{\phi}| &\leq C_1 \frac{h^{(m_1+m_2)}}{p^{(s_1+s_2+4)}} ||w||_{s_1+4} ||\psi||_{s_1+4} \\ &\leq C_1 C_2 C_3 \frac{h^{(m_1+m_2)}}{p^{(s_1+s_2+4)}} ||f||_{s_1} ||\zeta||_{s_2} \end{aligned} \tag{2.19}$$

where $m_i = \min(p-1, s_i+2)\ (i=1,2)$. Likewise (2.14) gives

$$E(w-\tilde{w}) \leq C_1 C_2^2 \frac{h^{2m_1}}{p^{2(s_1+2)}} ||f||_{s_1}^2 .$$

We mentioned in §2.2 and 2.3 that the relevant ζ's could be made arbitrarily smooth (i.e. n in (2.5b) etc. was arbitrary). For large values of n this could become laborious. Note that (2.19) allows discontinuities at meshpoints without any adverse effects on the accuracy of $\tilde{\Phi}$. What is important is that ζ be smooth in the interior of each of the I_k. In the h-version of the finite element method, there would, at least from our analysis, seem to be no reason to proceed any further than the stage at which $s_2 + 2 = p-1$. At this stage $|\Phi-\tilde{\Phi}| = O(h^{m_1 + p-1})$, and this rate would not be improved by increasing s_2. Comparing this with $E(w-\tilde{w}) = O(h^{2m_1})$, we see, as was previewed in §1, that the error $|\Phi-\tilde{\Phi}|$ is at least of the same order as the energy of the error in the finite element solution. For the p-version there would seem to be no such limit. We may increase s_2 indefinitely always improving the convergence rate for $\tilde{\Phi}$ as we go. Here we have $|\Phi-\tilde{\Phi}| = O(p^{-(s_1+s_2+4)})$ and $E(w-\tilde{w}) = O(p^{-2(s_1+2)})$.

Of course, actual computations must be carried out working from only a limited range of non zero h's and finite p's. In such a setting, the asymtotic rate of convergence alone is not necessarily a good indication of an approximation's accuracy. As (2.19) shows, the error in $\tilde{\Phi}$ is related not only to p and h, but also to $||\zeta||_{s_2}$ and the constants C_i. As s_2 increases, the numerical values of these quantities may also increase dramatically with the net effect that there is a loss of accuracy in $\tilde{\Phi}$. In practice, it is usually more important to ensure that the numerical value of $||\zeta||_{s_2}$ is reasonable, than to construct ζ's with high orders of continuity. Recall also our comment earlier that blending function techniques are generally superior to cut-off function methods in this respect. Another important practical consideration is the choice of a finite

element mesh. As (2.16) shows the accuracy of $\hat{\phi}$ is related to the approximability of both w and ψ. So, an optimal mesh for calculating $\hat{\phi}$ would be one which was, in some way, simultaneously good for both the original problem (2.1)/(2.2) and the auxillary problem (2.15a). We shall explore this question in the third of this series of papers. The sorts of concerns touched upon in this paragraph are of great importance in two dimensional problems. For many such problems there is no complete analogue to 2.18), and we are denied the luxury of being able to make ψ as smooth as we wish.

§3 Two Dimensional Problems

§3.1 Formulation of the problem

To illustrate the ideas of §1.2 in a two-dimensional setting, we shall consider in some detail the simple model example of

$$\nabla^2 w - kw = f \qquad \text{in } \Omega \text{ , a polygonal region,} \tag{3.1}$$

with the boundary condition

$$w = 0 \qquad \text{on } \partial\Omega \text{ , the boundary of } \Omega \text{ .} \tag{3.2}$$

Here we suppose that $k \geq 0$ and assume, for simplicity, that k is a constant and f a smooth function. The problem (3.1)/(3.2) could, for example, be thought of as describing a polygonal membrane on an elastic support, that is fixed along its edges. We shall be concerned with evaluating the following quantities which are related to w :

(i) The displacement $\Phi_1 = \Phi_1(w) = w(\bar{x})$ at a point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ in Ω .

(ii) The stress $\Phi_2 = \Phi_2(w) = \nabla w \cdot \hat{n}(\bar{y})$ at a point $\bar{y} = (\bar{y}_1, \bar{y}_2)$ on $\partial\Omega$, which is "far" from a corner point. (The case of $\bar{y}$ "close" to a corner point will be discussed in our second paper.)

§3.2 An integral for Φ_1

Let $\phi(x)$ be an arbitrary function defined and sufficiently smooth on $\Omega - \{\bar{x}\}$, which vanishes on $\partial\Omega$. For $\varepsilon > 0$, small enough, denote by S_ε a disc with centre $\bar{x}$ and radius ε which lies in Ω . Multiply (3.1) by ϕ and integrate over $\Omega - S_\varepsilon$. Using Green's Theorem, we obtain

$$\int_{\Omega - S_\varepsilon} f\,\phi \, dA = \int_{\partial S_\varepsilon} (\nabla w \cdot \hat{n}\phi - \nabla\phi \cdot \hat{n}\, w)\, ds + \int_{\Omega - S_\varepsilon} L[\phi] w \, dA \,, \tag{3.3}$$

where $L[\cdot] = \nabla^2(\cdot) - k(\cdot)$ and $\hat{n}$ denotes the unit normal on ∂S_ε pointing towards $\bar{x}$. Now, impose the extra conditions

$$(3.4)\qquad \left.\begin{aligned} \phi(x) &= (2\pi)^{-1}\log \bar{r}(x) + 0(1) \\ \nabla\phi(x) &= \nabla((2\pi)^{-1}\log \bar{r}(x)) + o(\bar{r}(x)^{-1}) \end{aligned}\right\} \qquad (\text{as } x \to \bar{x})$$

where $\bar{r}(x) = ((x_1-\bar{x}_1)^2 + (x_2-\bar{x}_2)^2)^{\frac{1}{2}}$. Then, in the limit as $\varepsilon \to 0$, (3.3) yields

$$(3.5)\qquad \ell_1(w) = w(\bar{x}) = -\int_\Omega L[\phi]w\, dA + \int_\Omega f\phi\, dA$$

Note that the integrals appearing on the right hand side of (3.5) are possibly improper. We see that (3.5) is precisely in the form required by (1.1) with $\zeta = -L[\phi]$ and $R = \int_\Omega f\,\phi\, dA$. Notice also that were ϕ the influence function (Green's function) for (3.1)/(3.2), then the first integral on the right hand side of (3.5) vanishes. In general, of course, the influence function is not available.

Just as in §2, it will turn out that from a numerical viewpoint, it is important for ζ to be a smooth function. The problem of selecting a suitable ζ , or what is the same thing, of choosing ϕ appropriately can be thought of as having two aspects. Firstly, ensuring that $L[\phi]$ is smooth in the immediate neighborhood of $\bar{x}$; and secondly, of imposing the boundary conditions on ϕ in such a way that no unsmooth behavior of ϕ is introduced.

Let us talk in more detail about these points as they relate to our model problem. It is easy to verify that if

$$(3.6)\qquad \tilde{\phi}(x) = (2\pi)^{-1}(1 + \tfrac{k}{4}\bar{r}(x)^2)\log \bar{r}(x) ,$$

then $L[\tilde{\phi}] = 0(\bar{r}^2 \log \bar{r})$ in the vicinity of $\bar{x}$. Now $\tilde{\phi}$ has the required asymtotic behaviour (3.4), however, it does not vanish on $\partial\Omega$, and so cannot be used directly for ϕ in (3.5). Let us suppose for a moment that $\bar{x}$ is not too close to $\partial\Omega$. Then, as for the one-dimensional case, there are

a number of techniques for modifying $\tilde{\phi}$. We could, for instance, proceed in one of the following ways:

(i) Let $X(x)$ be a smooth "cut off" function which vanishes on $\partial\Omega$, but is a constant, equal to 1, in a neighborhood of $\bar{x}$. Then $\phi(x) = X(x)\,\tilde{\phi}(x)$ satisfies all our requirements.

(ii) Let $\phi_0(x)$ be a smooth "blending" function on Ω which agrees with $\tilde{\phi}(x)$ on $\partial\Omega$. We could then take $\phi(x) = \tilde{\phi}(x) - \phi_0(x)$.

(iii) Use a combination of the above two techniques--"cut off" functions to handle part of the boundary, "blending" functions for the remainder.

In the case that $\bar{x}$ is very close to $\partial\Omega$, technique (i) is not the proper method to use, as the "cut off" function X would then have large derivatives in the region between $\bar{x}$ and $\partial\Omega$. Methods (ii)/(iii) provide better ways of handling this case. Note however, that though $L[\phi]$ is smooth near $\bar{x}$, $\tilde{\phi}$ is not. For an arbitrary "blending" function ϕ_0, agreeing with $\tilde{\phi}$ on $\partial\Omega$ near $\bar{x}$, there is no reason to expect that $L[\phi_0]$ be smooth. One way around this, at least in the case when $\bar{x}$, though close to the boundary of Ω, is far from a corner point of the boundary, is to formally extend $\tilde{\phi}$ across the straight line segment of the boundary closest $\bar{x}$. Now let ψ be the reflection of this extension back into Ω. Then $L\psi$ is smooth, and ψ agrees with $\tilde{\phi}$ on the straight line segment of the boundary closest $\bar{x}$. Standard "blending" techniques may then be used to deal with the remaining three sides of Ω. There are obvious extensions of the above ideas to domains with curved boundaries.

§3.3 An integral for Φ_2

For definiteness, suppose that $\bar{y}$ lies on the straight line segment $((1,-1), (1,1))$ which forms part of $\partial\Omega$. This time, let ϕ be an arbitrary, sufficiently smooth function defined on Ω, which vanishes on $\partial\Omega - \{\bar{x}\}$. For $\varepsilon > 0$, small enough, denote by $S_\varepsilon^+(\bar{x})$ a half disc with centre $\bar{x}$ and radius ε, which lies in Ω. Multiply (3.1) by ϕ and integrate over $\Omega - S_\varepsilon^+$. Using Green's Theorem

we obtain

$$(3.7)\qquad \int_{\Omega - S_\varepsilon^+} f\,\phi\,dA = \int_{\Gamma_\varepsilon} (\nabla w\cdot\hat{n}\phi - \nabla\phi\cdot\hat{n}w)ds + \int_{\Omega - S_\varepsilon^+} L[\phi]\,w\,dA\ ,$$

where Γ_ε denotes the circular portion of the boundary of S_ε^+, and as usual $\hat{n}$ denotes a unit normal pointing towards the centre of S_ε^+ (see Fig. 1). Now if we impose the extra conditions

$$(3.8)\qquad \left.\begin{aligned} \phi(x) &= \frac{1}{\pi}\frac{\cos\bar{\theta}(x)}{\bar{r}(x)} + o(\bar{r}(x)^{-1}) \\ \nabla\phi(x) &= \nabla\left(\frac{1}{\pi}\frac{\cos\bar{\theta}(x)}{\bar{r}(x)}\right) + o(\bar{r}(x)^{-2}) \end{aligned}\right\} \quad \begin{array}{l}(\text{as } x \to \bar{y} \\ \text{from within } \Omega\ )\end{array}$$

where $\bar{r}(x)$, $\bar{\theta}(x)$ are plane-polar coordinates centered on $\bar{y}$ (see Fig. 1), then, in the limit as $\varepsilon \to 0$, (3.7) gives

$$(3.9)\qquad \zeta_2(w) = \frac{\partial w}{\partial x_1}(\bar{y}) = \int_\Omega L[\phi]\,w\,dA - \int_\Omega f\,\phi\,dA\ .$$

This is in the form of (1.1) with $\zeta = L[\phi]$ and $R = \int_\Omega f\,\phi\,dA$. In general the integrals on the right hand side of (3.9) will be improper.

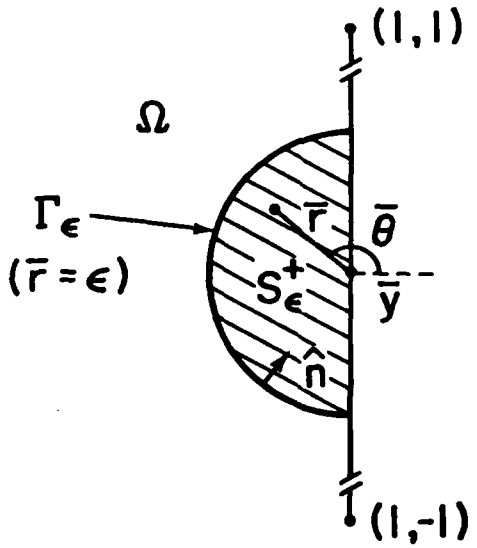


Figure 1: The region S_ε^+.

As usual we should choose ϕ such that ζ is smooth. This problem can be approached in an analagous fashion to that outlined in §3.2. In our case it can be verified that if

$$\tilde{\phi}(x) = \frac{\partial}{\partial x_1}\left(\frac{1}{\pi}\left(1 + \frac{k}{4}\,\bar{r}(x)^2\right)\log \bar{r}(x)\right) \quad , \tag{3.10}$$

then $L[\tilde{\phi}]$ is smooth about $\bar{x}$ (in fact, $L[\tilde{\phi}] = 0\,(\bar{r}(x)\log \bar{r}(x))$. Now $\tilde{\phi}$ has the necessary asymtotic and boundary behavior near $\bar{y}$, but it does not vanish everywhere on $\partial\Omega$. To construct suitable ϕ's based on $\tilde{\phi}$ we may use obvious adaptations of the "cut off" or "blending" techniques outlined in §3.2.

§3.4 More general problems than (3.1)/(3.2)

The methods of §3.2 and §3.3 may be applied to problems more general than (3.1)/(3.2) (e.g. problems in elasticity, problems with non-constant coefficients or non-homogeneous boundary conditions). We shall not go into details here. Let us however, just mention that for the problem (3.1) with the essential boundary condition (3.2) replaced by the natural boundary condition $\nabla w \cdot \hat{n} = 0$, there would have been no need in §3.2 to impose any boundary condition on ϕ. The expression (3.5) in this case would have to include an additional term, namely, a line integral around $\partial\Omega$.

§3.5 The accuracy of the approximations $\tilde{\phi}_1$, $\tilde{\phi}_2$

Suppose that we set up a finite element model of the problem (3.1)/(3.2). In the usual way let us partition Ω into elements $E_1, E_2, \ldots, E_n$ say (we do not need to be specific about the shapes of the elements), and assume that on each element we represent $\tilde{w}$ by a polynomial of degree p. The demand of conformity requires that these polynomials be continuous across the inter-element boundaries, and vanish on $\partial\Omega$. Denote by S the set of all such finite element functions. Let h_j be a characteristic linear dimension of E_j, and set $h = \max_j h_j$.

The finite element solution $\tilde{w}$ satisfies

$$(3.11) \qquad \int_\Omega (\nabla\tilde{w} \nabla v + k \tilde{w} v)\, dA = -\int_\Omega f v\, dA$$

for all v from S; in addition, we have

$$(3.12) \qquad \int_{\Omega} (\nabla(w-\tilde{w})\ \nabla v + k(w-\tilde{w})v)\ dA = 0$$

for all finite element functions v in S. Defining the strain energy expression by

$$E(\cdot) = \int_{\Omega} (\nabla(\cdot)^2 + k(\cdot)^2)\, dA\ ,$$

we have

$$(3.13) \qquad E(w-\tilde{w}) \leq \min_{v \in S} E(w-v) \quad .$$

In line with the general procedure outlined in §1 we consider approximations $\tilde{\phi}_1 = \tilde{\phi}_1(\tilde{w})$ and $\tilde{\phi}_2 = \tilde{\phi}_2(\tilde{w})$ to ϕ_1 and ϕ_2. In either case, we make an error of the form

$$(3.14) \qquad e = \phi - \tilde{\phi} = \int_{\Omega} \zeta\ (w-\tilde{w})\ dA \quad .$$

Now, just as for the one-dimensional case, introduce an auxiliary function $\psi(x)$ which satisfies

$$\nabla^2\psi - k\psi = -\zeta$$

$$\psi = 0 \quad \text{on } \partial\Omega \quad ,$$

or equivalently,

$$\int_{\Omega} (\nabla\ \psi\ \nabla u + k\ \ \psi\ u)dA = \int_{\Omega} \zeta\ u\ dA$$

for all functions u which vanish on $\partial\Omega$ and for which $E(u)$ is finite. We may certainly choose $u = w-\tilde{w}$ to obtain from (3.14)

$$e = \int_{\Omega} \nabla\ \psi\ \nabla(w-\tilde{w}) + k\ \psi(w-\tilde{w})\ dA \quad ,$$

and using (3.12) we see that for any finite element function v from S

$$|e| = \left| \int_{\Omega} (\nabla (\psi - v) \nabla (w - \tilde{w}) + k (\psi - v) (w - \tilde{w})) \, dA \right|$$

$$\leq E(\psi - v)^{\frac{1}{2}} E(w - \tilde{w})^{\frac{1}{2}}$$

So on choosing v to minimize $E(\psi - v)$, and recalling (3.13), we have

$$|e| \leq \min_{v \in S} E(\psi - v)^{\frac{1}{2}} \min_{v^* \in S} E(w - v^*)^{\frac{1}{2}} \tag{3.15}$$

This estimate is telling us, exactly as did (2.16) in the one-dimensional case, that the accuracy of $\tilde{\phi}$ depends on how well both the solution w of the original problem, and the solution ψ of the auxiliary problem, can be approximated in the energy norm by the finite element functions in S.

If we try to obtain rates of convergence for $\tilde{\phi}$, we come up against some important differences between the one and two dimensional cases. In general, the analog of (2.18) holds only if the boundary of Ω is smooth. If $\partial\Omega$ is not smooth then (2.18) must be modified to account for some special singular terms that arise because of corners of $\partial\Omega$ (see [4]). These singular terms govern the smoothness and approximability of w and ψ. The analog of estimate (2.17) is also more complicated in the two dimensional case (see [3]). Nonetheless, if the mesh has the proper level of refinement around the corners of $\partial\Omega$, then similar results to those in the one-dimensional case can be achieved if the rate of convergence is now measured with respect to the number of degrees-of-freedom rather than p and h. To go into further details is beyond the scope of this paper.

§4 A Numerical Example

§4.1 Formulation of the example

As a practical demonstration of the methods discussed in §3, we shall consider some numerical results for the problem modelling a square, uniformly loaded membrane which is fixed along its edges. More specifically, we deal with the problem governed by the differential equation

$$\nabla^2 w = -1 \quad \text{on } \Omega = (-1,1)^2 \tag{4.1}$$

and boundary conditions

$$w = 0 \quad \text{on the boundary } \partial\Omega \text{ of } \Omega\,. \tag{4.2}$$

We shall employ the theory of §3 for the calculation of approximate values for:

(I) The displacement at the centre of the membrane: $\phi_1 = \phi_1(w) = w(0)$

(II) The stress at the point $P(1,0)$: $\phi_2 = \phi_2(w) = \frac{\partial w}{\partial x_1}(P)$.

By the method of separation of variables, an infinite series representation of w can be found. Using this series the following exact values (accurate to 5 significant figures) can be calculated:

$$E(w) = \int_\Omega |\nabla w|^2 dA = .56231$$

$$\phi_1(w) = w(0) = -.29469$$

$$\phi_2(w) = \frac{\partial w}{\partial x_1}(P) = .67528$$

Let us also note that the solution w is relatively smooth (in fact, it has square integrable second derivatives, though not square integrable third derivatives).

§4.2 The finite element approximation

We shall consider a simple finite element model of (4.1)/(4.2). Namely, bilinear elements on a square uniform mesh. By the symmetry of the problem, we need only actually calculate using the quarter-segment OQRP of Ω (see

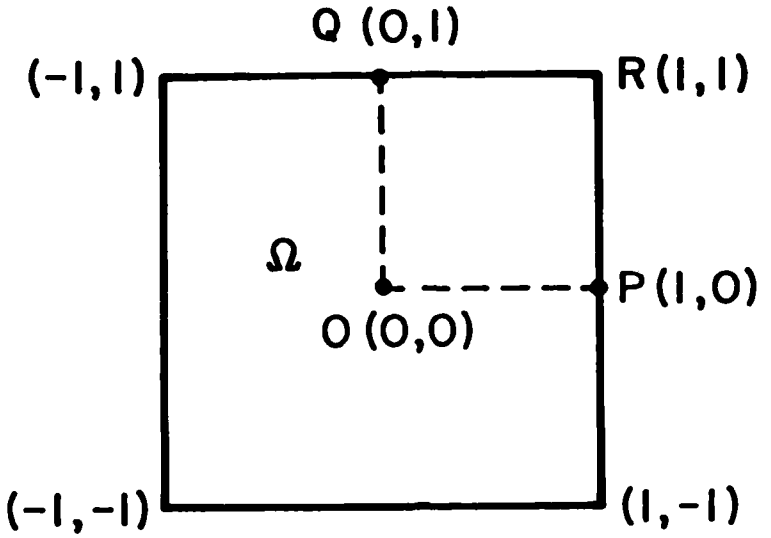


Figure 2; The region of the model problem.

Figure 2). For this problem we expect the following rates of convergence:

$O(h^2)$ for the energy $E(\tilde{w})$

$O(h)$ for the energy norm

$O(h^2)$ for the displacement $\tilde{w}(0)$

$O(h)$ for the stress $\frac{\partial \tilde{w}}{\partial x_1}$ (P)

where, as usual, h denotes the length of the side of an element. Using a uniform mesh and elements of degree 2 or higher we would obtain

$O(h^3)$ for the energy

$O(h^{2.5-\varepsilon})$ for the displacement $\tilde{w}(0)$

and $O(h^{1.5-\varepsilon})$ for the stress $\frac{\partial \tilde{w}}{\partial x_1}$(P)

where $\varepsilon > 0$ is an arbitrary small number. For the h-p-version, it is possible by suitable refinement about the corner points to obtain arbitrarily large orders of convergence with respect to the number of degrees-of-freedom (see [3]).

§4.3 Calculation of $\tilde{\tilde{\phi}}_1(\tilde{w})$

In accord with the theory developed in §3.2, we use the formula

$$\tilde{\tilde{\phi}}_1(\tilde{w}) = \int_\Omega \nabla^2\phi\ \tilde{w}\ dA - \int_\Omega \phi\ dA\ ,$$

where ϕ takes the generic form

$$\phi = X(x_1,x_2)\ [\tfrac{1}{2\pi}\ \log\ (x_1^{\,2} + x_2^{\,2})^{\frac{1}{2}} - \phi_o(x_1,x_2)]\ .$$

We consider two choices for ϕ :

Case (a): $X(x_1,x_2) = \overline{X}(x_1)\overline{X}(x_2)$ where

$$\overline{X}(t) = \begin{cases} 1 & 0 \le |t| \le \frac{1}{2} \\ 1 - 8(|t| - \frac{1}{2})^3 & \frac{1}{2} < |t| \le 1 \end{cases}$$

(see Figure 3 , and

$\phi_0(x_1,x_2) = 0$

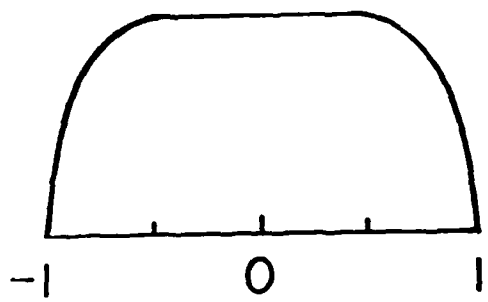


Figure 3: Cutoff function used in the evaluation of $\tilde{\phi}_1$, case (a).

Case (b): $X(x_1^2,x_2) = 1$, and

$$\phi_0(x_1,x_2) = \frac{1}{2\pi}\left(\log\left(\frac{(1+x_1^2)(1+x_2^2)}{2}\right)^{\frac{1}{2}}\right)$$

In Case (a) we have employed a cut off function technique to enforce the boundary conditions on ϕ , while in Case (b) a blending function method has been used. The first integral in the foregoing formula for $\tilde{\phi}_1$ may be calculated by numerical quadrature. (We used Gaussian quadrature.) The second integrand is singular at 0 . This integral may be evaluated analytically. However, it is also possible to calculate it numerically by the following procedure: Choosing ρ such that $\nabla^2\rho = 1$ (e.g. $\rho = \frac{1}{4}(x_1^2+x_2^2)$), integration by parts gives

$$\int_\Omega \phi\ \nabla^2\rho\ dA = \int_{\partial\Omega} \phi\ \nabla\rho\cdot\hat{n}\ ds - \int_{\partial\Omega} \nabla\phi\cdot\hat{n}\ \rho\ ds + \int_\Omega \nabla^2\phi\ \rho\ dA + \rho(0,0)$$

All the integrals on the right hand side are nonsingular, and may be readily evaluated by numerical means.

The results of the computations are shown in the middle section of Table 1. For comparison, we also list the value of the finite element solution $\tilde{w}$ at 0. Notice that $\tilde{w}(0)$ and both cases of $\tilde{\phi}_1$ all show an $O(h^2)$ rate of convergence. This is as we would expect. Observe also the superiority of the post-processed value in Case (b) over Case (a). This is in line with our previous comment that blending function techniques can usually be expected to perform better than cut off function methods. (Looking at the definition of X in Case (a) and examining Figure 3 shows that indeed X changes quite rapidly in the region $|x_2| > \frac{1}{2}$. In terms of the arguments we presented in §3.5, we should therefore not expect the corresponding ψ to be as well approximated from within our finite element subspace as it would be in Case (b)--

No. of elements in quarter segment (uniform mesh).		4	16	64	Exact Value
Energy norm error in $\tilde{w}$ $= \left(\frac{E(w-\tilde{w})}{E(w)}\right)^{1/2}$		30.1%	15.2%	7.62%	
$\tilde{w}(0)$ (relative % error)		(*).310714 (5.4%)	.298393 (1.3%)	.295596 (.31%)	
$\phi_1(\tilde{w})$ (relative % error)	(a)	.268783 (8.8%)	.287205 (2.5%)	.292751 (.65%)	.29469
	(b)	.287306 (2.5%)	.292829 (.63%)	.294220 (.16%)	
$\frac{\partial \tilde{w}}{\partial x}(P)$ (relative % error)		.482142 (29%)	.565480 (16%)	.616687 (8.7%)	
$\tilde{\phi}_2(\tilde{w})$ (relative % error)	(a)	.64758 (4.1%)	.67197 (.49%)	.67463 (.096%)	.67528
	(b)	.66623 (1.3%)	.67313 (.32%)	.67477 (.076%)	
	(c)	.66482 (1.5%)	.67276 (.37%)	.67468 (.089%)	

TABLE 1. Table of the results of the numerical calculations.
((*): negative signs have been supressed in this table)

see (3.15)).

The fact that the accuracies of $\tilde{w}(0)$ and $\tilde{\gamma}_1(\tilde{w})$ are comparable in this example is a consequence of our using bilinear elements. Nonetheless, Table 1 shows that in Case (b), the $\tilde{\gamma}$ values are twice as accurate as the $\tilde{w}(0)$ values for the same number of elements. Putting this another way, for the same accuracy the "direct" displacement method would require twice as many elements as the post-processing approach of Case (b).

§4.3 Calculation of $\tilde{\gamma}_2(\tilde{w})$

In the case of our model problem the theory of §3.3 leads to

$$\tilde{\gamma}_2(\tilde{w}) = \int \nabla^2 \phi \, \tilde{w} \, dA + \int \phi \, dA$$

where ϕ takes the generic form

$$\phi = X(x_1,x_2) \quad (\tfrac{1}{2}) \left(\frac{x_1-1}{(x_1-1)^2+x_2^2} - \phi_o(x_1,x_2) \right) .$$

We shall treat three cases:

Case (a):

$$X(x_1,x_2) = \begin{cases} 1 & \tfrac{1}{2} \le x_1 \le 1 \\ 8(6x_1^4 - 8x_1^3 + 3x_1^2) & 0 \le x_1 < \tfrac{1}{2} \\ 0 & -1 \le x_1 < 0 \end{cases}$$

(See Figure 4(i))

$$\phi_o(x_1,x_2) = \frac{x_1-1}{(x_1-1)^2 + 1}$$

Case (b):

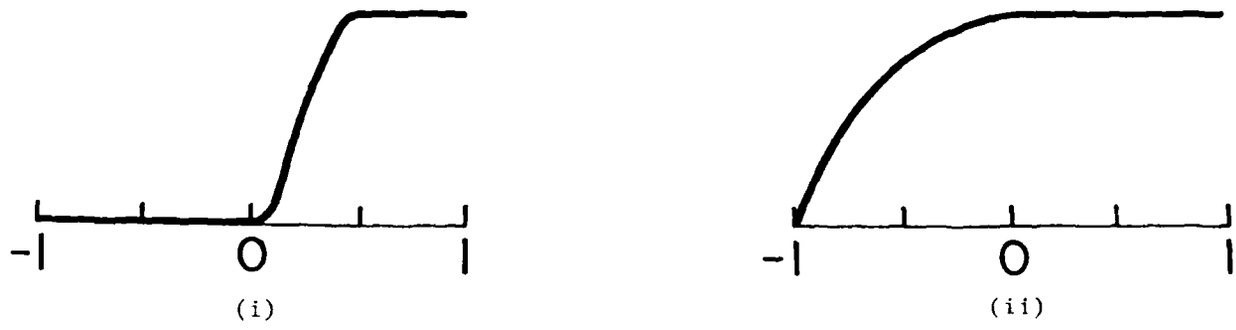
$$X(x_1,x_2) = \begin{cases} 1 & 0 \le x_1 \le 1 \\ 1 - |x_1|^3 & -1 \le x_1 < 0 \end{cases}$$

(See Figure 4(ii))

$$\phi_0 = \frac{x_1-1}{(x_1-1)^2 + 1}$$

Case (c): $\chi(x_1,x_2) = 1$

$$\tilde{\phi}_c = \frac{x_1-1}{(x_1-1)^2+1} + (x_1-1)\left(\frac{1}{4+x_2^2} - \frac{1}{5}\right)$$



(i) (ii)

Figure 4: Cut off functions used in the evaluation of $\tilde{\phi}_2$
(i) case (a)
(ii) case (b)

Cases (a) and (b) correspond to our using a blending function technique to satisfy the boundary condition on the edges $x_2 = \pm 1$, and a cut off function method to handle the edge $x_1 = -1$. In Case (c), a blending function method is used to handle the entire boundary. Concerning the actual evaluation of $\tilde{\phi}_2(\tilde{w})$, the same comments made in §4.2 about $\tilde{\phi}_1(\tilde{w})$ apply here also.

The results of the calculations are shown in the lower part of Table 1, where, for comparison, we have also listed the corresponding values of $\frac{\partial \tilde{w}}{\partial x_1}(P)$. In contrast to the situation for the displacements, we see that the post-processed values for $\tilde{\phi}_2$ are markedly more accurate than the "direct" value $\frac{\partial \tilde{w}}{\partial x_1}(\tilde{P})$. We see, as theory predicts, an $O(h)$ rate of convergence for the "direct" value, but an $O(h^2)$ rate for $\tilde{\phi}_2$.

Bibliography

[1] D. M. Parks, A Stiffness Derivative Finite Element Technique for Determination of Crack Tip Stress Intensity Factors, International Journal of Fracture, Vol. 10, No. 4, 1974, pp 487-502.

[2] J. R. Rice, A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks, Journal of Applied Mechanics, Vol. 35, 1968, pp 379-386.

[3] I. Babuška, M. Dorr, Error Estimates for the Combined h and p Versions of the Finite Element Method, Num. Math., Vol. 37, 1981, pp 257-277.

[4] P. Grisvard, Boundary Value Problems in Non-Smooth Domains, University of Maryland, Department of Mathematics, Lecture Notes No. 19, 1980.