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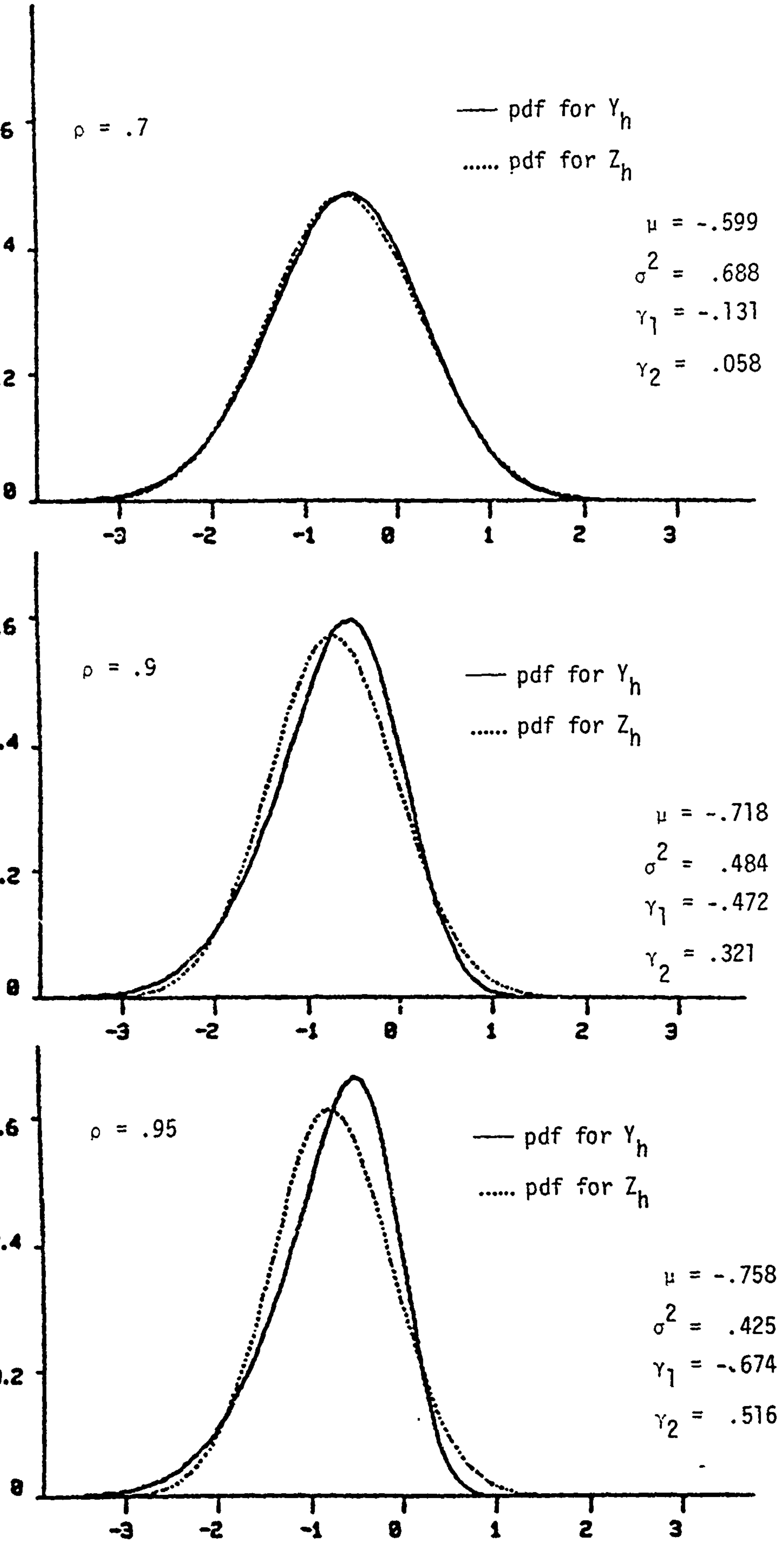
Key Words: Bivariate Normal, Screening, Selection, Skewness, Kurtosis.

## ABSTRACT

The bivariate normal distribution function may be expressed as the product of a marginal normal distribution times a conditional distribution. By approximating this conditional distribution, we obtain a simple method for approximating bivariate normal probabilities. When the correlation falls in the interval [-.5, .5], the maximum absolute error in our approximation is always less than .0008. The conditional distribution that we approximate is referred to as a 'normal conditioned on a truncated normal' distribution and is related to screening and selection problems.

Figure 1: Comparison of pdf for $Y_h$ and $Z_h$($\rho$ = .7, .9, and .95; h = 0)



## INTRODUCTION

The bivariate normal distribution is frequently employed as a model for screening procedures, where a product is either accepted or rejected based on some secondary measurement that is correlated with the performance characteristic (see, e.g., Owen, McIntire and Seymour, 1975). When screening is utilized, the proportion of all items which are rejected even though they are good and the proportion of accepted parts which are in fact defective are of interest. The purpose of this paper is to provide simple formulae for approximating these probabilities. The results that follow should also be of interest in other settings where bivariate normal probabilities are required but the potential user does not have ready access to a computer.

Tables of the bivariate normal distribution are like those of the univariate normal in that the variates are standardized, i.e., each variate has its mean subtracted and the difference is divided by the standard deviation. Hence, we start with $X$ and $Y$ having a joint (standardized) bivariate normal distribution with correlation $\rho$. We propose a simple approximation for the cumulative distribution function (cdf) $F(\cdot, \cdot; \rho)$ of $(X, Y)$. For any constants $h$ and $k$, the probability $F(h, k; \rho)$ may be factored into a marginal probability times a conditional probability, i.e.,

$$F(h, k; \rho) = \Pr[X \leq h] \cdot \Pr[Y \leq k | X \leq h]. \quad (1)$$

We propose approximating the conditional distribution of $Y$, given $X \leq h$, by a normal distribution. In this way, $F(h, k; \rho)$ can be approximated as the product of two univariate normal probabilities.

Mallows (1959) obtained an approximation for $F(h, k; \rho)$ which requires simple computations and evaluation of univariate normal probabilities and percentiles. For the values of h, k and $\rho$ that Mallows examined, the maximum error was .0076. The approximation we present is simpler than Mallow's formulae and is more accurate unless $|\rho|$ exceeds .8. El Lozy (1982) provides a simple way to compute probabilities of the special form $F(h, \rho h; \rho)$. El Lozy's method is most accurate for $\rho$ close to 1. In contrast, our approximation has its greatest accuracy when $|\rho|$ is less than .5.

Before we present the approximation, we investigate the conditional distribution of Y, given $X \leq h$. This distribution, which we label the normal conditioned on a truncated normal, is of interest in itself since it can represent the distribution of the performance variable in the accepted population after screening with respect to an upper specification limit.

## NORMAL CONDITIONED ON A TRUNCATED NORMAL

It is well-known that the distribution of Y conditioned on X = h is normal with mean $\rho h$ and variance $1 - \rho^2$. That is, Y is said to have a conditional normal distribution. However, the distribution of Y conditional on $X \leq h$ is not normal (unless $\rho = 0$), but has a probability density function (pdf) given by

$$G'(y)G[(h-\rho y)/(1-\rho^2)^{1/2}]/G(h), \qquad (2)$$

where $G'(\cdot)$ and $G(\cdot)$ denote the standard normal pdf and cdf, respectively.

Let $Y_h$ denote the random variable with pdf (2). The mean of $Y_h$ is

$$\mu = -\rho G'(h)/G(h) \qquad (3)$$

and the variance is

$$\sigma^2 = 1 + \rho h\mu - \mu^2. \qquad (4)$$

Higher moments are given by

$$\mu'_n = (n-1)\mu'_{n-2} + \mu[\sum_{i=0}^{n-1}(1-\rho^2)^{i/2}(\rho h)^{n-1-i}\binom{h-1}{i}E(Z^i)] \qquad (5)$$

where $\mu_n' = E(Y_h^n)$ and where $E(Z^i) = 0$ if i is odd, is equal to one if i is zero or two, and is equal to $(i-1)!2^{-(i-2)/2}[\{(i-2)/2\}!]^{-1}$ otherwise. Expressions (3) - (5) may be obtained from Johnson and Kotz (1972, pp. 112, 114).

Although the distribution of $Y_h$ differs from the normal, unless $|\rho|$ is near one, the distribution of $Y_h$ can be adequately approximated by a normal pdf. Let $Z_h$ denote a normal random variable with mean given by (3) and variance given by (4). Figure 1 gives a comparison of the pdf's of $Y_h$ and $Z_h$ for h = 0 and $\rho$ = .7, 9, and .95. We have also computed the coefficient of skewness $\gamma_1 = \mu_3/\mu_2^{3/2}$ and the measure of kurtosis $\gamma_2 = \mu_4/\mu_2^2 - 3$, where $\mu_n = E[(Y_h-\mu)^n]$. For $|\rho| \leq .5$, the pdf's of $Y_h$ and $Z_h$ are virtually indistinguishable. However, as $|\rho|$ approaches 1, the pdf of $Y_h$ approaches the pdf of a normal truncated above h if $\rho > 0$ and below -h if $\rho < 0$.

[Figure 1 here]

The similarity of the distribution of $Y_h$ to a normal distribution has implications for screening problems. For example, consider the situation where an aircraft part is inspected for fractures using X-rays. Suppose lifetime (L) and maximum crack length (MCL) are modeled using a bivariate normal distribution with $\rho = -.7$. The average and standard deviation (SD) for L are taken to be 8000 hours and 1000 hours, respectively. Suppose that we reject any part with a crack exceeding 1 SD above average for MCL (i.e., we reject about 16% of the parts). Then using (3) and (4) (with h = 1, $\rho = -.7$), $\mu = .201$ and $\sigma = .905$ and the distribution of lifetime for accepted parts has a mean of $8000 + 1000\mu = 8201$ hours and an SD of $1000\sigma$ = 905 hours. The fifth percentile for L before selection was 6355 hours. Using the normal to approximate the distribution of L for the accepted parts, we find that, after screening, only 2% of the parts should fail before 6355 hrs. [Expressions (3) - (5) are for computing the moments of the standardized variate Y, conditional on $X \leq h$. The calculations above illustrate how to convert from $\mu$ and $\sigma$ to the mean and SD in original units.]

## APPROXIMATING $F(h, k; \rho)$

Visual comparison of the distributions of $Y_h$ and $Z_h$ suggests that, whenever $|\rho|$ is small, we may accurately approximate $F(h, k; \rho)$ by the product

$$B(h, k; \rho) = G(h) \cdot G[(k-\mu)/\sigma] \tag{6}$$

[where $\mu$ and $\sigma$ are as defined in (3) and (4)], since $\Pr[Y \leq k | X \leq h] = \Pr[Y_h \leq k]$ may be approximated by $\Pr[Z_h \leq k] = G[(k-\mu)/\sigma]$.

As we investigated the accuracy of $B(h, k; \rho)$ for approximating $F(h, k; \rho)$ for various h, k and $\rho$, we found that the error was less when h was negative and when $|k| < |h|$. Therefore, to approximate the bivariate normal probability $F(c, d; r)$, the error of approximation is minimized by the following scheme:

1. Choose c and d so that $|c| \geq |d|$. This can always be done, since $F(c, d; r) = F(d, c; r)$.

2a. If $c \leq 0$, set $h = c$, $k = d$ and $\rho = r$ and approximate $F(c, d; r)$ by $B(h, k; \rho)$; or,

2b. If $c > 0$, set $h = -c$, $k = d$ and $\rho = -r$ and approximate $F(c, d; r) = G(d) - F(-c, d; -r)$ by $G(k) - B(h, k; \rho)$.

For example, to approximate the probability $F(.2, 1; .7)$, set $c = 1$, and following 2b, $h = -1$, $k = .2$ and $\rho = -.7$. Then $\mu = 1.0676$, $\sigma = .77946$,

$B(-1, 2; -.7) = G(-1) \cdot G[(.2-\mu)/\sigma] = .02108$, and the approximation is $G(.2) - .02108 = .55818$. The exact value is .55842, so our approximation error is only .00024.

Using this scheme, the maximum error incurred in approximating $F(c, d; r)$ for any c and d is:

| $\lvert r \rvert$ | Maximum $\lvert$error$\rvert$ |
|---|---|
| .1 | .00001 |
| .2 | .00006 |
| .3 | .00019 |
| .4 | .00043 |
| .5 | .00079 |
| .6 | .00121 |
| .7 | .00164 |
| .8 | .00282 |
| .9 | .00765 |

## EXAMPLE

In closing, we illustrate the usefulness of this approximation with the example of Lipow and Eidemiller (1964). Rocket motor cases are known to have a strength S that is normally distributed with mean 700 pounds/inch$^2$ (psi) and an SD of 100 psi. It is also known that peak rocket operating pressure P is normally distributed with mean 500 psi and SD 100 psi. (P and S are independently distributed.) When a rocket is fired, if the peak pressure exceeds case strength, i.e., if $\Delta = S-P$ is negative, the case will rupture and the rocket will fail. Thus, the proportion of cases that will rupture when fired is $Pr[\Delta < 0] = G(-\sqrt{2}) = .0786$ (since

$\Delta$ is normally distributed with mean 200 psi and SD 141.42 psi). In order to decrease the frequency of failure due to rupture, the motor cases are pre-tested to a pressure of 600 psi, and discarded if they rupture. Hence, the probability of rupture for a case which passes the test is $\Pr[\Delta < 0, S \geq 600]/\Pr[S \geq 600]$. Now $\Pr[\Delta < 0, S \geq 600] = \Pr[Y < -\sqrt{2}, X \geq -1]$ with $\rho = 1/\sqrt{2}$, which also equals $F(-\sqrt{2}, 1; -\rho)$. To approximate $F(-\sqrt{2}, 1; -\rho)$, we compute

$$B(-\sqrt{2}, 1; -.7071) = G(-\sqrt{2})G[(1 - 1.319)/.761] = .0265.$$

From this bivariate probability, we can easily construct the table of probabilities:

| | $S < 600$ | $S \geq 600$ | Totals |
|---|---|---|---|
| $\Delta < 0$ | .0521 | .0265 | .0786 |
| $\Delta \geq 0$ | .1066 | .8148 | .9214 |
| Totals | .1587 | .8413 | 1.0000 |

Hence, the proportion of failures for the cases that passed inspection is .0265/.8413 = .0315. By screening, we have reduced the failure rate from 7.86% to 3.15%. However, 10.66% of the motor cases would have fired successfully, but were discarded after pre-testing.

The true value for $F(-\sqrt{2}, 1; -.7071)$ is .0267, so the error of the above approximation was .0002. This example illustrates both the simplicity and accuracy of the approximation for bivariate normal probabilities presented in this paper.

## ACKNOWLEDGEMENT

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The bivariate normal distribution function may be expressed as the product of a marginal normal distribution times a conditional distribution. By approximating this conditional distribution, we obtain a simple method for approximating bivariate normal probabilities. When the correlation falls in the interval [-.5, .5], the maximum absolute error in our approximation is always less than .0008. The conditional distribution that we approximate is referred to as a 'normal conditioned on a truncated normal' distribution and is related to screening and selection problems.

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