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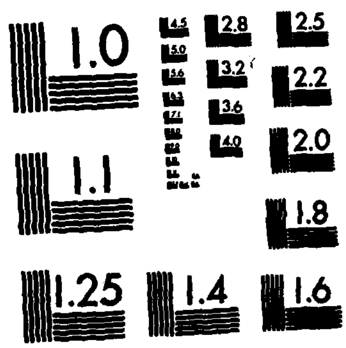

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THE ROLE OF FAR-FIELD BOUNDARY CONDITIONS IN NUMERICAL SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
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NOTICE

This technical report has been reviewed and is approved for publication.

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The problem of prescribing farfield boundary conditions on the outer computational domain boundary is discussed. Several boundary conditions which have been used in the literature are compared with two newly proposed sets. The proposed boundary conditions are shown to be non-reflecting and superior in both accuracy and time to convergence. Graphs showing the numerical results for the different types of boundary conditions are displayed and the wave motion demonstrated.

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FINAL REPORT

THE ROLE OF FAR-FIELD BOUNDARY CONDITIONS IN NUMERICAL SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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FOREWORD

This report is the result of work carried out in the Computational Aerodynamics Group, Aerodynamics and Airframe Branch, Aeromechanics Division, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories. The work was performed by Dr P. Joseph McKenna, summer SCEEE Research Associate, Mr Jeff Graham of AFWAL/FIMM, and directed by Dr Wilbur L. Hankey under contract No. F49620-79-C-0038. Dr McKenna accomplished the investigation from May 1981 through September, 1981, while on leave from the University of Florida.

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I. INTRODUCTION:

Attempting to determine fluid flow around an obstacle requires the solution of the Navier-Stokes equations;

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \qquad (1)$$

where

$$u = \begin{matrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{matrix} \qquad E = \begin{matrix} \rho u \\ \rho u^2 - \sigma_{xx} \\ \rho uv - \zeta_{xy} \\ \rho ue - u\sigma_{xx} - v\zeta_{xy} - \dot{q}_x \end{matrix}$$

$$F = \begin{matrix} \rho v \\ \rho uv - \zeta_{xy} \\ \rho v^2 - \sigma_{yy} \\ \rho ue - v\sigma_{yy} - u\zeta_{xy} - \dot{q}_y \end{matrix}$$

$$\sigma_{xx} = -p - (2/3)\ \mu\vec{\nabla}\cdot\vec{u} + 2\mu\ u_x \qquad \dot{q}_x = k\ T_x$$

$$\zeta_{xy} = \mu(u_y + v_x) \qquad \dot{q}_y = k\ T_y$$

$$\sigma_{yy} = -p - (2/3)\ \mu\vec{\nabla}\cdot\vec{u} + 2\mu\ u_y \qquad e = cvT + \frac{u^2+v^2}{2}$$

$$p = \delta RT \quad \mu = \mu(T)$$

Here, the four dependent variables ρ, u, v, and e represent the physical quantitive of density, x- and y- components of velocity, and internal energy.

Even a casual glance at this system of equations shows that it has many extremely difficult aspects. It is a system of four equations in the four basic unknowns, which is of mixed parabolic hyperbolic type. It is in two space dimensions and in addition is nonlinear. Any one of these problems would be sufficient to make obtaining an analytic solution difficult, so it is fair to say that an analytic solution is out of the question. Thus these problems are "solved" by numerical methods.

What this means of course is subject to some interpretation. What usually happens is that a number of approximations are made to the original problem, in order to arrive at a discrete approximation which may then be solved by computer. Most obviously, "space" is interpreted, not as a continuum, but as a finite number of grid points. The value of pressure for example at a grid point is then used to calculate the value at nearby points, by means of a few terms in a Taylor series. Derivatives are approximated by finite differences. If the flow is around a body, then one assumes that velocity is zero on the surface of the cylinder, and the surface temperature is prescribed. The assumption is that if the grid is fine enough, and if time steps are sufficiently small, then this discrete model is a reasonable approximation of the original equations (1).

A method for solving supersonic flows that was found by experience to work well was MacCormack's alternating direction explicit scheme. Thus, this was a logical method to be experimented with for subsonic flow. However, the transition was not easy. The MacCormack method demands certain fictitious (or numerical) boundary conditions due to the difference algorithm which are not physically present, and which were arrived at by a process of computational experimentation. These methods did not work in the case of subsonic flow, although they did in the case of supersonic flow.

II. OBJECTIVES:

The purpose of this project was to explain thus anomaly, and to arrive at improved far-field boundary conditions for the subsonic case. To do this, we will first review the theory for a single wave equation, then for systems of linear hyperbolic partial differential equations, and finally, we shall discuss how numerical experiments with the Navier-Stokes equation confirm the predictions based on the elementary theory.

III REVIEW OF NUMERICAL THEORY FOR ONE EQUATION:

Obviously the full system of equations (1) is too complicated to achieve a great deal with sophisticated mathematical analysis. Typically, mathematicians will work with simpler equations which in one way or another resemble the original system. One tries to use the mathematical insights of the simpler situation in the context of the more complicated one. This is obviously not one unbroken chain of reasoning, but more a process of educated guesswork.

Surprisingly, one equation which provides a great deal of insight is the simple wave equation

$$u_t + au_x = 0 \tag{2}$$

We consider this equation on the interval $[0,1]$. On the real line, the solutions to this equation would be waves running from left to right with velocity a, which are constant along the lines $x - at = d$, $d \varepsilon R$

Thus, if we consider this equation on the region $\{(x, t)\ 0 \leq x \leq 1,\ t \geq 0\}$ then it will be analytically determined by the initial value $U(x, 0) = f(x)$ $0 \leq x \leq 1$ and the boundary value $u(o, t) = g(t)$. With these conditions, the initial boundary value problem is well posed. It is impossible to prescribe boundary conditions at $x = 1$ instead of $x = 0$ without either (a) limiting the initial values or (b) causing discontinuities. All of this makes perfect physical sense. In order to know what happens in a wave situation, we must give information on the initial conditions and also on the waves entering at the inflow point.

However, the MacCormack scheme which is equivalent to the Lax-Wendrof scheme requires some knowledge of u (1, t) Indeed the value for u_{J-1}^{n+1} is given in terms of u_{J-2}^{u}, u_{J-1}^{n} and u_{J}^{n}. Therefore, the boundary point x = 1, (j = J) requires special treatment of some sort. This special treatment is called a "numerical boundary condition" or compatibility condition. The imposed can have an enormous impact on the successful numerical solutions of the problem.

Pherhaps the best two recent summaries on this simple equation were accomplished by Kreiss [5] and Gottlieb and Turkel [2].

Kreiss is basically concerned with what can go wrong. He observes for example that if you overspecify, i.e. just make a guess at what u(1, t) is going to be and then simply prescribe it, convergence to a steady state may or may not take place, depending on whether the number of grid points is even or odd. If this happens with this simple case obviously you don't want to try it in a more complicated case. (We shall have more to say on this later).

One method that works well is to use u_x (1, t) = 0, or in finite difference form $u_J^n = u_{J-1}^n$. A slight error in u at the outflow point is made, but since the flow is from left to right, this error does not propagate back into the x-domain. This is proved analytically in Parter [8]. About the worst mistake that can be made is not to specify u at the incoming boundary. For example, one might confuse the inflow and outflow boundary and prescribe $u_1^n = u_2^n$, and $u_J = M$ all n. In this case as the space step and time step become small this conveys to a steady state on an interval [0,T] where the steady state is determined by the initial conditions at the inflow point.

A moment's reflection ought to convince us of how undesirable this is, since we do not know the correct steady state, our initial conditions are bound to be different from the correct solution. However, what we seem to converge to actually depends on the choice of the initial conditions. Again, we shall have more to say about this when we discuss computational solutions of the Navier-Stokes equations.

Another excellent paper on the same subject is by Gottlieb and Turkel [2]. In this paper, a comprehensive review of many different numerical boundary conditions is given. For example, in addition to the ones already mentioned at the outflow, one might consider $u_{xx} = 0$, which numerically is $u_{J-2} - 2u_{J-1} + u_J = 0$ or even $u_t + au_x = 0$ where $u_J^{n+1} = u_J^n \; a(\Delta t/\Delta x) (u_{J-1}^n - u_J^n)$. This corresponds to a one-sided "upwind" difference approximation at the outflow boundary.

Their conclusion is that the upwind difference appears best, although the convergence of the scheme with $u_x = 0$ is just as fast (but less accurate).

IV. WELL-POSED BOUNDARY CONDITIONS FOR LINEAR SYSTEMS:

The simplest systems of two linear wave equations to consider is

$$u_t + au_x = 0$$

$$v_t - bv_x = 0 \qquad (3)$$

where $a_1 > 0_1 \; b > 0$, $0 \leq x \leq 1$, and $t \geq 0$. In this case the waves in u travel from left to right with velocity a, and the waves in v travel from right to left with velocity b. Clearly u must be given initially and at the left boundary; whereas the values of v must be given at the right boundary. This is analytically necessary in order to have enough information to solve the problem. However, there is a minor complication. The incoming value of u may be given at $x = 0$ in terms of v (which, after all, is determined there), and the incoming value of v at 1 may be given in terms of u. Thus, the initial boundary value problem (3) is well posed if the initial conditions are

$$u(x, 0) = f(x) \qquad v(x, 0) = g(x)$$

and the boundary conditions are

$$u(o, t) = F_1(t) + c_1 v(0, t)$$
$$v(1, t) = G_1(t) + c_2 u(1, t) \tag{4}$$

Notice that if $c_1 = c_2 = 0$, then the problems are completely uncoupled, and each is a copy of the single equation in section IV. For a moment, consider the case where $F_1(t) = G_1(t) = 0$, and $c_1 = c_2 = 1$. In this case, we get a different phenomenon. Let us assume that u is identically zero initially, and that v is identically 1. Then the square wave in v travels to the left, causing u to be non-zero at the in-flow boundary, and thereby propagating from left to right with velocity 1. When u reached $x = 1$ it would in turn influence v by the boundary condition (4) and a new disturbance would propogate in v from right to left. This is the reason that this boundary condition is called a "reflective boundary condition".

Notice that if both u and v travel from left to right, i.e. if

$$u_t + au_x = 0$$
$$v_t + bv_x = 0$$

where $a > 0$, $b > 0$, $0 \leq x \leq 1$ and $t \geq 0$, then this problem does not arise. The initial conditions of u and v must be specified as must the boundary conditions at $x = 0$. No reflections or coupling is allowed to take place.

For a general hyperbolic system of the form

$$\vec{u}_t + A\vec{u}_x = 0 \tag{5}$$

one must diagonalize the matrix A, i.e. one must find a matrix T so that

$$T\,A\,T^{-1} = D$$

where D is a diagonal matrix. One then makes the substitution W = TU and the equation (5) transforms to

$$\vec{W}_t + D\,\vec{W}_x = 0$$

The number of positive eigenvalues of D identifies the right-running variables W_i and the negative ones identify left running variables. At this point, we emphasize that the only way to be familiar with the wave nature of (5) is to look, not at the physical variables $\vec{u}$ in which the problem was originally presented, but instead at the new variables W which are linear combinations of the old ones. Only then is the wave structure apparent.

If k is the number of positive eigenvalues, and if W_I represents the k-vector of these k coordinates of W_I and W_{II} is the n-k other coordinates, then unless W_I and W_{II} are independently prescribed at the left and right boundaries respectively then we may get reflective boundary conditions. For well-posedness, it is sufficient that at x = 0, W_I be given (possibly depending on W_{II}) and that at x = 1, W_{II} be given with possible dependence on W_I. Only then can acceptable boundary conditions be formulated in terms of the physical variables U.

V THE NAVIER-STOKES EQUATIONS AND CHARACTERISTIC VARIABLES:

We now begin our discussion of the equations of gas dynamics. We will neglect viscosity for the purposes of this analysis. We will assume that the flow is one-dimensional and subsonic and that the deviations from free-stream solutions are small. This will allow us to neglect second order terms.

There are many forms of this equation, but the one most suitable for the present discussion is

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \qquad (6)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)\,u & \gamma-1 \\ (\gamma-1)u^3-\frac{\gamma eu}{\rho} & \frac{\gamma e}{\rho}-\frac{3}{2}(\gamma-1)u^2 & \gamma u \end{pmatrix}$$

and

$$U = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}$$

or in terms of physical variables

$$\frac{\partial \tilde{U}}{\partial t} + \tilde{A}\frac{\partial \tilde{U}}{\partial x} = 0 \qquad (7)$$

where

$$\tilde{A} = M^{-1}AM$$

and

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -u/p & 1/p & U \\ (\frac{\gamma-1}{2})u^2 & (1-\gamma)u & (\gamma-1) \end{pmatrix}$$

Here we make the key assumption that deviations from the free stream are going to be sufficiently small that we can treat the entries in the matrix $\hat{A}$ as being approximately constant (at least locally). Denote these frozen variables by a 0-subscript. We then make the substitution

$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/c_o^2 \\ 0 & 1 & 1/\rho_o c_o \\ 0 & -1 & 1/\rho_o c_o \end{pmatrix} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \qquad (8)$$

and when this is substituted into (7) we obtain

$$\frac{\partial W_1}{\partial t} + u_o \frac{\partial W_1}{\partial x} = 0$$

$$\frac{\partial W_2}{\partial t} + (u+c_o) \frac{\partial W_2}{\partial x} = 0$$

$$\frac{\partial W_3}{\partial t} + (u_o-c_o) \frac{\partial W_3}{\partial x} = 0$$

Notice now how this breaks down into two separate cases. On the one hand, if flow is supersonic then all wave motion is in the left to right direction. In this case <u>all</u> analytic boundary conditions ought be prescribed at the left hand side and only numerical boundary conditions prescribed at the right hand side.

Since the substitution (8) is equivalent to

$$\rho = K_1 + (\rho_o/2c_o)(K_2 + K_3)$$

$$u = 1/2\ (K_2 - K_3)$$

$$p = \rho_o c_o/2\ (K_2 + K_3)$$

it follows that prescribing all physical variables at the inflow and prescribing $\partial\rho/\partial x = \partial u/\partial x = \partial p/\partial x = 0$ at the outflow is sound in terms of analytical and numerical requirements.

However, we must now consider the case of subsonic flow. In this case the situation is completely different. Here, two of the variables W_1 and W_2 go left to right with velocities u and u+c respectively, whereas one of the variables runs right to left with velocity c_o-u_o. While the variables W_2 and W_3 have no clear physical significance. Yet it is only by considering these variables that the full wave structure of the equations (6) or (7) can be understood. Thus, one would be led to predict, for small deviations from free stream conditions, that the best boundary conditions would be for an interval (o, L)

$$W_1 (o,t) = K_1 \qquad \frac{dW_1}{dx} (L,t) = 0$$

$$W_2 (o,t) = K_2 \qquad \frac{dW_2}{dx} (L,t) = 0 \tag{9}$$

$$\frac{dW_3}{dx} (o,t) = 0 \qquad W_3 (L,t) = K_3$$

Note the curious aspect of these boundary conditions. In order to prescribe the numerical values K_1 and K_2, we need to know accurately all three physical variables at some distance to the left. However, only the two combinations K_1 and K_2 are prescribed. This can be summarized by saying that while we have used all three pieces of information upstream, we have done so in such a way that one degree of freedom remains, thus allowing the waves in W_3 to exit without problems.

On the basis of the linearized model, various other combinations would be well-posed. For example, it is possible to prescribe K_3 in terms of either K_1 or K_2 at the outflow $x = L$. Thus at the outflow one may prescribe

$$W_2 (L,t) = F_3 (t) + c_1 W_1 (L,t) + c_2 W_2 (L,t)$$

For example if $c_1 = 0$, $c_2 = 1$, then this amounts to putting

$$u (L,t) = 1/2 F_3 \tag{10}$$

i.e., we prescribe velocity at the outflow.

Alternatively, we might take $c_1 = 0$, $c_2 = -1$ and we would get

$$p (L,t) = ((\rho_o c_o)/2)F_3 \tag{11}$$

i.e. we prescribe velocity at the outflow. Many other combinations are possible, but as remarked in section IV, all these will cause errors in the initial data to be reflected back into the medium as waves running from right to left. For example, we would predict that an error in W_2 would be reflected back as an error in W_1 if we use boundary condition (10). As we shall see, this is exactly what happens.

At the inflow end, we may prescribe W_1 and W_2 in terms of W_3. Thus the following boundary conditions are well posed;

$$W_1\ (o,t) = F_1 + c_1\ W_3\ (o,t) \tag{12a}$$

$$W_2\ (o,t) = F_2 + c_2\ W_3\ (o,t) \tag{12b}$$

For example, choosing $c_2 = +1$ in (11b) corresponds to

$$u\ (o,t) = (1/2)\ F_2$$

(i.e. prescribing u at the inflow) and $c_2 = -1$ corresponds to

$$p\ (o,t) = (\rho_o c_1/2)\ F_2$$

(i.e. prescribing p). One can prescribe the combination (u,ρ) by first choosing $c_2 = 1$ (thereby prescribing u) and then choosing $c_1 = \rho_o/c_o$, thereby prescribing ρ in terms of a given F, and a prescribed u(o,t). Since (11a) and (11b) reduce to

$$u(1 + c_2) + \frac{p}{\rho_o c_o}\ (1 - c_2) = F_2$$

$$\rho - \frac{p}{\rho_o c_o}\ (\rho_o/c_o - c_1) + c_1 u = F_1$$

about the only condition we cannot prescribe is u(o,t), p(o,t), since there is no choice of c_1, c_2 to eliminate ρ from these equations.

Again, we emphasize that each of these boundary conditions is <u>reflecting</u>, i.e. deviations from the free stream in the initial data get reflected back as waves in W_1 and W_2 and then travel back downstream. About the worst thing that can be done is to prescribe reflecting boundary conditions at the inflow $x = 0$ and the outflow $x = L$. In this case errors can keep being reflected up and down the region, never being allowed to exit. This prevents convergence to a steady state and may even give rise to fictitious periodic oscillations.

We conclude this section with a review of the conclusions on boundary conditions. Based on the linear model, boundary conditions (9) seem optimal. Any other prescription of the physical variables, although well posed, causes reflections of the deviation from the true solution. If for example, only the physical variable p_∞ is known at the outflow then it is possible to prescribe p at the outflow, in such a manner that the problem remains well posed. We emphasize that this will cause errors to propagate upstream, thereby slowing the process of convergence to steady state. This may not be too bad, so long as the upstream boundary conditions are not also reflecting. On the other hand, if they are, then convergence to free stream may never occur.

VI. DISCUSSION OF NUMERICAL RESULTS - NONLINEAR COUPLING:

The one dimensional Navier-Stokes equations were solved with an alternative direction explicit MacCormack scheme, on a one-dimensional net with forty grid points. The code was an exact one-dimensional version of a three-dimensional code which had proved successful in many supersonic studies [6], [11]. There were two questions to answer. The first was, given that equation (8) is correct for infinitesimally small deviations from a constant free steam, how correct is it when deviations of an intermediate (of the order of 10%) magnitude are there instead? It would be too much to hope that the variables W_1, W_2, and W_3, remain uncoupled, but we should be able to get an idea of the order of magnitudes of the coupling involved. The second problem of course, is to assess the influence of the various types of boundary conditions commonly employed. We will deal with the latter problem in section VII.

To do this, we considered a uniform free stream situation, with pressure equal to 2000 lbs/ft^2, velocity equal to 548 ft/sec and density equal to 0.0023 $slugs/ft^3$. We created a deviation from the steady state condition in a variety of ways as in [9] , [10], and then watched the progress (or lack of it) to a steady state. A variety of different wrong initial conditions were used. One type was to impose a 10% deviation in one of the variables K_1, K_2, K_3 at the points $\{x_j, j=18,19,20,21,22\}$. We would then watch the disturbance, graphically as it propagated up or down stream. Another possiblity was to put in uniformly wrong initial conditions where some or all of the characteristic variables W_1, W_2, W_3 are perturbed throughout by a percentage error of 10%. Each plot then showed the percentage error, with the different curves representing the progress of time as one ascends the plot. The curves are plotted every twenty five time steps when $\Delta t = (.9)(\Delta x)/(u+c)$. We also point out the percentage errors in W_1, W_2, and W_3 at the end of the run, so as to obtain information on relative accuracy and speed of convergence of the various methods.

Figure 1*gives the results of an experiment, which is ideal in terms of the linear theory. An initial disturbance in W_3 the left-running

*Figures are located at end of report.

characteristic variable is given (graphed as K_3) and we observe deflections in the variables W_1 W_2 W_3. As one can readily see from the pictures, the disturbance propagates upstream rapidly until convergence is reached (.1% agreement with physical variables), which takes place within 130 time steps. This gives us six curves. Note that although a good deal of undershoot and overshoot in W_3 becomes apparent, there is no significant interaction with W_1 or W_2. The same situation appears with initial disturbances in W_1, the slow-moving right running wave. From this experiment it appears that deviations in either W_1 or W_3 will not effect either of the other two variables. However, as shown in Figures 2 and 3, when initial disturbances are in W_2, an entirely different situation exists. Figure 2 shows a uniform initial disturbance in W_2 of minus ten percent, while W_1 and W_3 are left undisturbed. Initially the wave in W_2, propagates rapidly out of the medium. Indeed, after fifty time steps it is essentially gone from the picture. However, this does not happen without affecting the other two variables. Notice how, in the top graph, a large disturbance is left in W_1 after fifty time steps and in W_3, we have that W_3 values one almost constant at minus eight percent. However, once the W_2 wave has made its exit, the other two variables uncouple and resume their normal wave motion, and the error can be seen propagating out of the solution in the usual way convergence is attained within three hundred iterations. These particular results illustrate what became increasingly clear throughout the series; perturbations in W_2 had a large effect on W_1 and W_3 whereas if W_2 was not perturbed, W_1 and W_3 behaved as if uncoupled. Perturbations in W_1 and W_3 had little effect on W_2. No qualitative explanation of this phenomenon is known at this time.

Figure 3 shows the same phenomenon. Here an error of -10% is made in W_2 whereas errors of +10% are made in W_1 and W_3. Again, we see that unitl the W_2 wave exists, there are massive disturbances in the wave structure of W_1 and W_2. As soon as W_2 exists, (after 75 interations) the regular wave structive reasserts itself and errors propagate out in a predictable wave-like manner. Again, convergence takes approximately three hundred and twenty five iterations. It seems clear that this is optimal given the limited wave velocity, so we can deduce that these affects are due to the nonlinear coupling. Thus, even after the W_2 wave exists it will take at least a miximum time of $\{L/(u-c), L/u\}$ seconds for the resulting errors to propagate out of the system.

VII. DISCUSSION OF NUMERICAL RESULTS - BOUNDARY CONDITIONS:

The linear "small deflection" theory predicts that the best boundary conditions would be the prescription of the charateristic variables at their point of entry with some form of (stable) numerical boundary condition for the point of exit. Here are two such schemes

INFLOW	OUTFLOW
$W_1\ (o,t) = K_1$	$\dfrac{dW_1}{dx}\ (L,t) = 0$
$W_2\ (o,t) = K_2$	$\dfrac{dW_2}{dx}\ (L,t) = 0$ (13)
$\dfrac{dW_3}{dx}\ (o,t) = 0$	$W_3\ (L,t) = K_3$

Here K_1 and K_2 are numbers calculated from the known values of u, p, ρ at the inflow and K_3 is calculated from the known values of u and p at the outflow. Notice, however, the one "degree of freedom" is left at the inflow point. This allows the variables to adjust but in compensating ways. The boundary conditions in the code are usually in terms of the physical variables so we translate (13) to physical variables.

INFLOW

$$p_1 = \frac{\rho_o c_o}{2} [K_2 - u_2 - (1/\rho_o c_o) p_2]$$

$$u_1 = 1/2 [K_2 + u_2 - (1/\rho_o c_o) p_2]$$

$$\rho_1 = K_1 + (\rho_o/2c_o) [K_2 - u_2 + (1/\rho_o c_o) p_2]$$

OUTFLOW

$$u_N = 1/2 [u_{N-1} + (1/\rho_o c_o) p_{N-1} + K_3]$$

$$p_N = (\rho_o c_o/2) [K_3 + u_{N-1} + (1/\rho_o c_o) p_{N-1}]$$

$$\rho_N = (\rho_o/2c_o) [K_3 + u_{N-1}] - [1/(2 c_o^2)] p_{N-1} + \rho_{N-1}$$

This set of boundary conditions is predicted to work well in the linear studies of one equation, occurring in [2] and [3]. We shall call these boundary conditions the "no-change characteristic boundary conditions". Another possiblity, suggested by one-D analogues in [2], is the following:

INFLOW

$$W_1 (o,t) = K_1$$

$$W_2 (o,t) = K_2$$

$$\frac{\partial W_3}{\partial t} + (u-c)_1 \frac{\partial W_3}{\partial x} = 0$$

OUTFLOW

$$\frac{\partial W_1}{\partial t} + u_N \frac{\partial W_1}{\partial x} = 0$$

$$\frac{\partial W_2}{\partial t} + (u+c)_N \frac{\partial W_2}{\partial x} = 0 \qquad (14)$$

$$W_3 (L,t) = K_3$$

where derivatives in the x-variable are downwind at the inflow and upwind at the outflow and forward in time. The numbers K_1, K_2, K_3 are prescribed as before. In terms of the physical variables these translate into

INFLOW

$$u_1^{n+1} = 1/2\ [K_2 + u_1^n - (1/\rho_o c_o)p_1^n + (u_o - c_o)(\frac{\Delta t}{\Delta x})[u_1^n - u_2^n + (1/\rho_o c_o)(p_2^n + p_1^n)]]$$

$$p_1^{n+1} = (\rho_o c_o/2)[K_2 + (1/\rho_o c_o)p_1^n - u_1^n(u_o - c_o)(\Delta t/\Delta x)\ [u_2^n - u_1^n + (1/\rho_o c_o)(p_1^n + p_2^n)]]$$

$$\rho_1^{n+1} = K_1 + (\rho_o/2c_o)[K_2 + (1/\rho_o c_o)p_1^n - u_1^n + (u_o - c_o)(\Delta t/\Delta x)[u_2^n - u_1^n + (1/\rho_o c_o)(p_1^n - p_2^n)]]$$

OUTFLOW

$$u_N^{+1} = (1/2)[u_N^n + \frac{1}{\rho_o c_o}\ p_N^n - K_3 + (\Delta x/\Delta x)(u_o + c_o)[u_{N-1}^n - u_N^n + (1/\rho_o c_o)(p_{N-1}^n + p_N^n)]]$$

$$p_N^n = (\rho_o c_o/2)[K_3 + u_N^n + (1/\rho_o c_o)p_N^n + (\Delta t/\Delta x)(u_o + c_o)[u_{N-1}^n - u_N^n + (1/\rho_o c_o)(p_{N-1}^n - p_N^n]]$$

$$\rho_N^{n+1} = (\varrho/2c_o)[K_3 + u_N^n + (1/\rho_o c_o)p_N^n + (\Delta t/\Delta x)(u_o + c_o)[u_{N-1}^n - u_N^n + (1/\rho_o c_o)(p_{N-1}^n - p_N^n]]$$

$$+ \rho_N^n - (1/c_o^2)p_N^n + (\Delta t/\Delta x)u_o[\rho_{N-1}^n - \rho_N^n + (1/c_o^2)(p_N^n - p_{N-1}^n)]$$

We shall call these the "windward difference characterstic boundary conditions". Note: u_o, c_o can be different values at inflow and outflow. The performance of the code with either of these was analyzed by posing initial conditions in which there was a disturbance in one or more of the characteristic variables either locally at the center of the grid or uniformly throughout the grid, of the order of 10%.

Thus figure 1 shows what happens if the disturbance in only in the third characteristics variable locally using the windward characteristic variables.

Figure 3 shows the effect of a plus +10% error in the initial conditions W_1 and W_3 and a -10% error in W_2. (The first curve from the bottom is the initial state of the variable, and the others are the states at intervals of 25 iterations). In figure 2, we have an initial disturbance in W_2 of -10% with no initial disturbance in W_1 or W_3. The pictures look essentially the same as figure 1. There is considerable nonlinear interaction until the

W_2 wave exists, and then uncoupled wave motion to the right in the first variable (W_1) and to the left in the third variable (W_3). There are no reflections when the W_1 and W_3 waves exit and convergence is reached in 300 iterations.

These computations were made using the windward differencing characteristic boundary conditions, although the same results were obtained with the no change characteristic boundary conditions.

In figure 4, we show the effect of a local disturbance at the center of the grid in the W_2 variable with the second set of B.C.'s and in figure 5 we show a speeded up version (every 50 iterations) of the same disturbance with the first set of B.C.'s. The third and fifth graphs on figure 4 are almost exactly the same as the second and fourth on figure 5. Figure 5 shows convergence being reached in 300 iterations.

We conclude that either of the first two sets of boundary conditions give optimal convergence since convergence cannot take place until the wave in W_2 exists (very quickly) and the residual (nonlinear) effects of W_2 on W_3 can exit upstream. If they can do this without any reflections, then the convergence is essentially optimal.

Sometimes, it is objected that in a wind tunnel experiment, the only variable known downstream is pressure and that we are requiring too much information in prescribing K_3, which demands a knowledge of p and u at the outflow. Suppose, then we just prescribe p_∞ at the outflow using the otherwise successful conditions of $\frac{\partial W_1}{\partial x} = \frac{\partial W_2}{\partial x} = 0$ as complementary numerical boundary conditions. Then we could have, for example

INFLOW

$$W_1(o,t) = K_1$$

$$W_2(o,t) = K_2$$

$$\frac{dW_3}{dx} = (o,t) = 0$$

OUTFLOW

$$p = p_\infty$$

$$\frac{dW_1}{dx}(L,t) = 0 \qquad (15)$$

$$\frac{dW_2}{dx} = (L,t) = 0$$

The inflow boundary conditions are precisely those of (12). The outflow boundary conditions (used by Steger [12]) are

$$p_N = p_\infty$$

$$\rho_N = (1/c_o^2)(p_\infty - p_{N-1}) + \rho_{N-1}$$

$$u_N = (1/\rho_o c_o)(p_{N-1} - p_\infty) + u_{N-1}$$

The predictions of section VI are clear. The fact that p is prescribed means that when a wave in W_2 comes downstream, it exits by adjusting the u values at $x = x_N$. This in turn causes disturbances in $W_3 = -u+(1/\rho_o c_o)p_\infty$ which cause a reflected wave upstream. This wave can be seen in figure 6. Since the upstream B.C. is non-reflecting, this means that the left running wave will exit without incidence. When compared with boundary conditions (13) or (14) it is obviously less desirable because of the magnitude of the reflection in W_3. However it does converge in approximately 300 iterations, which is again almost optimal. The effect of these large oscillations in more complicated geometries may prove undesirable, however.

We briefly review our progress so far. Two sets of non-reflecting boundary conditions have been produced, both of which give optimal convergence but which rely on a great deal of information at both ends. The infomation however is used in such a way as to allow additional degrees of freedom for the waves to exit without repeated reflections. One reflecting and one non-reflecting boundary condition can be combined to obtain almost optimal convergence at the cost of some large left running reflections, whose effect in more complicated geometries remains uncertain. While B.C. 13 was expected to be less accurate than B.C. 14, little evidence for this has been uncovered, except at the boundaries. Time dependent periodic flows (e.g. self excited oscillations) may prefer B.C. 14 however. We now consider some of the other boundary conditions which have been tried previously in the literature.

First we consider the case of reflecting boundary conditions. These occur in several places in the literature, for example in [10] and [12] and have been discussed in section VI. As we have seen, these arise from prescribing combinations of characteristic variables such as pressure downstream, and other combinations (perhaps to density and velocity upstream). In [10] Rudy and Strikwerda considered (among many others) the boundary conditions

INFLOW	OUTFLOW	
$u = u_\infty$	$\frac{du}{dx} = 0$	
$T = T_\infty$	$\frac{d\rho}{dx} = 0$	(16)
$\frac{dW_3}{dx} = 0$	$p = p_\infty$	

and in [12] Steger uses

$u = u_\infty$	$\frac{dW_1}{dx} = 0$	
$\rho = \rho_\infty$	$\frac{dW_2}{dx} = 0$	(17)
$\frac{dW_3}{dx} = 0$	$p = p_\infty$	

Figure 7 shows the effects of an initial local disturbance in W_3 on both of these sets of boundary conditions, (16) on the left and (17) on the right. Note that first there is only a disturbance in the bottom picture. By iteration 75 (fourth curve up from the bottom) we can see reflections in both W_2 and W_1 although the W_2 deiration is a little harder to see. By iteration 125 it can be seen that the W_2 wave has travelled downstream and is reflected back upstream in W_3. These reflections continued (somewhat smeared out) for at least 2000 iterations. Perhaps most startling is (at least at the beginning) how similar they are. A conclusion may be drawn from B.C.'s (15), (16), and (17). Prescribing physical values instead of characteristic values gives rise to reflection. If the reflections can occur only at one end, this does not impede convergence. However, reflecting conditions at both ends can be disastrous. This accounts for the "spurious pressure waves" mentioned by Moretti [7].

The case of prescription at the wrong end is now considered. In this case, the variables u, p, ρ are prescribed at the inflow and the conditions

$$\frac{du}{dx} = \frac{d\rho}{dx} = \frac{dp}{dx} = 0$$

are prescribed at the outflow. In this case, square wave disturbances which affected only the interiors of the domain exited as in figure 1, with some minor oscillations. However, when a uniformly wrong initial condition was imposed, convergence was very slow with large oscillations, and the solution converged to the wrong values, with errors of as much as 27%. The converged value was a function of the initial condition at the outflow end, as predicted in Gustafson and Kreiss [3]. The resulting graphs are given in figure 8.

Related to the above problem is the method of *over prescription of boundaries.* This method is mentioned in [9] as giving good results although the authors caution against it on the grounds of small oscillations being present. In fact the situation is much more serious. If initial waves in the interior of the domain are used with the initial conditions correct near the outflow, then the solution converges rapidly as the travelling waves exit without reflections and with minor oscillations. However, if the initial data is uniformly wrong with an error intially at the outflow point, then W_1 and W_2 converge rapidly but W_3 accumulates huge errors of the order of 70%. Eventually when W_1 and W_2 are converged, the correct value is propagated upwind in W_3 but taking large amounts of time to converge because of the large errors near the inflow point. Indeed, as W_2 becomes more accurate downstream the inaccuracies become much larger (140%) upstream. Particularly in a time dependent problem or in a problem with more complicated geometries, this could be truly disastrous. It points to another fact: if a new boundary condition is being tested, it is not sufficient to consider initial value perturbations from free stream which are non-zero in the interior only. In this case, we might have drawn totally wrong conclusions from the time taken for convergence.

In [10], a separate non-reflecting boundary condition is proposed. This boundary condition alters the value of K_3, increasing it if the computed value of p is less than p_∞, and decreasing it if the computed value of p is greater than p_∞. Some thought shows that this cannot be optimal. Indeed, we can choose a variation from the intial conditions in which p is less than p_∞, but because us is smaller than u_∞, computed W_3 is actually larger than $W_{3,\infty}$. In this case, the proposed non-reflecting boundary condition of [10] could introduce more errors into the system, by increasing W_3 at the outflow point even more.

VIII. RECOMMENDATIONS:

We have completed an initial study of the one-dimensional Navier-Stokes equations and their far-field boundary conditions; and developed a one-dimensional code to provide numerical solutions. We have used this code to evaluate the impact of a variety of different boundary conditions upon the successful numerical solution of these problems. Several outstanding problems remain to be studied.

First one should examine the usefulness of the two recommended boundary condition sets in time dependent situations. It may well be that the difference between no-change characteristic boundary conditions and windward-difference characteristic boundary conditions will prove to be highly significant important in a time dependent problem. If so, this would be significant since a high proportion of work currently done at the Flight Dynamics Laboratory Computational Aerodynamics Group is of this type.

Recommendation for future efforts is to implement these characteristic boundary conditions in two and three-dimensional codes. This is presently in progress. As this is done, there is no doubt that further refinements will need to be considered.

In the discussion of our results, we noted, but did not stress, the following problem. Truly accurate time dependent solutions only exist if the C.F.L. number is close to 1. The time step is limited by a stability requirement that the C.F.L. number be less than 1. The problem with this is that the C.F.L. number is calculated on the basis of the fastest moving wave, which is the fastest to leave the region. After a relatively short amount of time, only the slow-moving waves are left. This ought (at least for some cycle of iterations) allow us to use a larger time step. This has not yet been thoroughly explored and we recommend that a study be made of this problem This could result in substantial savings of expenditure of computer time.

More work needs to be done to properly understand the nonlinear actions between the characteristic waves. This problem is not understood at this time and a proper understanding might give a clue to correct posing of intial conditions in such a way as to minimize start-up shocks, as observed in figures 4 and 5.

Another problem to be investigated on the one-dimensional code is whether up-dating the coefficients ρ_o c_o u_o to the $(n+1)^{st}$ time step values significantly accelerates convergence to steady state or improves accuracy. Either result would improve computational efficiency in the future.

Finally, we need to evaluate how appropriate these sets of boundary conditions are in the presence of stationary shock waves. This can be done first on a one-dimensional model problem.

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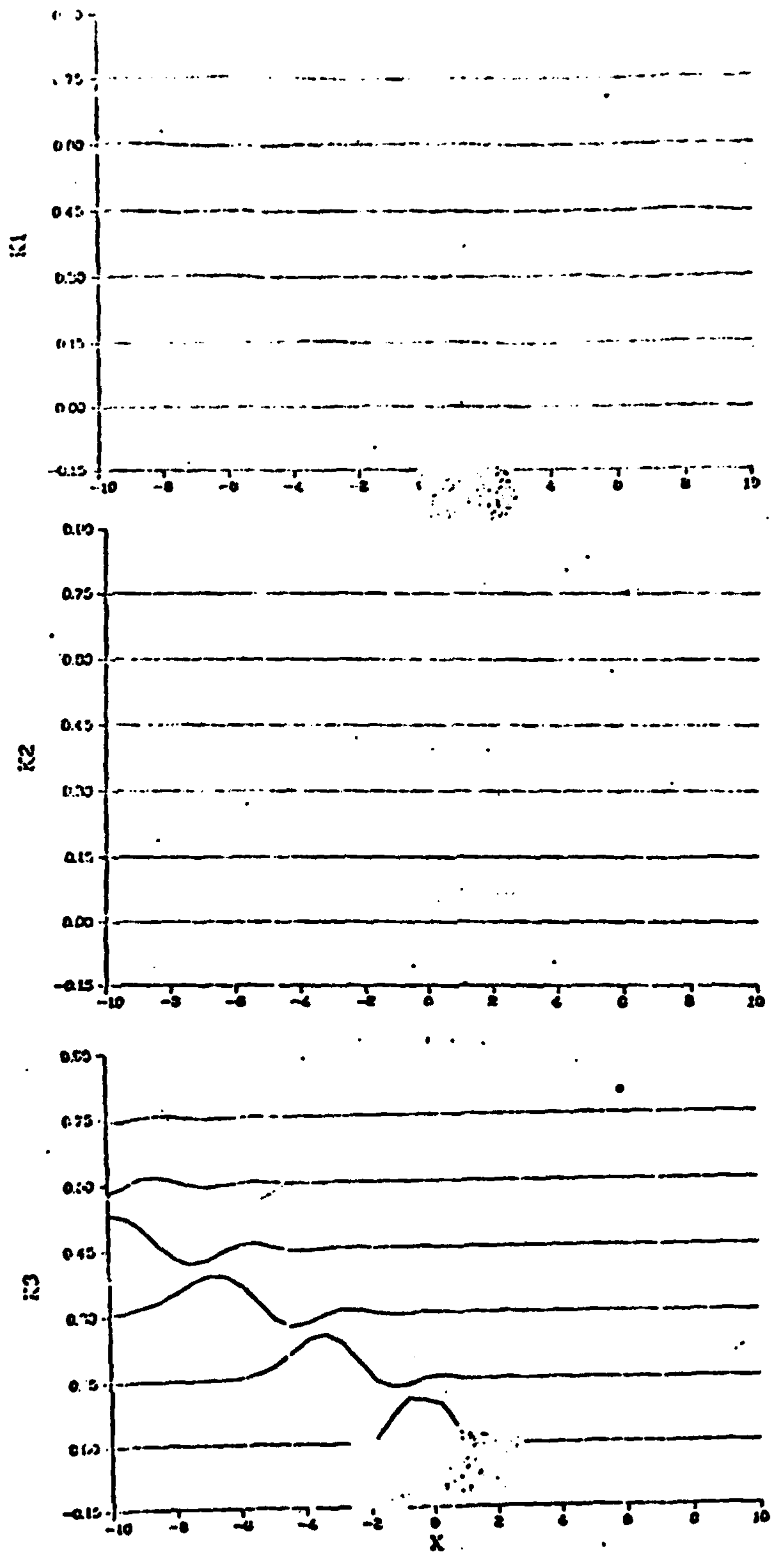

Figure 1 Initial local disturbance in characteristic variable K3

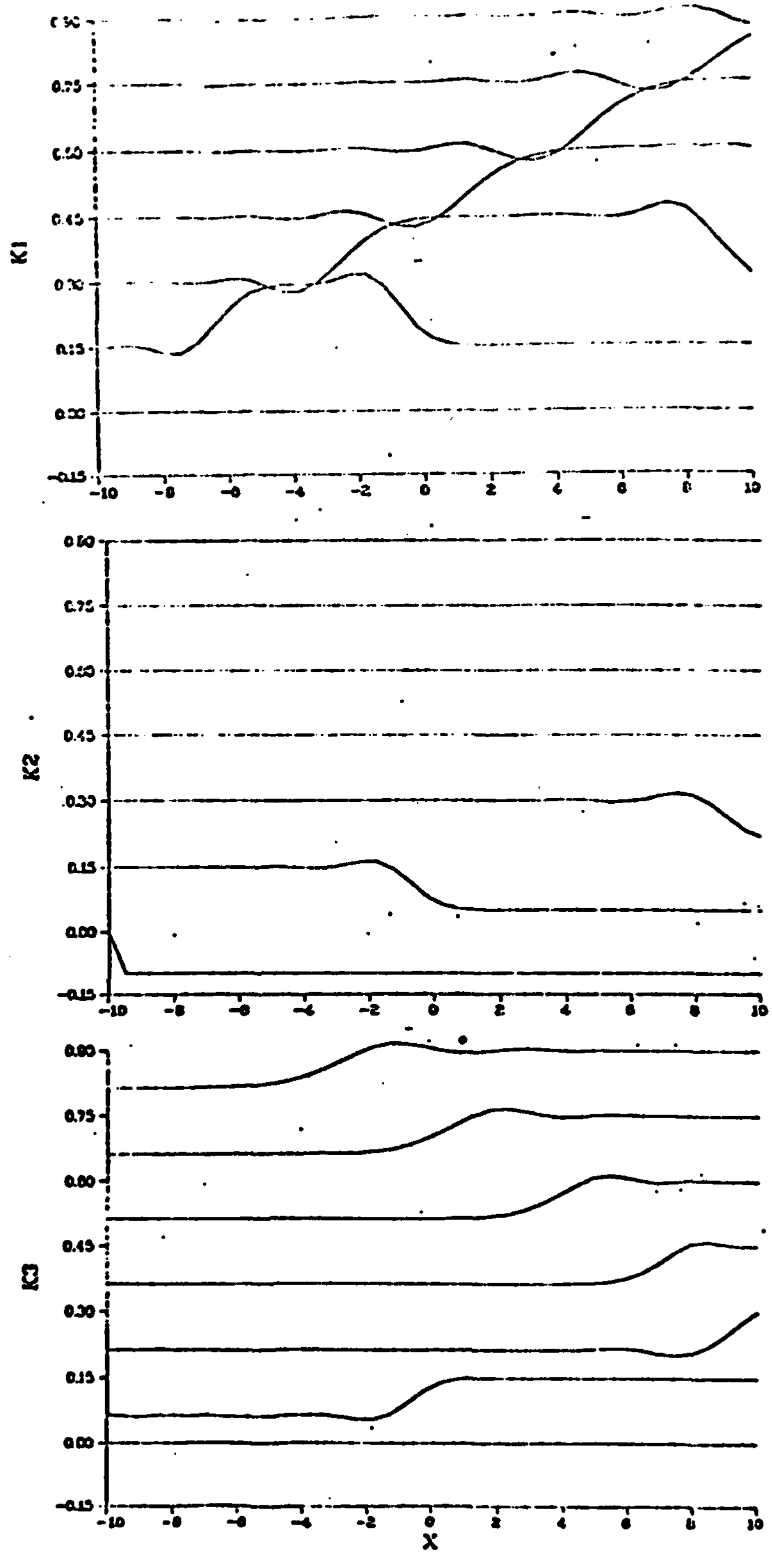

Figure 2 Initial uniform disturbance in K2

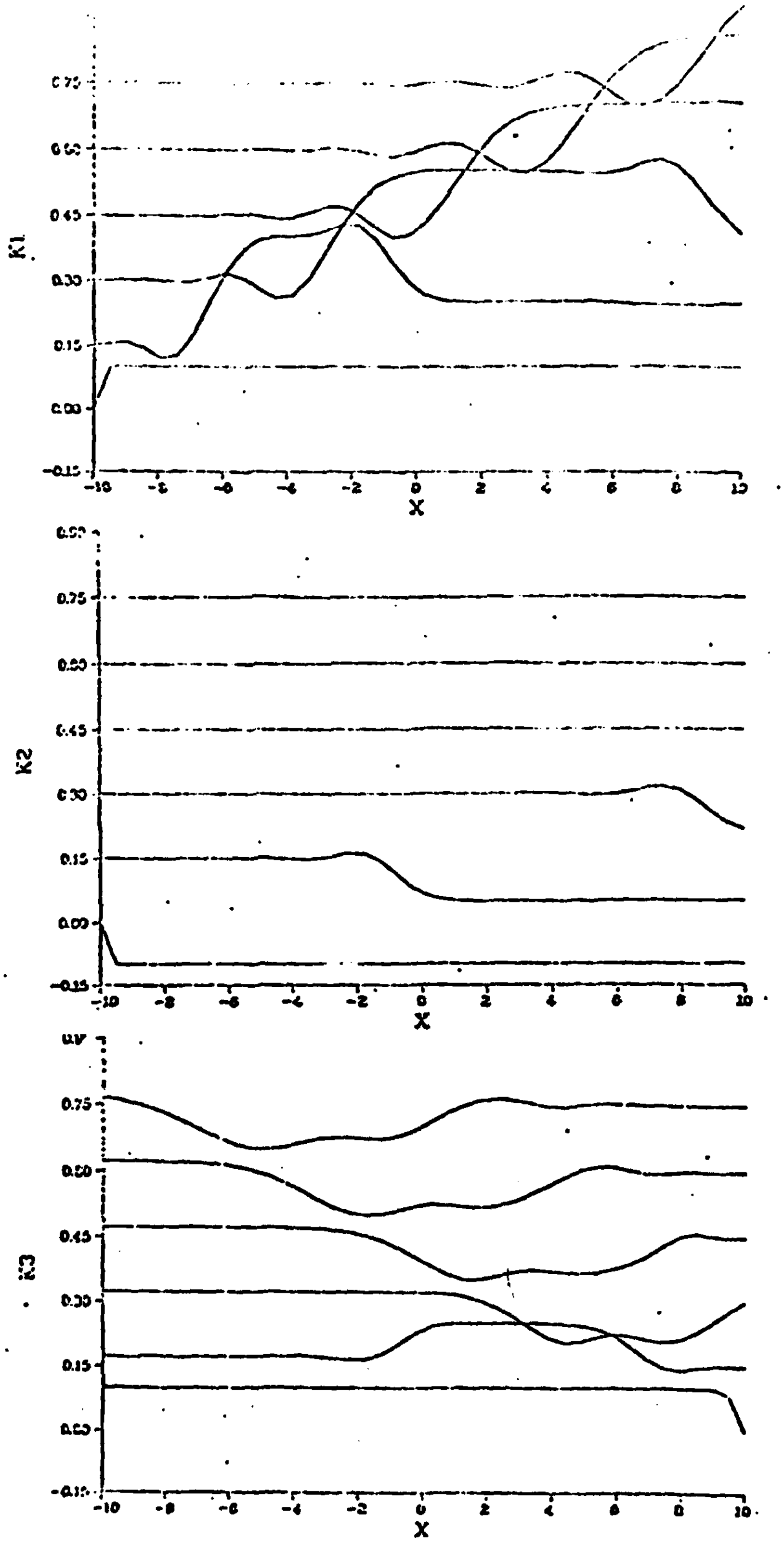

Figure 3 Initial uniform disturbances in all characteristic variables

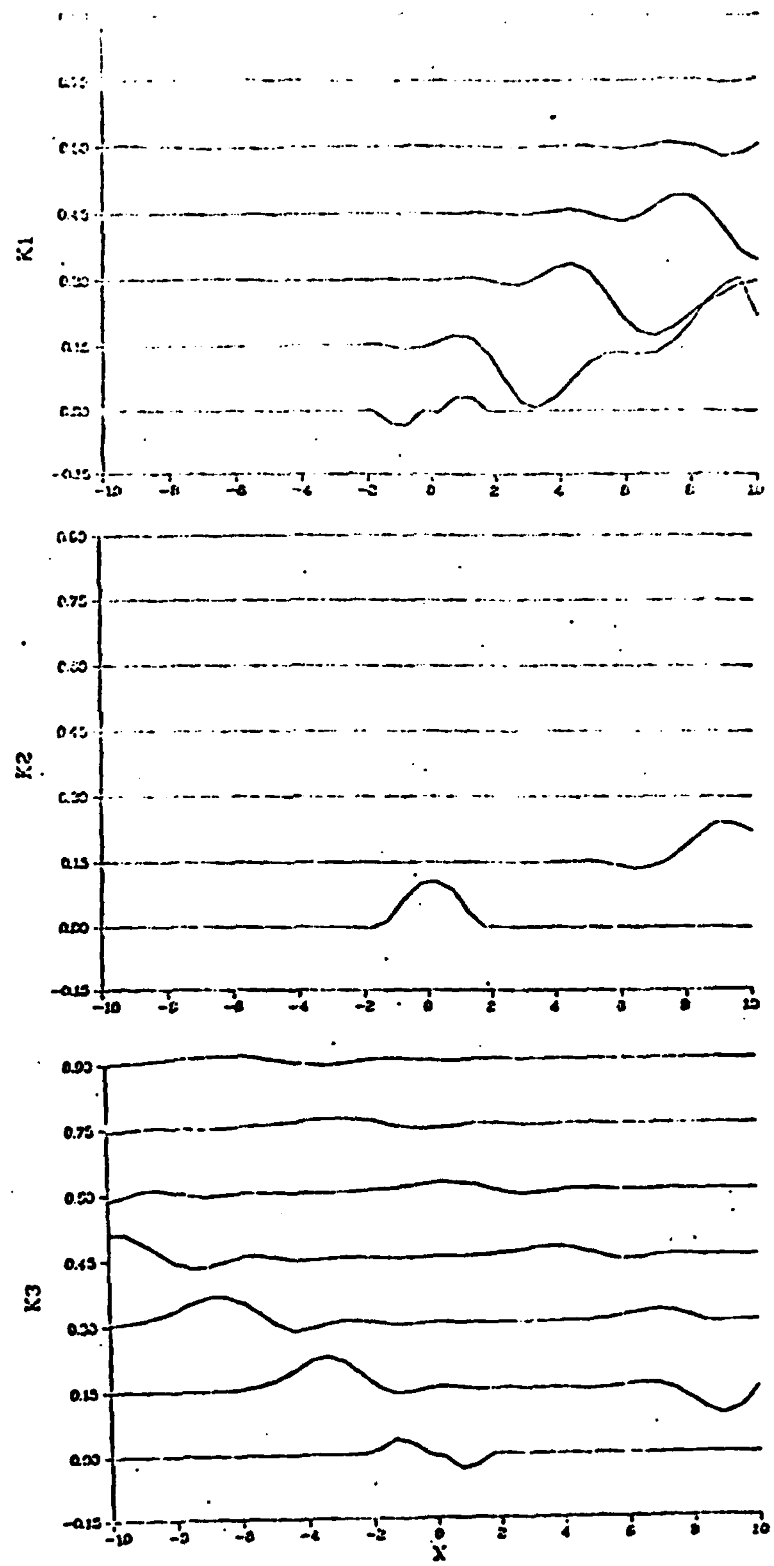

Figure 4 Initial local disturbance in K2, boundary conditions set (13)

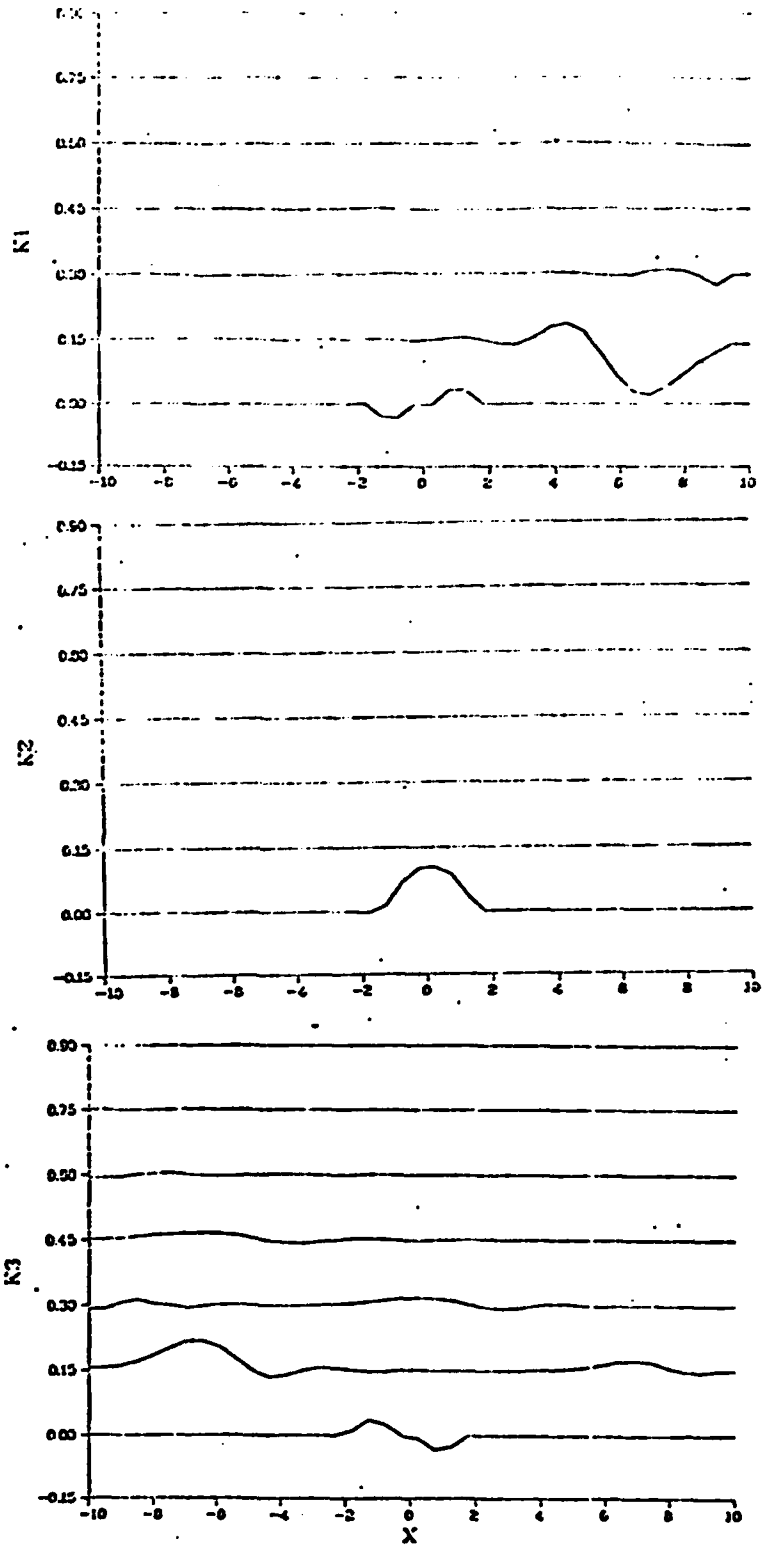

Figure 5 Initial local disturbance in K2, boundary conditions set (13)

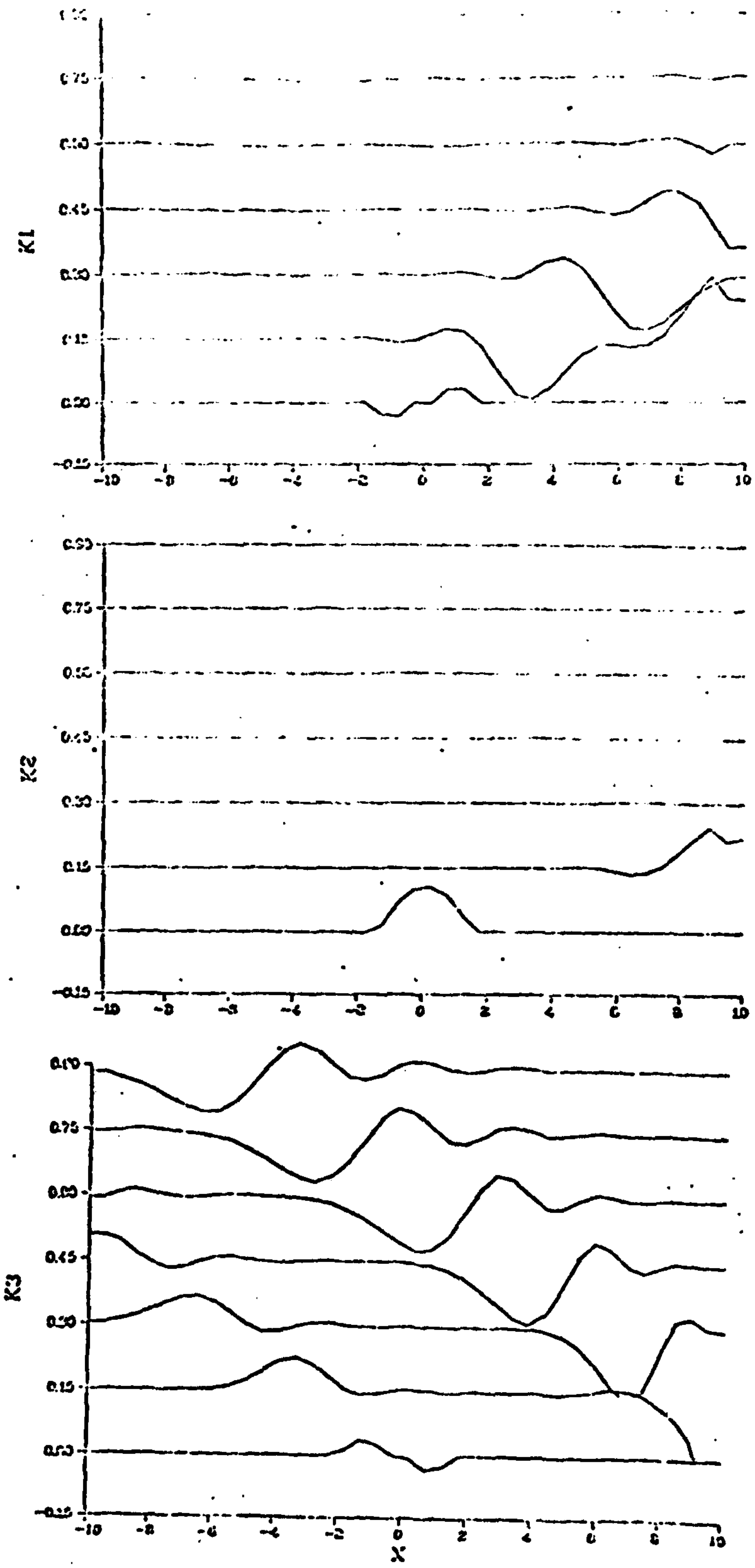

Figure 6 Initial local disturbance in K2, boundary conditions set (14)

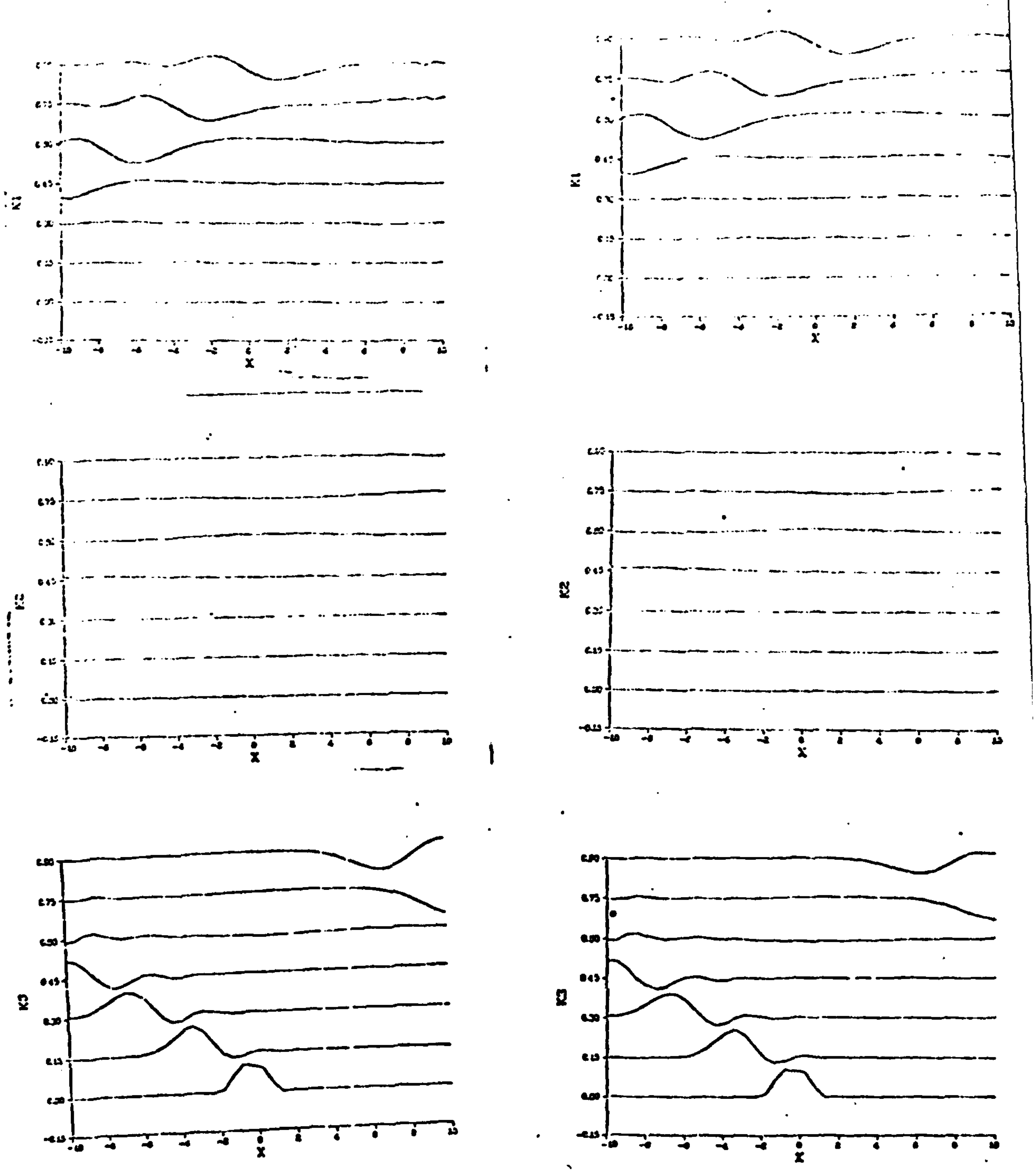

Figure 7   Initial local disturbance in K3, boundary conditions set (15) (left) and set (16) (right)

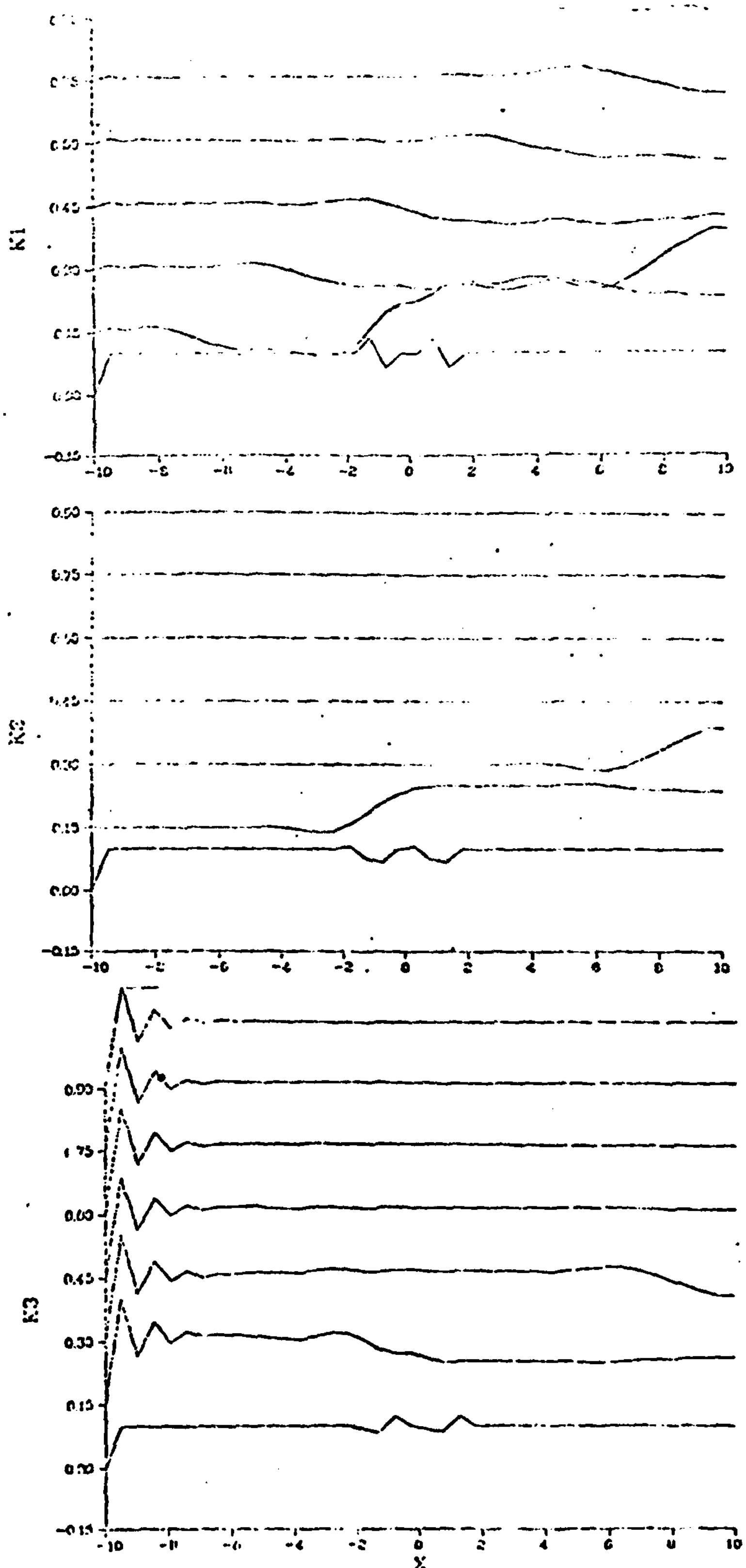

Figure 8 Initial uniform disturbance in all characteristic variables boundary conditions prescribed at wrong end

*USGPO:1983-659-062-