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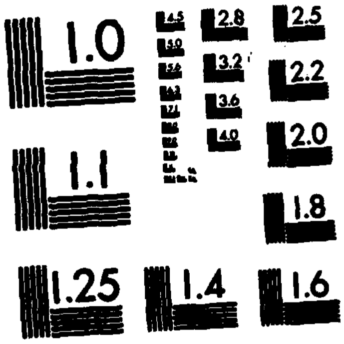

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CRITICAL DAMPING IN CERTAIN LINEAR CONTINUOUS DYNAMIC SYSTEMS

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1. INTRODUCTION

The most commonly used formulas for the analysis of fiber-reinforced and laminated beams are those given by the familiar "law of mixtures" ~~[1]~~. In their most general form, these formulas are derived by the methods of elementary strength-of-materials theory, on the basis of the Bernoulli-Euler assumptions. In general, of course, these formulas are only approximate: for example, when calculating the deflection of the simplest type of laminated beam, namely a sandwich, it is known that the inclusion of the effect of shear is essential ~~[2].~~ Because of their simplicity, however, it is desirable to use them whenever one may do so with sufficient accuracy. It is therefore the purpose of the present work to examine the validity of the simple formulas, and to determine how, and when, they should be corrected. More specifically, it will be shown that the generalized law of mixtures is exact for the case of uniform, uni-directionally reinforced beams under certain simple spanwise bending moment distributions, and a means of determining the required corrections when these conditions are not precisely met will be developed.

The work presented here is the counterpart, for reinforced beams, of that of [3]. Analogous developments have been carried out to show that the elementary formulas were valid if the derivations from uniform depth are small and smooth along the span, respectively in [4] for homogeneous and in [5] for fiber-reinforced beams. The basic method in these works was first introduced in [6] to study the effect of spanwise variations of temperature distributions on the validity of the elementary formulas for the case of rectangular beams, and was extended in [7,8] to beams of arbitrary cross-section.

In the analysis presented here the inhomogeneous beam is considered, at the outset, as one composed of a single material but with variable properties. The basic formulation of the problem thus contains spatial partial derivatives of the elastic constants, which are meaningless at the interfaces between adjacent layers, since sudden jumps in the material properties occur there. As one important conclusion of the present work, it will however be shown that the final results, both for stress and for deformations, can be expressed solely in terms of integrals, rather than derivatives, of the moduli: the difficulties which might arise because of discontinuities in material properties have therefore been eliminated, and the results obtained are valid and can be directly used for reinforced beams.

It will be seen that the present theory leads rather quickly to fairly cumbersome results, particularly if the geometry of the beam is at all complex. For this reason, it would be desirable to carry out, in subsequent research, a certain number of crucial examples, and thus to be able to reach some hopefully simple criteria for the identification of cases when corrections to the elementary formulas are indeed necessary. Furthermore, the development of a computer program for the numerical calculation of stresses and deformations from the techniques developed here would be useful as well. Lacking this additional work, it is felt that the results which follow present a valuable step towards the accurate and practical analysis of laminated beams and all circumstances. Some further details on the present work, and in particular on the numerical results, may be found in [9].

2. ELEMENTARY RESULTS

We consider in this paper a rectangular beam (occupying the space $0<x<L$, $-c<y<c$, $0<z<w$) unidirectionally reinforced, i.e., with $E = E(y)$ and $\nu = \nu(y)$. The elementary formula for the axial stress is the generalized law-of-mixtures, i.e.,

$$\sigma_{xx} = -\alpha E T + \frac{E}{D}\left\{(P+P_T)\left[\oint y^2 E\,dA - y\oint y E\,dA\right] + (M+M_T)\left[y\oint E\,dA - \oint y E\,dA\right]\right\} \tag{1}$$

The corresponding axial displacement u is easily calculated from the equation $u = \int_0^x (\sigma_{xx}/E + \alpha T)\,dx$, while the curvature is

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{D}\left[(P+P_T)\oint y E\,dA + (M+M_T)\oint E\,dA\right] \tag{2}$$

where

$$D = \oint E\,dA \oint y^2 E\,dA - \left(\oint y E\,dA\right)^2 ; \qquad \int_c^y = \int \;;\quad \int_{-c}^{c} = \oint \tag{3}$$

T is the temperature, dA = wdy and the other symbols have the usual meanings. Obvious simplifications occur, for example, if the origin of coordinates is chosen at the "centroid," i.e., if $\oint EydA = 0$.

3. BASIC THEORY

Under the normal thinness assumptions ($c/L \ll 1$, $w/L \ll 1$) and with $E = E(x,y)$ and $\nu = \nu(x,y)$, the problem requires [6] the solution for the Airy stress function $\varphi(x,y)$ from the equation

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\varphi}{\partial y^2}-\frac{\nu}{E}\frac{\partial^2\varphi}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2\varphi}{\partial x^2}-\frac{\nu}{E}\frac{\partial^2\varphi}{\partial y^2}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\frac{\partial^2\varphi}{\partial x\partial y}\right)=-\nabla^2(\alpha T) \qquad (4)$$

under boundary conditions to be presently discussed. The solution may be written [4,5] in the form

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \ldots \qquad (5)$$

if the quantities $\varphi_i(x,y)$ satisfy the equations:

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\varphi_1}{\partial y^2}\right) = -\frac{\partial^2(\alpha T)}{\partial y^2} \qquad (5a)$$

$$\begin{aligned}\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\varphi_2}{\partial y^2}\right) = &-\frac{\partial^2(\alpha T)}{\partial x^2} - 2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\frac{\partial^2\varphi_1}{\partial x\partial y}\right)\\ &+\frac{\partial^2}{\partial x^2}\left(\frac{\nu}{E}\frac{\partial^2\varphi_1}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\nu}{E}\frac{\partial^2\varphi_1}{\partial x^2}\right)\end{aligned} \qquad (5b)$$

$$\begin{aligned}\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\varphi_i}{\partial y^2}\right) = &-2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\frac{\partial^2\varphi_{i-1}}{\partial x\partial y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\nu}{E}\frac{\partial^2\varphi_{i-1}}{\partial y^2}\right)\\ &+\frac{\partial^2}{\partial y^2}\left(\frac{\nu}{E}\frac{\partial^2\varphi_{i-1}}{\partial x^2}\right)-\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2\varphi_{i-2}}{\partial x^2}\right) \quad \text{for } i \geq 3\end{aligned} \qquad (5c)$$

The boundary conditions depend on the details of the load application. We will consider for simplicity here only the isothermal case of a transverse load $q(x) = -d^2M/dx^2$ applied on $y = c$ (case I of [3]); the boundary conditions are then:

$$\sigma_{yy}(x,c) = \frac{q}{w} \;;\; \sigma_{yy}(x,-c) = \sigma_{xy}(x,\pm c) = 0 \tag{6}$$

and are all met if one sets

$$\frac{\partial^2 \varphi_1(x,c)}{\partial x^2} = \frac{q}{w} \;;\; \frac{\partial^2 \varphi_1}{\partial x^2}(x,-c) = \frac{\partial^2 \varphi_1(x,\pm c)}{\partial x \partial y} = 0 \tag{6a}$$

$$\varphi_i = \frac{\partial \varphi_i}{\partial y} = 0 \quad \text{on } y = \pm c\,,\; i \geq 2 \tag{6b}$$

4. STRESS COMPONENTS

Integration of eq. (5a) with T = 0 gives, after use of (6a) and of integration by parts,

$$\varphi_1 = \frac{M}{wD}\left[\oint E\,dy\left(y\int yE\,dy - \int y^2 E\,dy\right) - \oint yE\,dy\left(y\int E\,dy - \int yE\,dy\right)\right] \tag{7}$$

It is evident that $\partial^2\varphi_1/\partial y^2 = \sigma_{xx1}$ (i.e., the first term in a series similar to (5) for the axial stress) is identical with the elementary one of eq. (1). The first terms of similar series for the other stress components are

$$\sigma_{xy1} = -\frac{\partial^2 \varphi_1}{\partial x \partial y} = -\frac{1}{wD}\frac{dM}{dx}\left[\oint E\,dy \int Ey\,dy - \oint Ey\,dy \int E\,dy\right] \tag{8a}$$

$$\sigma_{yy1} = \frac{\partial^2 \varphi_1}{\partial x^2} = \frac{1}{wD}\frac{d^2M}{dx^2}\left[\oint E\,dy\left(y\int Ey\,dy - \int Ey^2\,dy\right) - \oint Ey\,dy\left(y\int E\,dy - \int Ey\,dy\right)\right] \tag{8b}$$

Further terms of the series are obtained in a straightforward manner by integration of (5b,c), under conditions (6b), integration by parts again being used to eliminate all partial derivatives of the moduli with respect to y. The result tends to be rather complicated: for example, φ_2 is given in Appendix A, and is indeed expressed solely in terms of moduli and their integrals, rather than derivatives with respect to y.

For the important special use of symmetrical reinforcements (i.e., with $\oint yE\,dy = \oint y\nu\,dy = 0$, such as, for example, an ordinary sandwich beam) the stress components take the form:

$$I^*\sigma_{xx} = MEy + \frac{d^2M}{dx^2}\Big[-2E\int\frac{1+\nu}{E}\int yE\,dy\,dy + \nu Ey\int y\nu\,dy - E\int \nu y^2\,dy + \nu y\int yE\,dy - \nu\int y^2E\,dy + C_1Ey + C_2E\Big] + \dots \tag{9a}$$

$$I^*\sigma_{xy} = -\frac{dM}{dx}\int yE\,dy - \frac{d^3M}{dx^3}\Big[-2\int E\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy + \int yE\int y\nu\,dy\,dy -$$

$$-\int E\int y^2\nu\,dy\,dy + \int y\nu\int yE\,dy\,dy - \int\nu\int y^2E\,dy\,dy + \tag{9b}$$

$$+ C_1\int yE\,dy + C_2\int E\,dy\Big] + \dots$$

$$I^*\sigma_{yy} = \frac{d^2M}{dx^2}\Big(y\int yE\,dy - \int y^2E\,dy\Big) + \frac{d^4M}{dx^4}\Big[-2y\int E\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy +$$

$$+ y\int Ey\int y\nu\,dy\,dy + 2\int yE\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy -$$

$$-\int Ey^2\int y\nu\,dy\,dy - y\int E\int y^2\nu\,dy\,dy + \int Ey\int y^2\nu\,dy\,dy + \tag{9c}$$

$$+ y\int\nu y\int yE\,dy\,dy - \int\nu y^2\int yE\,dy\,dy - y\int\nu\int y^2E\,dy\,dy + \int\nu y\int y^2E\,dy\,dy +$$

$$+ C_1\Big(y\int yE\,dy - \int y^2E\,dy\Big) + C_2\Big(y\int E\,dy - \int yE\,dy\Big)\Big] + \dots$$

where the weighted moment of inertia $I^* = w\oint Ey^2dy$ and where

$$C_1 = \frac{1}{\oint y^2 E\,dy}\Big(2\oint yE\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy - \oint Ey^2\int y\nu\,dy\,dy + \oint Ey\int y^2\nu\,dy\,dy - \oint \nu y^2\int yE\,dy\,dy + \oint \nu y\int y^2E\,dy\,dy\Big) \tag{10}$$

$$C_2 = \frac{1}{\oint E\,dy}\Big(2\oint E\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy - \oint Ey\int y\nu\,dy\,dy + \oint E\int y^2\nu\,dy\,dy - \oint \nu y\int yE\,dy\,dy + \oint \nu\int y^2E\,dy\,dy\Big)$$

A discussion of these results will be given in a later section.

It is important to show that is is indeed possible to obtain expressions to all values of i which are free of derivatives of the moduli with respect to y. To prove this in general one may proceed by induction, i.e., by showing that, if any value of i exists for which φ_{i-1} and φ_{i-2} and their first two derivatives with respect to y can be expressed in a form which is free of such derivatives, then φ_i will itself be free of them. This follows immediately by rewriting (5c) explicitly for $\frac{1}{E}\frac{\partial^3\varphi_i}{\partial y^3}$ and noting that it is, by our hypothesis, free of moduli derivatives; it then follows that $\frac{\partial\varphi_i}{\partial y}$ and φ_i have the same character. But we have already noted that φ_1 and φ_2 likewise have this property, and thus the proof is complete.

5. DISPLACEMENTS

The strain components are found, once the stresses are known, directly from Hooke's law; from these, integration of the strain-displacement relations in the usual manner [3] gives the desired displacement components. The series for the axial displacement u is of the form

$$I^* u = y\int_0^x M\,dx + \frac{dM}{dx}\left[\ \right] + \frac{d^3M}{dx^3}\left[\ \right] + \ldots \tag{11}$$

where the second and third terms (i.e., u_2 and u_3) are given explictly in Appendix A. The expression for the curvature $\partial^2 v/\partial x^2$ is as follows:

$$-I^* \frac{\partial^2 v}{\partial x^2} = M + \frac{d^2M}{dx^2}\left(\int y \nu\, dy + C_1\right) + \ldots \tag{12}$$

6. DISCUSSION OF RESULTS

Several observations regarding the results obtained above may be made:

1. All results obtained for non-homogeneous beams contain no spatial derivations of the moduli. They are thus in the desired form, and no computational difficulties arise at the interfaces between component materials.

2. At each interface the shear stresses, the stresses normal to the interface and the transverse displacements are continuous (as they should be), but the axial normal stress and the axial displacement are not.

3. The results are all in series form, successive terms of which are proportional to progressively higher spanwise derivations of the bending moment and on higher powers of the thickness dimensions. They are therefore entirely analogous to the corresponding ones for homogeneous beams, and hence the

conclusions well known for that case can be immediately carried over to reinforced beams. In particular, note that the law-of-mixtures (i.e., the first term) is complete solution for the axial stress and for the curvature if the bending moment is of the form $M = a + bx$. The sum of the first and second terms is the complete solution of M is at most a cubic polynomial in x, and in general the accuracy of the elementary formulas increases as one considers thinner and thinner beams and smoother and smoother moment distributions. For further discussions of the meaning of "smooth" and of further conclusions, the reader is referred to the previous treatments (e.g., [3] or [7]). Of course, all the present results reduce to the known ones valid for homogeneous beams if simplified by taking both E and ν to be constants.

4. In the calculation of stresses in sandwich and multi-layer beams, the terms other than the first one, as can be expected, are of very minor importance. This is, of course, not true in the calculation of deflections, where the second term, for example, includes the effect normally referred to as shearing deformations and is therefore of substantial magnitude when one of the component materials is weak in shear. Such matters as the bending rigidity of sandwich faces, and the normal-stress carrying capacity of the sandwich core are automatically included in the present theory; they are normally (cf. [2]) neglected in sandwich analyses, and their effect can therefore be estimated with the aid of the present formulas.

5. In complete analogy again with homogeneous beam results, the correction terms in the series are often more important for thermal-stress analysis than for isothermal beams [7]. This occurs because non-trivial cases in which the first term vanishes can arise, as for example, that of a free multi-layer beam under a temperature linear with y, if the various layers have different moduli but the same coefficient of expansion. In such cases of course the entire stress is given by the

"correction" terms, and they cannot therefore be neglected, although no detailed consideration of the thermal case is included here.

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APPENDIX A

The expression for φ_2 for the case in which $E = E(y)$ and $\nu = \nu(y)$ is obtained by the process described in Section 3 to be as follows:

$$
\begin{aligned}
\varphi_2 = \frac{1}{wD}\frac{d^2M}{dx^2}\Big\{\Big[& \oint E\,dy\Big(y\int Ey\int y\nu\,dy\,dy - 2y\int E\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy + \\
& + 2\int yE\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy - \int Ey^2\int y\nu\,dy\,dy - y\int E\int y^2\nu\,dy\,dy + \\
& + \int Ey\int y^2\nu\,dy\,dy + y\int y\nu\int yE\,dy\,dy - \int\nu y^2\int yE\,dy\,dy - \\
& - y\int\nu\int y^2E\,dy\,dy + \int\nu y\int y^2E\,dy\,dy\Big) + \oint yE\,dy\Big(\int Ey^2\int\nu\,dy\,dy + \\
& + 2y\int E\int\frac{1+\nu}{E}\int E\,dy\,dy\,dy - 2\int yE\int\frac{1+\nu}{E}\int E\,dy\,dy\,dy - \\
& - y\int Ey\int\nu\,dy\,dy + y\int E\int y\nu\,dy\,dy - \int Ey\int y\nu\,dy\,dy - \qquad\text{(A1)} \\
& - y\int\nu y\int E\,dy\,dy + \int\nu y^2\int E\,dy\,dy + y\int\nu\int yE\,dy\,dy - \int\nu y\int yE\,dy\,dy\Big)\Big] + \\
+ \frac{1}{D}\Big[& \oint E\,dy\Big[\oint E\,dy\Big(y\int yE\,dy - \int y^2E\,dy\Big) - \oint yE\,dy\Big(y\int E\,dy - \\
& - \int yE\,dy\Big)\Big]\Big[2\oint yE\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy - \oint Ey^2\int y\nu\,dy\,dy + \\
& + \oint Ey\int y^2\nu\,dy\,dy - \oint\nu y^2\int yE\,dy\,dy + \oint\nu y\int y^2E\,dy\,dy\Big] + \\
& + \oint yE\,dy\Big[\oint E\,dy\Big(y\int yE\,dy - \int y^2E\,dy\Big) - \oint yE\,dy\Big(y\int E\,dy - \\
& - \int yE\,dy\Big)\Big]\Big[-2\oint yE\int\frac{1+\nu}{E}\int E\,dy\,dy\,dy + \oint Ey^2\int\nu\,dy\,dy - \\
& - \oint Ey\int y\nu\,dy\,dy + \oint\nu y^2\int E\,dy\,dy - \oint\nu y\int yE\,dy\,dy\Big] +
\end{aligned}
$$

$$+\oint E\,dy\Big[\oint yE\,dy\Big(y\int yE\,dy-\int y^2E\,dy\Big)-\oint y^2E\,dy\Big(y\int E\,dy-$$
$$-\int yE\,dy\Big)\Big]\Big[-2\oint E\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy+\oint Ey\int y\nu\,dy\,dy-$$
$$-\oint E\int y^2\nu\,dy\,dy+\oint \nu y\int yE\,dy\,dy-\oint\nu\int y^2E\,dy\,dy\Big]+$$
$$+\oint yE\,dy\Big[\oint yE\,dy\Big(y\int yE\,dy-\int y^2E\,dy\Big)-\oint y^2E\,dy\Big(y\int E\,dy-$$
$$-\int yE\,dy\Big)\Big]\Big[2\oint E\int\frac{1+\nu}{E}\int E\,dy\,dy\,dy-\oint Ey\int\nu\,dy\,dy+$$
$$+\oint E\int \nu y\,dy\,dy-\oint \nu y\int E\,dy\,dy+\oint\nu\int yE\,dy\,dy\Big]\Big]\Big\}$$

Here

$$D=\oint y^2E\,dy\oint E\,dy-\Big(\oint Ey\,dy\Big)^2 \tag{A2}$$

The stress components obtained from φ_2 are now easily calculated (cf. [9]), but are not explicitly listed here for the sake of brevity. They form the first-order correction to be applied to the stress components listed in eqs. (7) and (8). It is easily seen that all these stress components are themselves free of derivatives with respect to y.

Eq. (A1) and the corresponding stress components reduce identically to those of [3] for the special case of homogeneous beams. Simplifications resulting in the case of symmetrically placed reinforcements are given in eq. (9).

The displacement components are obtained by means of the steps outlined in the text. For the axial components, the result is:

$$I^*u=y\int_0^x M\,dx-\frac{dM}{dx}\Big[2\int\frac{1+\nu}{E}\int yE\,dy\,dy-y\int y\nu\,dy+$$
$$+\int y^2\nu\,dy-\frac{y}{\oint y^2E\,dy}\Big(\oint Ey\int y^2\nu\,dy\,dy+2\oint yE\int\frac{1+\nu}{E}\int yE\,dy\,dy\,dy-$$

$$- \oint E y^2 \int y \nu \, dy \, dy - \oint \nu y^2 \int y E \, dy \, dy + \oint \nu y \int y^2 E \, dy \, dy \Big)$$

$$- \frac{1}{\oint E \, dy} \Big(2 \oint E \int \frac{1+\nu}{E} \int y E \, dy \, dy \, dy - \oint E y \int y \nu \, dy \, dy +$$

$$+ \oint E \int y^2 \nu \, dy \, dy - \oint \nu y \int y E \, dy \, dy + \oint \nu \int y^2 E \, dy \, dy \Big) \Big] +$$

$$+ \frac{d^3 M}{dx^3} \Big[2 \iint \nu \int \frac{\nu+1}{E} \int y E \, dy \, dy \, dy \, dy - \iint y \nu \int y \nu \, dy \, dy \, dy +$$

$$+ \iint \nu \int y^2 \nu \, dy \, dy \, dy - \iint \frac{\nu(\nu^2-1)}{E} \int y E \, dy \, dy \, dy +$$

$$+ \iint \frac{\nu^2-1}{E} \int y^2 E \, dy \, dy \, dy + 2 \int \frac{1+\nu}{E} \int E y \int \nu y \, dy \, dy \, dy -$$

$$- 4 \int \frac{1+\nu}{E} \int E \int \frac{1+\nu}{E} \int y E \, dy \, dy \, dy \, dy - 2 \int \frac{1+\nu}{E} \int E \int y^2 \nu \, dy \, dy \, dy +$$

$$+ 2 \int \frac{1+\nu}{E} \int \nu y \int y E \, dy \, dy \, dy - 2 \int \frac{1+\nu}{E} \int \nu \int y^2 E \, dy \, dy \, dy - \qquad \text{(A3)}$$

$$- \Big(\iint y \nu \, dy \, dy + 2 \int \frac{1+\nu}{E} \int y E \, dy \, dy \Big) \frac{1}{\oint y^2 E \, dy} \times$$

$$\times \Big(2 \oint y E \int \frac{1+\nu}{E} \int y E \, dy \, dy \, dy - \oint E y^2 \int y \nu \, dy \, dy +$$

$$+ \oint E y \int y^2 \nu \, dy \, dy - \oint \nu y^2 \int y E \, dy \, dy + \oint \nu y \int y^2 E \, dy \, dy \Big) -$$

$$- \Big(\iint \nu \, dy \, dy + 2 \int \frac{1+\nu}{E} \int E \, dy \, dy \Big) \frac{1}{\oint E \, dy} \Big(2 \oint E \int \frac{1+\nu}{E} \int y E \, dy \, dy \, dy - \oint \nu y \int y E \, dy \, dy$$

$$- \oint E y \int y \nu \, dy \, dy + \oint E \int y^2 \nu \, dy \, dy + \oint \nu \int y^2 E \, dy \, dy \Big) \Big] +$$

$$+ \ldots$$

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