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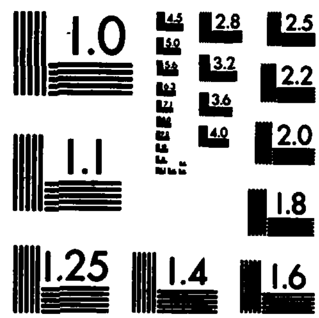

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Risk-efficient estimation of the mean exponential survival time under random censoring

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Abstract: The paper proposes a sequential estimator $\hat{\theta}$ of the parameter θ of an exponential distribution when the data is censored. Without any further conditions, it is shown that $\hat{\theta}$ is asymptotically risk efficient when the loss is measured by the squared error loss of estimation of θ plus a linear cost function of the number of observations. In addition, it is shown that $\hat{\theta}$ is asymptotically normal as the cost per observation goes to zero.

Key Words and Phrases: censored data, exponential distribution, and sequential estimation.

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1. Introduction.

In several longitudinal investigations the estimation of the mean survival time is of basic importance. This is usually based on the data gathered from a sample of $n(\geq 1)$ identical units on test as in a clinical trial or life test. However, it is often the case in these survival studies that the lifetimes of the specimens are not completely observable due to random withdrawals and consequent loss to follow-up. Accordingly, for each survival time X we envisage a competing censoring time Y, but the only datum available is (Z,δ) where $Z = \min(X,Y)$ and $\delta = 1$ whenever $X \leq Y$ and $\delta = 0$ otherwise. We shall assume X to be independent of Y and $Z \geq 0$ almost surely (a.s.) for each θ.

Suppose $\{(Z_i,\delta_i):1 \leq i \leq n\}$ is a random sample of size n with lifetimes X_i having common exponential survival function $F(t) = \exp(-t/\theta)$, $t > 0$ with $\theta(> 0)$ unknown and the censoring times Y_i having common (unknown) survival function G. We consider the sequence of estimators $\{\hat{\theta}_n\}$ of θ given by

$$\hat{\theta}_n = \begin{cases} (\sum_{i=1}^n Z_i)/(\sum_{i=1}^n \delta_i) & \text{if } \sum_{i=1}^n \delta_i \geq 1 \\ 0 & \text{if } \sum_{i=1}^n \delta_i = 0 . \end{cases} \qquad [1.1]$$

$\hat{\theta}_n$ is easily seen to converge a.s. to θ as $n \to \infty$ for each θ.

The loss incurred in estimation of θ by $\hat{\theta}_n$ is

$$L_n(a,c) = a(\hat{\theta}_n - \theta)^2 + cn \qquad [1.2]$$

where $a(> 0)$ is a known constant and $c(> 0)$ is the cost per unit observation. With the definition [1.1] we can show that

$$E(\hat{\theta}_n - \theta)^2 = (a\theta^2/E\delta)n^{-1} + o(n^{-1}) \qquad [1.3]$$

whence the risk associated with the estimation scheme becomes

$$R_n(a,c) = EL_n(a,c) = (a\theta^2/E\delta)n^{-1} + cn + o(n^{-1}) .$$

Therefore, in order to minimize this risk (with respect to n) we must take a sample of size n_c^0 with subsequent minimum risk R_c^0 where (for small c)

$$R_c^0 \sim 2cn_c^0 \quad \text{and} \quad n_c^0 \sim (a\theta^2/cE\delta)^{\frac{1}{2}} . \qquad [1.4]$$

Since θ is the unknown parameter which we wish to estimate, this minimum risk R_c^0 and optimal sample size n_c^0 are also unknown and thus we are led naturally to explore a sequential scheme for estimating θ . In this article we propose one such procedure in which the sample size is given by a random <u>stopping number</u> $N(c)$ and the associated risk $R_c^* = EL_{N(c)}$ satisfies $R_c^*/R_c^0 \to 1$ as $c \to 0$, that is if our scheme is <u>asymptotically risk-efficient</u>.

Sequential procedures analogous to the one outlined here have been considered, in the absence of censoring, by several researchers beginning with the pioneering work of Robbins (1959) for the estimation of the mean of a normal population. This was later extended by Starr (1966) and Starr and Woodroofe (1969). The study of sequential point estimation of the exponential mean is taken up in Starr and Woodroofe (1972). Gardiner and Susarla (1982) make the first examination of the problem of sequential estimation of the mean survival time in the presence of censoring when both the underlying survival time and censoring distributions are unspecified. When censoring is absent, similar procedures for estimation of functionals of an unspecified distribution are discussed in Sen and Ghosh (1981), Ghosh and Mukhopadhayay (1978) with some extensions and refinements by Chow et. al. (1981, 1982). Throughout the rest of the paper, the terms involving the random variable $[\sum_{i=1}^n \delta_i = 0]$ are left out without any further indication since $P[\sum_{i=1}^n \delta_i = 0]$ goes to zero at an exponential rate and all our scale factors will be like n^{-a} with $a > 0$.

2. The Sequential Procedure

Let $m(\geq 1)$ be the initial sample size. Then motivated by [1.4] define the stopping number N_c ($\equiv N(c)$) by

$$N_c = \min\{n \geq m : n \geq (\frac{a}{c})^{\frac{1}{2}}(\hat{\sigma}_n + n^{-\gamma})\} \tag{2.1}$$

where $\gamma > 0$ is a constant and $\hat{\sigma}_n^2$ is an appropriate estimator of $\sigma^2(\theta) \equiv \theta^2/E\delta = \theta^3(\int_0^\infty FG)^{-1}$. Notice that by the ordinary Central Limit Theorem we have from [1.1]

$$n^{\frac{1}{2}}(\hat{\theta}_n - \theta) \to_{\mathcal{D}} N(0,\sigma^2(\theta)). \tag{2.2}$$

An appropriate estimator of $\sigma^2(\theta)$ based on $\{(Z_i,\delta_i):1 \leq i \leq n\}$ is $\hat{\sigma}_n^2$ where

$$\hat{\sigma}_n^2 = \hat{\theta}_n^3/\overline{Z}_n = \{\overline{Z}_n^2/\overline{\delta}_n^3\}[\overline{\delta}_n \neq 0] \tag{2.3}$$

with the overscore denoting the usual corresponding sample mean and [A] denoting the indicator of the set A . Observe that for each fixed θ , $\hat{\sigma}_n^2$ converges a.s. to $\sigma^2(\theta)$ as $n \to \infty$.

Our proposed sequential scheme will take $N(c)$ observations and estimate θ by $\hat{\theta}_{N(c)}$ with resulting risk

$$R_c^* = a\,E(\hat{\theta}_{N(c)} - \theta)^2 + c\,E(N(c)) . \tag{2.4}$$

The main result of this paper is

<u>Theorem 1.</u> <u>(Risk Efficiency)</u>. For each $\gamma > 0$,

$$R_c^*/R_c^0 \to 1 \quad \text{as} \quad c \downarrow 0 .$$

Furthermore, we can easily demonstrate

<u>Theorem 2.</u> <u>(Asymptotic Normality of $\hat{\theta}_{N(c)}$)</u>. As $c \downarrow 0$, for each θ $\{\sqrt{N(c)}\,(\hat{\theta}_{N(c)} - \theta)/\hat{\sigma}_{N(c)}\}$ converges in distribution to a standard normal random variable.

The proofs of these theorems are outlined in the next section. We begin with a demonstration of the expansion [1.3] and a lemma concerning the rate of growth of $N(c)$ as $c \downarrow 0$.

3. Proofs.

We begin with a Lemma on left truncated inverse binomial moments.

Lemma 1. Let V be a binomial random variable with parameters n, p . Then for any positive integers k, ℓ

$$E(V^{-k}[V \geq \ell]) \leq K(np)^{-k}$$

where the constant K does not depend on n and p .

Proof. Write $n^{[k]} = n(n-1)\ldots\ldots(n-k+1)$ and $n^{-[k]} = (n^{[k]})^{-1}$. Then

$$\sum_{x=0}^{n}\{(x+k)^{-[k]}\, n^{[x]}\, p^x(1-p)^{n-x}/x!\}$$

$$= (n+k)^{-[k]} \sum_{x=0}^{n}\{(n+k)^{[x+k]}\, p^x(1-p)^{n-x}/(x+k)!\}$$

$$\leq (n+k)^{-[k]}\, p^{-k}$$

Therefore since $(x+k)^{[k]}/x^k$ is decreasing with respect to x ,

$$E(V^{-k}[V \geq \ell] \leq (\ell+k)^{[k]}\, \ell^{-k}\, E\{(V+k)^{-[k]}[V \geq \ell]\}$$

$$\leq (\ell+k)^{[k]}\, \ell^{-k}\, (n+k)^{-[k]}\, p^{-k}$$

$$\leq K(np)^{-k} .$$

We now turn to the crucial expansion given in [1.3]. To this end note that from [1.1] we can write

$$\hat{\theta}_n - \theta = U_n V_n \qquad [3.1]$$

where

$$U_n = (\bar{Z}_n - \theta\bar{\delta}_n);\; V_n = \bar{\delta}_n^{-1}[\bar{\delta}_n \neq 0] , \qquad [3.2]$$

We then have

Lemma 2. For any $p \geq 1$, $E|U_n|^{2p} = O(n^{-p})$ and $E|V_n|^p = O(1)$.

Proof. Since δ is either 0 or 1 and $0 \le Z \le X$ a.s., all moments of δ and Z are finite. From [3.2], we see that U_n is the average of n iidrv's with zero mean and variance $\theta^2 E\delta (= \theta\ EZ)$. Then by the Marcinkiewicz-Zygmund-Hölder inequality we obtain

$$E|U_n|^{2p} \le k_1 n^{-p}\ E|Z - \theta\delta|^{2p} = O(n^{-p}) .$$

(Here, and throughout this sequel k_j is a generic constant not depending on n).

For the second part of this lemma we apply Lemma 1 to the binomial random variable $n\bar{\delta}_n$, from which the required statement follows immediately. This concludes the proof of Lemma 2.

Before we proceed further, we point out that [1.3] follows from the equality $\hat{\theta}_n - \theta = (E\delta)^{-1}\ U_n\{1 - V_n(\bar{\delta}_n - E\delta)\}$ where U_n and V_n are defined by [3.2].

The next Lemma gives a convergence rate (as $c \downarrow 0$) for the stopping number N_c of [2.1]. First note that $N_c \ge (\frac{a}{c})^{1/2(1+\gamma)}$ a.s. so that $N_c \uparrow \infty$ a.s. as $c \downarrow 0$. In the sequel int $[x]$ denotes the greatest integer $\le x$ and $k_1, k_2, \cdots$ denote constants independent of c, but could depend on G and θ.

Lemma 3. Let $0 < \varepsilon < 1$ be arbitrary. Then as $c \downarrow 0$

$$\left.\begin{aligned} &P[N_c \le \operatorname{int}[n_c^0(1 - \varepsilon)]] = O(c) \\ &\text{and} \\ &P[N_c > \operatorname{int}[n_c^0(1 + \varepsilon)]] = O(c) . \end{aligned}\right\} \qquad [3.3]$$

Proof. Write $b = (\frac{a}{c})^{\frac{1}{2}}$ and $n_{1c} = \operatorname{int}[b^{1/c1+\gamma)}]$, $n_{2c} = \operatorname{int}[n_c^0(1 - \varepsilon)]$. On the set $[N_c \le n_{2c}]$ we have that $n \ge b\hat{\sigma}_n$ for some $n \in \{n_{1c}, \cdots\cdots, n_{2c}\}$. Therefore

$$P[N_c \le n_{2c}] \le P[\bigcup_{n=n_{1c}}^{n_{2c}} [\hat{\sigma}_n \le bn^{-1}]]$$

$$\le P[\bigcup_{n=n_{1c}}^{n_{2c}} [\hat{\sigma}_n^2 - \sigma^2 \le b^{-2}n_{2c}^2 - \sigma^2]] \text{ , where } \sigma^2 = \sigma^2(\theta) = \theta^2/E\delta \text{ ,}$$

$$\le P[\bigcup_{n=n_{1c}}^{n_{2c}} [|\hat{\sigma}_n^2 - \sigma^2| \ge \sigma^2 \varepsilon(2 - \varepsilon)]]$$

$$= P[\max_{n_{1c} \le n \le n_{2c}} |\hat{\sigma}_n^2 - \sigma^2| \ge \eta] \text{ , for some } \eta > 0 \text{ .} \qquad [3.4]$$

Now $\hat{\sigma}_n^2 - \sigma^2 = a_n(\overline{Z}_n - EZ) - b_n(\overline{\delta}_n - E\delta)$ [3.5]

where

$$a_n = (\overline{Z}_n + EZ)\overline{\delta}_n^{-3} [\overline{\delta}_n \ne 0] \text{ and}$$

$$b_n = (EZ)^2(E\delta)^{-3}(\overline{\delta}_n^2 + \overline{\delta}_n E\delta + (E\delta)^2)\overline{\delta}_n^{-3} [\overline{\delta}_n \ne 0] \text{ .}$$

Since $a_n \to 2\theta(E\delta)^{-2} = a_0$ a.s. and $b_n \to 3\theta^2(E\delta)^{-2} = b_0$ a.s. as $n \to \infty$, we obtain from [3.5] that the righthand side of [3.4] may be bounded by

$$P[\max_{n_{1c} \le n \le n_{2c}} |\overline{Z}_n - EZ| > \eta] + P[\max_{n_{1c} \le n \le n_{2c}} |\overline{\delta}_n - E\delta| > \eta] \qquad [3.6]$$

for an $\eta > 0$. Each of the two terms in [3.6] are handled in the same fashion. Observe that $\{|\overline{Z}_n - EZ| : n_{1c} \le n \le n_{2c}\}$ is a reverse submartingale to which the Kolmogorov maximal inequality applies. Thus

$$P[\max_{n_{1c} \le n \le n_{2c}} |\overline{Z}_n - EZ| \ge \eta] \le k_2 \, E|\overline{Z}_{n_{1c}} - EZ|^{4+4\gamma} = O(c) \text{ ,}$$

where k_1 is a constant depending only on γ. Thus combining [3.4] through [3.6] yields the first statement of Lemma 3. The second is proven in much the same way and so the proof of the lemma is terminated.

We may utilize the expansion in [3.5] to obtain the uniform integrability of $\{N_c/n_c^0 : 0 < c \le c_0\}$ for some c_0 sufficiently small; from this $E\{N_c/n_c^0\} \to 1$ as $c \downarrow 0$ will follow.

To this end note from [2.1] that

$\hat{\sigma}_{N(c)}/\sigma \le N_c/n_c^0 \le \hat{\sigma}_{N(c)-1}/\sigma + (N(c) - 1)^{-\gamma}/\sigma + (b\sigma)^{-1}$.

Then since $N_c \uparrow \infty$ a.s. (as $c \downarrow 0$) , we obtain $N_c/n_c^0 \longrightarrow 1$ a.s.,

Furthermore

$$(N_c/n_c^0)^2 \le 8\{(\hat{\sigma}_{N(c)-1}/\sigma)^2 + \sigma^{-2}(N(c) - 1)^{-2\gamma} + (b\sigma)^{-2}\} .$$

Thus in order to show $E(\sup_{c \le c_0} (N_c/n_c^0)^2) < \infty$ for some c_0 sufficiently small, we need only examine $E(\sup_{n \ge m} |\hat{\sigma}_n^2/\sigma^2 - 1)$ for sufficiently large m . However, from [3.5] and the convergences $a_n \longrightarrow a_0$, $b_n \longrightarrow b_0$ a.s. it suffices to establish the finiteness of $E(\sup_{n \ge m} |\overline{Z}_n - EZ|)$ and $E(\sup_{n \ge m} |\overline{\delta}_n - E\delta|)$. The second is trivial, while the first is bounded by four times $E|\overline{Z}_m - EZ|^2$ since $\{|\overline{Z}_n - EZ| : n \ge m\}$ is a reverse submartingale.

<u>Proof of Theorem 2</u>. Recall [3.1] and [3.2]. From the central limit theorem $n^{\frac{1}{2}}U_n \longrightarrow_{\mathcal{D}} N(0,\theta^2 E\delta)$, for each $\theta > 0$. Also $V_n \longrightarrow (E\delta)^{-1}$ a.s. Hence $n^{\frac{1}{2}}(\hat{\theta}_n - \theta) \longrightarrow_{\mathcal{D}} N(0,\sigma^2)$. We have noted that $\hat{\sigma}_n^2 \longrightarrow \sigma^2$ a.s. (as $n \longrightarrow \infty$) and $N_c \uparrow \infty$ as $c \downarrow 0$. Hence, by [3.8], [3.13], and [3.15] below, we get

$$\sqrt{N(c)}\ (\hat{\theta}_{N(c)} - \theta)/\hat{\sigma}_{N(c)} \longrightarrow_{\mathcal{D}} N(0,1) \quad \text{as} \quad c \downarrow 0 .$$

<u>Proof of Theorem 1</u>. In view of the fact that $E\{N_c/n_c^0\} \longrightarrow 1$, [1.4] and [2.4] we must verify that

$$\lim_{c \downarrow o} (cn_c^0)^{-1} E(\hat{\theta}_{N(c)} - \theta)^2 = 1 . \qquad [3.7]$$

Recall the expansion [3.3]. We first show

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{(\hat{\theta}_{N(c)} - \theta)^2[N_c \le n_{2c}]\} = 0 . \qquad [3.8]$$

Now

$$E\{(\hat{\theta}_{N(c)} - \theta)^2[N_c \le n_{2c}]\} \le k_3 \{EU_{N(c)}^2[N_c \le n_{2c}] + EW_{N(c)}^2[N_c \le n_{2c}]\} , \qquad [3.9]$$

where

$$W_n = U_n V_n(\overline{\delta}_n - E\delta)$$

Now applying the maximal inequality to the reverse submartingale $\{|\bar{Z}_n - \theta\bar{\delta}_n| : n_{1c} \le n \le n_{2c}\}$ in the third step and Lemma 3 we have

$$E\{U^2_{N(c)}[N_c \le n_{2c}]\} \le E(\max_{n_{1c}\le n\le n_{2c}} U^2_n[N_c \le n_{2c}])$$

$$\le E^{1/(1+\gamma)}(\max_{n_{1c}\le n\le n_{2c}} |U_n|^{2+2\gamma})(P[N_c \le n_{2c}])^{\gamma/(1+\gamma)}$$

$$\le k_4 E^{1/(1+\gamma)}|U^{2+2\gamma}_{n_{1c}}|\ 0\ (c^{\gamma/(1+\gamma)})$$

$$= 0(n^{-1}_{1c})0(c^{\gamma/1+\gamma}) .$$

Since $n_{1c} \sim c^{-1/2(1+\gamma)}$ and $(cn^0_c)^{-1} \sim c^{-\frac{1}{2}}$ we have that

$$\lim_{c\downarrow 0}(cn^0_c)^{-1}\ E\{U^2_{N(c)}[N_c \le n_{2c}]\} = 0 . \qquad [3.10]$$

Similarly $E\{W^2_{N(c)}[N_c \le n_{2c}]\} = \sum_{n=n_{1c}}^{n_{2c}} E\{W^2_n[N_c = n]\}$

$$\le (\sum_{n=n_{1c}}^{n_{2c}} E\ W^{2p}_n)^{1/p}(P[N_c \le n_{2c}])^{1-1/p} , \ p > 1 . \qquad [3.11]$$

Now $E\ W^{2p}_n = E\{U^{2p}_n\ V^{2p}_n(\bar{\delta}_n - E\delta)^{2p}\}$

$$\le E^{1/3}\{U^{6p}_n\}E^{1/3}\{V^{6p}_n\}E^{1/3}(\bar{\delta}_n - E\delta)^{6p}$$

$$= \{0(n^{-3p})0(1)0(n^{-3p})\}^{1/3} = 0(n^{-2p}) ,$$

Thus the rhs in [3.11] is $0(n^{-2+1/p}_{1c}\ c^{1-1/p}) = 0(c^h)$, where $h = \{2(1+\gamma)\}^{-1}(2 - 1/p) + (1 - 1/p) > \frac{1}{2}$ whenever $p > (2\gamma + 3)/(3 + \gamma)$. So

$$\lim_{c\downarrow 0}(cn^0_c)^{-1}\ E\{W^2_{N(c)}\ [N_c \le n_{2c}]\} = 0 \qquad [3.12]$$

Collecting our results [3.10] and [3.12] we obtain [3.8]. An entirely analogous argument using the second part of Lemma 3 will show

$$\lim_{c\downarrow 0}(cn^0_c)^{-1}\ E\{\hat{\theta}_{N(c)} - \theta)^2[N_c > n_{3c}]\} = 0 , \qquad [3.13]$$

where $n_{3c} = [n^0_c(1 + \varepsilon)]$. Therefore, in order to verify [3.7] we are left with proving

$$\lim_{c\downarrow 0}(cn_c^0)^{-1}\, E\{(\hat\theta_{N(c)} - \theta)^2[n_{2c} < N_c \le n_{3c}]\} = 0\ . \qquad [3.14]$$

However, from [1.3] $\lim_{c\downarrow 0}(cn_c^0)^{-1}\, E(\hat\theta_{n_c^0} - \theta)^2 = 1$, and further from the inequality

$$E\{(\hat\theta_{N(c)} - \hat\theta_{n_c^0})(\hat\theta_{n_c^0} - \theta)[n_{2c} < N_c \le n_{3c}]\}$$

$$\le E^{\frac{1}{2}}\{(\hat\theta_{N(c)} - \hat\theta_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\}\, E^{\frac{1}{2}}(\hat\theta_{n_c^0} - \theta)^2\ ,$$

we see that it suffices to establish

$$\lim_{c\downarrow 0}(cn_c^0)^{-1}\, E\{(\hat\theta_{N(c)} - \hat\theta_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\} = 0\ . \qquad [3.$$

Once again we utilize the expression [3.3] from which we get

$$\hat\theta_{N(c)} - \hat\theta_{n_c^0} = (E\delta)^{-1}\{U_{N(c)} - U_{n_c^0}) - (W_{N(c)} - W_{n_c^0})\}\ . \qquad [3.16]$$

Observe that

$$E\{(\hat\theta_{N(c)} - \hat\theta_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\}$$

$$\le k_5[E\{(U_{n(c)} - U_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\} \qquad [3.17]$$

$$+ E\{W_{N(c)} - W_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\}]$$

Now

$$E\{(U_{N(c)} - U_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\}$$

$$[3.18]$$

$$\le \max\,[E\{\max_{n_{2c}<n\le n_c^0}(U_n - U_{n_c^0})^2\}\ ,\ E\{\max_{n_c^0<n\le n_{3c}}(U_n - U_{n_c^0})^2\}]\ .$$

Since $\{U_n : n_{2c} < n \le n_c^0\}$ is a reverse martingale, $\{(U_n - U_{n_c^0})^2 : n_{2c} < n \le n_c^0\}$ is a reverse submartingale to which the maximal inequality will be applied.

This yields

$$E\{\max_{n_{2c}<n\le n_c^0}(U_n - U_{n_c^0})^2\} \le 4\, E\{U_{n_{2c}} - U_{n_c^0}\}^2$$

$$= 4\,\{EU^2_{n_{2c}} - EU^2_{n_2^0}\} \qquad [3.19]$$

Now $EU_n^2 = (\theta^2 E\delta)n^{-1}$, $n_{2c} = \mathrm{int}[n_c^0(1-\varepsilon)]$ and $cn_c^{0^2} = O(1)$. Therefore the rhs of [3.19] is of order $(cn_2^0)\varepsilon(1-\varepsilon)^{-1}O(1)$. Hence, by choosing ε arbitrarily small we have that

$$\lim_{c\downarrow 0}(cn_c^0)^{-1}E\{\max_{n_{2c}<n\le n_c^0}(U_n - U_{n_c^0})^2\} = 0\ . \qquad [3.20]$$

The second term in [3.18] is handled in an entirely analogous fashion and will yield the result paralleling [3.20]. So

$$\lim_{c\downarrow 0}(cn_c^0)^{-1}\, E\{(U_{N(c)} - U_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\} = 0\ . \qquad [3.21]$$

Finally we must dispose of the second term in [3.17]. This is easily handled using the bound $EW_n^2 = O(n^{-2})$ and the simple inequalities

$$E\{(W_{N(c)} - W_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\}$$

$$\le 4\,\{EW^2_{n_c^0} + \sum_{n=n_{2c}}^{n_{3c}} EW_n^2\} = O(n_c^{0^{-2}}) + O(n_c^{0^{-1}})\varepsilon(1-\varepsilon)^{-2}\ .$$

Then on selecting ε arbitrarily small we obtain

$$\lim_{c\downarrow 0}(cn_c^0)^{-1}\, E\{(W_{N(c)} - W_{n_c^0})^2[n_{2c} < N_c \le n_{3c}]\} = 0\ . \qquad [3.22]$$

Therefore in conjunction with [3.17], [3.21], and [3.22] yield [3.15]. This concludes the proof of Theorem 1.

4. Concluding Remarks.

To start with, it is worth pointing out that the results are true even if G is degenerate at any point T. That is, the results hold even in the fixed point truncation of X_i. Although the results are stated here only when $Y_1, \cdots, Y_n, \cdots$ are i.i.d., similar results hold even if $Y_1, \cdots, Y_n, \cdots$ are assumed to be independent only. In the latter case however, some sort of condition like $\varliminf_{n\to\infty} \int_0^\infty F\overline{G}_n > 0$, with $n\,\overline{G}_n = \sum_{i=1}^{n} G_i$, needs to be imposed.

The final comment concerns the damping factor $n^{-\gamma}$ introduced in the definition (2.1) of the stopping random variable N_c. Without this factor, the sequential estimator $\hat{\theta}_{N_c}$ may not be asymptotically risk efficient as the following argument suggests. Define

$$N_{c,1} = \min\{n \ge m : n \ge (\tfrac{a}{c})^{\frac{1}{2}}\, \hat{\sigma}_n\} \qquad [4.1]$$

where $\hat{\sigma}_n$ is defined by [2.3]. With these definitions, it can be seen that if the censoring distribution G has positive mass p at zero, then the risk $R^*_{c,1}$ of the sequential procedure $\hat{\theta}_{N_{c,1}}$ is at least $\theta^2 p^m$ and $R^*_{c,1}/R^0_c \longrightarrow \infty$ as $c \downarrow 0$ and hence $\hat{\theta}_{N_{c,1}}$ is asymptotically totally risk inefficient. Thus the factor $n^{-\gamma}$ in the definition of N_c appears to be important.

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The paper proposes a sequential estimator $\hat{\theta}$ of the parameter $\hat{\theta}$ of an exponential distribution when the data is censored. Without any further conditions, it is shown that $\hat{\theta}$ is asymptotically risk efficient when the loss is measured by the squared error loss of estimation of $\hat{\theta}$ plus a linear cost function of the number of observations. In addition, it is shown that $\hat{\theta}$ is asymptotically normal as the cost per observation to zero.