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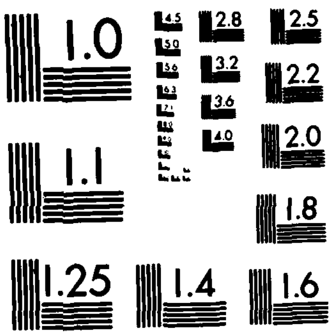

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SEMI-VALUES OF POLITICAL ECONOMIC GAMES

by

Abraham Neyman

Technical Report No. 366

February, 1982

A REPORT OF THE
CENTER FOR RESEARCH ON ORGANIZATIONAL EFFICIENCY
STANFORD UNIVERSITY

Contract ONR-N00014-79-C-0685, United States Office of Naval Research

THE ECONOMICS SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
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1. Introduction

Semi-values are defined in Dubey and Weber [1981] where characterization of the semi-values is given for two basic spaces; the space of all finite games, and the space of "differentiable" non-atomic games, i.e., pNA. In the purely economic situation, we usually encounter games in pNA (or in pNAD); but in many political economic situations, as in the Aumann-Kurz models of power and taxation [1977a], [1977b], we face games which are the products of weighted majority games by games in pNA. These games are members of other spaces which contain pNA and which we will refer to as spaces of political economic games. In this paper we will characterize all semi-values on spaces of political economic games. Section 3 presents a characterization of all continuous semi-values on a typical class of political economic games, followed by a detailed proof. In Section 4, we introduce further results without proofs. The proofs of the results in Section 4 are more involved than that of Section 3, but actually are based on the same ideas and thus we decided to omit them from our paper.

2. Preliminaries

Most of the definition and notations are according to Aumann and Shapley [1974]. Let $(I,\mathcal{C})$ be a measurable space isomorphic to $([0,1],\mathcal{B})$,

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*This work was supported by the Office of Naval Research Contract ONR-N00014-79-C-0685 at the Institute for Mathematical Studies in the Social Sciences, Stanford University.

where $\mathcal{B}$ is the σ-field of Borel subsets of $[0,1]$. A <u>set function</u> (or <u>game</u>) is a function $v:\mathcal{C} \to \mathbb{R}$ with $v(\phi) = 0$. A set function v is <u>monotonic</u> if for all T and S in $\mathcal{C}$, $T \subset S \Rightarrow v(T) \leq v(S)$. The space of all set functions on $(I,\mathcal{C})$ that are the difference of two monotonic set functions is denoted BV. The space of all bounded finitely additive set functions is denoted FA, and its subspace of all non-atomic measures is denoted NA. If $Q \subset BV$ then Q^+ denotes the subset of Q of all monotonic set functions. A mapping $\Psi:Q \to BV$ is <u>positive</u> if $\Psi(Q^+) \subset BV^+$. Let G denote the group of antomorphism of $(I,\mathcal{C})$. Each θ in G induces a linear mapping θ^* of BV onto itself that is given by $(\theta^* v)(S) = v(\theta S)$ for all S in $\mathcal{C}$. A subset Q of BV is <u>symmetric</u> if for each θ in G, $\theta^* Q \subset Q$.

Let Q be a symmetric subspace of BV. A <u>semi-value</u> on Q is a positive linear mapping ψ from Q into FA that satisfies:

(2.1) ψ is symmetric, i.e., $\psi\theta^* = \theta^*\psi$ for all θ in G.

(2.2) if $v \in Q \cap FA$ then $\psi v = v$.

The <u>bounded variation norm</u> of a set function v in BV is defined by $\|v\| = \inf(u(I) + w(I))$ where the infinum ranges over all pairs of monotonic set function u,w with $v = u - w$. A nondecreasing sequence of sets in $\mathcal{C}$ of the form Ω: $S_0 \subset S_1 \subset \ldots \subset S_n$ is called a chain. The variation of v over a chain Ω is defined by $\|v\|_\Omega = \sum_{i=1}^{n} |v(S_i) - v(S_{i-1})|$. It is known [3,Proposition 4.1] that $\|v\| = \sup\|v\|_\Omega$ where the supremum is taken over all chains Ω. If Q is a subspace of BV and $\psi:Q \to BV$ is linear then $\|\psi\|$ is defined as $\sup\{\|\psi v\|:v \in Q, \|v\| = 1\}$.

The space pNA is the closed subspace of BV that is generated by powers of nonatomic measures.

Let $\mathcal{I}$ denote the family of all measurable functions from I to [0,1] (measurable with respect to the σ-fields $\mathcal{C}$ and $\mathcal{B}$). There is a partial order on $\mathcal{I}$: $f \geq g$ if $f(s) \geq g(s)$ for all s in I. A real valued function w on $\mathcal{I}$ with $w(0) = 0$ is called an <u>ideal set function</u>; it is called <u>monotonic</u> if $f \geq g$ implies $w(f) \geq w(g)$. For every ideal set function w we denote by $\|w\|$ the supremum of $\sum_{i=1}^{n} |w(f_i) - w(f_{i-1})|$ taken over all increasing sequences $f_0 \leq f_1 \leq \dots \leq f_n$ of ideal set functions. The indicator function of a set S in $\mathcal{C}$ is denoted χ_S i.e., $\chi_S(s) = 1$ if $s \in S$ and equals 0 if $s \notin S$. We will sometimes write S for χ_S, t for $t\chi_I$ and tS for $t\chi_S$.

It is known [3,Theorem G] that there is a unique linear mapping that associates with each set function v in pNA an ideal set function v^* such that $(vw)^* = v^*w^*$ for all v,w in pNA, v^* is monotonic wherever v is in pNA^+, $\|v\| = \|v^*\|$, and such that $\mu^*(f) = \int_I f d\mu$ for all μ in NA and all f in $\mathcal{I}$.

Denote $\partial v^*(t,S) = (d/d\tau)v^*(t + \tau S)_{\tau=0}$. By theorem H of [3] we know that for each v in pNA and each S in $\mathcal{C}$, the derivative $\partial v^*(t,S)$ exists for almost all t in [0,1] and is integrable over [0,1] as a function of t.

We denote by W the set of non-negative functions g in $L_\infty([0,1])$ with $\int_0^1 g(t)dt = 1$.

The characterization of the class of semi-values on pNA is given in Dubey, Neyman and Weber ([1981], Theorem 2).

Theorem 2.3. ([1981], Theorem 2). For each g in W the mapping ψ_g: pNA $\to$ FA that is given by

$$(\psi_g v)(S) = \int_0^1 \partial v^*(t,S)g(t)dt$$

is a semi-value. Moreover, every semi-value on pNA is of this form. The map $g \to \psi_g$ of W onto the class of semi-values on pNA is a linear isometry.

Define DIAG to be the set of all v in BV satisfying: there exists a positive integer k, a k-dimensional vector ξ of probability measures in NA, and a neighborhood U in $\mathbb{R}^k$ of the diagonal $[0,\xi(I)]$ such that if $\xi(S) \in v$ then $v(S) = 0$. A semi-value ψ on a symmetric subspace Q of BV is diagonal if $\psi v = 0$ for all $v \in Q \cap$ DIAG.

Proposition 2.4. Continuous semi-values are diagonal.

Proof. The proof in Nayman [1977] that continuous values are diagonal does not make use of the efficiency axiom and therefore the same proof works here.

Another result which will be used in the proofs of the present paper is:

Proposition 2.5. Let Q be a symmetric subspace of BV, and let ψ be a semi-value on Q. If $\mu \in NA^+$ and f is defined on the range of μ with $f \circ \mu \in Q$, then $\psi(f \circ \mu) = a\mu$ for some constant a in $\mathbb{R}$.

Proof. Follows from the proof of proposition 6.1 in Aumann and Shapley [1974].

3. Characterization of the Semi-Values on a Class of Political Economic Games

In the purely economic situation, we are usually encountered with games in pNA (or in pNAD - the closed linear space generated by pNA and DIAG) but in many political economic situations we face games of the form $v = uq$ where q is in pNA and u is a jump function with respect to a given NA probability measure μ, i.e.,

$$u(S) = \begin{cases} 1 & \text{if } \mu(S) \geq \alpha \\ 0 & \text{if } \mu(S) < \alpha \end{cases} .$$

Such games arose for instance in models for taxation (See Aumann-Kurz [1977a], [1977b]. We denote by u*pNA the minimal linear symmetric space containing pNA and all games of the form uq where $q \in$ pNA and α is a fixed number in (0,1).

Theorem A. For any pair (a,g), $a \in \mathbb{R}^{+}$, $g \in W$, there is a semi-value $\psi_{(a,g)}$ on u*pNA such that for any $q \in$ pNA

(3.1) $$(\psi_{(a,g)}q)(S) = \int_0^1 g(t)\partial q^*(t,S)dt$$

and

(3.2) $$(\psi_{(a,g)}(uq))(S) = aq^*(\alpha)\mu(S) + \int_\alpha^1 g(t)\partial q^*(t,S)dt \quad .$$

Moreover, any contⁱnous semi-value on u*pNA is of that form. The mapping $(a,g) \to \psi_{(a,g)}$ is 1-1 and $\|\psi_{(a,g)}\| = \max(a, \|q\|_\infty)$.

The proof of the theorem is accomplished in several stages. First we shall state and prove a result on the range of vector of members of pNA. This is a generalization of a result of Dvoretzky, Wald and Wolfowitz ([1951], p. 66, Theorem 4).

Lemma 3.3. Let ν be a finite dimensional vector of measures in NA, and let m be a positive integer. Then for each m-tuple $f_1,\ldots,f_m$ of ideal sets such that $f_1 + \ldots + f_m = 1 = \chi_I$, and each k-tuple $q_1,\ldots,q_k$ of members of pNA, and each $\varepsilon > 0$ there is a partition $(T_1,\ldots,T_m)$ of I, T_i in $\mathcal{C}$ such that for all $A \subset \{1,\ldots,m\}$ and all $1 \leq j \leq k$

$$\nu\Big(\bigcup_{i\in A} T_i\Big) = \int\Big(\sum_{i\in A} f_i\Big)d\nu$$

and

$$\Big|q_j\Big(\bigcup_{i\in A} T_i\Big) - q_j^*\Big(\sum_{i\in A} f_i\Big)\Big| < \varepsilon .$$

Remark: The same result holds if pNA is replaced by pNA' (replace in the proof $\| \ \|$ by $\| \ \|'$).

Proof. From the definition of pNA it follows that for each $1 \leq j \leq k$ there exists a polynomial v_j of NA-measures; $v_j = P_j(\mu_1^j,\ldots,\mu_{n_j}^j)$ with $\|q_j - v_j\| < \varepsilon$. By

$$\nu(T_i) = \int f_i d\nu$$

and

$$\mu_\ell^j(T_i) = \int f_i d\mu_\ell^j$$

From the finite additivity of members of NA, we deduce that for each $A \subset \{1,\ldots,m\}$, $1 \leq j \leq k$ and $1 \leq \ell \leq n_j$ $\nu(T(A)) = \int f(A)d\nu$ and $\mu_\ell^j(T(A)) = \int f(A)d\mu_\ell^j$ where $T(A) = \bigcup_{i \in A} T_i$ and $f(A) = \sum_{i \in A} f_i$. From the last equalities and the properties of the mapping $v \to v^*$, it follows that also $v_j(T(A)) = v_j^*(f(A))$.

Thus

$$|q_j(T(A)) - q_j^*(f(A))| \leq |q_j(T(A)) - v_j(T(A))| + |v_j(T(A)) - v_j^*(f(A))| +$$

$$+ |v_j^*(f(A)) - q_j^*(f(A))| \leq \|q_j - v_j\| + 0 + \|v_j^* - q_j^*\|$$

and as $\|v^*\| = \|v\|$ for each $v \in pNA$, $|q_j(T(A)) - q_j^*(f(A))| < 2\varepsilon$. This completes the proof of Lemma 3.3. Q.E.D.

We will use in our proof the following immediate corollary of Lemma 3.3.

Corollary 3.4. Let ν be a finite dimensional vector of measures in NA, and let m be a positive integer. Then for each m-tuple $f_1,\ldots,f_m$ of ideal sets such that $f_1 \leq f_2 \leq \ldots \leq f_m$, and each k-tuple $q_1,\ldots,q_k$ of set functions in pNA, and each $\varepsilon > 0$ there is an m-tuple $T_1, T_2, \ldots, T_m$ of sets in C such that $T_1 \subset \ldots \subset T_m$ and for all $1 \leq j \leq k$ and all $1 \leq i \leq i' \leq m$

$$\nu(T_i) = \int f_i d\nu$$

and

$$|q_j(T_i) - q_j^*(f_i)| < \varepsilon \quad .$$

and

$$|q_j(T_{i'} \setminus T_i) - q_j^*(f_{i'} - f_i)| < \varepsilon$$

<u>Proof</u>. Follows by applying lemma 3.3 to the m+1-tuple $f_1, f_2 - f_1, \ldots, f_m - f_{m-1}, 1-f_m$.

We will proceed in order to show that (3.1) and (3.2) define a unique linear symmetric operator from u*pNA into FA. For this we shall need the following lemma.

<u>Lemma 3.5</u>: If $w = v + \sum_{i=1}^{n} (\theta_i^* u) q_i$ is monotonic, where $v \in pNA$, $q_i \in pNA$ and $\theta_i \in G, i = 1, \ldots, n,$ then for any $S \in C$ and $g \in L_\infty, g \geq 0$

$$(3.6) \quad \int_0^\alpha g(t) \partial v^*(t,S) dt \geq 0$$

$$(3.7)\quad \int_{\alpha}^{1} g(t)\partial v^{*}(t,S)dt + \sum_{i=1}^{n}\int_{\alpha}^{1} g(t)\partial q_i^{*}(t,S)dt \geq 0$$

and

$$(3.8)\quad \sum_{i=1}^{n} \mu(\theta_i S) q_i^{*}(\alpha) \geq 0 \quad .$$

Proof. Assume that w is monotonic. For proving (3.6) it .. ough to show that for any t with $0 < t < \alpha$ for which $\partial v^{*}(t,S)$ is defined, $\partial v^{*}(t,S) \geq 0$. Let $0 < t < \alpha$ and let $0 < h$ be such that $t + h < \alpha$. For such t and h, $t + hS \leq t + h$ and therefore $(\theta_i^{*}\mu)^{*}(t + hS) \leq (\theta_i^{*}\mu)^{*}(t + h) = t + h < \alpha$ for each $i = 1,\ldots,n$.

For any $\varepsilon > 0$ we could apply corollary 3.4 to the vector $\nu = (\theta_1^{*}\mu,\ldots,\theta_n^{*}\mu)$ of nonatomic measures, the 2-tuple t, $t + hS$ and the set function v in pNA to show the existnce of two sets T_1, T_2 in $\mathcal{C}$ with $T_1 \subset T_2$ and such that $(\theta_i^{*}\mu)(T_1) = (\theta_i^{*}\mu)(t) \equiv t < \alpha$, $(\theta_i^{*}\mu)(T_2) = (\theta_i^{*}\mu)(t + hS) < \alpha$ for all $i = 1,\ldots,n$ and such that $|v^{*}(t) - v(T_1)| < \varepsilon$ and $|v^{*}(t + hS) - v(T_2)| < \varepsilon$. Therefore, on the one hand, $(\theta_i^{*}u)(T_1) = (\theta_i^{*}u)(T_2) = 0$ which implies that $w(T_2) - w(T_1) = v(T_2) - v(T_1)$, and on the other hand, $v(T_2) - v(T_1) \leq v^{*}(t + hS) - v^{*}(t) + 2\varepsilon$. Altogether $v^{*}(t + hS) - v^{*}(t) \geq w(T_2) - w(T_1) - 2\varepsilon$. As w is monotonic we deduce that $v^*(t + hS) - v^*(t) \geq -2\varepsilon$, and as this holds for any $\varepsilon > 0$ we conclude that $v^*(t + hS) - v^*(t) \geq 0$ and therefore $\partial v^*(t,S) \geq 0$ for any $0 < t < \alpha$ for which $\partial v^*(t,S)$ is defined. This completes the proof of (3.6).

For proving (3.7) it is enough to prove that for any t with $\alpha < t < 1$ for which all the derivatives $\partial v^*(t,S)$ and $\partial q_i^*(t,S)$ exist,

$$\partial v^*(t,S) + \sum_{i=1}^{n} \partial q_i^*(t,S) \geq 0$$

Applying corollary 3.4 to the vector $(\theta_1^*\mu,\ldots,\theta_n^*\mu)$ of nonatomic measures, the 2-tuple t, $t + hS$ (where $0 < h$ is such that $t + h < 1$) and the members $v,q_1,\ldots,q_n$ in pNA we have for every $\varepsilon > 0$ two sets $T_1 \subset T_2$ in $\mathcal{C}$ such that for every $1 \leq i \leq n$, $\alpha < t = (\theta_i^*\mu)^*(t) = (\theta_i^*\mu)(T_1) \leq (\theta_i^*\mu)(T_2)$ and $|q_i(T_1) - q_i^*(t)| < \varepsilon$, $|q_i(T_2) - q_i^*(t + hS)| < \varepsilon$, and $|v^*(t) - v(T_1)| < \varepsilon$, $|v^*(t + hS) - v^*(T_2)| < \varepsilon$.

Therefore,

$$w(T_2) - w(T_1) = v(T_2) - v(T_1) + \sum_{i=1}^{n} q_i(T_2) - q_i(T_1)$$

$$\leq v^*(t + hS) - v^*(t) + \sum_{i=1}^{n} q_i^*(t + hS) - q_i^*(t) + 2(n + 1)\varepsilon \quad .$$

Again as this holds for all $\varepsilon > 0$ and as w is monotonic, it follows that

$$\partial v^*(t,S) + \sum_{i=1}^{n} \partial q_i^*(t,S) \geq 0 \quad ,$$

which completes the proof of (3.7).

The proof of (3.8) will make use of

<u>Lemma 3.9.</u> Let $\mu_1,\ldots,\mu_n$ be nonatomic probability measures and $q_1,\ldots,q_m$ set functions in pNA. Then for every $0 < \alpha < 1$ and every $\varepsilon > 0$ and every $1 \leq k \leq n$ there are two sets T_1, T_2 in $\mathcal{C}$, $T_1 \subset T_2$ such that for all $1 \leq i \leq n$ and for all $1 \leq j \leq m$

$$|q_j(T_2) - q_j^*(\alpha)| < \varepsilon \quad , \quad |q_j(T_2) - q_j^*(\alpha)| < \varepsilon$$

$$|q_j(T_2 \setminus T_1)| < \varepsilon$$

$$\mu_i(T_2 \setminus T_1) < \varepsilon$$

$$\mu_i(T_2) \geq \alpha > \mu_i(T_1) \text{ iff } \mu_i = \mu_k \quad .$$

<u>Proof</u>. Let $K = \{1 \leq i \leq n : \mu_i = \mu_k\}$. By Lyapunov's theorem there is T in $\mathcal{C}$ such that $\mu_i(T) = \mu_k(T)$ iff $\mu_i = \mu_k$. Let $b = \mu_k(T)$. Observe that for sufficiently small $\gamma > 0$, $\alpha + \gamma(T-b)$ is an ideal set and that

$$\mu_i^*(\alpha + \gamma(T - b)) = \alpha \text{ iff } i \in k \text{ (i.e.,iff } \mu_i = \mu_k) \quad .$$

If $(f_r)_{r=1}^{\infty}$ is a sequence in $\mathcal{I}$ that converges uniformly to f in $\mathcal{I}$ then for every q in pNA, $q^*(f_r)$ converges to $q^*(f)$. (All that is needed for that conclusion is that $\mu^*(f_r)$ converges to $\mu(f)$ for every nonatomic measure μ). Therefore there is $\gamma > 0$ sufficiently small so that

$\alpha + \gamma(T - b)$ is an ideal set and such that for all $1 \leq j \leq m$, $|q_j^*(\alpha + \gamma(T - b)) - q_j^*(\alpha)| < \varepsilon/3$. Fix such a $\gamma > 0$, and observe that $\beta(\alpha + \gamma(T - b)) \to \alpha + \gamma(T - b)$ as $\beta < 1$ converges to 1. Therfore for sufficiently large $\beta < 1$, for all $1 \leq j \leq m$

$$|q_j^*((1 - \beta)(\alpha + \gamma(T - b)))| < \varepsilon/3 \quad ,$$

$$|q_j^*(\beta(\alpha + \gamma(T - b))) - q_j^*(\alpha + \gamma(T - b))| < \varepsilon/3 \quad .$$

and thus

$$|q_j^*(\beta(\alpha + \gamma(T - b))) - q_j^*(\alpha)| < 2\varepsilon/3$$

As $\mu_i^*(\alpha + \gamma(T - b)) = \alpha$ iff $i \in K$, it follows that for sufficiently large $\beta < 1$ we also have

$$\mu_i^*(\alpha + \gamma(T - b)) \geq \alpha > \mu_i^*(\beta(\alpha + \gamma(T - b))) \text{ iff } i \in K \quad ,$$

Fix such a $\beta < 1$ and apply corollary 3.4 to the 2-tuple $\beta(\alpha + \gamma(T - b) < (\alpha + \gamma(T - b))$ with $\varepsilon/3$, the vector $(\mu_1, \ldots, \mu_n)$ of nonatomic measures and the members $q_1, \ldots, q_m$ of pNA to show the existence of $T_1, T_2 \in C$ with $T_1 \subset T_2$,

$$\mu_i(T_1) = \mu_i^*(\beta(\alpha + \gamma(T - b))) \qquad 1 \leq i \leq n$$

$$\mu_i(T_2) = \mu_i^*(\alpha + \gamma(T - b)) \qquad 1 \leq i \leq n$$

$$|q_j^*(\alpha + \gamma(T - b)) - q_j(T_2)| < \varepsilon/3 \qquad 1 \le j \le m$$

$$|q_j^*(\beta(\alpha + \gamma(T - b))) - q_j(T_1)| < \varepsilon/3 \qquad 1 \le j \le m$$

$$|q_j^*((1 - \beta)(\alpha + \gamma(T - b))) - q_j(T_2 \setminus T_1)) < \varepsilon/3$$

Altogether, we conclude that for all $1 \le i \le n$, $1 \le j \le m$

$$|q_j(T_2) - q_j^*(\alpha)| < \varepsilon/3 + \varepsilon/3 < \varepsilon$$

$$|q_j(T_1) - q_j^*(\alpha)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 < \varepsilon$$

$$|q_j(T_2 \setminus T_1)| < \varepsilon/3 + \varepsilon/3 < \varepsilon \quad ,$$

and $\qquad \mu_i(T_2) \ge \alpha > \mu_i(T_1)$ iff $\mu_i = \mu_k$

which completes the proof of Lemma 3.9.

We return now to the proof of (3.8) of Lemma 3.5. Observe that it is sufficient to prove that for every $1 \le k \le n$ if $K(k)$ denotes the set of all $1 \le i \le n$ with $\theta_i^*\mu = \theta_k^*\mu$ then $\sum_{i \in K(k)} q_i^*(\alpha) \ge 0$. Apply lemma 3.9 to the nonatomic probability measures $\theta_1^*\mu,\ldots,\theta_n^*\mu$ and the set functions $v,q_1,\ldots,q_n$ in pNA to show the existence of T_1,T_2 in $\mathcal{C}$ with $T_1 \subset T_2$ and such that for all $1 \le i \le n$,

$$|v(T_1) - v^*(\alpha)| < \varepsilon \quad , \quad |v(T_2) - v^*(\alpha)| < \varepsilon$$

$$|q_i(T_1) - q_i^*(\alpha)| < \varepsilon \quad , \quad |q_i(T_2) - q_i^*(\alpha)| < \varepsilon$$

$$\theta_i^*\mu(T_2) \geq \alpha > \theta_i^*\mu(T_1) \quad \text{iff} \quad i \in K(k) \quad .$$

Therefore

$$w(T_2) - w(T_1) \leq v(T_2) - v(T_1) + \sum_{i\in K(k)} q_i(T_2) + \sum_{i\notin K(k)} |q_i(T_2) - q_i(T_1)|$$

$$\leq 2\varepsilon + \sum_{i\in K(k)} q_i^*(\alpha) + 2\varepsilon n$$

$$\leq \sum_{i\in K(k)} q_i^*(\alpha) + 2(n+1)\varepsilon \quad .$$

As this holds for every $\varepsilon > 0$ the assumption that w is monotonic implies that $\sum_{i\in K(k)} q_i^*(\alpha) \geq 0$ which completes the proof of lemma 3.5.

Lemma 3.10: Let g be in W and a in R^+. Then (3.1) and (3.2) defines (uniquely) a semi value $\psi_{(a,g)}$ on u^*pNA.

Proof. Any element w in u^*pNA is of the form $w = v + \sum_{i=1}^{n} (\theta_i^* u) q_i$, $\theta_i \in G$, v, $q_i \in pNA$. By linearity and symmetry, it follows from (3.1) and (3.2) that

$$\text{(3.9)} \qquad \psi_{(a,g)} w(S) = \int_0^1 g(t)\partial v^*(t,S)dt + \sum_{i=1}^{n} \int_\alpha^1 g(t)\partial q_i^*(t,S)dt$$

$$+ \sum_{i=1}^{n} a q_i^*(\alpha)(\theta_i^*\mu)(S) \quad .$$

We have to show that $\psi_{(a,g)}$ is well defined, i.e., that it is independent of the representation of w. Because of the linearity it is enough to show that if $w = 0$ then $\psi_{(a,g)}w = 0$. If $w = 0$ then by lemma (3.4) we conclude that $\psi_{(a,g)}w(S) \geq 0$, and that $\psi_{(a,g)}(-w)(S) = -\psi_{(a,g)}w(S) \geq 0$ which means that $\psi_{(a,g)}w = 0$. Linearity and symmetry of $\psi_{(a,g)}$ follows from the definition. The finite additivity of $\partial q^*(t,S)(q \in pNA)$ as well as that of $\theta_i^*\mu$ implies that $\psi_{(a,g)}w$ is finitely additive. Positivity of $\psi_{(a,g)}$ follows now from lemma (3.4) and the finite additivity of $\psi_{(a,g)}w$. Obviously $u*pNA$ is reproducing; hence that positivity of $\psi_{(a,g)}$ and the finite additivity of $\psi_{(a,g)}w$ implies that $\psi_{(a,g)}w$ is in FA whenever w is in $u*pNA$. Now let $w \in (u*pNA) \cap FA$. We have to show that $\psi_{(a,g)}w = w$. Without loss of generality we may assume that $w = v + \sum_{i=1}^{n} (\theta_i^*\mu)q_i$ where $v \in pNA$ and, $q_i \in pNA$ and $\theta_i^*\mu = \theta_j^*\mu$ iff $i = j$. First we shall show that $q_k^*(\alpha) = 0$ for each k, $1 \leq k \leq n$. Let $1 \leq k \leq n$ be given. Applying lemma 3.9 to the nonatomic probability measures $\theta_1^*\mu, \ldots, \theta_n^*\mu$, the set functions $v, q_1, \ldots, q_n$ in pNA we have for every $0 < \varepsilon$ two sets $T_1, T_2 \in C$, $T_1 \subset T_2$ and such that for all $1 \leq i \leq n$ and

$$\theta_i^*\mu(T_2) \geq \alpha > \theta_i^*\mu(T_1) \quad \text{iff} \quad i = k$$

$$|v(T_2) - v(T_1)| < \varepsilon$$

$$|v(T_2 \setminus T_1)| < \varepsilon$$

$$|q_i(T_2) - q_i(T_1)| < \varepsilon$$

and

$$\theta_i^* \mu(T_2 \smallsetminus T_1) < \varepsilon \quad .$$

Assuming $\varepsilon < \alpha$ we find that

$$|w(T_2 \smallsetminus T_1)| = |v(T_2 \smallsetminus T_1)| < \varepsilon$$

On the other hand,

$$|w(T_2) - w(T_1)| \geq q_k(T_2) - |v(T_2) - v(T_1)|$$

$$- \sum_{i=1}^{n} |q_i(T_2) - q_i(T_1)|$$

$$\geq |q_k^*(\alpha)| - \varepsilon - \varepsilon - n\varepsilon = q_k^*(\alpha) - (n + 2)\varepsilon \quad .$$

The assumption that w is finitely additive will imply that $\varepsilon > |w(T_2 \smallsetminus T_1)| = |w(T_2) - w(T_1)| \geq |q_k^*(\alpha)| - (n + 2)\varepsilon$, i.e., that $|q_k^*(\alpha)| \leq (n + 3)\varepsilon$. As this is true for every $0 < \varepsilon < \alpha$ we conclude that $q_k^*(\alpha) = 0$. Let S be in $\mathcal{C}$, with $\mu(\theta_i S) < \alpha$. In that case $w(S) = v(S)$, and by using the finite additivity of w and lemma 3.3, we see that $v^*(hS) = h(v(S))$ for any rational $0 \leq h \leq 1$ and then by continuity of v^* we deduce that $v^*(hS) = hv(S)$ for any real h, $0 \leq h \leq 1$. Therefore $\partial v^*(0,S) = v(S)$. Now, let $0 < t < \alpha$, and let $S \in \mathcal{C}$ be given. Again using lemma 3.3 to the vector measure $\theta_i^* \mu$ $1 \leq i \leq n$, and the game $v \in pNA$ and the 3-tuple $hS, t, 1 - t - hS$ $h < \alpha - t$ we have for any $\varepsilon > 0$ a partition (T_1, T_2, T_3) of I with $|v(T_1) - v^*(hS)| < \varepsilon$, $|v(T_2) - v^*(t)| < \varepsilon$ and $|v(T_1 \cup T_2) - v^*(t + hS)| < \varepsilon$ and $\theta_1^* \mu(T_1 \cup T_2) < \alpha$. Hence $w(T_1 \cup T_2) = v(T_1 \cup T_2)$, $w(T_1) = v(T_1)$ and $v(T_2) = v(T_2)$. Therefore, using the finite additivity of w we have $v(T_1 \cup T_2) - v(T_2) = v(T_1) - v(\emptyset) = v(T_1)$, and as $|v(T_1 \cup T_2) - v^*(t + hS)| < \varepsilon$, $|v(T_2) - v^*(t)| < \varepsilon$ and $|v(T_1) - v^*(hS)| < \varepsilon$ $|[v^*(t + hS) - v^*(t)] - v^*(hS)| < 3\varepsilon$ and as this holds for any $\varepsilon > 0$, $v^*(t + hS) - v^*(t) = v^*(hS) = hv(S)$ and therefore $\partial v^*(t,S)$ exists and equals $v(S)$. In a similar way, by using lemma 3.3 to the vector measure $\theta_i^* \mu$, $1 \leq i \leq n$, and the games $v \in pNA$, q_i, $1 \leq i \leq n$ and the 3-tuple $hS, t, 1 - t - hS, h < 1 - t$ we can prove that for $\alpha < t < 1$ $\partial(\sum_{i=1}^{n} q_i)^*(t,S) = v(S)$. Therefore as $\int_0^1 g(t)dt = 1$ we conclude that $\psi_{(a,g)} w(S) = v(S) = w(S)$ whenever S is in C with $\mu(\theta_i S) < \alpha$. For S in C there exists always a partition $S = S_1 \cup \ldots \cup S_k$ with S_i $i = 1, \ldots, k$ in C and $\mu(\theta_i S_j) < \alpha$ $1 \leq i \leq n$, $1 \leq j \leq k$. Therefore by the finite additivity of w as well as that of ψw we have $\psi_{(a,g)} w(S) = \sum_{i=1}^{k} \psi_{(a,g)} w(S_i) = \sum_{i=1}^{k} w(S_i) = w(S)$ which completes the proof of lemma 3.10.

<u>Lemma 3.11</u>. Let g be in W and a in R^+. Then the semi-value $\psi_{(a,g)}$ on $u{*}pNA$ defined by (3.1) and (3.2) is continuous and $\|\psi_{(a,g)}\| = \max\{a, \|g\|_{L_\infty}\}$.

<u>Proof.</u> Let w be in $u{*}pNA$. Without loss of generality we may assume that $w = v + \sum_{i=1}^{n} (\theta_i^* u) q$ where v is in pNA, $q_i \in pNA$ for $1 \leq i \leq n$ and $\theta_i^* \mu = \theta_j^* \mu$ iff $i = j$, $(1 \leq i \leq j \leq n)$.

$$\|\psi_{(a,g)}w\| = \sup_{S\in C}|(\psi_{(a,g)}w(S)| + |(\psi_{(a,g)}w)(1-S)|$$

Therefore we have to prove that the right hand side is at most $\max\{a,\|g\|_{L_\infty}\}\|w\|$. As

$$+ |(\psi_{(a,g)}w)(S)| = |a\sum_{i=1}^{n} q_i^*(\alpha)(\theta_i^*\mu)(S) + \int_0^\alpha \partial v^*(t,S)g(t)dt$$

$$+ \int_\alpha^1 \partial(v + \sum_{i=1}^{n} q_i)^*(t,S)dt|$$

$$\leq a\sum_{i=1}^{n}|q_i^*(\alpha)|(\theta_i^*\mu)(S) + \|g\|_{L_\infty}(\int_0^\alpha|\partial v^*(t,S)|dt$$

$$+ \int_\alpha^1|\partial(v + \sum_{i=1}^{n} q)^*(t,S)|dt)$$

$$\leq \max\{a,\|g\|_{L_\infty}\} [\int_0^\alpha|\partial v^*(t,S)|dt$$

$$+ \int_\alpha^1|\partial(v + \sum_{i=1}^{n} q)^*(t,S)|dt + \sum_{i=1}^{n}|q_i^*(\alpha)|(\theta_i^*\mu)(S)]$$

it is sufficient to prove that

$$\|w\| \geq \sum_{i=1}^{n} |q_i^*(\alpha)| + \int_0^\alpha (|\partial v^*(t,S)| + |\partial v^*(t,1-S)|)dt +$$

(3.12)

$$\int_\alpha^1 (|\partial(v + \sum_{i=1}^{n} q_i)^*(t,S)| + |\partial(v + \sum_{i=1}^{n} q_i)^*(t,1-S)|)dt \quad ,$$

First assume that v and q_i, $1 \leq i \leq n$ are polynom als in nonatomic probability measures. For every integer $k > 2$ we will construct a chain Ω_k so that $\|w\|_{\Omega_k}$ will converge as $k \to \infty$ to the right hand side of (3.12).

Observe that there is f in I with $(\theta_i^*\mu)^*(f) = (\theta_j^*\mu)^*(f)$ iff $i = j$. We may assume that $1/2 \leq f \leq 1$. (Otherwise replace f by $(1 + f)/2$). For every $k > 1$ let ℓ be the largest integer with $\ell < \alpha k$. Without loss of generality we may assume that for $1 \leq i,j \leq n$ $(\theta_i^*\mu)^*(f) > (\theta_j^*\mu)^*(f)$ iff $i < j$. Therefore for each $1 \leq i \leq n$ there is a (unique) $\beta_i = \beta_i(k)$ with $0 < \beta_i \leq 2/k$ and $(\theta_i^*\mu)^*(\ell/k + \beta_i f) = \alpha$. Obviously all the β_i's are different and $0 < \beta_i < \beta_j \leq 2/k$ whenever $1 \leq i < j \leq n$. Define $(g_i)_{i=1}^n$ by

$$g_i = \ell/k + \beta_i f$$

and define $(\bar{f}_i)_{i=0}^{2k+n-3}$ by

$$\overline{f}_i = \begin{cases} \frac{i}{2k} & \text{if } i \leq 2 \text{ is an even integer,} \\ \frac{i-1}{2k} + \frac{S}{k} & \text{if } i < 2\ell \text{ is an odd integer,} \\ \frac{\ell}{k} + g_{i-2\ell} & \text{if } 2\ell < i \leq 2\ell + n \ . \\ \frac{i+3-n}{2k} & \text{if } 2\ell + n \leq i \text{ and } i - n \text{ is an odd integer.} \\ \frac{i+2-n}{2k} + \frac{S}{k} & \text{if } 2\ell + n < i \text{ and } i - n \text{ is an even integer.} \end{cases}$$

Apply corollary 3.4 to the vector $(\theta^*_1\mu,\ldots,\theta^*_u\mu)$ of nonatomic measures, the members $v, q_1,\ldots,q_n$ of pNA and $\varepsilon = 1/k(2k+n)$ to construct a chain Ω_k: $(T_i)_{i=0}^{2k+n-3}$ $(T_0 \subseteq T \subseteq \ldots \subseteq T_{2k+n-3})$ such that for all $1 \leq j \leq n$ and for all $0 \leq i \leq 2k + n - 3$,

$$(\theta^*_j\mu)(T_i) = (\theta^*_j\mu)(\overline{f}_i)$$

$$|q_j(T_i) - q^*_j(\overline{f}_i)| < \frac{1}{k(2k+n)}$$

$$|v(T_i) - v^*(\overline{f}_i)| < \frac{1}{k(2k+n)}$$

Denote by Ω^1_k the subchain $(T_i)_{i=0}^{2\ell}$, Ω^2_k the subchain $(T_i)_{i=2\ell}^{2\ell+n}$ and Ω^3_k - the subchain $(T_i)_{i=2\ell+n+1}^{2k+n-3}$. Then

$$\|w\|_{\Omega_k} \geq \|w\|_{\Omega_k^1} + \|w\|_{\Omega_k^2} + \|w\|_{\Omega_k^3} .$$

As $(\theta_j^*\mu)(T_{2\ell}) = (\theta_j^*\mu)(\bar{f}_{2\ell}) = \ell/k < \alpha$ for all $1 \leq j \leq n$,

$$\|v\|_{\Omega_k^1} = \sum_{i=1}^{2\ell} |v(T_i) - v(T_{i-1})| \geq \sum_{i=1}^{2\ell} |v^*(\bar{f}_i) - v^*(\bar{f}_{i-1})| - \frac{2\ell}{k(2k+n)} .$$

As v is a polynomial in nonatomic measures, $\sum_{i=1}^{2\ell} |v^*(\bar{f}_i) - v^*(\bar{f}_{i-1})|$ converges as $k \to \infty$ to $\int_0^\alpha |\partial v^*(t,S)| + |\partial v^*(t,1-S)|)dt$ (see for instance p.45, 46 of [3] or observe that for $1 \leq i \leq 2\ell$ $|v^*(\bar{f}_i) - v^*(\bar{f}_{i-1})| = |\partial v^*(i/2k,\bar{S})|/k + o(1/k)$ where $\bar{S} = S$ if i is odd and $\bar{S} = 1 - S$ if i is even). Thus $\liminf_{k\to\infty} \|w\|_{\Omega_k^1} \geq \int_0^\alpha (|\partial v^*(t,S)| + |\partial v^*(t,1-S)|)dt.$

Similarly $\liminf_{k\to\infty} \|w\|_{\Omega_k^3} \geq$ $|\partial(v + \sum_{j=1}^n q_j)^*(t,S)| + |\partial(v + \sum_{j=1}^n q_j)^*(t,1-S)|)dt.$

We turn now to the estimation of $\|w\|_{\Omega_k^2}$. For each fixed $1 \leq j \leq n$, $(\theta_i^*\mu)(T_{j+2\ell}) \geq \alpha > (\theta_i^*\mu)(T_{j+2\ell-1})$ iff $i = j$. Thus

$$|w(T_{j+2\ell}) - w(T_{j+2\ell-1})| \geq |q_j(T_{j+2\ell})| - |v(T_{j+2\ell}) - v(T_{j+2\ell-1})|$$

$$- \sum_{i=1}^{n} |q_i(T_{j+2\ell}) - q_i(T_{j+2\ell-1})|$$

Thus $\|w\|_{\Omega_k^2} \geq \sum_{j=1}^n q_j(T_{j+2\ell}) - \|v\|_{\Omega_k^2} - \sum_{j=1}^n \|q_j\|_{\Omega_k^2}$

For each fixed $1 \le j \le n$, $q_j(T_{j+2\ell}) \to q_j^*(\alpha)$ as $k \to \infty$ and $\|q_j\|_{\Omega_k^2} \to 0$ as $k \to \infty$, and also $\|v\|_{\Omega_k^2} \to 0$ as $k \to \infty$ and thus $\liminf_{k\to\infty} \|w\|_{\Omega_k^3} \ge \sum_{j=1}^n |q_j^*(\alpha)|$. Altogether we conclude that

$$\|w\| \ge \liminf \|w\|_{\Omega_k} \ge \liminf \|w\|_{\Omega_k^1} + \liminf \|w\|_{\Omega_k^2} + \liminf \|w\|_{\Omega_k^3} \ge$$

$$\ge \sum_{i=1}^n |q_i^*(\alpha)| + \int_0^\alpha (|\partial v^*(t,S)| + |\partial v^*(t,1-S)|)dt +$$

$$\int_\alpha^1 (|\partial(v + \sum_{i=1}^n q_i)^*(t,S)| + |\partial(v + \sum_{i=1}^n q_i)^*(t,1-S)|)dt$$ which proves (3.12) in the case that v and q_i are polynomials in nonatomic measures. For the general case let $\varepsilon > 0$ and approximate v and q_i by polynomials of NA-measures $\bar{v}$ and $\bar{q}_i$ respectively with $\|v - \bar{v}\| < \varepsilon$ $|q_i - \bar{q}_i| < \varepsilon$, and let $\bar{w} = \bar{v} + \sum_{i=1}^n (\theta_i^* u)\bar{q}_i$. As $\|\theta_i^* u\| = 1$ and $\|v_1\|\|v_2\| \le \|v_1\|\|v_2\|$ for all v_1, v_2 in BV, $\|\bar{w} - w\| \le \|\bar{v} - v\| + \sum_{i=1}^n \|(\theta_i^* u)(q_i - \bar{q}_i)\| \le (n+1)\varepsilon$.

Using lemma 23.1 of [3] we have for all $S \in C$,

$$\int_0^\alpha |\partial \bar{v}^*(t,S) - \partial v^*(t,S)|dt \le \|v - \bar{v}^*\| \le \varepsilon \quad \text{and}$$

$$\int_\alpha^1 |\partial(\bar{v} + \sum_{i=1}^n \bar{q}_i)^*(t,S) - \partial(v + \sum_{i=1}^n q_i)^*(t,S)|dt \le (n+1)\varepsilon \quad .$$

Also $|q_i^*(\alpha) - \bar{q}_i^*(\alpha)| \le \|q_i^* - \bar{q}_i^*\| \le \varepsilon$. Altogether,

$$(n+1)\varepsilon + \|w\| \geq \|\tilde{w}\| \geq \sum_{i=1}^{n} |q_i^*(\alpha)| - n\varepsilon + \int_0^\alpha |\partial v^*(t,S)|dt - \varepsilon$$

$$+ \int_0^\alpha |\partial v^*(t,1-S)|dt - \varepsilon + \int_\alpha^1 |\partial(v + \sum_{i=1}^{n} q_i)^*(t,S)|dt - (n+1)\varepsilon$$

$$+ \int_\alpha^1 |\partial(v + \sum_{i=1}^{n} q_i)^*(t,1-S)|dt - (n+1)\varepsilon \quad .$$

As this is true for all $\varepsilon > 0$, (3.12) is proved which completes the proof of lemma 3.11.

<u>Proof of Theorem A:</u> We have already seen that for a in R^+ and g in W, (3.1) and (3.2) define (uniquely) a (continuous) semi-value $\psi_{(a,g)}$ on $u*pNA$. Now we have to show that any continuous semi-value on $u*pNA$ is of that form. Let ψ be a continuous semi-value on $u*pNA$. In particular, ψ induces a semi-value on pNA and therefore by theorem 2.3 there is g in W with

$$(3.13) \quad \psi v(S) = \int_0^1 g(t)\partial v^*(t,S)dt \quad \text{for each } v \text{ in } pNA \quad .$$

Let ν be a probability measure in NA, and k a positive integer. For any $\delta > 0$, $\delta < (1/2)\min\{\alpha, 1-\alpha\}$ define $F_\delta\colon [0,1] \to R^+$ by

$$F_\delta(x) = \begin{cases} 0 \text{ if } |x-\alpha| \geq 2\delta \\ 1 \text{ if } |x-\alpha| \leq \delta \\ 1 - 1/\delta\,(|x-\alpha| - \delta) \text{ if } \delta < |x-\alpha| < 2\delta. \end{cases}$$

and define $\tilde{v}_\delta$ by:

(3.14) $$\tilde{v}_\delta = (F_\delta o\nu)(F_\delta o\mu)(\nu^k - \mu^k)$$

First we shall show that

(3.15) $$\|u\tilde{v}_\delta\| \leq 32k\delta \quad .$$

Define $U = \{S \in \mathcal{C} : 0 \leq \mu(S) - \alpha < 2\delta,\ |\nu(S) - \alpha| \leq 2\delta\}$. Then for S in U, $\mu(S) \geq \alpha$ and therefore $u(S) = 1$ and also for S in U, $|\nu(S) - \mu(S)| \leq 4\delta$ and thus for S in U, $|\nu^k(S) - \mu^k(S)| \leq 4\delta k$, and $|\tilde{v}_\delta(S)| \leq 4\delta k$. For every S in $\mathcal{C} \setminus U$ either $\mu(S) < \alpha$ and thus $u(S) = 0$ or $|\nu(S) - \alpha| > 2\delta$ and thus $\tilde{v}_\delta(S) = 0$. In any case for $S \notin U$, $(u\tilde{v}_\delta)(S) = 0$. Let $\Omega: S_0 \subset S_1 \subset \ldots \subset S_L$ be a chain. Let i_0 be the first index for which $S_{i_0} \in U$ and let j_0 be the last index for which $S_{j_0} \in U$. Then from the definition of U it follows that $S_i \in U$ iff $i_0 \leq i \leq j_0$. Therefore, as $(u\tilde{v}_\delta)(S) = 0$ whenever $S \notin U$, and $|(u\tilde{v}_\delta)(S)| \leq 4\delta k$ whenever $S \in U$ we deduce that

$$\|u\tilde{v}_\delta\|_{\Omega} \equiv \sum_{i=1}^{L} |(u\tilde{v}_\delta)(S_i) - u\tilde{v}_\delta(S_{i-1})|$$

$$= \sum_{i=i_0}^{j_0+1} |(u\tilde{v}_\delta)(S_i) - (u\tilde{v}_\delta)(S_{i-1})|$$

$$= |u\tilde{v}_\delta(S_{i_0})| + |u\tilde{v}_\delta(S_{j_0})| + \sum_{i=i_0+1}^{j_0} |u\tilde{v}_\delta(S_i) - u\tilde{v}_\delta(S_{i-1})|$$

$$\leq 8\delta k + \sum_{i=i_0+1}^{j_0} |(u\tilde{v}_\delta)(S_i) - (u\tilde{v}_\delta)(S_{i-1})| \quad .$$

$$\sum_{i=i_0+1}^{j_0} |\tilde{v}_\delta(S_i) - \tilde{v}_\delta(S_{i-1})| = \sum_{i=i_0+1}^{j_0} |(F_\delta o\nu)(F_\delta o\mu)(S_i)(\nu^k - \mu^k)(S_i)$$

$$- (F_\delta o\nu)(F_\delta o\mu)(S_i)(\nu^k - \mu^k)(S_{i-1}) + (F_\delta o\nu)(F_\delta o\mu)(S_i)(\nu^k - \mu^k)(S_{i-1})$$

$$- (F_\delta o\nu)(F_\delta o\mu)(S_{i-1})(\nu^k - \mu^k)(S_{i-1})| \leq$$

$$\leq \max_{S\in U} |(F_\delta o\nu)(F_\delta o\mu)(S)| \sum_{i=i_0+1}^{j_0} |(\nu^k - \mu^k)(S_i) - (\nu^k - \mu^k)(S_{i-1})|$$

$$+ \max_{S\in U} |(\nu^k - \mu^k)(S)| \sum_{i=i_0+1} |(F_\delta o\nu)(F_\delta o\mu)(S_i) - (F_\delta o\nu)(F_\delta o\mu)(S_{i-1})| \quad .$$

But $\max|(F_\delta o\nu)(F_\delta o\mu)(S)| \leq 1$ and

$$\sum_{i=i_0+1}^{j_0} |(\nu^k - \mu^k)(S_i) - (\nu^k - \mu^k)(S_{i-1})| \leq (\nu^k + \mu^k)(S_{j_0}) -$$

$$- (\nu^k + \mu^k)(S_{i_0}) \leq 8\delta k$$

and

$$\max_{S \in U} |(\nu^k - \mu^k)(S)| \leq 4k\delta$$

and

$$\|(F_\delta o\nu)(F_\delta o\mu)\| \leq \|F_\delta o\nu\| \|F_\delta o\mu\| \leq 4 \quad .$$

Therefore $\sum_{i=i_0+1}^{j_0} |\tilde{v}_\delta(S_i) - \tilde{v}_\delta(S_{i-1})| \leq 8\delta k + (4\delta k)4 = 24k\delta,$

hence $\|u\tilde{v}_\delta\|_{\Omega} \leq 32k\delta$. As this holds for any chain Ω (3.15) is proved. Define $G: [0,1] \to R^+$ by

$$G_\delta(x) = \begin{cases} 0 & \text{if } x \geq \alpha + 2\delta \\ 1 & \text{if } x \leq \alpha + \delta \\ 1 - \frac{1}{\delta}(x - \alpha - \delta) & \text{if } \alpha + \delta < x \leq \alpha + 2\delta \end{cases}$$

and define $\bar{v}_\delta$ by

(3.16) $$\bar{v}_\delta = (G_\delta o\nu)(G_\delta o\mu)(\nu^k - \mu^k) \quad .$$

First observe that $\bar{v}_\delta \in pNA$ (although $(G\circ\nu)(G\circ\mu) \notin pNA$). Define $\mathcal{D}$ to be the diagonal neighborhood defined by

$$\mathcal{D} = \{S: |\mu(S) - \nu(S)| < \delta\} \quad .$$

Let $S \in \mathcal{D}$ and denote $v = \nu^k - \mu^k$; then $u(v - \tilde{v}_\delta)(S) = (v - \bar{v}_\delta)(S)$, because if $\mu(S) < \alpha$ and $S \in \mathcal{D}$ then $\nu(S) < \alpha + \delta$ and therefore $\bar{v}_\delta(S) = v(S)$ and of course then $(u(v - \tilde{v}_\delta))(S) = 0 = (v - \bar{v}_\delta)(S)$, and if $\mu(S) \geq \alpha$ then $u(v - \tilde{v}_\delta)(S) = (v - \tilde{v}_\delta)(S)$, and $\nu(S) \geq \alpha - \delta$. But for $x \geq \alpha - \delta$, $G_\delta(x) = F_\delta(x)$ which yield that $(v - \bar{v}_\delta)(S) = (v - \tilde{v}_\delta)(S)$, whenever $S \in D$ with $\mu(S) \geq \alpha$. Thus we have seen that

(3.17)
$$\begin{aligned} &u(v - \tilde{v}_\delta) \text{ coincides with } v - \bar{v}_\delta \text{ on a diagonal neighborhood,} \\ &v - \bar{v}_\delta \in pNA,\ u(v - \tilde{v}_\delta) \in u{*}pNA \quad . \end{aligned}$$

As ψ is continuous proposition (2.4) and (3.17) implies that

$$\psi(u(v - \tilde{v}_\delta)) = \psi(v - \bar{v}_\delta) \quad . \tag{3.18}$$

Now we claim that

$$\partial\bar{v}^*_\delta(t,S) = \begin{cases} 0 & \text{if } t > \alpha + 2\delta \\ \partial v^*(t,S) & \text{if } t < \alpha + \delta \end{cases} \tag{3.19}$$

To prove (3.19) observe that if $t > \alpha + 2\delta$ and $h > 0$ then

$[(G_\delta o\nu)(G_\delta o\mu)(\nu^k - \mu^k)]^*(t) = 0 = [(G_\delta o\nu)(G_\delta o\mu)(\nu^k - \mu^k)]^*(t + hS)$ and if $0 < t < \alpha + \delta$ and $h \leq 0$ with $t + h > 0$ then $[(G_\delta o\nu)(G_\delta o\mu)(\nu^k - \mu^k)^*(t + hS) = 0$. As $v - \overline{v}_\delta$ is in pNA, (3.13) and (3.18) implies that

$$(3.20)\quad |\psi(v - \overline{v}_\delta)(S) - \int_{\alpha+2\delta}^{1} \partial v^*(t,S)g(t)dt| \leq |\int_{\alpha+\delta}^{\alpha+2\delta} g(t)|\partial(v - \overline{v}_\delta)^*(t,S)dt| \underset{\delta\to 0}{\to} 0$$

If we let $\delta \to 0$, (3.20), (3.18) and (3.15) imply that

$$(3.21)\quad \psi(u(\nu^k - \mu^k))(S) = \int_\alpha^1 g(t)\partial(\nu^k - \mu^k)^*(t,S)dt \quad .$$

Observe that $u \in u{*}pNA$. By proposition 2.5 $\psi u = a\mu$, and by the positivity of ψ, $a \in R^+$. Now let B be the subset of pNA of all games q for which

$$(6.23)\quad \psi(uq) = \psi_{(a,g)}(uq) \quad .$$

By (3.21) $\nu^k - \mu^k \in B$. Observe that $u\mu^k - \alpha^k u$ is in pNA and hence $\psi(u\mu^k - \alpha^k u)(S) = \int_\alpha^1 g(t)\partial(\mu^k)^*(t,S)dt$ and $\psi(\alpha^k u) = \alpha^k a\mu$. Therefore it is easily verified that $\mu^k \in B$. But B is obviously a linear subspace of pNA and therefore as it contains μ^k and $\nu^k - \mu^k$ it contains ν^k for any probability measure in NA and hence any polynomial in NA^+ measures. As both ψ and $\psi_{(a,g)}$ are continuous and $\|uq\| \leq \|u\|\|q\|$ it follows that B is closed, thus B = pNA. Now as both ψ and $\psi_{(a,g)}$ are linear and continuous we deduce that they coincide on u*pNA, which completes the proof of theorem A. Q.E.D.

4. Further Results and Remarks.

We are able to characterize the set of all continuous semi-values on many other important spaces, like bv'NA and bv'NA*pNA. As the proof uses similar methods to those presented in the former sections we will just give a sample of results.

Notations: Let X be a linear subspace (not necessarily closed) of the Banach space bv' (the space of all functions $f: [0,1] \to \mathbb{R}$ with $f(0) = 0$ such that f is of bounded variation continuous at zero and 1, endowed with the total variation norm). We denote by $W(X)$ the subset of the dual $\overline{X}^*$ (of the closure $\overline{X}$ of X) of all elements x^* satisfying: (1) For each monotonic nondecreasing f in X, $x^*(f) \geq 0$; (2) If X contains the function h defined by $h(x) = x$, then $x^*(h) = 1$. The subspace of all absolutely continuous elements in bv' is denoted ac'. For each $0 < x < 1$ define $f_x: [0,1] \to \mathbb{R}$ by $f_x(y) = 0$ iff $y < x$ and $f_x(y) = 1$ iff $y \geq x$ and $\overline{f}_x: [0,1] \to \mathbb{R}$ by $\overline{f}_x(y) = 0$ iff $y \leq x$ and $\overline{f}_x(y) = 1$ iff $y > x$. The subspace of bv' generated by the functions $f_x(\overline{f}_x)$ is denoted by rj'(ℓj'), and that generated by all jump functions (i.e., by rj' and ℓj') is denoted by j'. If $X \subset$ bv' we denote by XNA the linear symmetric space generated by game of the form $f \circ \mu$, $f \in X$ and μ is a probability measure in NA.

Theorem 4.1: Let X be a subspace of bv'. There is a 1-1 linear isometry from $W(X)$ onto the continuous semi-values on XNA; for each $x^* \in W(X)$ the semi-value ψ_{x^*} on XNA is given by

$$\psi_{x^*}(f \circ \mu) = x^*(f)\mu$$

Remarks:

(a) $W(ac') = W$ and therefore Theorem 7.1 can be considered a generalization of Theorem 2.3 ($ac'NA$ is dense in pNA).

(b) $W(rj')$ is identified with all bounded functions $a: (0,1) \to R^+$; for $0 < x < 1$ $x^*(a)(f_x) = a(x)$ and $\|x^*(a)\| = \sup_{0<x<1} a(x)$. Each of the continuous semi-values on $rj'NA$ can be extended to a semi-value on its closure: However, there are discontinuous semi-values on $rj'NA$; they can be obtained by omitting the boundness condition on a.

(c) $W(j')$ is identified with all pairs of bounded functions $a, b : (0,1) \to R^+$ where for $0 < x < 1$, $x^*(a,b)(f_x) = a(x)$ and $x^*(a,b)(\bar{f}_x) = b(x)$. We have $\|x^*(a,b)\| = \sup_{0<x<1} \{a(x),b(x)\}$.

Notations: If Q_1 and Q_2 are linear symmetric subspaces of BV we denote by $Q_1 \circledast Q_2$ the linear symmetric space generated by games of the form $v_1 v_2$ where $v_i \in Q_i$ $(i = 1,2)$, and the space $Q_1 * Q_2$ is defined as the linear symmetric space generated by $Q_1 \circledast Q_2$, Q_1 and Q_2.

Theorem 4.2. For each pair (a,g), $a: (0,1) \to R^+$ and $g \in W = W(ac')$ there is a semi-value $\psi_{(a,g)}$ on $rj'NA*pNA$ given by:

$$(4.3) \quad \psi_{(a,g)}(v) = \psi_g v \quad \text{whenever} \quad v \in pNA$$

$$(4.4) \quad \psi_{(a,g)}((f_x \circ \mu)v)(S) = a(x)v^*(x)\mu(s) + \int_x^1 g(t)\partial v^*(t,S)dt$$

whenever $v \in pNA$, $0 < x < 1$ and μ is a probability measure in NA. The semi-value $\psi_{(a,g)}$ is continuous iff a is bounded. Moreover, any continuous semi-value on $rj'NA*pNA$ is of that form. $\psi_{(a,g)}$ can be extended to a semi-value on rj'NA*pNA iff a is bounded and then

$$\|\psi_{a,g)}\| = \max \left(\sup_{0<x<1} a(x), \|g\|_{L_\infty} \right) .$$

<u>Remark</u>: Similar results hold for the spaces ℓj'NA*pNA and j'NA*pNA (in the second case the semi-values are associated with triples (a,b,g)).

<u>Theorem 4.5</u>: For each pair (a,g), a: $(0,1) \to \mathbb{R}^+$ and $g \in L^+_B(0,1)$ there is a semi-value $\psi_{(a,g)}$ on rj'NA ⊛ pNA given by (4.4). This semi-value is continuous if and only if a is bounded. Moreover, any continuous semi-value is of that form.

<u>Remarks</u>:

(a) The semi-values on rj'NA*pNA differ from those on rj'NA * pNA since NA ⊄ rj'NA ⊛ pNA while NA ⊂ rj'NA*pNA.

(b) The proof of Theorems 4.2 and 4.5 are similar to that of Theorem A.

(c) The fact that (a,0) is a semi-value on rj'NA ⊛ pNA is easy to prove (see lemma 3.5(3.8)) and actually makes use only on the property of pNA of having a continuous extension to ideal sets satisfying lemma 3.3. Thus it follows that the existence of such semi-values is valid for any space of the form rj'NA ⊛ Q where Q has such an extension. If Q is such a space satisfying: there exist α: $(0,1) \to R^+\setminus\{0\}$ s.t. for each $v \in Q$ and $0 < x < 1$ $v^*(x) = \alpha(x)v^*(1)$ then by setting $a(x) = 1/\alpha(x)$, $\psi_{(a,0)}$ is a value on rj'NA ⊛ Q. However, these values are discontinuous, whenever α is not bounded away from 0.

(d) For every g in W which is continuous there is a semi-value on DIFF (for definition see Mertens) which is defined in the same way as the value is defined on DIFF. The proof is essentially the same as in Merten's proof of the existence of a value on DIFF.

Footnotes

1/ Along the proof α and μ stand for the fixed scalar and the probability measure, respectively, that are used in the definition of the set function u.

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