AD-A123 292 A CHARACTERIZATION OF BINARY COMPLETELY SYMMETRIC CONNECTED BLOCK DESIGNS..(U) ILLINOIS UNIV AT CHICAGO CIRCLE DEPT OF MATHEMATICS A S HEDAYAT 1982 1/1

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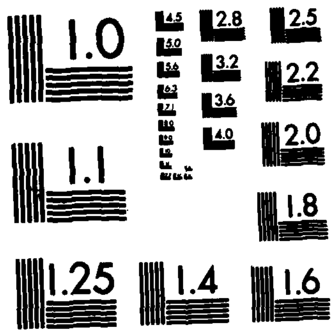

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A CHARACTERIZATION OF BINARY COMPLETELY SYMMETRIC CONNECTED BLOCK DESIGNS WITH $v = b$ BASED ON $n \equiv 0(\mathrm{MOD}\ v)$ EXPERIMENTAL UNITS

by

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Chief, ... Information Division

AMS 1980 Subject Classification: Primary 62K10; secondary 05B05.
Key words and phrases: Completely symmetric information matrix, C-matrix, connected, binary design, block design, incomplete block, BIB design.

*Research is supported by Grant AFOSR 80-0170.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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4. TITLE (and Subtitle) A CHARACTERIZATION OF BINARY COMPLETELY SYMMETRIC CONNECTED BLOCK DESIGNS WITH v = b BASED ON n = o(MOD v) EXPERIMENTAL UNITS		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A.S. Hedayat		8. CONTRACT OR GRANT NUMBER(s) AFOSR-80-0170
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics University of Illinois at Chicago Chicago IL 60680		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate Air Force Office of Scientific Research Bolling AFB DC 20332		12. REPORT DATE 1982
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION ST. 4ENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY .(TES

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1. INTRODUCTION

Suppose v treatments are to be compared via n experimental units grouped into b blocks with k_j experimental units in block j. An arrangement of the v treatments into the n experimental units is called a design. Let D be the collection of all such designs and let r_{di} be the number of experimental units which design d assigns to treatment i. We associate the following matrix

$$C_d = R_d - N_d K^{-1} N_d'$$

with design d, where R_d and K_d stand for diagonal matrices with diagonal elements $r_{d1}, r_{d2}, \ldots, r_{dv}$ and $k_1, k_2, \ldots, k_b$ respectively and $N_d = (n_{dij})$, represents the $v \times b$ incidence matrix of the treatments versus blocks. Throughout the paper we shall adopt the following standard notation. A' and $\mathrm{Tr}\, A$ for the transpose and the trace of the matrix A respectively; I for the identity matrix and J for a matrix with all entries equal to one. The dimensions of the matrices will be clear from the context.

The matrix C_d plays an important role in the statistical analysis of the data collected under design d whenever the model of response is the usual homoscedastic linear additive model which specifies that the expectation of an observation on treatment i in block j is $\mu + \alpha_i + \beta_j$ (general mean+treatment effect+block effect), where μ, α_i and β_j are unknown constants, $1 \leq i \leq v$, $1 \leq j \leq b$ with the additional assumption that the n observations are uncorrelated with common variance σ^2. Under this model the reduced normal equations for estimating the vector of treatment effects can be written as $C_d\hat{\alpha} = Q_d$, where $Q_d = (T_{d1}, T_{d2}, \ldots, T_{dv})' - N_d K^{-1}(B_{d1}, B_{d2}, \ldots, B_{db})'$. Here T_{di} stands for the total of all observations obtained under treatment i and B_{dj} stands for the total of all observations produced in block j. We recall that if $q'\hat{\alpha}$ is an estimable linear contrast in treatment effects then under design d the BLU estimator of $q'\alpha$ is $q'\hat{\alpha} = q'C_d^- Q_d$ with variance $q'\hat{\alpha} = q'C_d^- q\sigma^2$. Here A^- is a generalized inverse of A, that is, any matrix A^- as long as $AA^-A = A$. For this reason C_d is known as the information matrix of the estimable treatment contrasts.

As we see from the definition of C_d each d uniquely determines its associated C_d matrix. Fortunately, in general the converse is not true, that is, there are different designs which yield identical information matrices. Therefore, in such a situation the statistician has more than one design at his disposal for his client who insists on running a design with a certain form for the information matrix of his design. One particular pattern which yields a great deal of optimality is complete symmetry by which we mean C_d to be of the form $a_1I + a_2J$. Kiefer (1975) proved that any design which yields a completely symmetric C_d matrix is universally optimal if $\text{Tr}\, C_d$ is maximum in the class of competing designs. In this paper we shall prove that if $b = v$ and $n \equiv 0 \pmod v$ tnen C_d will be completely symmetric if and only if d is a symmetric BIB design. The sufficiency of the condition is obvious. To prove its necessity we need to recall some basic old results and prove some new ones which we shall do in Section 2. We would like to point out that our result contains the interesting result of Rao (1966), who proved that an equireplicate binary balanced design with $b = v$ is a symmetric BIB design. Specifically, we shall show that to obtain the conclusion of Rao (1966) we can start with a much weaker assumption. Namely, rather than assuming that d is equally replicated we only have to assume that the number of available experimental units, n, is a multiple of v. Of course, somewhere in the process of the proof we will see that, under the given conditions, our assumptions and the assumptions of Rao (1966) are indeed equivalent. But _a priori_ we did not have to start with the strong condition of equal replications. A result has to be established (Proposition 2.6) in order to enable us to weaken the _a priori_ condition or to enlarge the class of competing designs.

2. CHARACTERIZATION

In the sequel for simplicity we shall drop the subscript d to the matrices and quantities associated with design d. In Propositions 2.1-2.6 we establish some results needed to prove our result stated in Theorem 2.1 which was given originally in Hedayat (1972). The stated results can be easily verified whenever the proof is omitted.

Proposition 2.1. If d is a completely symmetric connected design with

$$C = a_1 I + a_2 J$$

then,

$$a_1 = \mathrm{Tr}C/(v-1) \quad \text{and} \quad a_2 = -\mathrm{Tr}C/v(v-1).$$

Proposition 2.2. If d is a binary completely symmetric connected design with

$$C = a_1 I + a_2 J$$

then,

$$a_1 = (n-b)/(v-1) \quad \text{and} \quad a_2 = -(n-b)/v(v-1).$$

Definition 2.1. d is said to be a proper design if $k_j = k$.

Since for a proper design $n = bk$ then we have

Proposition 2.3. If d is a binary, proper and completely symmetric connected design, then

$$C = \frac{b(k-1)}{v-1} I - \frac{b(k-1)}{v(v-1)} J.$$

Given that d is completely symmetric, then in some settings we can make useful inferences about R and K.

Proposition 2.4. If d is a binary, proper and completely symmetric connected design, then d is equireplicated.

Proof: $C_{ii} = r_i (1 - \frac{1}{k})$ since d is binary and proper. Also, $C_{ii} = (n-b)/v$ since d is completely symmetric. Thus, $r_i = n/v$.

Proposition 2.5. If d is a binary, proper, equireplicated and completely symmetric connected design, then d is a BIB design.

Proof: On one hand

$$C_{ii'} = -1(k) \sum_{j=1}^{b} n_{ij} n_{i'j}.$$

One the other hand

$$C_{ii'} = -(n-b)/v(v-1),$$

and hence

$$\lambda_{ii'} = \sum_{j=1}^{b} n_{ij} n_{i'j} = r(k-1)/(v-1).$$

Proposition 2.6. If d is a binary and completely symmetric connected design with $b = v$ then d is equireplicated if $n \equiv 0 (\mathrm{mod}\ v)$.

Proof: Since d is binary and completely symmetric, then

$$(1) \qquad C_{ii} = (n-b)/v = (n-v)/v = (n/v) - 1.$$

On the other hand,

$$(2) \qquad C_{ii} = r_i - \sum_{j=1}^{b} \frac{n_{ij}}{k_j} .$$

(1) and (2) imply that

$$r_i = r - 1 + \sum_{j=1}^{b} \frac{n_{ij}}{k_j} \quad \text{where} \quad r = n/v.$$

Since r_i is an integer, this implies that $\sum_{j=1}^{b} \frac{n_{ij}}{k_j}$ is an integer greater than or equal to one. However, we observvce that

$$\sum_{i=1}^{v} \left[\sum_{j=1}^{b} \frac{n_{ij}}{k_j} \right] = \sum_{j=1}^{b} \frac{1}{k_j} \sum_{i=1}^{v} \frac{1}{k_j} (k_j) = b = v,$$

that is, v $(=b)$ terms ≥ 1 add up to v which implies that each term $\equiv 1$. Therefore, $r_i = r - 1 + 1 = r;\ i = 1,2,\ldots,v.$

Theorem 2.1. If d is a binary and completely symmetric connected design with $b = v$ and $n \equiv 0 (\text{mod } v)$, then d is a symmetric BIB design.

Proof: By Proposition 2.6, it is sufficient to prove that d is proper.

$$C = R - NK^{-1}N' = rI - NK^{-1}N', \qquad n = vr$$

$$= \frac{n-b}{v-1} I - \frac{n-b}{v(v-1)} J$$

$$= \frac{v(r-1)}{v-1} I - \frac{r-1}{v-1} J .$$

Thus

$$NK^{-1}N' = rI - \frac{v(r-1)}{v-1} I + \frac{r-1}{v-1} J$$

$$= \frac{v-r}{v-1} I + \frac{r-1}{v-1} J .$$

Since

$$\left[\frac{v-r}{v-1} I + \frac{r-1}{v-1} J\right]^{-1} = \frac{v-1}{v-r} I - \frac{r-1}{r(v-r)} J$$

we have

$$[NK^{-1}N']^{-1} = (N')^{-1} KN^{-1} = \frac{v-1}{v-r} I - \frac{r-1}{r(v-r)} J.$$

By premultiplying by N' and pastmultiplying by N the both sides of the above relation we obtain

$$K = \frac{v-1}{v-r} N'N - \frac{r-1}{r(v-r)} N'JN.$$

Note that $J = 11'$ and $N'1 = (k_1, \ldots, k_b)$. Since K is a diagonal matrix with the i^{th} diagonal entry equal to k_i we thus observe that:

(3) $k_i = k_i(v-1)/(v-r) - k_i^2(r-1)/r(v-r),$

$i,j = 1,2,\ldots,b;\ i \neq j.$

(4) $0 = \lambda_{ij}(v-1)/(v-r) - k_i k_j(r-1)/r(v-r)$

Because $k_i > 0$, we conclude from (3) that $k_i = r$. Consequently, (4) implies that $\lambda_{ij} = r(r-1)/(v-1)$. This completes the proof.

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7. AUTHOR(s) A. Hedayat		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics, University of Illinois, Chicago Circle, Box 4348, Chicago, Illinois 60680		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research Bolling AFB, Washington, D.C. 20332		12. REPORT DATE March 1982
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
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