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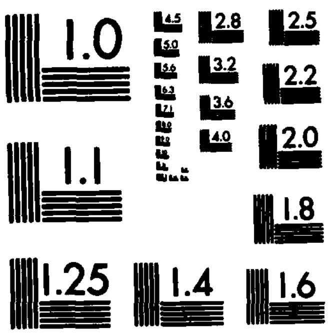
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[illegible] SPACE GREEN'S FUNCTIONS OF THE REDUCED WAVE EQUATION

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FREE-SPACE GREEN'S FUNCTION OF THE REDUCED WAVE EQUATION

BY

E A SKELTON

Summary

The governing differential equations are solved by the method of separation of variables for the separate cases of Cartesian, cylindrical and spherical coordinate systems. The resulting integral/series expansions of the Green's functions for outgoing waves are then reduced to their standard and simple form. In addition to bringing together, in tabular form, a number of frequently used Green's functions this memorandum also provides an introduction to their derivation.

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LIST OF SYMBOLS

(x,y,z)	Cartesian coordinates
(r,ϕ,z)	Cylindrical polar coordinates
$(R,\theta\ \phi)$	Spherical polar coordinates
$G(\underline{R},\underline{R}_o)$	Three-dimensional Green's function
$G_F(\underline{R},\underline{R}_o)$	Three-dimensional free-space Green's function
	$=G_F(x,y,z,x_o,y_o,z_o)$ in Cartesian coordinates
	$=G_F(r,\phi,z,r_o,\phi_o,z_o)$ in cylindrical polar coordinates
	$=G_F(R,\theta,\phi,R_o,\theta_o,\phi_o)$ in spherical polar coordinates
$G_F(\underline{r},\underline{r}_o)$	Two-dimensional free-space Green's function
	$=G_F(x,y,x_o,y_o)$ in Cartesian coordinates
	$=G_F(r,\phi,r_o,\phi_o)$ in polar coordinatinates
$G_F(x,x_o)$	One-dimensional free-space Green's function
$\delta(\underline{R}-\underline{R}_o)$	Three-dimensional Dirac delta function
$\delta(x-x_o)$	One-dimensional Dirac delata function
J_n	Bessel function of the first kind, order n
H_n	Hankel function of the first kind, order n, $H_n(x)=J_n(x)+iY_n(x)$
K_n	Bessel function of imaginary argument, order n
j_n	Spherical Bessel function of the first kind, order n
	Note $j_n(x)=\sqrt{(\pi/2x)}J_{n+1/2}(x)$
P_n	Legendre function of the first kind, degree n
P_n^m	Associated Legendre function of the first kind of degree n and order m. Note $P_n^m=P^m_{-(n+1)}$; $P_n^m=0$, $m>n$
e_n	$e_o=1$; $e_n=2$, $n=1,2,3,\ldots$
k	$(=\omega/c)$ wavenumber
$\|z-z_o\|$	$=(z-z_o)\,\mathrm{sgn}(z-z_o)$
$\|\underline{R}-\underline{R}_o\|$	$=\sqrt{[(x-x_o)^2+(y-y_o)^2+(z-z_o)^2]}$
	$=\sqrt{[r^2+r_o^2-2rr_o\cos(\phi-\phi_o)+(z-z_o)^2]}$
	$=\sqrt{[R^2+R_o^2-2R_oR\cos(\psi)]}$, $\cos(\psi)=\cos\theta\cos\theta_o+\sin\theta\sin\theta_o\cos(\phi-\phi_o)$
$\|\underline{r}-\underline{r}_o\|$	$=\sqrt{[(x-x_o)^2+(y-y_o)^2]}$
	$=\sqrt{[r^2+r_o^2-2rr_o\cos(\phi-\phi_o)]}$

INTRODUCTION

The Green's functions of the reduced wave equation are defined as the solutions of the inhomogeneous equation

$$(\nabla^2+k^2)G(\underline{R},\underline{R}_o) = -4\pi\delta(\underline{R}-\underline{R}_o) \tag{1.1}$$

which satisfy the appropriate linear boundary conditions. In the absence of boundaries the solution is the free-space Green's function, $G_F(\underline{R},\underline{R}_o)$, which takes the particularly simple form

$$G_F(\underline{R},\underline{R}_o) = \exp(ik|\underline{R}-\underline{R}_o|)/|\underline{R}-\underline{R}_o| \tag{1.2}$$

for outgoing waves in three-dimensions. The time dependence factor, $\exp(-i\omega t)$, which is assumed throughout, is omitted from all equations. It is evident that this Green's function is the outgoing wave solution of the reduced wave equation when the excitation is a point source. It is also evident that reciprocity applies, viz G_F is unaltered when the positions of the source and the observer are interchanged.

The reduced wave-equation with general source distribution $F(\underline{R})$ inside a volume V

$$(\nabla^2+k^2)p(\underline{R}) = -4\pi F(\underline{R}) \tag{1.3}$$

is easily solved by the principle of superposition when the Green's function is known. The solution is

$$p(\underline{R}) = \int_{V'} F(\underline{R}')G(\underline{R},\underline{R}')d^3\underline{R}' \tag{1.4}$$

The determination of a Green's function usually proceeds by assuming that it is the sum of the free-space Green's function, G_F, and a scattering term which satisfies the homogeneous equation and whose amplitude is determined by the boundary conditions. In this case the homogeneous form of equation (1.1) is usually solved by the method of separation of variables, which results in the scattering term being represented by integral/series transforms. There is therefore a requirement to expand the free-space Green's function, G_F, in a similar form in order to find the amplitud of the scattered wave.

The expansions of the free-space Green's function in the various coordinate systems are developed in texts too numerous to reference here. It is the purpose of this memorandum to obtain the expansions in a simple way, without being especially rigorous. It is hoped that the expansions will be of particular interest to researchers who are using Green's function techniques for perhaps the first time, or to those who would find a collection of free-space Green's function representations to be useful. For reference, the Appendix summarizes the representations derived herein.

2. PROBLEM FORMULATION

The free-space Green's function considered here is the outgoing wave solution of equation (1.1), which is expressed in Cartesian, cylindrical and spherical coordinate systems respectively, Figure 1, as

$$[\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2+k^2]\,G(\underline{R},\underline{R}_o) = -4\pi\delta(x-x_o)\delta(y-y_o)\delta(z-z_o) \qquad (2.1)$$

$$[\partial^2/\partial r^2+(1/r)\partial/\partial r+(1/r^2)\partial^2/\partial\phi^2+\partial^2/\partial z^2+k^2]\,G(\underline{R},\underline{R}_o)$$
$$= -4\pi\delta(r-r_o)\delta(\phi-\phi_o)\delta(z-z_o)/r \qquad (2.2)$$

$$[\partial^2/\partial R^2+(2/R)\partial/\partial R+(1/R^2)\partial^2/\partial\theta^2+(\cot\theta/R^2)\partial/\partial\theta$$
$$+(1/R^2\sin^2\theta)\partial^2/\partial\phi^2+k^2]\,G(\underline{R},\underline{R}_o)$$
$$= -4\pi\delta(R-R_o)\delta(\theta-\theta_o)\delta(\phi-\phi_o)/R^2\sin\theta \qquad (2.3)$$

The evaluation of the three-dimensional free-space Green's function in each coordinate system proceeds as follows. First, the differential equation (1.1) is solved by the method of separation of variables which leads to a general integral/series transform representation of the Green's function. Secondly, this transform representation is evaluated to obtain the Green's function in a form appropriate to outgoing waves. Thirdly, the transform representation is evaluated to recover the Green's function in its simple form, equation (1.2). Finally the particular cases of the two-dimensional and one-dimensional free-space Green's function are analysed.

3. CARTESIAN COORDINATES

3.1 Fourier Transform Solution

The three-dimensional Fourier integral transform representation

$$G_F(\underline{R},\underline{R}_o) = (1/8\pi^3)\iiint_{-\infty}^{\infty} \bar{G}_F(\alpha,\beta,\varepsilon,\underline{R}_o)\exp[i\alpha x+i\beta y+i\varepsilon z]\,d\alpha d\beta d\varepsilon \qquad (3.1)$$

and its inverse

$$\bar{G}_F(\alpha,\beta,\varepsilon,\underline{R}_o) = \iiint_{-\infty}^{\infty} G_F(\underline{R},\underline{R}_o)\exp[-i\alpha x-i\beta y-i\varepsilon z]\,dxdydz \qquad (3.2)$$

are used to obtain a general solution of the reduced wave equation (2.1), with point source excitation, in Cartesian coordinates as

$$G_F(\underline{R},\underline{R}_o) = (1/2\pi^2)\iiint_{-\infty}^{\infty} \frac{\exp[i\alpha(x-x_o)+i\beta(y-y_o)+i\varepsilon(z-z_o)]\,d\alpha d\beta d\varepsilon}{\alpha^2+\beta^2+\varepsilon^2-k^2} \qquad (3.3)$$

The Green's function G_F and its Fourier transform $\bar{G}_F$ are considered as distributions in the sense of Schwartz [4] because consideration must be given to the choice of integration contours in their evaluation to ensure that the outgoing wave solution alone is obtained.

3.2 Plane Boundary, z=constant

In many cases of interest there are plane boundaries, perpendicular to the z-axis, say, on which boundary conditions must be satisfied. In such cases it is advantageous to carry out the ε-integration. This integration may be accomplished by choosing a contour which depends on the sign of $z-z_o$, viz for $z-z_o \geq 0$ the contour is closed in the upper half plane, while for $z-z_o \leq 0$ the contour is closed in the lower half plane. These contours are shown in Figure 2.

When $k^2 > \alpha^2 + \beta^2$ the integrand has poles on the real axis and the portion of the contour along the real axis must be indented to avoid passing through them. The integral representation of the Green's function, equation (3.3), is a general solution which is arbitrary with respect to a solution of the homogeneous wave equation corresponding to incoming free-waves generated at infinity. It is necessary to indent the contour so that outgoing waves only are present. This 'radiation condition' may be simulated mathematically by allowing for energy dissipation in the medium by setting $k \equiv k+ik'$, corresponding to a complex sound velocity $c \equiv c-ic'$, so that any disturbance originating at infinity is negligible in any finite region of interest. The 'real' poles are thus forced off the real axis to allow an unambiguous evaluation of the integral by the residue theorem. The dissipation term is then allowed to tend to zero to show that this k'-prescription is equivalent to indenting the contour above and below the negative and positive poles respectively. The following integrals are obtained

$$G_F(\underline{R},\underline{R}_o) = (i/2\pi) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1/\gamma)\exp\left[i\alpha(x-x_o) + i\beta(y-y_o) + i\gamma(z-z_o)\right] d\alpha d\beta \qquad z \geq z_o \tag{3.4}$$

$$G_F(\underline{R},\underline{R}_o) = (i/2\pi) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1/\gamma)\exp\left[i\alpha(x-x_o) + i\beta(y-y_o) - i\gamma(z-z_o)\right] d\alpha d\beta \qquad z \leq z_o \tag{3.5}$$

or more generally

$$G_F(\underline{R},\underline{R}_o) = (i/2\pi) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1/\gamma)\exp\left[i\alpha(x-x_o) + i\beta(y-y_o) + i\gamma|z-z_o|\right] d\alpha d\beta \tag{3.6}$$

for all z, where

$$\gamma = \sqrt{(k^2-\alpha^2-\beta^2)}, \quad \mathrm{Im}(\gamma) \geq 0, \quad \mathrm{Re}(\gamma) \geq 0$$

3.3 Reduction to Standard Form

The standard form of the Green's function equation (1.2) may be obtained by evaluating the double integral in equation (3.6). This is accomplished by first transforming the variables to polar coordinate systems, via the substitutions

$$\alpha = s.\cos\theta \qquad \beta = s.\sin\theta \qquad d\alpha d\beta = s ds\, d\theta$$

$$x-x_o = t.\cos\phi \qquad y-y_o = t.\sin\phi$$

The identity [2]

$$\int_0^{2\pi} \exp\left[its.\cos(\theta-\phi)\right] d\theta = 2\pi J_o(ts)$$

is then used to reduce the resulting double integral to the single integral

$$G_F(\underline{R},\underline{R}_o) = \int_0^{\infty} \frac{J_o(ts)\exp\left[-|z-z_o|\surd(s^2-k^2)\right] s ds}{\surd(s^2-k^2)} \qquad (3.7)$$

The integrand has a branch point at s=k, and the contour must be indented to avoid it. The inclusion of a dissipation term, as in the case with a plane boundary, allows the branch point to be lifted above the real axis. In the limiting case of no dissipation this is equivalent to indenting the contour below the branch point, Figure 3. With this choice of contour for the integration, ensuring that the outgoing wave solution alone is obtained, equation (3.7) is a standard integral [1] which reduces to

$$G_F(\underline{R},\underline{R}_o) = \frac{\exp\left[ik\{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2\}^{\frac{1}{2}}\right]}{\left[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2\right]^{\frac{1}{2}}}$$

which is the simple form of the free-space Green's function in Cartesian coordinates

4. CYLINDRICAL COORDINATES

4.1 Transform Solution

The Green's function is represented by a Fourier series transform in the angle ϕ, a Fourier integral transform in the axial coordinate z, and a Hankel transform in the radial coordinate r, viz

$$G_F(\underline{R},\underline{R}_o) = (1/2\pi)\sum_{n=-\infty}^{\infty} \exp(in\phi) \int_{-\infty}^{\infty}\int_0^{\infty} \bar{G}_F(\xi,n,\alpha,\underline{R}_o) J_n(\xi r)\exp(i\alpha z)\xi d\xi d\alpha \qquad (4.1)$$

where the inverse transform is

$$\bar{G}_F(\xi,n,\alpha,\underline{R}_o) = (1/2\pi)\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} G_F(\underline{R},\underline{R}_o)J_n(\xi r)\exp(-in\phi-i\alpha z)\,r\,dr\,d\phi\,dz \quad (4.2)$$

The above transform representations are used to obtain a general solution of the reduced wave equation (2.2), with point source excitation. After simplification, using the differential equation satisfied by the Bessel function, this general solution is

$$G_F(\underline{R},\underline{R}_o) = (1/\pi)\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_o)]\int_{-\infty}^{\infty}\int_0^{\infty}\frac{J_n(\xi r_o)J_n(\xi r)\exp[i\alpha(z-z_o)]\,\xi\,d\xi\,d\alpha}{\xi^2-(k^2-\alpha^2)} \quad (4.3)$$

4.2 Cylindrical Boundary r=constant

In many cases of interest in this geometry there are cylindrical boundaries present, defined by r=constant, on which boundary conditions must be satisfied. In such cases it is advantageous to carry out the ξ-integration. The integrand has a pole on the real axis for $k^2>\alpha^2$, and the contour must be indented to avoid it. Introducing a dissipation factor, as before, to obtain the outgoing wave solution only, allows the pole to be lifted above the real axis or equivalently the contour to be indented below the pole. The standard integral [1]

$$\int_0^{\infty}\frac{J_n(\xi r_o)J_n(\xi r)\xi\,d\xi}{\xi^2-\gamma^2} = \begin{cases}(\pi i/2)J_n(\gamma r_o)H_n(\gamma r) & r \geq r_o>0\\ (\pi i/2)J_n(\gamma r)H_n(\gamma r_o) & r_o\geq r>0\end{cases}$$

with $\mathrm{Im}(\gamma=\sqrt{(k^2-\alpha^2)})>0$, then enables the free-space Green's function to be represented as

$$G_F(\underline{R},\underline{R}_o) = (i/2)\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_o)]\int_{-\infty}^{\infty}J_n(\gamma r_o)H_n(\gamma r)\exp[i\alpha(z-z_o)]\,d\alpha \qquad r\geq r_o>0 \quad (4.4)$$

$$G_F(\underline{R},\underline{R}_o) = (i/2)\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_o)]\int_{-\infty}^{\infty}J_n(\gamma r)H_n(\gamma r_o)\exp[i\alpha(z-z_o)]\,d\alpha \qquad r_o\geq r>0 \quad (4.5)$$

with $\gamma=\sqrt{(k^2-\alpha^2)}$ and with $\mathrm{Im}(\gamma)\geq 0$, $\mathrm{Re}(\gamma)\geq 0$.

4.3 Source or Observer on z-axis

In the special case of the source being positioned on the z-axis, it is evident that the Green's function must be axisymmetric. Hence, the n=0 term alone contributes to the field which is obtained from equation (4.3) as

$$G_F(\underset{\sim}{R},\underset{\sim}{R}_o) = (1/\pi)\int_{-\infty}^{\infty}\int_{o}^{\infty}\frac{J_o(\xi r)\exp[i\alpha(z-z_o)]\,\xi d\xi d\alpha}{\xi^2-\gamma^2} \qquad (4.6)$$

The ξ-integration is facilitated by extending the range of integration to the entire real axis by the use of the formula

$$J_o(\xi r) = (1/2)\,[H_o(\xi r)-H_o(-\xi r)] \qquad (4.7)$$

to give

$$G_F(\underset{\sim}{R},\underset{\sim}{R}_o) = (1/2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H_o(\xi r)\exp[i\alpha(z-z_o)]\,\xi d\xi d\alpha}{\xi^2-\gamma^2} \qquad (4.8)$$

Figure 3 shows the contour used to evaluate the ξ-integral. The indentation at the origin is necessary because of the branch point of the function $H_o(\xi r)$. The contribution from this indentation can easily be shown to be zero when the small argument expansion of the Hankel function, viz $H_o(x)=(2i/\pi)\ell n(x)$, is used in its evaluation. The integrand also has poles on the real axis when $\gamma^2>0$. The required indentations are again chosen by the inclusion of a small dissipation term, $k\equiv k+ik'$, to give, on applying the residue theorem,

$$G_F(\underset{\sim}{R},\underset{\sim}{R}_o) = (i/2)\int_{-\infty}^{\infty} H_o(\gamma r)\exp[i\alpha(z-z_o)]\,d\alpha \qquad (4.9)$$

a result which is identical to equation (4.4), in the special case of $r_o=0$.

In the case of the observer being positioned on the z-axis the analysis is identical, as would be expected by reciprocity, and yields

$$G_F(\underset{\sim}{R},\underset{\sim}{R}_o) = (i/2)\int_{-\infty}^{\infty} H_o(\gamma r_o)\exp[i\alpha(z-z_o)]\,d\alpha \qquad (4.10)$$

Equations (4.4) and (4.5) are therefore valid for all values of r and r_o such that r and r_o are not both zero.

4.4 Plane Boundary z=constant

In cases where a plane boundary perpendicular to the z-axis is present, on which boundary conditions must be satisfied, it is advantageous to perform the α-integration in equation (4.3). The contours used for this are identical to those used for the ε-integration in Section 3, and give the result

$$G_F(\underline{R},\underline{R}_o) = i \sum_{n=-\infty}^{\infty} \exp[in(\phi-\phi_o)] \int_0^{\infty} \frac{J_n(\xi r)J_n(\xi r_o)\exp[i|z-z_o|\sqrt{(k^2-\xi^2)}]\,\xi d\xi}{\sqrt{(k^2-\xi^2)}} \tag{4.11}$$

4.5 Reduction to Standard Form

Equations (4.4) and (4.5) can be reduced to standard form, equation (1.2), by making use of one of the Bessel function addition theorems [3], viz,

$$H_o(\gamma r_1) = \begin{cases} \sum_{n=-\infty}^{\infty} \exp[in(\phi-\phi_o)]\,H_n(\gamma r_o)J_n(\gamma r) & r_o \geqslant r > 0 \\ \sum_{n=-\infty}^{\infty} \exp[in(\phi-\phi_o)]\,H_n(\gamma r)J_n(\gamma r_o) & r \geqslant r_o > 0 \end{cases}$$

to give

$$G_F(\underline{R},\underline{R}_o) = (i/2) \int_{-\infty}^{\infty} H_o(\gamma r_1)\exp[i\alpha(z-z_o)]\,d\alpha \tag{4.12}$$

where

$$r_1 = \sqrt{[r^2 + r_o^2 - 2r\,r_o \cos(\phi-\phi_o)]} = \sqrt{[(x-x_o)^2 + (y-y_o)^2]}$$

This standard integral [2,p.710] reduces to the simple form of the free-space Green's function for outgoing waves, equation (1.2).

5. SPHERICAL COORDINATES

5.1 Transform Solution

The Green's function is represented by a Fourier Series transform in the angle ϕ, an associated Legendre Series transform in the angle θ, and a spherical Hankel transform in the radial coordinate r, viz

$$G_F(\underline{R},\underline{R}_o) = \sum_{m=-\infty}^{\infty} \exp(im\phi) \sum_{n=-\infty}^{\infty} P_n^m(\cos\theta) \int_0^{\infty} \bar{G}_F(\xi,n,m,\underline{R}_o)\,j_n(\xi R)\,\xi^{3/2}d\xi \tag{5.1}$$

where the inverse transform is

$$\bar{G}_F(\xi,n,m,\underline{R}_o) = (2n+1)(n-m)!\xi^{1/2}/(4\pi^2(n+m)!) \cdot$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} G_F(\underline{R},\underline{R}_o)\exp(-im\phi)j_n(\xi R)P_n^m(\cos\theta)R^2\sin\theta\, dR d\theta d\phi \tag{5.2}$$

The above transform representations are used to obtain a general solution of the reduced wave equation (2.3) in spherical coordinates which, after simplification using the differential equation satisfied by the spherical Bessel function, is

$$G_F(\underline{R},\underline{R}_o) = (1/\pi)\sum_{n=-\infty}^{\infty}\sum_{m=-|n|-1}^{|n|+1} \frac{(2n+1)(n-m)!\,P_n^m(\cos\theta)P_n^m(\cos\theta_o)\exp[im(\phi-\phi_o)]}{(n+m)!} \cdot$$

$$\int_0^{\infty} \frac{j_n(\xi R_o)j_n(\xi R)\xi^2 d\xi}{\xi^2-k^2} \tag{5.3}$$

5.2 Spherical Boundary

In cases of interest in this geometry there are spherical boundaries present, defined by R=constant, on which boundary conditions must be satisfied. In such cases it is advantageous to carry out the ξ-integration in equation (5.3). The identity

$$\int_0^{\infty} \frac{j_n(\xi R_o)j_n(\xi R)\xi^2 d\xi}{\xi^2-k^2} = \frac{\pi}{2\sqrt{(RR_o)}}\int_0^{\infty} \frac{J_{n+1/2}(\xi R_o)J_{n+1/2}(\xi R)\xi d\xi}{\xi^2-k^2} \tag{5.4}$$

enables this integration to be accomplished by the methods of Section 4 to give the outgoing wave representation for $0<R_o<R$

$$G_F(\underline{R},\underline{R}_o) = [\pi i/4\sqrt{(RR_o)}]\sum_{n=-\infty}^{\infty}(2n+1)J_{n+1/2}(kR_o)H_{n+1/2}(kR) \cdot$$

$$\sum_{m=-|n|-1}^{|n|+1} \frac{(n-m)!P_n^m(\cos\theta)P_n^m(\cos\theta_c)\exp[im(\phi-\phi_o)]}{(n+m)!} \tag{5.5}$$

while for $0<R<R_o$ it is only necessary to interchange R and R_o. The above equation may be simplified to

$$G_F(\underline{R},\underline{R}_o) = [\pi i/2\surd(RR_o)] \sum_{n=o}^{\infty} e_n(2n+1)J_{n+1/2}(kR)H_{n+1/2}(kR_o).$$

$$\sum_{m=o}^{n} \frac{(n-m)!\,P_n^m(\cos\theta)P_n^m(\cos\theta_o)\cos[m(\phi-\phi_o)]}{(n+m)!} \tag{5.6}$$

5.3 Source or Observer at the Origin

In the special case of the source being positioned at the origin it is evident that the Green's function is spherically symmetric. Hence the (n=0, m=0) and (n=-1, m=0) terms alone contribute to the field which is obtained from equation (5.3) as

$$G_F(\underline{R},\underline{R}_o) = (2/\pi)\int_o^{\infty} \frac{j_o(\xi R)\xi^2 d\xi}{\xi^2 - k^2} \tag{5.7}$$

This integral is of standard form when the spherical Bessel function is replaced by a Bessel function of order one-half and the contour is indented below the pole at $\xi=k$. It is [2, p.687]

$$G_F(\underline{R},\underline{R}_o) = \surd(-2ik/\pi R)K_{1/2}(-ikR)$$

which can be expressed [2, p.967] as

$$G_F(\underline{R},\underline{R}_o) = \exp(ikR)/R \tag{5.8}$$

In the separate case of the observer being positioned at the origin, the analysis is identical and gives the result expected by reciprocity, viz equation (5.8) with R and R_o interchanged.

5.4 Reduction to Standard Form

Equation (5.6) can be reduced to the standard form of the Green's function equation (1.2) by making use of the addition theorems [2, p.1013 and p.980] for the Legendre and spherical Bessel functions, viz

$$P_n(\cos\psi) = \sum_{m=o}^{n} \frac{e_n(n-m)!\,P_n^m(\cos\theta)P_n^m(\cos\theta_o)\cos[m(\phi-\phi_o)]}{(n+m)!} \tag{5.9}$$

$$\exp(ik|\underline{R}-\underline{R}_o|)/|\underline{R}-\underline{R}_o| = [\pi i/2\surd(RR_o)] \sum_{n=o}^{\infty} (2n+1)J_{n+1/2}(kR_o)H_{n+1/2}(kR)P_n(\cos\psi)$$

$$R > R_o \tag{5.10}$$

where $\cos\psi = \cos\theta\cos\theta_o + \sin\theta\sin\theta_o\cos(\phi-\phi_o)$

is the cosine of the angle between the source and observer.

6. ONE AND TWO DIMENSIONAL REPRESENTATIONS

6.1 General

The analysis necessary to obtain the expansions of the one and two dimensional free-space Green's functions, and their subsequent reduction to the simple standard forms follows the same procedure as that used to solve the three-dimensional problem. This section therefore contains a minimum of analytical detail.

6.2 Cartesian Coordinates: (x,y)

The two-dimensional Fourier transform and its inverse are used to obtain the general representation of the free-space Green's function in two-dimensions as

$$G_F(\underline{r},\underline{r}_o) = (1/\pi) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp[i\alpha(x-x_o)+i\beta(y-y_o)]\, d\alpha d\beta}{\alpha^2+\beta^2-k^2} \qquad (6.1)$$

In the presence of the plane boundary y=constant, the β-integration is performed to give the free-space Green's function as

$$G_F(\underline{r},\underline{r}_o) = i \int_{-\infty}^{\infty} \frac{\exp[i\alpha(x-x_o) + i\sqrt{(k^2-\alpha^2)}\,|y-y_o|]\, d\alpha}{\sqrt{(k^2-\alpha^2)}} \qquad (6.2)$$

This integral may be written as

$$G_F(\underline{r},\underline{r}_o) = 2 \int_{0}^{\infty} \frac{\cos[\alpha(x-x_o)]\exp[-|y-y_o|\sqrt{(\alpha^2 + (-ik)^2)}]\, d\alpha}{\sqrt{(\alpha^2 + (-ik)^2)}}$$

which is of standard form [2, p.498] when damping in the system is considered, and gives

$$G_F(\underline{r},\underline{r}_o) = 2K_o(-ik[(x-x_o)^2 + (y-y_o)^2]^{1/2})$$

which reduces to the free-space Green's function in its simple form, viz

$$G_F(\underline{r},\underline{r}_o) = \pi i H_o(k[(x-x_o)^2+(y-y_o)^2]^{1/2}) \qquad (6.3)$$

$$= \pi i H_o(k|\underline{r}-\underline{r}_o|) \qquad (6.4)$$

6.3 Polar Coordinates: (r,ϕ)

The Fourier series and Hankel transforms in the angle ϕ and the radius r, respectively, and their inverses are used to obtain the general representation of the free-space Green's function in two dimensions as

$$G_F(\underline{r},\underline{r}_o) = 2\sum_{n=-\infty}^{\infty} \exp\left[in(\phi-\phi_o)\right] \int_o^{\infty} \frac{J_n(\xi r)J_n(\xi r_o)\xi d\xi}{\xi^2-k^2} \tag{6.5}$$

In the presence of a circular boundary, the ξ-integration is performed as

$$G_F(\underline{r},\underline{r}_o) = \pi i \sum_{n=-\infty}^{\infty} \exp\left[in(\phi-\phi_o)\right] H_n(kr)J_n(kr_o) \qquad r \geqslant r_o \geqslant 0 \quad r \neq 0$$

(6.6)

$$G_F(\underline{r},\underline{r}_o) = \pi i \sum_{n=-\infty}^{\infty} \exp\left[in(\phi-\phi_o)\right] H_n(kr_o)J_n(kr) \qquad r_o \geqslant r \geqslant 0 \quad r_o \neq 0$$

(6.7)

a result which reduces immediately, via a Bessel function summation theorem [3], to the simple form of the two-dimensional free-space Green's function equation (6.4).

6.4 One-Dimensional Representation

The one-dimensional Fourier transform and its inverse are used to obtain the general representation of the free-space Green's function in one-dimension as

$$G_F(x,x_o) = 2\int_{-\infty}^{\infty} \exp\left[i\alpha(x-x_o)\right]/(\alpha^2-k^2)d\alpha \tag{6.8}$$

which is immediately integrable to the standard and simple form of the one-dimensional free-space Green's function, viz

$$G_F(x,x_o) = (2\pi i/k)\exp\left[ik(x-x_o)\right] \qquad x > x_o \tag{6.9}$$

$$G_F(x,x_o) = (2\pi i/k)\exp\left[ik(x_o-x)\right] \qquad x < x_o \tag{6.10}$$

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APPENDIX

GREEN'S FUNCTIONS IN THREE DIMENSIONS

General Form

$$G_F(\underline{R},\underline{R}_o) = \exp(ik|\underline{R}-\underline{R}_o|)/|\underline{R}-\underline{R}_o|$$

$$G_F(\underline{R},\underline{R}_o) = (1/2\pi^2) \iiint_{-\infty}^{\infty} \frac{\exp[\, i\alpha(x-x_o) + i\beta(y-y_o) + i\varepsilon(z-z_o)]\, d\alpha d\beta d\varepsilon}{\alpha^2+\beta^2+\varepsilon^2-k^2}$$

$$G_F(\underline{R},\underline{R}_o) = (1/\pi) \sum_{n=-\infty}^{\infty} \exp[\, in(\phi-\phi_o)] \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{J_n(\xi r)J_n(\xi r_o)\exp[\, i\alpha(z-z_o)]\, \xi d\xi d\alpha}{\xi^2+\alpha^2-k^2}$$

$$G_F(\underline{R},\underline{R}_o) = (1/\pi) \sum_{n=-\infty}^{\infty} \sum_{m=-|n|-1}^{|n|+1} (2n+1)(n-m)!\, P_n^m(\cos\theta) P_n^m(\cos\theta_o) \exp[\, im(\phi-\phi_o)]/(n+m)!$$

$$\int_{0}^{\infty} \frac{j_n(\xi R_o) j_n(\xi R)\xi^2 d\xi}{\xi^2 - k^2}$$

Cartesian Coordinates - Plane Boundary (z=constant)

$$G_F(\underline{R},\underline{R}_o) = (i/2\pi) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1/\gamma)\exp[\, i\alpha(x-x_o) + i\beta(y-y_o) + i\gamma(z-z_o)]\, d\alpha d\beta$$

with $\gamma = \surd(k^2-\alpha^2-\beta^2)$, for $(z-z_o)>0$. Interchange z and z_o when $(z-z_o)<0$.

Cylindrical Coordinates - Cylindrical Boundary (r=constant)

$$G_F(\underline{R},\underline{R}_o) = \begin{cases} (i/2) \displaystyle\sum_{n=-\infty}^{\infty} \exp[\, in(\phi-\phi_o)] \int_{-\infty}^{\infty} J_n(\gamma r_o)H_n(\gamma r)\exp[\, i\alpha(z-z_o)]\, d\alpha \\ (i/2) \displaystyle\sum_{n=o}^{\infty} e_n \cos[\, n(\phi-\phi_o)] \int_{-\infty}^{\infty} J_n(\gamma r_o)H_n(\gamma r)\exp[\, i\alpha(z-z_o)]\, d\alpha \end{cases}$$

with $\gamma=\sqrt{(k^2-\alpha^2)}$, for $r>r_o\geq 0$, $r\neq 0$. Interchange r and r_o when $r_o>r\geq 0$, $r_o\neq 0$.

Cylindrical Coordinates - Plane Boundary (z=constant)

$$G_F(\underline{R},\underline{R}_o) = \begin{cases} i \sum_{n=-\infty}^{\infty} \exp[in(\phi-\phi_o)] \int_0^{\infty} \frac{J_n(\xi r)J_n(\xi r_o)\exp[i(z-z_o)\sqrt{(k^2-\xi^2)}]\,\xi d\xi}{\sqrt{(k^2-\xi^2)}} \\ \\ i \sum_{n=o}^{\infty} e_n \cos[n(\phi-\phi_o)] \int_0^{\infty} \frac{J_n(\xi r)J_n(\xi r_o)\exp[i(z-z_o)\sqrt{(k^2-\alpha^2)}]\,\xi d\xi}{\sqrt{(k^2-\xi^2)}} \end{cases}$$

for $(z-z_o)\geq 0$. Interchange z and z_o when $(z-z_o)\leq 0$.

Spherical coordinates - Spherical Boundary (R=constant)

$$G_F(\underline{R},\underline{R}_o) = \begin{cases} (\pi i/4\sqrt{(RR_o)}) \sum_{n=-\infty}^{\infty} (2n+1)\, J_{n+1/2}(kR)H_{n+1/2}(kR_o)\,. \\ \quad \sum_{m=-|n|-1}^{|n|+1} (n-m)!\, P_n^m(\cos\theta)P_n^m(\cos\theta_o)\exp[im(\phi-\phi_o)]/(n+m)! \\ \\ (\pi i/2\sqrt{(RR_o)}) \sum_{n=o}^{\infty} e_n(2n+1)J_{n+1/2}(kR)H_{n+1/2}(kR_o)\,. \\ \quad \sum_{m=o}^{n} (n-m)!\, P_n^m(\cos\theta)P_n^m(\cos\theta_o)\cos[m(\phi-\phi_o)]/(n+m)! \end{cases}$$

for $0<R<R_o$. Interchange R and R_o when $0<R_o<R$.

GREEN'S FUNCTIONS IN TWO DIMENSIONS

General Form

$$G_F(\underline{r},\underline{r}_o) = \pi i\, H_o(k|\underline{r}-\underline{r}_o|)$$

$$G_F(\underline{r},\underline{r}_o) = (1/\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp[i\alpha(x-x_o)+i\beta(y-y_o)]\,d\alpha d\beta}{\alpha^2+\beta^2-k^2}$$

$$G_F(\underline{r},\underline{r}_o) = 2\sum_{n=-\infty}^{\infty} \exp[in(\phi-\phi_o)] \int_0^{\infty} \frac{J_n(\xi r)J_n(\xi r_o)\,\xi d\xi}{\xi^2-k^2}$$

Cartesian Coordinates - Plane Boundary (y=constant)

$$G_F(\underline{r},\underline{r}_o) = i \int_{-\infty}^{\infty} \frac{\exp\,[\,i\alpha(x-x_o) + i(y-y_o)\sqrt{(k^2-\alpha^2)}]\,d\alpha}{\sqrt{(k^2-\alpha^2)}}$$

for $y \geqslant y_o$. Interchange y and y_o when $y \leqslant y_o$.

Polar Coordinates - Circular Boundary (r=constant)

$$G_F(\underline{r},\underline{r}_o) = \begin{cases} \pi i \sum\limits_{n=-\infty}^{\infty} \exp\,[\,in(\phi-\phi_o)]\,H_n(kr)J_n(kr_o) \\ \\ \pi i \sum\limits_{n=o}^{\infty} e_n \cos\,[\,n(\phi-\phi_o)]\,H_n(kr)J_n(kr_o) \end{cases}$$

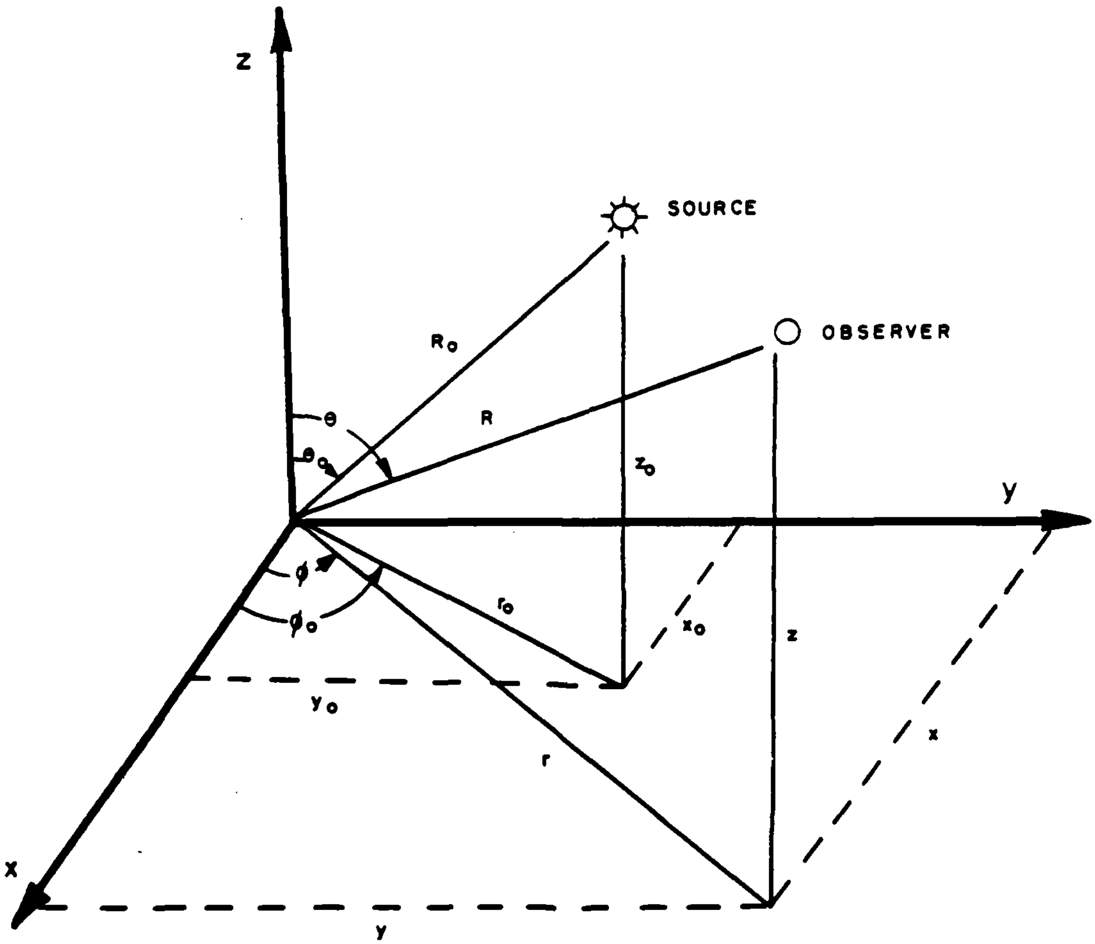
for $r>r_o$. Interchange r and r_o when $r<r_o$.

GREEN'S FUNCTION IN ONE DIMENSION

General Form

$$G_F(x,x_o) = (2\pi i/k)\exp(ik|x-x_o|)$$

$$G_F(x,x_o) = 2 \int_{-\infty}^{\infty} \frac{\exp\,[\,i\alpha(x-x_o)]\,d\alpha}{\alpha^2-k^2}$$



CARTESIAN		CYLINDRICAL		SPHERICAL
x	=	r . cos (φ)	=	R.sin (θ) cos (φ)
y	=	r . sin (φ)	=	R.sin (θ) sin (φ)
z	=	z	=	R.cos (θ)

FIG. 1 COORDINATE SYSTEMS

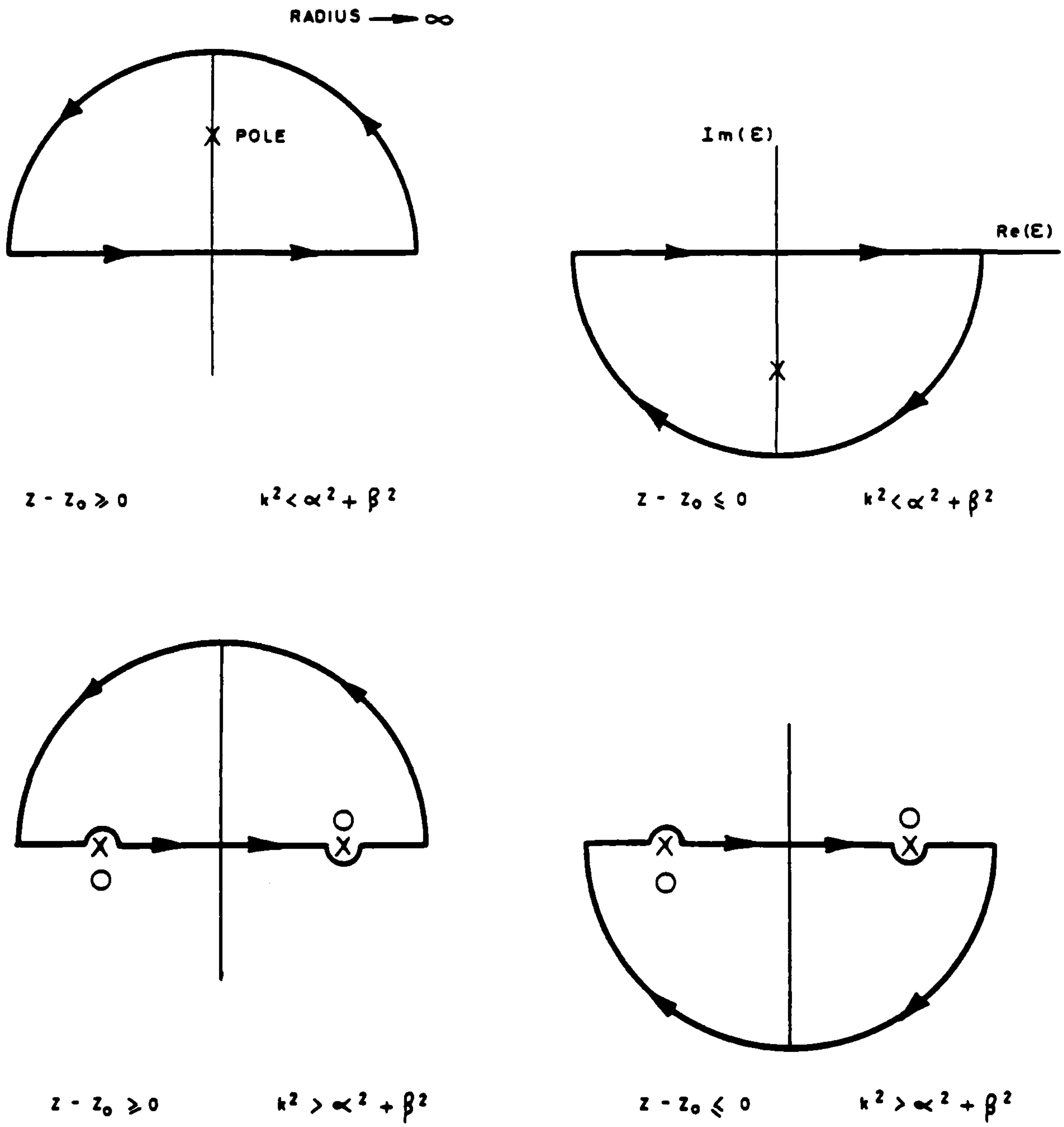


FIG. 2 CONTOURS OF INTEGRATION IN COMPLEX Ɛ- PLANE

O / X = POSITION OF POLE WITH / WITHOUT DAMPING

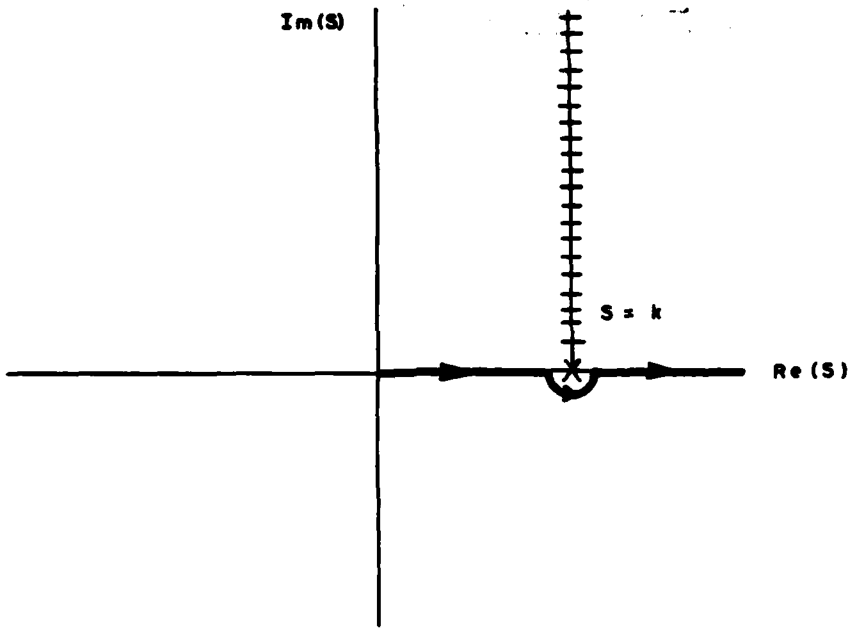


FIG. 3a INTEGRATION CONTOUR $\int_0^\infty \frac{Jo(ts)\exp[-|Z-Z_0|\sqrt{(s^2-k^2)}]\,s\,ds}{\sqrt{(s^2-k^2)}}$

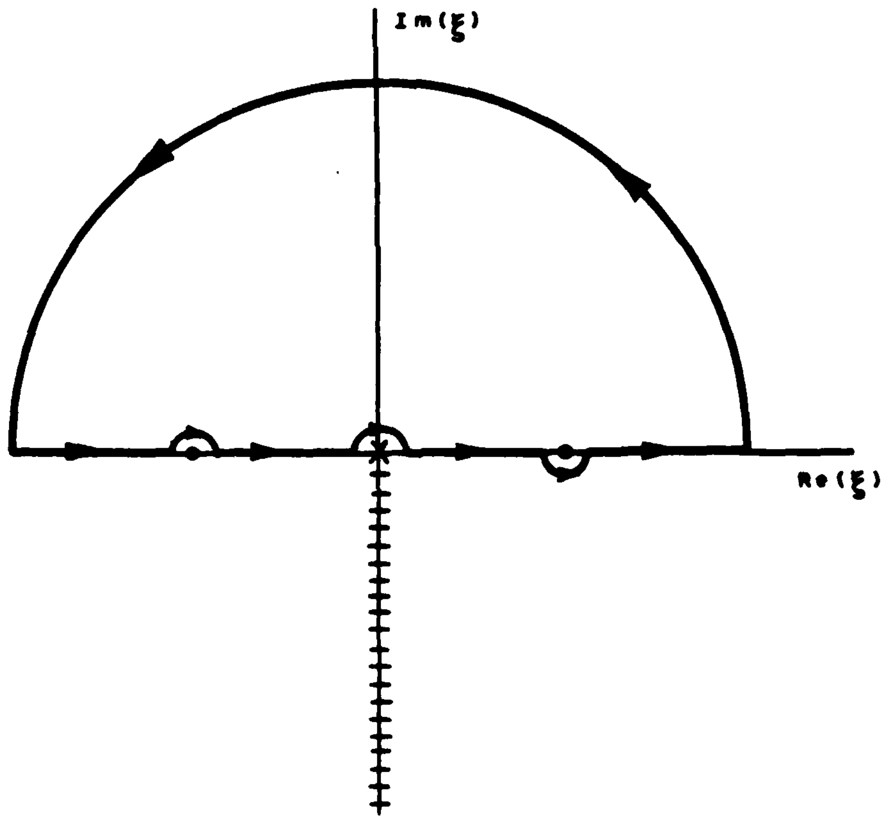


FIG. 3b INTEGRATION CONTOUR $\int_{-\infty}^{\infty} \frac{Ho(\xi r)\,\xi\,d\xi}{\xi^2-\gamma^2}$

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