Gradient Space Under Orthography and Perspective

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Abstract

Mackworth's gradient space has proven to be a useful tool for image understanding. However, descriptions of its important properties have been somewhat scattered in the literature.

This paper develops and summarizes the fundamental properties of the gradient space under orthography and perspective, and for curved surfaces. While largely a recounting of previously published results, there are a number of new observations, particularly concerning the gradient space and perspective projection. In addition, the definition and use of vector gradients as well as surface gradients provides concise notation for several results.

The properties explored in the paper include the orthographic and perspective projections themselves; the definition of gradients; the gradient space consequences of vectors (edges) belonging to one or more surfaces, and of several vectors being contained on a single surface; and the relationships between vanishing points, vanishing lines, and the gradient space.

The paper is intended as a study guide for learning about the gradient space, as well as a reference for researchers working with gradient space.

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Gradient Space Under Orthography and Perspective

Introduction

The *gradient space* has proven a useful tool for image understanding. Since its proposal by Mackworth [12] based on Huffman's *dual space* [5], the gradient space has been used for defining consistency of line-labelings [6, 7], relating surface orientation to image intensity [3, 4, 16], and relating surface orientation to image geometry [8, 9, 10, 13, 15].

The descriptions of important gradient space properties, however, have been scattered throughout the literature. In this paper, the gradient space is defined and its fundamental properties are summarized. This presentation is especially useful because its assignment of gradients to *vectors* as well as surfaces allows concise statements of important properties.

This paper is primarily a summary and re-statement of important gradient space properties, but also includes statements of some new properties. It is intended that the paper provide a reference for people working with the gradient space, as well as a study guide for researchers being introduced to the gradient space.

Preliminary Definitions

Coordinate System
In these equations, the coordinate system being used is Mackworth's [12]: the x and y axes in the scene are aligned in the image (x horizontal, y vertical), and the z axis points *towards* the viewer (i.e. a right-handed coordinate system) (figure 1). The *eye* (center of lens) is at the origin (0,0,0), and the *image plane* is $z = -1$ (i.e. the focal plane is $z = 1$, which is rotated around the origin to the image plane, $z = -1$, to preserve the sense of "up", "down", "left", and "right" from the scene).

Orthography
In *orthographic projection*, the scene point (x, y, z) is mapped onto the image point (x, y). Thus, the image point (x, y) represents the set of scene points (x, y, z) for all values of z.

Perspective
In *perspective projection*, the scene point (x, y, z) is mapped onto the image point $(-x/z, -y/z)$. The image point is the point at which a line through the origin (eye) and (x, y, z) intersects the image plane. The unit of measure in the coordinate system is the focal length of the camera lens. This is similar to Kender's coordinate system, but with the direction of the z-axis reversed [11]. An image point (x, y) corresponds to the set of scene points $(ax, ay, -a)$ for all values of a.
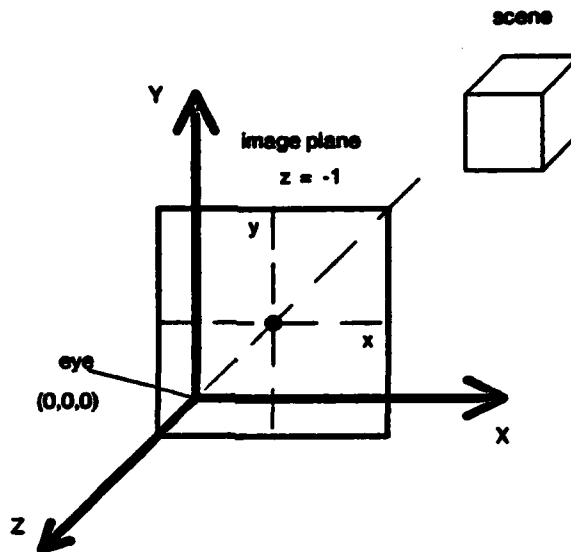
Figure 1: Coordinate System

Gradient Space and Orthography

In this section important relationships between surfaces, vectors, and gradients are described. In addition, several important observations concerning *orthographic projection* are noted.

1. **Definition of Surface Gradient**

   Suppose a surface is defined as $-z = f(x, y)$. Then its gradient (p, q) is defined by [12]:

$$(p, q) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (\frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y})$$

   The set of all gradients (p, q) is the *gradient space*.

   Corollary to this result is:

   1. In any direction u, the tangent vector to the surface is:

$$(\frac{dx}{du}, \frac{dy}{du}, -\frac{\partial f}{\partial x}\frac{dx}{du} - \frac{\partial f}{\partial y}\frac{dy}{du}) = (\frac{dx}{du}, \frac{dy}{du}, -p\frac{dx}{du} - q\frac{dy}{du})$$

   The tangent vectors in the directions $u = x$ and $u = y$ are thus $(1, 0, -p)$ and $(0, 1, -q)$ respectively. Their cross product, $(p, q, 1)$, is therefore a surface normal.

2. **Gradient of a Plane**

   Suppose a plane is defined by $Ax + By + Cz + D = 0$. Then its gradient is:

$$(p, q) = (\frac{A}{C}, \frac{B}{C})$$

Corollary to this result is:

1. Since D has no effect on p and q, parallel planes have the same gradient. Each point (p, q) thus represents the gradient for a family of parallel planes.

3. Gradient of a Vector

Suppose a vector is $(\Delta x, \Delta y, \Delta z)$. Then its gradient can be defined as:

$$(p, q) = (\frac{\Delta x}{\Delta z}, \frac{\Delta y}{\Delta z})$$

(This is not Huffman's *dual line* [5]; the dual is the line described below in section 7.)

Although the term *gradient* technically refers to a property of differentiable surfaces, it is used here for vectors because the gradient space can represent 3D orientation in general, not just orientation of surfaces.
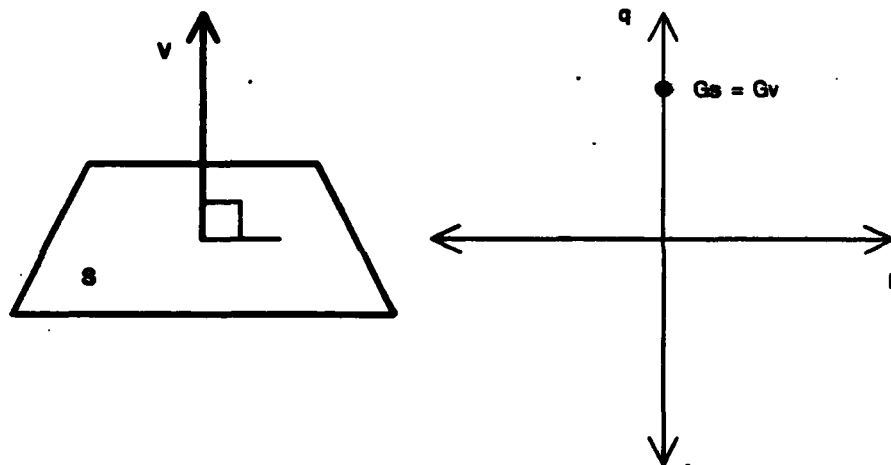

Figure 2: Gradient of Surface and Surface Normal

Corollaries of this result are:

1. Parallel vectors have the same gradient. The gradient of a line can be defined as the gradient of any vector contained in the line.

2. The gradient of a surface is the same as the gradient of its surface normal vectors (figure 2).

3. Under orthography, the vector $(\Delta x, \Delta y, \Delta z)$ is seen in the image as an edge $E = (\Delta x, \Delta y)$. If the vector's gradient is G, then $G = E / \Delta z$. The line in gradient space from the origin through G is thus parallel to the edge in the image (figure 3).
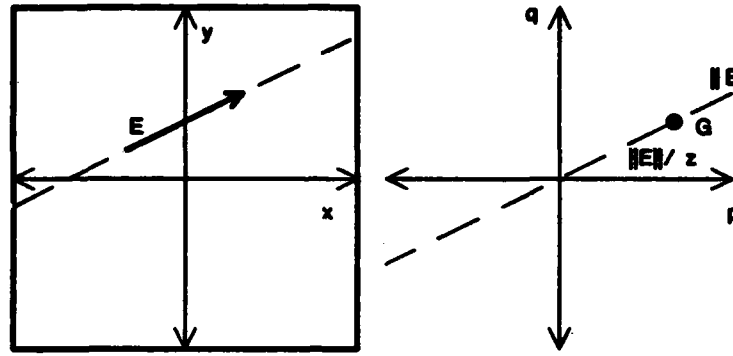
Figure 3: Gradient of a Vector Under Orthography

4. Vector Contained on a Surface

Suppose the vector (Δx, Δy, Δz) is contained on a surface whose gradient is (p, q).

Since the surface normal (p, q, 1) must be orthogonal to the vector,

$$(p, q, 1) \cdot (\Delta x, \Delta y, \Delta z) = 0$$
$$p\Delta x + q\Delta y + \Delta z = 0$$

Therefore,

$$(p, q) \cdot (\Delta x, \Delta y) = -\Delta z$$

Corollary to this result is:

1. Under orthography, the vector (Δx, Δy, Δz) is seen in the image as the edge E = (Δx, Δy). If it is contained on a surface with gradient G, then

$$G \cdot E = -\Delta z$$

This is one of the most important relations in orthography, since polyhedral scenes contain surfaces bounded by many edges.

5. Vector Contained on Two Surfaces

Suppose the vector (Δx, Δy, Δz) is the boundary between two surfaces with gradients G_1 = (p_1, q_1) and G_2 = (p_2, q_2) (figure 4). Further, define E to be (Δx, Δy).

Then:

$$-\Delta z = (p_1, q_1) \cdot (\Delta x, \Delta y) = (p_2, q_2) \cdot (\Delta x, \Delta y) = G_1 \cdot E = G_2 \cdot E$$
$$0 = (G_1 - G_2) \cdot E$$
$$(G_1 - G_2) \perp E$$

$G_1 - G_2$ is the vector from G_1 to G_2 in the gradient space. Thus, the vector E is perpendicular to the line containing G_1 and G_2 in gradient space.
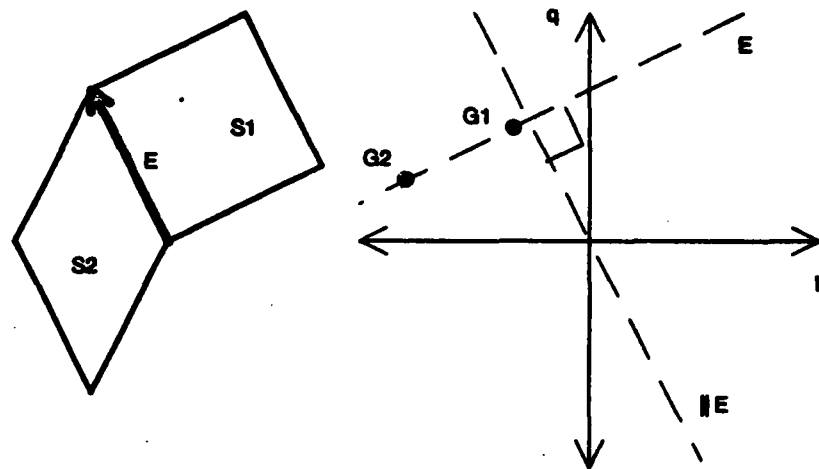
Figure 4: Vector Contained on Two Surfaces

As a corollary:

1. Under orthography, with the above definitions, the edge in the image is E. Therefore, the edge in the image is perpendicular to the line containing G_1 and G_2. (This is Mackworth's relation involving *connect edges* [12].)

6. Two Vectors Contained on a Surface

Suppose the two vectors $(\Delta x_1, \Delta y_1, \Delta z_1)$ and $(\Delta x_2, \Delta y_2, \Delta z_2)$ both lie on a surface whose gradient is $G = (p, q)$. Further, let $E_1 = (\Delta x_1, \Delta y_1)$ and $E_2 = (\Delta x_2, \Delta y_2)$ correspond to the x-y components of the two vectors (figure 5).
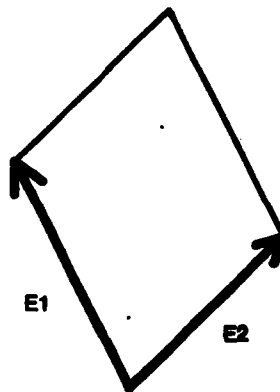

Figure 5: Two Vectors Contained on a Surface

Then:

$$- \Delta z_1 = E_1 \cdot G$$

$$-\Delta z_2 = E_2 \cdot G$$

These can be combined into a single matrix equation, in which the upper row is the first equation above, and the lower row is the second equation:

$$\begin{bmatrix} -\Delta z_1 \\ -\Delta z_2 \end{bmatrix} = \begin{bmatrix} E_1 \cdot G \\ E_2 \cdot G \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} G^T$$

$$G^T = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^{-1} \begin{bmatrix} -\Delta z_1 \\ -\Delta z_2 \end{bmatrix}$$

This expresses the surface gradient G as a function of the x, y, and z components of two vectors contained on the surface.

As a corollary:

1. Under orthography, vectors E_1 and E_2 have the same coordinates in the image that they have in the scene. So, given the Δz values for two vectors on a surface, the gradient of the surface can be found using the image.

7. Gradients of Perpendicular Vectors and Planes

Suppose two vectors $(\Delta x_1, \Delta y_1, \Delta z_1)$ and $(\Delta x_2, \Delta y_2, \Delta z_2)$ are perpendicular (in the *scene*), and that their gradients (as defined above) are $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ (figure 6). Then:
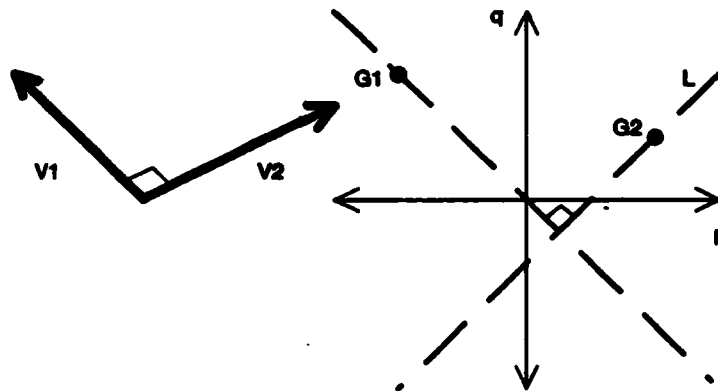

Figure 6:   Gradients of Perpendicular Vectors

$$(\Delta x_1, \Delta y_1, \Delta z_1) \cdot (\Delta x_2, \Delta y_2, \Delta z_2) = 0$$

$$\Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2 + \Delta z_1 \Delta z_2 = 0$$

Dividing by $\Delta z_1 \Delta z_2$,

$$p_1 p_2 + q_1 q_2 + 1 = 0$$

$$G_1 \cdot G_2 = -1$$

Suppose that G_1 is given. Then, the above equation is a line L in gradient space, which is the loci of the possible locations of G_2. In fact,

- L is perpendicular to the line from G_1 to the origin

- the distance from L to the origin is the reciprocal of the distance from G_1 to the origin

- L is on the opposite side of the origin from G_1.

The line L is Huffman's *dual line* (for a line in the image whose gradient is G_1) [5].

This result has two corollaries:

1. If two planes are orthogonal, their gradients obey the above relationship, since their surface normals are perpendicular.

2. If a plane contains a vector, the gradients of the plane and vector obey the above relationship, since the vector is perpendicular to the surface normal.

8. Rotation of the Image and Gradient Space

Suppose a surface is defined by $-z = I(x, y)$ and has gradient (p, q).

If we rotate the x-y axes around the z-axis through some angle θ to new $(x'\text{-}y')$ axes (figure 7), then:
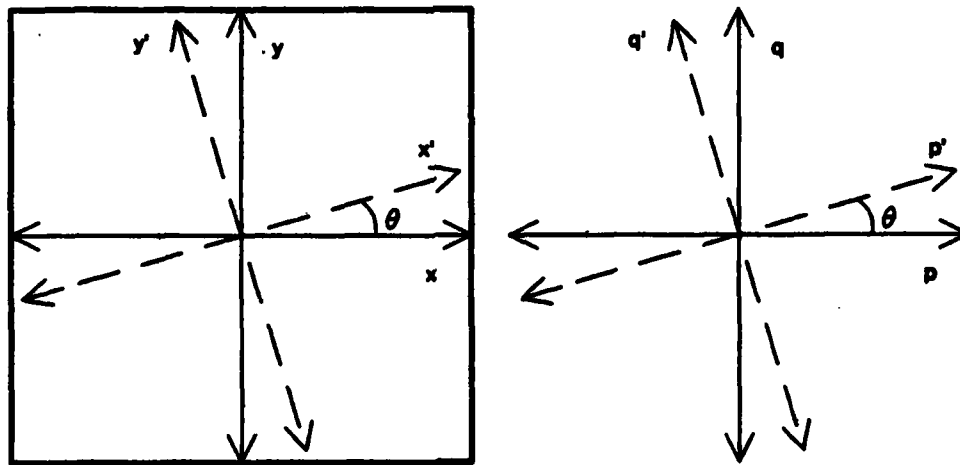

Figure 7: Rotating the Image and the Gradient Space

$$(x', y') = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$$

and

$$(x, y) = (x' \cos \theta - y' \sin \theta, x' \sin \theta + y' \cos \theta)$$

Now,

$$-z' = -z = f(x, y) = f(x' \cos \theta - y' \sin \theta, x' \sin \theta + y' \cos \theta)$$

Using

$$\frac{df(x, y)}{du} = \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du}$$

we have:

$$p' = \frac{\partial f}{\partial x'} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} = p \cos \theta + q \sin \theta$$

and

$$q' = \frac{\partial f}{\partial y'} = -\sin \theta \frac{\partial f}{\partial x} + \cos \theta \frac{\partial f}{\partial y} = -p \sin \theta + q \cos \theta$$

This amounts to rotating the p-q axes by θ to determine the p'-q' axes. Thus, rotation of the image corresponds to identical rotation of the gradient space.

Perspective

In this section, *perspective projection* is assumed and its consequences in gradient space are described. While most of the results were presented by Kender [11], some new results are included in sections 12 and 13. This paper does not deal with scene transformations (such as rotation) and their effects in the image under perspective projection [14], nor with camera models [1, 2]. Instead, it describes the relationship between the scene, the image, and gradient space under perspective projection.

9. Vanishing Point of a Line

Suppose a line in the scene is defined by $(x, y, z) + a (\Delta x, \Delta y, \Delta z)$ for all values of a, where (x, y, z) is any point on the line and $(\Delta x, \Delta y, \Delta z)$ is a direction vector of the line (i.e. any vector contained in the line). For any a, the corresponding point on the line is $(x + a\Delta x, y + a\Delta y, z + a\Delta z)$ and its image point P_a is:

$$P_a = (\frac{-x - a\Delta x}{z + a\Delta z}, \frac{-y - a\Delta y}{z + a\Delta z})$$

As a grows larger, the image point P_a converges to some point V in the image (figure 8):

$$V = \lim_{a \to \infty} P_a = (\frac{-\Delta x}{\Delta z}, \frac{-\Delta y}{\Delta z})$$

The image point V is called the *vanishing point* of the line.

The assumption has been made here that $\Delta z \neq 0$, i.e. that the line is not parallel to the image plane.
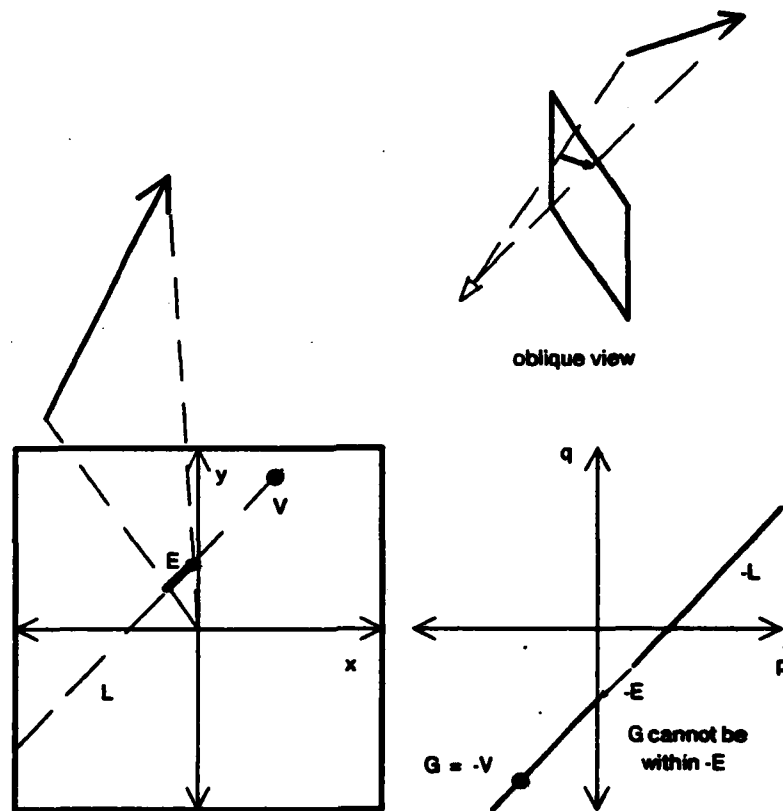
Corollaries of this definition are:

oblique view

Figure 8: Vanishing Point of a Line

1. If a line (or vector) has gradient G and vanishing point V, then $G = -V$.

2. Since V depends only on the direction vector (Δx, Δy, Δz), parallel lines have the same vanishing point. Thus, each point in the image is the vanishing point for a family of parallel lines.

3. If a line passes through the origin, then its vanishing point is the point at which it intersects the image plane ($z = -1$).

4. The vanishing point V of a vector must lie on the image line L containing the image of the vector (figure 8). The vector's gradient G must therefore lie on the line $-L$ in gradient space, where $-L$ is the line:

   - parallel to L

   - at an equal distance from the origin as L

   - on the opposite side of the origin from L.

Two additional observations concerning G are:

a. The vanishing point of a vector cannot be in the middle of its image, so if E is the image of the vector, then V cannot be within E and G cannot be within $-E$ (figure 8).

b. If the vector in the scene is parallel to the image plane, it has no vanishing point (i.e. V is infinitely far away on L); it has no gradient (i.e. G is infinitely far from the origin, in the direction parallel to L).

10. Vanishing Line of a Surface

Suppose a surface S has gradient G_S. For any vector L on S with gradient G_L, $G_S \cdot G_L = -1$, as shown in section 7. Since the vanishing point of L is $V_L = -G_L$ (by corollary 1 of section 9),

$$G_S \cdot V_L = 1$$

Suppose that G_S is given. Then the above equation defines a line V_S in the image, containing the vanishing points V_L for all vectors L contained on S (figure 9). In fact,
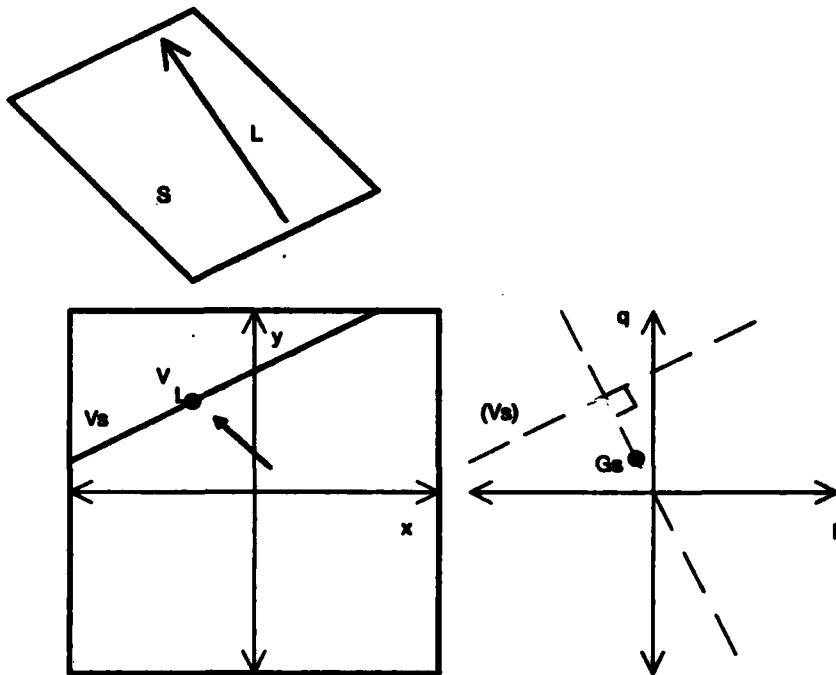

Figure 9: Vanishing Line and Gradient of a Surface

- V_S is perpendicular to the line from G_S to the origin

- the distance from V_S to the origin is the reciprocal of the distance from G_S to the origin

- V_S is on the same side of the origin as G_S.

The line V_S is called the *vanishing line* of the surface; it is the locus of vanishing points for all vectors on the surface.

Corollaries of this definition are:

1. Since V_S depends only on G_S, parallel surfaces have the same vanishing line. Thus, each line in the image is the vanishing line for a family of parallel surfaces.

2. If a surface passes through the origin, its vanishing line is the line along which it intersects the image plane ($z = -1$).

3. Suppose L is a line in the image. There exists a family of parallel surfaces for which L is the vanishing line. These surfaces all have the same gradient, which might be called the *vanishing gradient* for L, denoted G^v_L. Let L be the set of points (x,y) defined by the equation: $1 = ax + by = (a,b) \cdot (x,y)$. Then by section 10, since (x,y) is a spoint on L, (a,b) must be the gradient of the surfaces for which L is the vanishing line, i.e. $G^v_L = (a, b)$. Thus, for any line L in the image, we can determine the associated *vanishing gradient* G^v_L -- the gradient of the surfaces for which L is the vanishing line.

4. Suppose edge E is the image of some vector V with gradient G_V. If S is the surface through the origin and E, then E is the vanishing line of S (by corollary 2 above) and the gradient of S is G^v_E (by corollary 3 above). V must be contained on the surface S, so by corollary 2 of section 7,

$$G_V \cdot G^v_E = -1$$

This is the relationship between a vector and the vanishing gradient of its image.

11. Point Contained on a Surface

Suppose a surface S has gradient $G = (p, q)$ and intersects the z-axis at $z = D$ (i.e. the plane is defined by $px + qy + z - D = 0$). Let $P = (x, y)$ be a point in the image; it must correspond to some point X on surface S in the scene (figure 10).

Since the image of X is P, $X = (ax, ay, -a)$ for some value of a. Since X also lies on S,

$$p(ax) + q(ay) + (-a) - D = 0$$

Solving this equation for a yields $a = D / (px + qy - 1) = D / (P \cdot G - 1)$ and

$$X = \frac{D}{P \cdot G - 1} (x, y, -1)$$

X is sometimes called the *back-projection* of image point P onto surface S [11].

The quantity $D / (P \cdot G - 1)$ is the distance from X to the x-y plane ($z = 0$). If this value is negative, then X is not in the scene (i.e. it is behind the viewer), and P does not correspond to any point on the image of S.
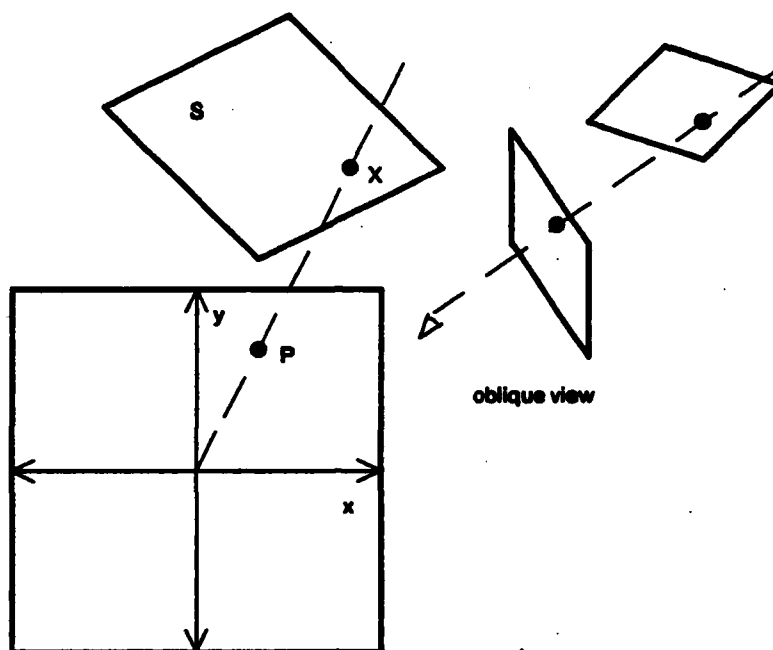
Figure 10: Back-Projection of a Point Onto a Surface

The assumption has been made here that $P \cdot G - 1 \neq 0$, i.e. that P does not lie on the vanishing line of S.

12. Vector Contained on a Surface

Suppose a vector V has gradient G_V and lies on a surface S with gradient G_S. Then by corollary 2 of section 7:

$$G_S \cdot G_V = -1$$

Now suppose that V is visible in the image as some edge E (figure 11). By corollary 4 of section 10, if G^V_E is the vanishing gradient of E,

$$G^V_E \cdot G_V = -1$$

These equations can be combined into a single matrix equation, in which the upper row is the first equation, and the lower row is the second equation:

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} G_S \cdot G_V \\ G^V_E \cdot G_V \end{bmatrix} = \begin{bmatrix} G_S \\ G^V_E \end{bmatrix} G_V^T$$

$$G_V^T = \begin{bmatrix} G_S \\ G^V_E \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Since G^V_E is determined by the edge E in the image, this equation relates the gradient G_V of a vector with the image E of the vector and the gradient G_S of a surface containing the
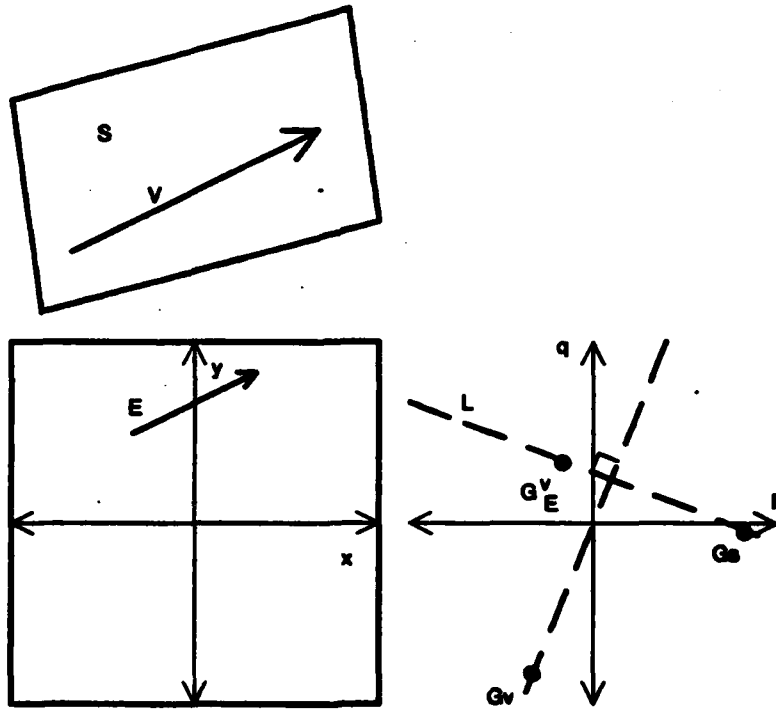
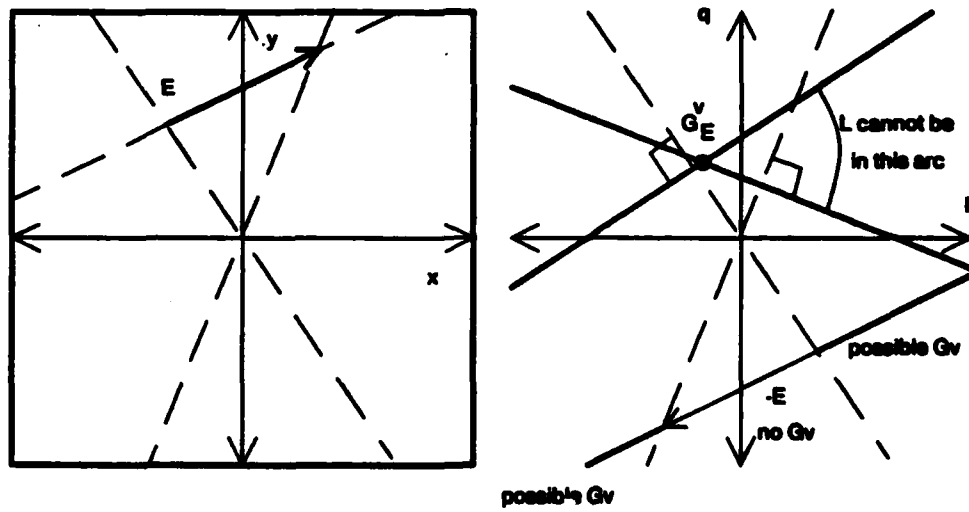Figure 11: Vector Contained on a Surface Under Perspective

Figure 12: Restrictions on the Slope of L

vector.

Corollary to this result is:

1. Combining the two equations in a different manner:

$$0 = (G_S \cdot G_V) - (G^V_E \cdot G_V) = (G_S - G^V_E) \cdot G_V$$

$$(G_S - G^V_E) \perp G_V$$

So, the line L in gradient space containing G_S and G^V_E is perpendicular to the line through the origin and G_V (figure 11).

2. There is an interesting restriction on line L in corollary 1. As shown in corollary 1, L must pass through G^V_E. Its slope depends on the gradient G_V of V. G_V is described in corollary 4 of section 9: it must lie on the *line* $-E$, but not within the *line segment* $-E$ corresponding to the edge itself. This constrains the orientation of L such that the line through G^V_E perpendicular to L cannot pass through the line segment $-E$ (figure 12). Hence, the position and length of an edge in the image constrain the gradients of surfaces containing the corresponding vector in the scene.

13. Vector Contained on Two Surfaces

Suppose a vector with gradient G_V is the boundary between two surfaces with gradients G_1 and G_2 (figure 13). If the vector appears as an edge E in the image, and G^V_E is the vanishing gradient of E, then by corollary 1 of result 12,
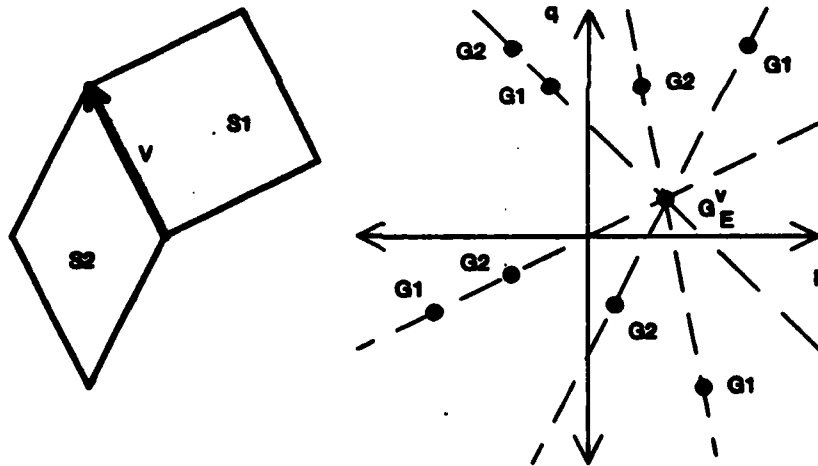

Figure 13: Vector Contained on Two Surfaces Under Perspective

$$(G_1 - G^V_E) \perp G_V$$

$$(G_2 - G^V_E) \perp G_V$$

So, $(G_1 - G^V_E) \parallel (G_2 - G^V_E)$, i.e. G^V_E, G_1, and G_2 are collinear in gradient space.

Since G^V_E was determined by the location of the edge E in the image, the constraint provided on G_1 and G_2 is that they lie on a line which passes through G^V_E. This line is the same as L in corollaries 1 and 2 of section 12, and its orientation is limited to certain

angles depending on the location and size of the image edge E.

This is the *connect edge* relation under perspective; it is the perspective counterpart to corollary 1 of section 5.

Curved Surfaces and Arcs

In these sections, the gradient is defined for an arc in the scene. Then, using calculus, the fundamental results are developed concerning curved arcs and surfaces. The results are very similar to those of sections 4, 5, and 6; this is because surface and arc gradients capture the same (first-order differential) information contained in tangent lines and planes, which obey the results of sections 4, 5, and 6.

14. Gradient of an Arc

Suppose an arc A is defined (in parametric form) by $(x(s), y(s), z(s))$. Then its gradient can be defined as:

$$G_A = (p,q) = (\frac{dx}{dz}, \frac{dy}{dz}) = (\frac{dx}{ds}\frac{ds}{dz}, \frac{dy}{ds}\frac{ds}{dz})$$

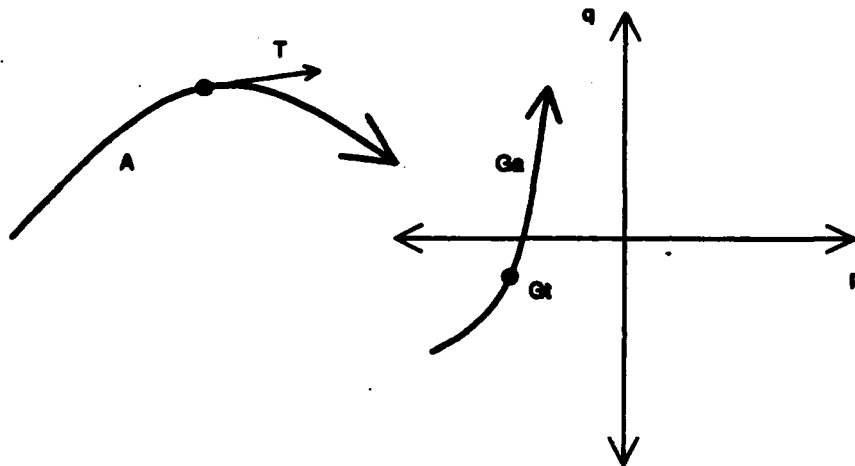Note that both p and q are themselves functions of s.

Figure 14: Gradient of an Arc and a Tangent Vector

Corollaries:

1. At any point on an arc defined as above, the tangent vectors are defined by:

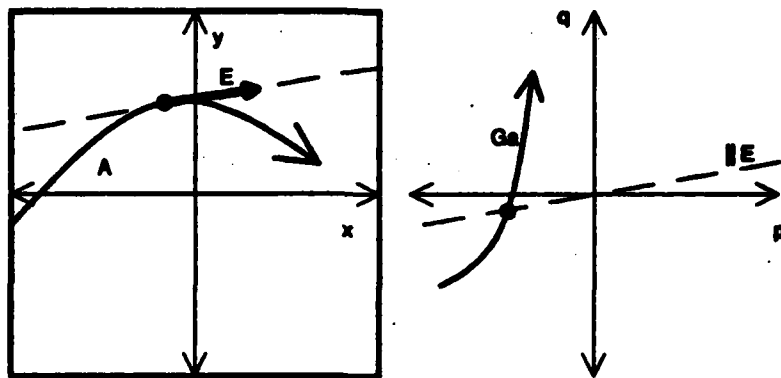$$T = a\,(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) \quad \text{for all } a$$

Figure 15: Gradient of an Arc Under Orthography

2. Thus, the gradient G_A of an arc A is the same as the gradient G_T of a tangent vector T to the arc (figure 14).

3. If the gradient of an arc is (p, q), then the vector $(p, q, 1)$ is tangent to the arc.

4. Under orthography, the tangent vector to an arc is visible as the edge $E = a$ $(dx/ds, dy/ds)$ for some value of a (figure 15). If the gradient of the arc is $G = (dx/dz, dy/dz)$, then $G = (E/a)(ds/dz)$. In other words, the line in gradient space from the origin to G is parallel to the tangent vector E seen in the image.

15. Arc Contained on a Surface

Suppose an arc $A = (x_A(s), y_A(s), z_A(s))$ is contained on a surface S defined by $-z = f(x, y)$. Let the arc gradient be $G_A = (p_A, q_A) = (dx_A/dz_A, dy_A/dz_A)$, and let the surface gradient be $G_S = (p_S, q_S) = (\partial f/\partial x, \partial f/\partial y)$. Then for all s,

$$-z_A(s) = f(x_A(s), y_A(s))$$

We can differentiate using the rule:

$$\frac{df(x,y)}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

to yield:

$$-\frac{dz_A}{ds} = \frac{\partial f}{\partial x}\frac{dx_A}{ds} + \frac{\partial f}{\partial y}\frac{dy_A}{ds} = p_S\frac{dx_A}{ds} + q_S\frac{dy_A}{ds} = G_S \cdot (\frac{dx_A}{ds}, \frac{dy_A}{ds})$$

Corollaries to this result are:

1. With the above definitions, the above equation can be multiplied by ds/dz_A to yield:

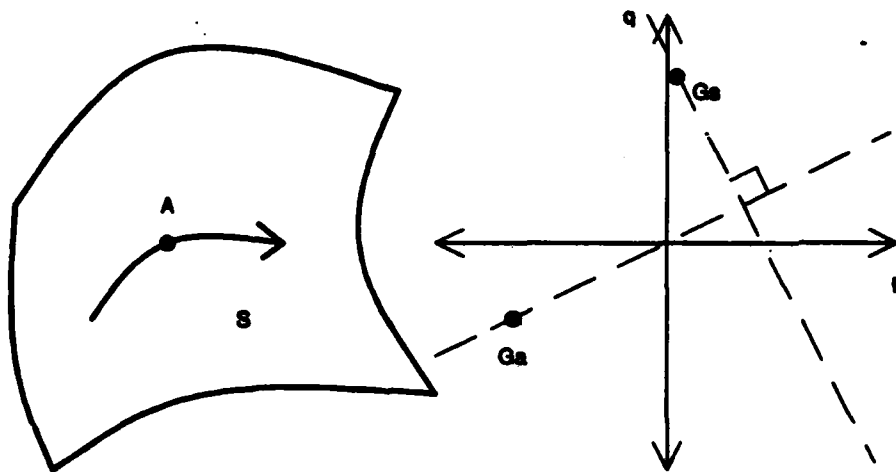$$-1 = p_S\frac{dx_A}{ds}\frac{ds}{dz_A} + q_S\frac{dy_A}{ds}\frac{ds}{dz_A} = p_S p_A + q_S q_A$$

Figure 16: Arc Contained on a Curved Surface

$$= G_S \cdot G_A$$

So, G_A and G_S obey the relationship of section 7 (figure 16).

2. If (Δx, Δy, Δz) is tangent to arc A at point X, and A is contained on a surface whose gradient at X is G, then $-\Delta z = G \cdot (\Delta x, \Delta y)$. Thus, the results of section 4 apply to the tangent vector to an arc.

16. Arc Contained on Two Curved Surfaces

Suppose an arc $A = (x(s), y(s), z(s))$ with gradient G_A is the boundary between two surfaces S_1 and S_2 with gradients G_1 and G_2 (figure 17). At any point on A, we have (by corollary 1 of section 15):
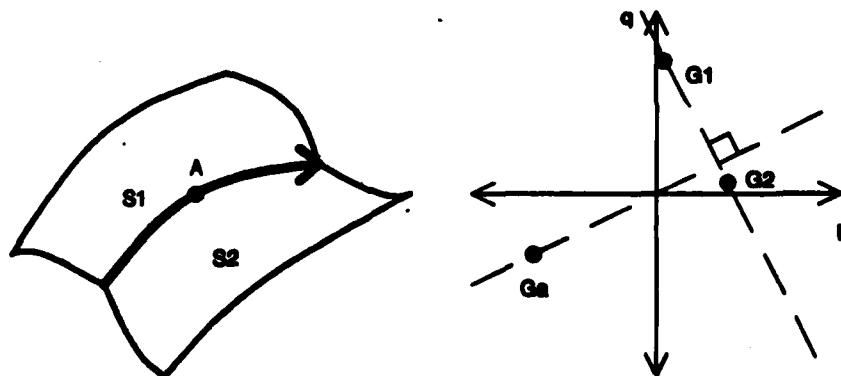

Figure 17: Arc Contained on Two Curved Surfaces

$$-1 = G_A \cdot G_1 = G_A \cdot G_2$$

$$0 = G_A \cdot (G_1 - G_2)$$

$$G_A \perp (G_1 - G_2)$$

So the line containing G_1 and G_2 in gradient space is perpendicular to the line from the origin to G_A.

Corollary to this is:

1. Under orthography, we can combine the above result with corollary 3 of section 14 to conclude that the line containing G_1 and G_2 in gradient space is perpendicular to the tangent to the arc in the image. This is the counterpart to the *connect edge* relationship for curved surfaces and arcs.

17. Two Arcs Contained on a Curved Surface

Suppose arcs A_1 and A_2 with gradients G_1 and G_2 are contained on a surface with gradient G_S (figure 18).
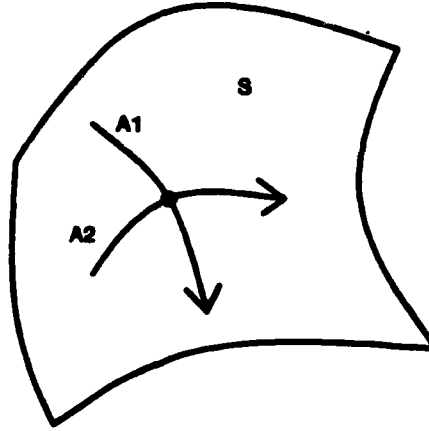
Figure 18: Two Arcs Contained on a Curved Surface

At a point of intersection of A_1 and A_2, we have (by corollary 1 to section 15):

$$-1 = G_1 \cdot G_S$$
$$-1 = G_2 \cdot G_S$$

These can be combined into a single matrix equation to yield:

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} G_S^T$$

$$G_S^T = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

This allows us to compute the gradient of a surface from the gradients of two intersecting arcs on the surface.

Corollary to this is:

1. Under orthography, suppose that edges E_1 and E_2 in the image are tangent to the images of arcs A_1 and A_2 at a point where they intersect, and that E_1 and E_2 correspond to scene vectors $(\Delta x_1, \Delta y_1, \Delta z_1)$ and $(\Delta x_2, \Delta y_2, \Delta z_2)$. If A_1 and A_2 are contained on a surface with gradient G_S, then:

$$-\Delta z_1 = E_1 \cdot G_S$$

$$-\Delta z_2 = E_2 \cdot G_S$$

$$G_S^T = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^{-1} \begin{bmatrix} -\Delta z_1 \\ -\Delta z_2 \end{bmatrix}$$

Thus, under orthography, the gradient of a surface can be computed from the Δz values for two vectors tangent to arcs on the surface, at a point where the arcs intersect.

Summary

In this paper, we have defined the gradient for surfaces, planes, arcs, and vectors, and we have seen that the gradient and Δz attribute of an object are mutually constrained. We have also seen that knowledge about the gradient (or Δz-component) of a surface can be used to determine the gradient of a vector or arc on that surface, and that knowledge about the gradients of two such vectors or arcs can be used to uniquely determine the surface gradient. In addition, features in the image can be combined with gradient or Δz information to yield three-dimensional reconstructions of scene objects, under both perspective and orthographic projections.

This collection of theorems and definitions includes a recounting of results from Huffman [5], Mackworth [12], and Kender [11] as well as some new notation and results. The definition and use of *arc* and *vector* gradients as well as *surface* gradients has provided a more concise notation for several of these results.

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