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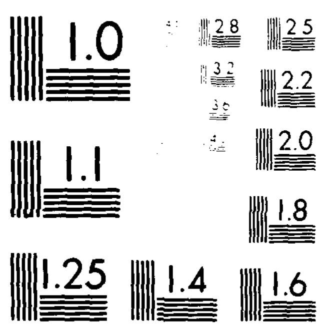

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ON THE THEORY OF MULTIVARIATE ELLIPTICALLY CONTOURED DISTRIBUTIONS AND THEIR APPLICATIONS

BY

T. W. ANDERSON and KAI-TAI FANG

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OFFICE OF NAVAL RESEARCH

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1. Introduction.

If the characteristic function of an n-dimensional random vector $\underset{\sim}{x}$ has the form $\exp(i\underset{\sim}{t}'\underset{\sim}{\mu})\phi(\underset{\sim}{t}'\underset{\sim}{\Sigma}\underset{\sim}{t})$, where $\underset{\sim}{\mu}$: $n \times 1$, $\underset{\sim}{\Sigma}$: $n \times n$, and $\underset{\sim}{\Sigma} \geq 0$, we say that $\underset{\sim}{x}$ is distributed according to an elliptically contoured distribution with parameters $\underset{\sim}{\mu}$, $\underset{\sim}{\Sigma}$, and ϕ, and we write $\underset{\sim}{x} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma},\phi)$.

The class of elliptically contoured distributions has been studied by several authors: Schoenberg (1938), Kelker (1970), Devlin, Gnanadesikan and Keltenring (1976), Kariya and Eaton (1977), Muirhead (1980), Cambanis, Huang and Simons (1981), and Anderson and Fang (1982).

Statisticians have been trying to extend the sample theory in multivariate analysis to the case of samples being dependent or the case of samples being from nonnormal populations. In this paper we consider sampling theory in which the distribution of the population belongs to the class of elliptically contoured distributions and the samples are dependent. According to this requirement multivariate elliptically contoured distributions are defined and some basic properties are discussed in Section 1 and Section 2. The distributions of some important statistics in the sampling theory (such as the correlation coefficients, the multiple correlation coefficients, Hotelling-T^2, the sample covariance matrix, the generalized variance, the quadratic forms, etc.) are obtained in Sections 3 to Section 6. As applications of the theory on multivariate elliptically contoured distributions we consider the model of multiple regression with the error matrix being distributed according to a multivariate elliptically contoured distribution.

Kariya and Eaton (1977), and Muirhead (1980) discussed the effects of elliptical distributions on some standard procedures involving correlation coefficients, but the model that we consider in this paper is different from theirs.

Throughout the paper, $N_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$ denotes the n-dimensional normal distribution with mean $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$; χ_k^2 denotes the chi-squared variable with k degrees of freedom; $B(\alpha_1,\alpha_2)$ denotes the Beta distribution with parameters α_1 and α_2; $D_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m)$ denotes the Dirichlet distribution with parameters $\alpha_1,\ldots,\alpha_m$; $F(k,\ell)$ denotes F-distribution with k and ℓ degrees of freedom, t_n denotes the t-distribution with n degrees of freedom; $W_p(\underset{\sim}{\Sigma},n)$ denotes the Wishart distribution with covariance matrix $\underset{\sim}{\Sigma}$: $p\times p$ and n degrees of freedom; $U_{p,m,n}$ denotes Wilks' statistic which is the ratio $|G|/|\underset{\sim}{G}+\underset{\sim}{H}|$, where $\underset{\sim}{G} \sim W_p(\underset{\sim}{\Sigma},n)$, $\underset{\sim}{H} \sim W_p(\underset{\sim}{\Sigma},m)$, and $\underset{\sim}{H}$ and $\underset{\sim}{G}$ are independent; $\underset{\sim}{I}_n$ denotes the $n\times n$ identity matrix; $\underset{\sim}{\varepsilon}_n$ denotes the $n\times 1$ vector with elements 1; $rk(\underset{\sim}{A})$ denotes the rank of the matrix $\underset{\sim}{A}$ and $\underset{\sim}{A}^-$ denotes a generalized inverse of $\underset{\sim}{A}$.

2. Definitions and Basic Properties.

Let $\underset{\sim}{X}$, $\underset{\sim}{M}$ and $\underset{\sim}{T}$ be $n\times p$ matrices. We express them in terms of elements, columns, and rows as

$$X = (x_{ij}) = (x_1, x_2, \ldots, x_p) = \begin{bmatrix} x'_{(1)} \\ \vdots \\ x'_{(n)} \end{bmatrix}, \quad x = \text{vec } X',$$

$$M = (\mu_{ij}) = (\mu_1, \mu_2, \ldots, \mu_p) = \begin{bmatrix} \mu'_{(1)} \\ \vdots \\ \mu'_{(n)} \end{bmatrix}, \quad \mu = \text{vec } M',$$

$$T = (t_{ij}) = (t_1, t_2, \ldots, t_p) = \begin{bmatrix} t'_{(1)} \\ \vdots \\ t'_{(n)} \end{bmatrix}, \quad t = \text{vec } T'.$$

Here $x = \text{vec } X' = (x'_{(1)}, x'_{(2)}, \ldots, x'_{(n)})'$ and with the same meaning for μ and t.

Definition 2.1. If the characteristic function of a random matrix X has the form

$$\text{(2.1)} \qquad \exp(i \sum_{j=1}^{n} t'_{(j)} \mu_{(j)}) \phi(t'_{(1)} \Sigma_1 t_{(1)}, \ldots, t'_{(n)} \Sigma_n t_{(n)}),$$

with $\Sigma_1, \ldots, \Sigma_n \geq 0$, we say that X is distributed according to a multivariate (rows) elliptically contoured distribution and write $X \sim MEC_{n \times p}\ (M; \Sigma_1, \ldots, \Sigma_n; \phi)$.

Obviously, when $n = 1$ the multivariate elliptically contoured distribution reduces to the common elliptically contoured distribution.

Let $u^{(q)}$ denote a random vector which is uniformly distributed on the unit sphere in $\mathbb{R}^q$ and $\Omega_q(\|t\|^2)$ denote its characteristic function.

Let $\Phi_{m_1,\ldots,m_n}$ be the class of all functions $\phi:[0,\infty)\times[0,\infty)\times\cdots\times[0,\infty)$ $=[0,\infty)^n \to \mathbb{R}$ such that $\phi(\|\underset{\sim}{t}_1\|^2,\ldots,\|\underset{\sim}{t}_n\|^2)$ is a characteristic function, where $\underset{\sim}{t}_1,\ldots,\underset{\sim}{t}_n$ are $m_1\times 1,\ldots,m_n\times 1$ vectors, respectively.

By a method similar to Schoenberg (1938), it can be shown that $\phi \in \Phi_{m_1,\ldots,m_n}$ if and only if

$$(2.2)\qquad \phi(u_1,\ldots,u_n) = \int_0^\infty \cdots \int_0^\infty \Omega_{m_1}(r_1^2u_1),\ldots,\Omega_{m_n}(r_n^2u_n)dF(r_1,\ldots,r_n)$$

for some distribution function $F(x_1,\ldots,x_n)$ on $[0,\infty)^n$. When $n=1$ (2.2) reduces to

$$(2.3)\qquad \phi(u) = \int_0^\infty \Omega_m(ur^2)dF(r)\ .$$

Schoenberg (1938) pointed out that $\Phi_m \supset \Phi_n$ if $m<n$. If the distribution function F of R is related to $\phi \in \Phi_n$ as in (2.3) with n substituted for m, then also $\phi \in \Phi_m$, $m<n$, and there exists a distribution function $F^*(x)$ of R^* being related to ϕ as in (2.3) with F^* substituted for F. Cambanis, Huang and Simons pointed out that $R^* \overset{d}{=} R\cdot b$, where $b \geq 0$, $b^2 \sim B(m/2,(n-m)/2)$ and b is independent of R. For convenience we denote these relationships by $R \leftrightarrow \phi \in \Phi_n$ and $R^* \overset{d}{=} R\cdot b_{m/2,(n-m)/2} \leftrightarrow \phi \in \Phi_m$

<u>Lemma 1.</u> $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\ldots,\underset{\sim}{\Sigma}_n;\phi)$ with $rk(\underset{\sim}{\Sigma}_j)=k_j$, $j=1,\ldots,n$, if and only if

$$(2.4)\qquad \underset{\sim}{X} \overset{d}{=} \underset{\sim}{M} + \left(R_1\underset{\sim}{A}_1'\underset{\sim}{u}_1^{(k_1)},\ldots,R_n\underset{\sim}{A}_n'\underset{\sim}{u}_n^{(k_n)}\right)',$$

where $R_1,\ldots,R_n$ are independent of $\underset{\sim}{u}_j^{(k_j)}$ $(j=1,\ldots,n)$, $\underset{\sim}{u}_1^{(k_1)},\ldots,\underset{\sim}{u}_n^{(k_n)}$ are independent, $\underset{\sim}{\Sigma}_j = \underset{\sim}{A}_j'\underset{\sim}{A}_j$ is a factorization of $\underset{\sim}{\Sigma}_j$, $j=1,\ldots,n$, and the joint distribution function $F(x_1,\ldots,x_n)$ of $(R_1,\ldots,R_n)$ is related to ϕ as

$$(2.5)\qquad \phi(u_1,\ldots,u_n) = \int_{[0,\infty)}\cdots\int_{[0,\infty)} \Omega_{k_1}(r_1^2u_1),\ldots,\Omega_{k_n}(r_n^2u_n)dF(r_1,\ldots,r_n),$$

and "$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$" denotes that the distribution of $\underset{\sim}{X}$ is the same as that of $\underset{\sim}{Y}$.

The proof is similar to one of Theorem 1 (Cambanis, Huang and Simons (1981)). The properties of the operation "$\overset{d}{=}$" are discussed by Anderson and Fang (1982). The following two properties are important in this paper:

(i) Assume that $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$ and $f_j(\cdot)$, $j=1,\ldots,m$, are Borel functions, then

$$\begin{pmatrix} f_1(\underset{\sim}{X}) \\ \vdots \\ f_m(\underset{\sim}{X}) \end{pmatrix} = \begin{pmatrix} f_1(\underset{\sim}{Y}) \\ \vdots \\ f_m(\underset{\sim}{Y}) \end{pmatrix} .$$

(ii) Assume that $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are $n\times p$ random matrices, z is a random variable and is independent of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$, respectively. If

$$(2.6)\qquad p(z > 0) = 1$$

then $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$ if and only if $z\underset{\sim}{X} \overset{d}{=} z\underset{\sim}{Y}$.

By using the first property and (2.4) we have

$$(2.7)\qquad (R_1^2,\ldots,R_n^2) \overset{d}{=} \left((\underset{\sim}{x}_{(1)}-\underset{\sim}{\mu}_{(1)})'\,\underset{\sim}{\Sigma}_1^-\,(\underset{\sim}{x}_{(1)}-\underset{\sim}{\mu}_{(1)}),\ldots,(\underset{\sim}{x}_{(n)}-\underset{\sim}{\mu}_{(n)})'\underset{\sim}{\Sigma}_n^-(\underset{\sim}{x}_{(n)}-\underset{\sim}{\mu}_{(n)})\right),$$

where $\underset{\sim}{A}^{-}$ is a generalized inverse of $\underset{\sim}{A}$. In particular, if $\underset{\sim}{\Sigma}_j = \underset{\sim}{I}_p$, $j = 1,2,\ldots,n$, then (2.7) becomes

$$(2.8) \qquad (R_1^2,\ldots,R_n^2) \overset{d}{=} (\|\underset{\sim}{x}_{(1)}\|^2,\ldots,\|\underset{\sim}{x}_{(n)}\|^2) ,$$

where $\|\underset{\sim}{x}_{(i)}\|^2 = \underset{\sim}{x}'_{(i)}\underset{\sim}{x}_{(i)}$, $i = 1,\ldots,n$, or

$$(2.8)' \qquad (R_1,\ldots,R_n) \overset{d}{=} (\|\underset{\sim}{x}_{(1)}\|,\ldots,\|\underset{\sim}{x}_{(n)}\|) .$$

In this case, if $p(\underset{\sim}{X}=\underset{\sim}{0}) = 0$, we have

$$(2.9) \qquad \left(\frac{\underset{\sim}{x}_{(1)}}{\|\underset{\sim}{x}_{(1)}\|},\ldots,\frac{\underset{\sim}{x}_{(n)}}{\|\underset{\sim}{x}_{(n)}\|}\right) \overset{d}{=} (\underset{\sim}{u}_1^{(p)},\ldots,\underset{\sim}{u}_n^{(p)}) .$$

When $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\ldots,\underset{\sim}{\Sigma}_n;\phi)$ with $\phi(u_1,\ldots,u_n) = \exp[-(u_1+\cdots+u_n)/2]$, the corresponding population is normal, i.e. $\underset{\sim}{x}_{(j)} \sim N_p(\underset{\sim}{\mu}_{(j)},\underset{\sim}{\Sigma}_j)$, $j=1,\ldots,n$, and $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)}$ are independent. Now $R_1^2,\ldots,R_n^2$ are independently distributed according to $\chi^2_{k_1},\ldots,\chi^2_{k_n}$, respectively. If the joint distribution of $R_1^2,\ldots,R_n^2$ is a multivariate chi-square distribution (or generalized Rayleigh distribution)(cf. Johnson and Kotz (1972), p. 220), then $\underset{\sim}{x}_{(j)} \sim N_p(\underset{\sim}{\mu}_{(j)},\underset{\sim}{\Sigma}_j)$, $j=1,\ldots,n$, but $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)}$ are dependent. By this method, we could generalize the theory of the distribution and estimation in the multiple normal population to the dependent case.

Suppose $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\ldots,\underset{\sim}{\Sigma}_n;\phi)$. From Lemma 1 $\underset{\sim}{X}_{(j)} \overset{d}{=} \underset{\sim}{\mu}_{(j)}+R_j\underset{\sim}{A}'_j\underset{\sim}{u}_j^{(k_j)}$, i.e. $\underset{\sim}{x}_{(j)} \sim EC_p(\underset{\sim}{\mu}_{(j)},\underset{\sim}{\Sigma}_j, \phi^*)$, where $\phi^* \in \Phi_{k_j} \leftrightarrow R_j$. It is easy to see that $\phi^*(u) = \phi(0,\ldots,0,u,0,\ldots,0)$ where u is in the j-th position. What is the marginal distribution of $\underset{\sim}{x}_j$? From (2.4) we have

$$(2.10) \qquad \underset{\sim}{x}_j \overset{d}{=} \underset{\sim}{\mu}_j + \begin{pmatrix} R_1 \underset{\sim}{a}_j^{(1)\prime} \underset{\sim}{u}_1^{(k_1)} \\ \vdots \\ R_n \underset{\sim}{a}_j^{(n)\prime} \underset{\sim}{u}_n^{(k_n)} \end{pmatrix}$$

where $\underset{\sim}{a}_j^{(\ell)}$ is the j-th column of $\underset{\sim}{A}_\ell$, $j = 1,\ldots,p$; $\ell = 1,\ldots,n$. Let $\underset{\sim}{\Sigma}_\ell = (\sigma_{ij}^{(\ell)})$ and $\underset{\sim}{y}_\ell = (\underset{\sim}{y}_{\ell 1},\ldots,\underset{\sim}{y}_{\ell k_\ell})' \sim N_{k_\ell}(\underset{\sim}{0}, \underset{\sim}{I}_{k_\ell})$, then

$$\underset{\sim}{a}_j^{(\ell)\prime} \underset{\sim}{u}_\ell^{(k_\ell)} \overset{d}{=} \underset{\sim}{a}_j^{(\ell)\prime} \underset{\sim}{y}_\ell / \| \underset{\sim}{y}_\ell \| \overset{d}{=} \sqrt{\sigma_{jj}^{(\ell)}}\, y_{\ell j} / \| \underset{\sim}{y}_\ell \|$$

as $\underset{\sim}{a}_j^{(\ell)\prime} \underset{\sim}{a}_j^{(\ell)} = \sigma_{jj}^{(\ell)}$, and $y_{\ell j}^2 / \| \underset{\sim}{y}_\ell \|^2 \sim B\left(\frac{1}{2}, \frac{k_\ell - 1}{2}\right)$, (cf. Anderson and Fang (1982)). Hence

$$(2.11) \qquad \underset{\sim}{x}_j = \underset{\sim}{\mu}_j + \begin{pmatrix} R_1 \sqrt{\sigma_{jj}^{(1)}}\, z_1 \\ \vdots \\ R_n \sqrt{\sigma_{jj}^{(n)}}\, z_n \end{pmatrix},$$

where $z_1,\ldots,z_n$ are independent, $z_j \geq 0$, $z_j^2 \sim B(\frac{1}{2}, \frac{k_j - 1}{2})$, $R_1,\ldots,R_n$ are independent of $z_1,\ldots,z_n$ and $(R_1,\ldots,R_n) \sim F(x_1,\ldots,x_n)$.

We believe some further results similar to the elliptically contoured distribution could be obtained by the method used by Cambanis, Huang and Simons (1981). In this paper we mainly pay attention to a specific subclass of multivariate elliptically contoured distributions which will be defined in the next section.

3. The Case in Which the Characteristic Function is Composed of Addition of Arguments.

From now on we assume

$$\phi(u_1,\dots,u_n) = \phi(u_1+\cdots+u_n), \tag{3.1}$$

and we still denote the multivariate elliptically contoured distribution by $MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\dots,\underset{\sim}{\Sigma}_n;\phi)$.

<u>Theorem 1</u>. $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\dots,\underset{\sim}{\Sigma}_n;\phi)$ if and only if $\underset{\sim}{x} \sim EC_{np}(\underset{\sim}{\mu},\underset{\sim}{V},\phi)$ with

$$\underset{\sim}{V} = \begin{bmatrix} \underset{\sim}{\Sigma}_1 & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Sigma}_2 & \cdots & \underset{\sim}{0} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{\Sigma}_n \end{bmatrix}. \tag{3.2}$$

<u>Proof</u>. If $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\dots,\underset{\sim}{\Sigma}_n;\phi)$, the characteristic function of $\underset{\sim}{X}$ is (cf. (2.1) and (3.1).)

$$\exp\left(i\sum_{j=1}^{n}\underset{\sim}{t}'_{(j)}\underset{\sim}{\mu}_{(j)}\right)\phi\left(\sum_{j=1}^{n}\underset{\sim}{t}'_{(j)}\underset{\sim}{\Sigma}_j\underset{\sim}{t}_{(j)}\right) = \exp(i\underset{\sim}{t}'\underset{\sim}{\mu})\phi(\underset{\sim}{t}'\underset{\sim}{V}\underset{\sim}{t}),$$

i.e. $\underset{\sim}{x} \sim EC_{np}(\underset{\sim}{\mu},\underset{\sim}{V},\phi)$. The "if" part is obvious and the theorem follows.

Theorem 1 shows us that for any $\phi \in \Phi_k$ we can construct a $\phi(u_1,\dots,u_n)$ by (3.1) such that $\phi \in \Phi_{k_1,\dots,k_n}$ with $k_1+\cdots+k_n = k$.

<u>Corollary</u>. $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma},\ldots,\underset{\sim}{\Sigma};\phi)$ if and only if $\underset{\sim}{x} \sim EC_{np}(\underset{\sim}{\mu},\underset{\sim}{V},\phi)$ with $\underset{\sim}{V} = \underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma}$.

<u>Definition 3.1</u>. If the random vector $(z_1,\ldots,z_m)'$ satisfies

$$(3.3) \qquad (z_1,\ldots,z_m) \overset{d}{=} R^2(d_1,\ldots,d_m) ,$$

where $(d_1,\ldots,d_{m-1}) \sim D_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m)$, $d_m = 1 - \Sigma_{i=1}^{m-1} d_i$, $R \geq 0$, $R \sim F(x)$ and R is independent of $d_1,\ldots,d_{m-1}$, then we write $(z_1,\ldots,z_m) \sim G_m(\alpha_1,\ldots,\alpha_{m-1},\alpha_m;\phi)$ and $(z_1,\ldots,z_{m-1}) \sim G_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m;\phi)$, where $\phi \in \Phi_n \leftrightarrow$ the distribution function $F(x)$ of R and $n = 2\,\Sigma_1^m \alpha_i$.

Anderson and Fang (1982) point out that if $P(R=0) = 0$ the density of $z_1,\ldots,z_{m-1}$ is

$$(3.4) \qquad \frac{\Gamma(\frac{n}{2})}{\prod_1^m \Gamma(\alpha_i)} \prod_{i=1}^{m-1} z_i^{\alpha_i - 1} \int_{\sqrt{\sum_1^{m-1} z_i}}^{\infty} r^{-(n-2)}\Big(r^2 - \sum_1^{m-1} z_i\Big)^{\alpha_m - 1} dF(r) ,$$

$$\text{for } z_1,\ldots,z_{m-1} \geq 0 .$$

Further if $\underset{\sim}{x} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma},\phi)$ has a density, which must have the form

$$(3.5) \qquad |\underset{\sim}{\Sigma}|^{-\frac{1}{2}} g((\underset{\sim}{x}-\underset{\sim}{\mu})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}))$$

for a suitable function $g(\cdot)$, then the density of R related to ϕ is

$$(3.6) \qquad f(r) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} r^{n-1} g(r^2) .$$

In this case the density of $G_m(\alpha_1,\ldots,\alpha_{m-1},\alpha_m;\phi)$ has the simpler form

$$\text{(3.7)} \qquad \frac{\pi^{n/2}}{\prod_1^m \Gamma(\alpha_i)} \prod_{i=1}^m z_i^{\alpha_i-1} \; g(\sum_1^m z_i) \; .$$

From Theorem 1, if $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\ldots,\underset{\sim}{\Sigma}_n;\phi)$ and $rk\underset{\sim}{\Sigma}_j = k_j$, $j = 1,\ldots,n$, then $\underset{\sim}{x} \sim EC_{np}(\underset{\sim}{\mu},\underset{\sim}{V},\phi)$ where $\underset{\sim}{V}$ is defined by (3.2). Hence, $\underset{\sim}{x}$ has the stochastic representation (cf. Cambanis, Huang and Simons (1981))

$$\text{(3.8)} \qquad \underset{\sim}{x} \overset{d}{=} \underset{\sim}{\mu} + R\underset{\sim}{B}'\underset{\sim}{u}^{(k)} \;, \quad k = \sum_{j=1}^n k_j \;,$$

where $\underset{\sim}{B}'\underset{\sim}{B} = \underset{\sim}{V}$ and $\underset{\sim}{B}'$ is an $np \times k$ matrix. On the other hand, there is the stochastic representation (2.4) for $\underset{\sim}{X}$; what is the relationship between R and $R_1,\ldots,R_n$, and between $\underset{\sim}{B}$ and $\underset{\sim}{A}_i$ $(i=1,\ldots,n)$?

<u>Theorem 2</u>. (i) $(R_1^2,\ldots,R_n^2) \sim G_n\left(\frac{k_1}{2},\ldots,\frac{k_{n-1}}{2}, \frac{k_n}{2};\phi\right)$, (ii)

$$\text{(3.9)} \qquad \underset{\sim}{B} = \begin{pmatrix} \underset{\sim}{A}_1 & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}_2 & \cdots & \underset{\sim}{0} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{A}_n \end{pmatrix} .$$

<u>Proof</u>. Comparing (2.4) and (3.8), the formula (3.9) follows and we have

$$R\underset{\sim}{u}^{(k)} \overset{d}{=} \begin{pmatrix} R_1 u_1^{(k_1)} \\ \vdots \\ R_n u_n^{(k_n)} \end{pmatrix} .$$

Let $\underset{\sim}{y}_{k\times 1} \sim N_k(\underset{\sim}{0},\underset{\sim}{I}_k)$ and $\underset{\sim}{y} = (\underset{\sim}{y}^{(1)'},\ldots,\underset{\sim}{y}^{(n)'})'$, where $\underset{\sim}{y}^{(1)},\ldots,\underset{\sim}{y}^{(n)}$ have $k_1,\ldots,k_n$ elements of $\underset{\sim}{y}$, respectively. Therefore

$$R\underset{\sim}{u}^{(k)} \overset{d}{=} R\begin{pmatrix} \frac{\underset{\sim}{y}^{(1)}}{\|\underset{\sim}{y}\|} \\ \vdots \\ \frac{\underset{\sim}{y}^{(n)}}{\|\underset{\sim}{y}\|} \end{pmatrix} = R\begin{pmatrix} \frac{\underset{\sim}{y}^{(1)}}{\|\underset{\sim}{y}^{(1)}\|} \frac{\|\underset{\sim}{y}^{(1)}\|}{\|\underset{\sim}{y}\|} \\ \vdots \\ \frac{\underset{\sim}{y}^{(n)}}{\|\underset{\sim}{y}^{(n)}\|} \frac{\|\underset{\sim}{y}^{(n)}\|}{\|\underset{\sim}{y}\|} \end{pmatrix} \overset{d}{=} \begin{pmatrix} R\frac{\|\underset{\sim}{y}^{(1)}\|}{\|\underset{\sim}{y}\|}\underset{\sim}{u}_1^{(k_1)} \\ \vdots \\ R\frac{\|\underset{\sim}{y}^{(n)}\|}{\|\underset{\sim}{y}\|}\underset{\sim}{u}_n^{(k_n)} \end{pmatrix},$$

because $\underset{\sim}{u}_j^{(k_j)} \overset{d}{=} \underset{\sim}{y}^{(j)}/\|\underset{\sim}{y}^{(j)}\|$, $j = 1,\ldots,n$, and $\underset{\sim}{y}^{(j)}/\|\underset{\sim}{y}^{(j)}\|$ is independent of $\|\underset{\sim}{y}^{(j)}\|$, and therefore $\underset{\sim}{y}^{(j)}/\|\underset{\sim}{y}^{(j)}\|$ is independent of $\|\underset{\sim}{y}^{(j)}\|/\|\underset{\sim}{y}\|$. Now

$$(R_1^2,\ldots,R_n^2) \overset{d}{=} R^2\left(\frac{\|\underset{\sim}{y}^{(1)}\|^2}{\|\underset{\sim}{y}\|^2},\ldots,\frac{\|\underset{\sim}{y}^{(n)}\|^2}{\|\underset{\sim}{y}\|^2}\right),$$

the theorem follows. Q.E.D.

<u>Corollary 1</u>. If $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2 = \cdots = \underset{\sim}{\Sigma}_n = \underset{\sim}{\Sigma} = \underset{\sim}{A}'\underset{\sim}{A}$, $rk(\underset{\sim}{\Sigma}) = \ell$, then $(R_1^2,\ldots,R_n^2) \sim G_n(\ell/2,\ldots,\ell/2,\ell/2;\phi)$ and

$$\underset{\sim}{B} = \begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A} & \cdots & \underset{\sim}{0} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{A} \end{pmatrix}.$$

<u>Definition 3.2</u>. If $\underset{\sim}{X} \sim MEC_{n\times p}(\underset{\sim}{M};\underset{\sim}{\Sigma}_1,\ldots,\underset{\sim}{\Sigma}_n;\phi)$, $\underset{\sim}{\Sigma}_1 = \cdots = \underset{\sim}{\Sigma}_n = \underset{\sim}{\Sigma}$ and $\underset{\sim}{M} = \underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}}'$, we write $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$.

Corollary 2. Assume $\underset{\sim}{X} \sim LEC_{n\times p}(\tilde{\mu},\underset{\sim}{\Sigma},\phi)$ and $rk(\underset{\sim}{\Sigma}) = \ell$, then

$$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\mu}' + \underset{\sim}{R}\underset{\sim}{U}\underset{\sim}{A} , \tag{3.10}$$

where $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$, $\underset{\sim}{A}$: $\ell \times p$, U: $n \times \ell$, vec $\underset{\sim}{U} = \underset{\sim}{u}^{(n\ell)}$, $R \leftrightarrow \phi \in \Phi_{n\ell}$ and R is independent of $\underset{\sim}{U}$.

Proof. From Theorem 1 and Theorem 2 we have $\underset{\sim}{x} \sim \underset{\sim\sim}{EC}_{np}(\underset{\sim}{\varepsilon}_n \otimes \tilde{\underset{\sim}{\mu}}, \underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma},\underset{\sim}{\phi})$. Let $\underset{\sim}{U} = (\underset{\sim}{u}_{(1)},\ldots,\underset{\sim}{u}_{(n)})'$, then

$$\underset{\sim}{x} \overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes \tilde{\underset{\sim}{\mu}} + R\begin{pmatrix} \underset{\sim}{A}' & \cdots & \underset{\sim}{0} \\ \vdots & & \vdots \\ \underset{\sim}{0} & \cdots & \underset{\sim}{A}' \end{pmatrix} \underset{\sim}{u}^{(n\ell)}$$

$$\overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes \tilde{\underset{\sim}{\mu}} + R\begin{pmatrix} \underset{\sim}{A}'\underset{\sim}{u}_{(1)} \\ \vdots \\ \underset{\sim}{A}'\underset{\sim}{u}_{(n)} \end{pmatrix} ,$$

because $\underset{\sim}{u}^{(n\ell)} = \text{vec } \underset{\sim}{U} \overset{d}{=} \text{vec } \underset{\sim}{U}'$. Thus

$$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes \tilde{\underset{\sim}{\mu}}' + R\begin{pmatrix} \underset{\sim}{u}'_{(1)}\underset{\sim}{A} \\ \vdots \\ \underset{\sim}{u}'_{(n)}\underset{\sim}{A} \end{pmatrix} = \varepsilon_n \otimes \tilde{\underset{\sim}{\mu}}' + \underset{\sim\sim\sim}{RUA} . \qquad \text{Q.E.D.}$$

In order to obtain the marginal distribution of $\underset{\sim}{X}$, we need the following Lemma which is from Cambanis, Huang and Simons (1981), and Anderson and Fang (1982).

Lemma 2. Assume $\underset{\sim}{y} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{I}_n,\phi)$, $\phi \in \Phi_n$ corresponds to R and C is an $n \times p$ matrix with $rk(\underset{\sim}{C}) = p < n$, then $\underset{\sim}{x} = \underset{\sim}{C}'\underset{\sim}{y} \sim EC_p(\underset{\sim}{C}'\underset{\sim}{\mu},\underset{\sim}{C}'\underset{\sim}{C},\phi)$, where $\phi \in \Phi_p \leftrightarrow R^* \overset{d}{=} Rb_{p/2,(n-p)/2}$.

Assume $\underset{\sim}{X} \sim LEC_{n\ p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$, $rk(\underset{\sim}{\Sigma}) = \ell$ and $\phi \in \Phi_{n\ell} \leftrightarrow R$, from Theorem 1 and Lemma 2 we immediately obtain the marginal distributions of rows and columns of $\underset{\sim}{X}$:

(1) $\underset{\sim}{x}_{(j)} \sim EC_p(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$, where $\phi \in \Phi_\ell \leftrightarrow R^* \overset{d}{=} Rb_{\ell/2,\ell(n-1)/2}$.

(2) $\underset{\sim}{x}_j \sim EC_n(\mu_j\underset{\sim}{\varepsilon}_n,\sigma_{jj}\underset{\sim}{I}_n,\phi)$, where $\phi \in \Phi_n \leftrightarrow R^* = Rb_{n/2,n(\ell-1)/2}$, μ_j is the j-th component of $\underset{\sim}{\tilde{\mu}}$ and $\underset{\sim}{\Sigma} = (\sigma_{ij})$.

The following corollaries concern the distributions of linear functions of $\underset{\sim}{X}$.

Corollary 1. Assume $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$, $rk(\underset{\sim}{\Sigma}) = \ell$, $\phi \in \Phi_{n\ell} \leftrightarrow R$, and $\underset{\sim}{B}$ is a $p \times q$ matrix with $rk(\underset{\sim}{B}) = \min(p,q)$, then

(1) $\underset{\sim}{X}\underset{\sim}{B} \sim LEC_{n\times q}(\underset{\sim}{B}'\underset{\sim}{\tilde{\mu}},\underset{\sim}{B}'\underset{\sim}{\Sigma}\underset{\sim}{B},\phi)$ with $\phi \in \Phi_{n\ell} \leftrightarrow R$ if $q \geq \ell$;

(2) $\underset{\sim}{X}\underset{\sim}{B} \sim LEC_{n\times q}(\underset{\sim}{B}'\underset{\sim}{\tilde{\mu}},\underset{\sim}{B}'\underset{\sim}{\Sigma}\underset{\sim}{B},\phi)$ with $\phi \in \Phi_{nq} \leftrightarrow R^* = Rb_{nq/2,n(\ell-q)/2}$ if $q < \ell$.

Proof. As $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$ with $rk(\underset{\sim}{\Sigma}) = \ell$, then

$$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}}' + R\underset{\sim}{U}\underset{\sim}{A} ,$$

where $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$. Thus

$$\underset{\sim}{X}\underset{\sim}{B} \overset{d}{=} (\underset{\sim}{\varepsilon}_n \otimes \tilde{\underset{\sim}{\mu}}')\underset{\sim}{B} + \underset{\sim}{R}\underset{\sim}{U}\underset{\sim}{A}\underset{\sim}{B}$$

$$= \varepsilon_n \otimes (\tilde{\underset{\sim}{\mu}}'\underset{\sim}{B}) + R\underset{\sim}{U}(\underset{\sim}{A}\underset{\sim}{B}) ,$$

or

$$\mathrm{vec}(\underset{\sim}{X}\underset{\sim}{B})' \overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes (\underset{\sim}{B}'\tilde{\underset{\sim}{\mu}}) + R \begin{pmatrix} \underset{\sim}{B}'\underset{\sim}{A}' & & \\ & \ddots & \\ & & \underset{\sim}{B}'\underset{\sim}{A}' \end{pmatrix} \underset{\sim}{u}^{(n\ell)} .$$

The corollary follows from Lemma 2 and Corollary 2 of Theorem 2. Q.E.D.

<u>Corollary 2</u>. Assume $\underset{\sim}{X} \sim LEC_{n\times p}(\tilde{\underset{\sim}{\mu}},\underset{\sim}{\Sigma},\phi)$, $rk(\underset{\sim}{\Sigma}) = \ell$, $\phi \in \Phi_{n\ell} \leftrightarrow R$, and $\underset{\sim}{B}$ is a $q \times n$ matrix with $rk\underset{\sim}{B} = q < n$. Then

(3.11) $$\mathrm{vec}(\underset{\sim}{B}\underset{\sim}{X}) \sim EC_{qp}(\tilde{\underset{\sim}{\mu}}\otimes(\underset{\sim}{B}\underset{\sim}{\varepsilon}_n),\ \underset{\sim}{\Sigma} \otimes (\underset{\sim}{B}\underset{\sim}{B}'),\phi) ,$$

where $\phi \in \Phi_{q\ell} \leftrightarrow R^* = Rb_{q\ell/2,(n-q)\ell/2}$.

<u>Proof</u>. From the assumption

$$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes \tilde{\underset{\sim}{\mu}}' + R\underset{\sim}{U}\underset{\sim}{A} ,$$

where $\underset{\sim}{U}$: $n \times \ell$, $\underset{\sim}{A}$:$\ell \times p$ and $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$. It can be shown that

$$\mathrm{vec}\ \underset{\sim}{X} \overset{d}{=} \tilde{\underset{\sim}{\mu}} \otimes \varepsilon_n + R \begin{pmatrix} \underset{\sim}{a}'_{(1)} \cdots & \underset{\sim}{a}'_{(1)} \\ \cdots & \\ \underset{\sim}{a}'_{(p)} \cdots & \underset{\sim}{a}'_{(p)} \end{pmatrix} \underset{\sim}{u}^{(n\ell)} \quad \text{with} \quad \underset{\sim}{A} = \begin{pmatrix} \underset{\sim}{a}'_{(1)} \\ \vdots \\ \underset{\sim}{a}'_{(p)} \end{pmatrix} .$$

Thus

$$\mathrm{vec}(\underset{\sim}{B}\underset{\sim}{X}) \overset{d}{=} \underset{\sim}{\tilde{\mu}} \otimes (\underset{\sim}{B}\underset{\sim}{\varepsilon}_n) + R \begin{bmatrix} \underset{\sim}{B} & \cdots & \underset{\sim}{0} \\ \vdots & & \vdots \\ \underset{\sim}{0} & \cdots & \underset{\sim}{B} \end{bmatrix} \begin{bmatrix} \underset{\sim}{a}'_{(1)} & & \\ & \ddots & \underset{\sim}{a}'_{(1)} \\ & \cdots & \\ \underset{\sim}{a}'_{(p)} & & \\ & \ddots & \underset{\sim}{a}'_{(p)} \end{bmatrix} \underset{\sim}{u}^{(n\ell)}$$

$$\equiv \underset{\sim}{\tilde{\mu}} \otimes (\underset{\sim}{B}\underset{\sim}{\varepsilon}_n) + RC'u^{(n\ell)} \quad \text{(say)} .$$

Using Lemma 2 $\mathrm{vec}(\underset{\sim}{B}\underset{\sim}{X}) \sim EC_{qp}(\underset{\sim}{\tilde{\mu}} \otimes (\underset{\sim}{B}\underset{\sim}{\varepsilon}_n), \underset{\sim}{C}'\underset{\sim}{C}, \phi)$ with $\phi \in \Phi_{q\ell} \leftrightarrow R^* = Rb_{q\ell/2,(n-q)\ell/2}$. The coroolary follows from $\underset{\sim}{C}'\underset{\sim}{C} = \underset{\sim}{\Sigma} \otimes (\underset{\sim}{B}\underset{\sim}{B}')$. Q.E.D.

Remark. The above two corollaries show us that the distribution of $\underset{\sim}{X}\underset{\sim}{B}$ still belongs to the class of the multivariate elliptically contoured distributions, but the distribution of $\underset{\sim}{B}\underset{\sim}{X}$ in general does not belong to this class unless $\underset{\sim}{B}\underset{\sim}{B}' = c^2\underset{\sim}{I}_q$ where c is a constant.

Theorem 3. Suppose that $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}}, \underset{\sim}{\Sigma}, \phi)$ with $\underset{\sim}{\Sigma} > 0$, then

(1) $R^2 \overset{d}{=} \mathrm{tr}\, \underset{\sim}{\Sigma}^{-1}\underset{\sim}{G}$ where $R \leftrightarrow \phi \in \Phi_{np}$ and

(3.12) $$\underset{\sim}{G} = \sum_{i=1}^{n} (\underset{\sim}{x}_{(i)} - \underset{\sim}{\tilde{\mu}})(\underset{\sim}{x}_{(i)} - \underset{\sim}{\tilde{\mu}})' = (\underset{\sim}{X} - \varepsilon_n \otimes \underset{\sim}{\tilde{\mu}}')'(\underset{\sim}{X} - \underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}}') ;$$

(2) The density of $\underset{\sim}{X}$ has the form of

(3.13) $$|\underset{\sim}{\Sigma}|^{-n/2}\, g(\mathrm{tr}\, \underset{\sim}{\Sigma}^{-1}\underset{\sim}{G})$$

if it exists.

<u>Proof</u>. As $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$, then $\underset{\sim}{x} = \text{vec } \underset{\sim}{X}' \sim EC_{np}(\underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}}, \underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma}, \phi)$. From Corollary 1 of Cambanis, Huang and Simons (1981)

$$R^2 \overset{d}{=} (\underset{\sim}{x}-\underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}})'(\underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma})^{-1}(\underset{\sim}{x}-\underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}})$$

$$= (\underset{\sim}{x}-\underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}})'(\underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma}^{-1})(\underset{\sim}{x}-\underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}})$$

$$= \sum_{i=1}^{n} (\underset{\sim}{x}_{(i)}-\underset{\sim}{\tilde{\mu}})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}_{(i)}-\underset{\sim}{\tilde{\mu}}) = \text{tr} \sum_{i=1}^{n} \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}_{(i)}-\underset{\sim}{\tilde{\mu}})(\underset{\sim}{x}_{(i)}-\underset{\sim}{\tilde{\mu}})'$$

$$= \text{tr } \underset{\sim}{\Sigma}^{-1}\underset{\sim}{G} .$$

If $\underset{\sim}{X}$ has a density, so does $\underset{\sim}{x}$ and the density of $\underset{\sim}{x}$ has the form of (cf. (3.5))

$$|\underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma}|^{-\frac{1}{2}} g((\underset{\sim}{x}-\underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}})'(\underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma})^{-1}(\underset{\sim}{x}-\underset{\sim}{\varepsilon}_n \otimes \underset{\sim}{\tilde{\mu}}))$$

$$= |\underset{\sim}{\Sigma}|^{-n/2} g(\text{tr } \underset{\sim}{\Sigma}^{-1}\underset{\sim}{G}). \quad \text{Q.E.D.}$$

<u>Corollary 1</u>. Under the supposition of Theorem 3, if $\underset{\sim}{X}$ has the density (3.13), then R has the density

$$\frac{2\pi^{\frac{1}{2}np}}{\Gamma(\frac{1}{2}np)} r^{np-1} g(r^2) . \tag{3.14}$$

The proof uses (3.6).

<u>Corollary 2</u>. Assume that $\underset{\sim}{x}_{(j)} \sim N_p(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma})$ with $\underset{\sim}{\Sigma} > 0$, $j = 1,\ldots,n$ and $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)}$ are independent, then $\text{tr } \underset{\sim}{\Sigma}^{-1}\underset{\sim}{G} \sim \chi^2_{np}$.

Proof. Take $\phi = \exp(-t/2)$ in Theorem 3. As we know $R^2 \sim \chi^2_{np}$, the corollary follows.

4. The Distributions of Correlation Coefficients, the Multiple Correlation Coefficients, and T^2.

Throughout the rest of this paper we assume $\underset{\sim}{X} \sim LEC_{n\times p}(\tilde{\underset{\sim}{\mu}},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and $\phi \in \Phi_{np} \leftrightarrow$ the distribution function F of R.

According to the common definition of the correlation coefficient r_{ij},

$$r_{ij} = \frac{(\underset{\sim}{x}_i-\bar{x}_i\underset{\sim}{\varepsilon}_n)'\ (\underset{\sim}{x}_j-\bar{x}_j\underset{\sim}{\varepsilon}_n)}{\|\underset{\sim}{x}_i-\bar{x}_i\underset{\sim}{\varepsilon}_n\|\cdot\|\underset{\sim}{x}_j-\bar{x}_j\underset{\sim}{\varepsilon}_n\|}$$

where $\bar{x}_i = \frac{1}{n}\,\underset{\sim}{\varepsilon}_n'\underset{\sim}{x}_i$, $i = 1,\ldots,p$. Let $\underset{\sim}{D} = \underset{\sim}{I}_n - \frac{1}{n}\,\underset{\sim}{\varepsilon}_n\underset{\sim}{\varepsilon}_n'$; then

$$r_{ij} = \frac{\underset{\sim}{x}_i'\underset{\sim}{D}\underset{\sim}{x}_j}{\sqrt{\underset{\sim}{x}_i'\underset{\sim}{D}\underset{\sim}{x}_i \cdot \underset{\sim}{x}_j'\underset{\sim}{D}\underset{\sim}{x}_j}}\,. \tag{4.1}$$

At first we consider $\underset{\sim}{x}_{(j)} \sim N_p(\tilde{\underset{\sim}{\mu}},\underset{\sim}{\Sigma})$, $j = 1,\ldots,n$, and suppose $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)}$ are independent. There exist $\underset{\sim}{y}_{(j)} \sim N_p(\underset{\sim}{0},\underset{\sim}{I}_p)$, $j = 1,\ldots,n$, $\underset{\sim}{y}_{(1)},\ldots,\underset{\sim}{y}_{(n)}$ independent such that

$$\underset{\sim}{x}_{(j)} \overset{d}{=} \tilde{\underset{\sim}{\mu}} + \underset{\sim}{A}'y_{(j)}\ ,\quad j=1,\ldots,n\ , \tag{4.2}$$

where $\underset{\sim}{\Sigma} = \underset{\sim}{A}'\underset{\sim}{A}$. Let

$$\underset{\sim}{Y} = \begin{pmatrix} \underset{\sim}{y}_{(1)}' \\ \vdots \\ \underset{\sim}{y}_{(n)}' \end{pmatrix}\ ,\ \underset{\sim}{A} = (\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_p)\ \text{ and }\ \tilde{\underset{\sim}{\mu}} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}\ ; \tag{4.3}$$

it is easy to see that

(4.4) $$\underset{\sim}{x}_i \overset{d}{=} \mu_i \underset{\sim}{\varepsilon}_n + \underset{\sim}{Y}\underset{\sim}{a}_i \ , \quad i = 1,\ldots,p \ .$$

Hence we have

(4.5) $$r_{ij} = \frac{\underset{\sim}{a}_i' \underset{\sim}{Y}' \underset{\sim}{D} \underset{\sim}{Y} \underset{\sim}{a}_j}{(\underset{\sim}{a}_i' \underset{\sim}{Y}' \underset{\sim}{D} \underset{\sim}{Y} \underset{\sim}{a}_i)^{\frac{1}{2}} (\underset{\sim}{a}_j' \underset{\sim}{Y}' \underset{\sim}{D} \underset{\sim}{Y} \underset{\sim}{a}_j)^{\frac{1}{2}}} \ .$$

Secondly, we come back to the case of $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$. From (3.10)

$$\underset{\sim}{x}_j \overset{d}{=} \mu_j \underset{\sim}{\varepsilon}_n + R \underset{\sim}{U} \underset{\sim}{a}_j \ ,$$

where $\underset{\sim}{U}$: $n \times p$ and vec $\underset{\sim}{U} \overset{d}{=} \underset{\sim}{u}^{(np)}$. Hence

$$\underset{\sim}{x}_i' \underset{\sim}{D} \underset{\sim}{x}_j \overset{d}{=} R^2 \underset{\sim}{a}_i' \underset{\sim}{U}' \underset{\sim}{D} \underset{\sim}{U} \underset{\sim}{a}_j \overset{d}{=} R^2 \underset{\sim}{a}_i' \underset{\sim}{Y}' \underset{\sim}{D} \underset{\sim}{Y} \underset{\sim}{a}_j / \mathrm{tr}\ \underset{\sim}{Y}' \underset{\sim}{Y}, \quad \forall\ i,j,$$

because $\underset{\sim}{U} \overset{d}{=} \underset{\sim}{Y}/(\mathrm{tr}\ \underset{\sim}{Y}'\underset{\sim}{Y})^{\frac{1}{2}}$, now (4.1) becomes

$$r_{ij} \overset{d}{=} \frac{\underset{\sim}{a}_i' \underset{\sim}{Y}' \underset{\sim}{D} \underset{\sim}{Y} \underset{\sim}{a}_j}{(\underset{\sim}{a}_i' \underset{\sim}{Y}' \underset{\sim}{D} \underset{\sim}{Y} \underset{\sim}{a}_i)^{\frac{1}{2}} (\underset{\sim}{a}_j' \underset{\sim}{Y}' \underset{\sim}{D} \underset{\sim}{Y} \underset{\sim}{a}_j)^{\frac{1}{2}}}$$

which is equivalent to the normal case by the properties (i) and (ii) of the operation "$\overset{d}{=}$" .

<u>Theorem 4</u>. Suppose that $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$, then the joint distribution of r_{ij}, $i = 1,\ldots,j-1$; $j = 2,\ldots,p$ is the same as the normal case where $\underset{\sim}{x}_{(j)} \sim N_p(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma})$, $j = 1,\ldots,n$, and $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(n)}$ are independent.

Corollary 1. Suppose that $\underset{\sim}{X} \sim LEC_{n\times p}(\tilde{\underset{\sim}{\mu}}, \underset{\sim}{I}_p, \phi)$ and $\underset{\sim}{R} = (r_{ij})$, then the joint density of r_{ij}, $i < j$ is

$$(4.6) \qquad \frac{[\Gamma(\frac{1}{2}m)]^p}{\Gamma_p(\frac{1}{2}n)} \, |\underset{\sim}{R}|^{\frac{1}{2}(m-p-1)} ,$$

where $\Gamma_p(\frac{m}{2}) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(\frac{m-i+1}{2})$ and $m = n-1$. In particular, the density of r_{ij} is

$$(4.7) \qquad \frac{\Gamma(\frac{1}{2}m)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}(m-1))} (1-r_{ij}^2)^{\frac{1}{2}(m-3)}$$

(cf. Anderson (1958), Section 7.6).

Corollary 2. Suppose that $\underset{\sim}{X} \sim LEC_{n\times p}(\tilde{\underset{\sim}{\mu}}, \underset{\sim}{\Sigma}, \phi)$ and $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}} \neq 0$, then the density of r_{ij} is

$$(4.8) \qquad \frac{2^{m-2}(1-\rho_{ij}^2)^{m/2}(1-r_{ij}^2)^{(m-3)/2}}{(m-2)!\,\pi} \sum_{\alpha=0}^{\infty} \frac{(2\rho_{ij}r_{ij})^{\alpha}}{\alpha!} \Gamma^2(\tfrac{1}{2}(m+\alpha)) .$$

(cf. Anderson (1958), Section 4.2).

Now we consider the distribution of the multiple correlation coefficient, for instance, the multiple correlation coefficient between the first variable and the rest of the variables. Denote (cf. (3.12)).

$$\underset{\sim}{G} = \begin{pmatrix} g_{11} & \underset{\sim}{G}_{12} \\ \underset{\sim}{G}_{21} & \underset{\sim}{G}_{22} \end{pmatrix} \text{ and } \underset{\sim}{\Sigma} = \begin{pmatrix} \sigma_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} ,$$

where g_{11} and σ_{11} are the first diagonal elements of $\underset{\sim}{G}$ and $\underset{\sim}{\Sigma}$, respectively. According to the definition of the sample multiple correlation coefficient

$$R_*^2 = \frac{\underset{\sim}{G}_{12}\underset{\sim}{G}_{22}^{-1}\underset{\sim}{G}_{21}}{g_{11}} .$$

It is easy to see that $\underset{\sim}{G} = \underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X}$, $g_{11} = \underset{\sim}{x}_1'\underset{\sim}{D}\underset{\sim}{x}_1$, $\underset{\sim}{G}_{12} = \underset{\sim}{G}_{21}' = \underset{\sim}{x}_1'\underset{\sim}{D}\underset{\sim}{X}_{(2)}$, and $\underset{\sim}{G}_{22} = \underset{\sim}{X}'_{(2)}\underset{\sim}{D}\underset{\sim}{X}_{(2)}$ where $\underset{\sim}{X}_{(2)} = (\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_p)$. By the same method as in the case of the correlation coefficients we have

$$R_*^2 = \frac{\underset{\sim}{a}_1'\underset{\sim}{Y}'\underset{\sim}{D}\underset{\sim}{Y}\underset{\sim}{A}_{(2)}[\underset{\sim}{A}_{(2)}'\underset{\sim}{Y}'\underset{\sim}{D}\underset{\sim}{Y}\underset{\sim}{A}_{(2)}]^{-1}\underset{\sim}{A}_{(2)}'\underset{\sim}{Y}'\underset{\sim}{D}\underset{\sim}{Y}\underset{\sim}{a}_1}{\underset{\sim}{a}_1'\underset{\sim}{Y}'\underset{\sim}{D}\underset{\sim}{Y}\underset{\sim}{a}_1} ,$$

where $\underset{\sim}{A}_{(2)} = (\underset{\sim}{a}_2,\ldots,\underset{\sim}{a}_p)$ for the normal case and the same expression for R_*^2 in the case of $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$.

<u>Theorem 5</u>. Suppose $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$.

(1) If

$$\bar{R}^2 \equiv \frac{\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{21}}{\sigma_{11}} = 0 ,$$

then $[R_*^2/(1-R_*^2)](n-p)/(p-1) \sim F(p-1,n-p)$.

(2) If $\bar{R}^2 > 0$, then the density of R_*^2 is

$$\frac{(1-R_*^2)^{\frac{1}{2}(n-p-2)}(1-\bar{R}^2)^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}(n-p))\Gamma(\frac{1}{2}(n-1))} \sum_{\alpha=0}^{\infty} \frac{(\bar{R}^2)^{\alpha}(R_*^2)^{\frac{1}{2}(p-1)+\alpha-1}\,\Gamma^2(\frac{1}{2}(n-1)+\alpha)}{\alpha!\,\Gamma(\frac{1}{2}(p-1)+\alpha)} .$$

(cf. Anderson (1958) Section 4.4)).

Suppose $\underset{\sim}{X} \sim LEC_{n\times p}(\tilde{\underset{\sim}{\mu}},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ we want to test

$$H_0\colon \tilde{\underset{\sim}{\mu}} = \underset{\sim}{\mu}_0 \quad \text{and} \quad H_1\colon \tilde{\underset{\sim}{\mu}} \neq \underset{\sim}{\mu}_0 .$$

Without loss of generality we can assume $\underset{\sim}{\mu}_0 = 0$. It is well-known that we can use Hotelling's T^2-test for testing the null hypothesis under the normal distribution. What is the distribution of T^2 in the case of the multiple elliptically contoured distributions?

According to the definition of T^2 we have

$$T^2 = n(n-1)\bar{\underset{\sim}{x}}'\underset{\sim}{G}^{-1}\bar{\underset{\sim}{x}} , \tag{4.9}$$

where

$$\bar{\underset{\sim}{x}} = \frac{1}{n}\sum_{j=1}^{n}\underset{\sim}{x}_{(j)} = \frac{1}{n}\underset{\sim}{X}'\underset{\sim}{\varepsilon}_n , \tag{4.10}$$

and $\underset{\sim}{G}$ is defined by (3.12). Using $\underset{\sim}{G} = \underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X}$ and (4.10) we have

$$T^2 = n(n-1)\left(\frac{1}{n}\underset{\sim}{\varepsilon}_n'\underset{\sim}{X}\right)(\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X})^{-1}\left(\frac{1}{n}\underset{\sim}{X}'\underset{\sim}{\varepsilon}_n\right) . \tag{4.11}$$

In the normal case we have $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{\varepsilon}_n \otimes \tilde{\underset{\sim}{\mu}}' + \underset{\sim}{Y}\underset{\sim}{A}$ with $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$ (cf. (4.2)-(4.4)). If the null hypothesis is true we have

$$T^2 = \frac{n-1}{n}\underset{\sim}{\varepsilon}_n'\underset{\sim}{Y}\underset{\sim}{A}(\underset{\sim}{A}'\underset{\sim}{Y}'\underset{\sim}{D}\underset{\sim}{Y}\underset{\sim}{A})^{-1}\underset{\sim}{A}'\underset{\sim}{Y}'\underset{\sim}{\varepsilon}_n . \tag{4.12}$$

By (3.10) and $\underset{\sim}{U} \overset{d}{=} \underset{\sim}{Y}/(\mathrm{tr}\,\underset{\sim}{Y}'\underset{\sim}{Y})^{1/2}$, when the null hypothesis is true we obtain the same expression as (4.12) for T^2 in the case of $\underset{\sim}{X} \sim LEC_{n\times p}(\tilde{\underset{\sim}{\mu}},\underset{\sim}{\Sigma},\phi)$.

Theorem 6. Suppose that $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and T^2 is defined by (4.9), if $\underset{\sim}{\tilde{\mu}} = 0$ the distribution of $[T^2/(n-1)][(n-p)/p]$ is $F(p,n-p)$.

5. The Distributions of the Sample Covariance Matrix and Generalized Variance.

5.1. The distribution of the sample covariance matrix. It is a well-known fact that the distribution of the sample covariance matrix of multivariate normal population is the Wishart distribution. If the sample $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(N)}$ is from the population of $N_p(\underset{\sim}{\tilde{\mu}},\underset{\sim}{\Sigma})$ and

$$\underset{\sim}{A} = \sum_{\alpha=1}^{N} (\underset{\sim}{x}_{(\alpha)}-\underset{\sim}{\bar{x}})(\underset{\sim}{x}_{(\alpha)}-\underset{\sim}{\bar{x}})' ,$$

then $\underset{\sim}{A} \overset{d}{=} \Sigma_{\alpha=1}^{n} \underset{\sim}{z}_\alpha \underset{\sim}{z}_\alpha'$ where $n = N-1$ and $z_{(1)},\ldots,z_{(n)}$ are i.i.d. distributed according to $N_p(\underset{\sim}{0},\underset{\sim}{\Sigma})$. Now we want to find the corresponding distribution for multivariate elliptically distributions. Assume $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ we want to obtain the distribution of

$$\underset{\sim}{W} = \sum_{\alpha=1}^{n} \underset{\sim}{x}_{(\alpha)}\underset{\sim}{x}_{(\alpha)}' = \underset{\sim}{X}'\underset{\sim}{X} = (w_{ij}) . \tag{5.1}$$

(1) Assume $\underset{\sim}{X}$ has a density. Theorem 3 shows us that the density of $\underset{\sim}{X}$ is

$$|\underset{\sim}{\Sigma}|^{-n/2}\, g(\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{X}'\underset{\sim}{X}) = |\underset{\sim}{\Sigma}|^{-n/2}\, g(\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}) \tag{5.2}$$

as $\underset{\sim}{\tilde{\mu}} = 0$ and $\underset{\sim}{W} = \underset{\sim}{G}$. By Lemma 13.3.1 (p. 319, Anderson (1958)), the density of $\underset{\sim}{W}$ (i.e. the density of $w_{11},\ldots,w_{1p},w_{22},\ldots,w_{2p},\ldots,w_{pp}$) is

$$(5.3)\qquad \frac{\pi^{np/2-p(p-1)/4}}{\prod_{\alpha=1}^{p}\Gamma(\frac{n-\alpha+1}{2})}\ |\underset{\sim}{W}|^{(n-p-1)/2}\ |\underset{\sim}{\Sigma}|^{-n/2}\ g(\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{W})\ .$$

In the normal case (5.3) reduces to the Wishart density.

(2) Assume $p(\underset{\sim}{X}=\underset{\sim}{0}) = 0$. In this case it is possible that $\underset{\sim}{X}$ does not have a density, but all the marginal distributions of $\underset{\sim}{X}$ will have densities (cf. Kelker (1970)). We consider the following interesting and important situations:

(A) Let

$$(5.4)\qquad \underset{\sim}{X}_k = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \quad \text{and} \quad \underset{\sim}{W}_k = \underset{\sim}{X}_k'\underset{\sim}{X}_k\ ,\ 1 \le k < p\ .$$

In order to obtain the density of $\underset{\sim}{X}_k$ we need the following Lemma.

<u>Lemma 3</u>. Assume $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and $p(\underset{\sim}{X}=\underset{\sim}{0}) = 0$. Let $\underset{\sim}{x} = (\underset{\sim}{x}_{(1)}',\underset{\sim}{x}_{(2)}')'$ where $\underset{\sim}{x}_{(1)}$ is an $m \times 1$ vector, $m < n$, then the density of $\underset{\sim}{x}_{(1)}$ is

$$(5.5)\qquad \frac{\Gamma(\frac{n}{2})|\underset{\sim}{\Sigma}_{11}|^{-\frac{1}{2}}}{\pi^{m/2}\,\Gamma(\frac{n-m}{2})} \int_{(\underset{\sim}{x}_{(1)}'\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{x}_{(1)})^{\frac{1}{2}}}^{\infty} r^{-(n-2)}(r^2-x_{(1)}'\Sigma_{11}^{-1}x_{(1)})^{(n-m)/2-1}\, dF(r),$$

where $\underset{\sim}{\Sigma}_{11}$ is the first principal minor of order m of $\underset{\sim}{\Sigma}$.

<u>**Proof**</u>. If $\underset{\sim}{\Sigma} = \underset{\sim}{I}_n$, the density of $\underset{\sim}{x}_{(1)}$ is

(5.6) $$\frac{\Gamma(\frac{n}{2})}{\pi^{m/2}\,\Gamma(\frac{n-m}{2})}\int_{\sqrt{\underset{\sim}{x}'_{(1)}\underset{\sim}{x}_{(1)}}}^{\infty} r^{-(n-2)}(r^2-\underset{\sim}{x}'_{(1)}\underset{\sim}{x}_{(1)})^{(n-m)/2-1}\,dF(r)\;.$$

(cf. (21), Cambanis, Huang and Simons (1981); the lower limit of the integral in that paper should be $\sqrt{u}$, not u). When $\underset{\sim}{\Sigma} \neq \underset{\sim}{I}_n$ the Lemma follows by using the transformation $\underset{\sim}{y} = \underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{x}$. Q.E.D.

Using Lemma 3 and noting the structure of $\underset{\sim}{X}_k$, the density of $\underset{\sim}{X}_k$ is

(5.7) $$\frac{\Gamma(\frac{np}{2})|\underset{\sim}{\Sigma}_k|^{-k/2}}{\pi^{nk/2}\,\Gamma(\frac{n(p-k)}{2})}\int_{\sqrt{\mathrm{tr}\underset{\sim}{\Sigma}_k^{-1}\underset{\sim}{X}'_k\underset{\sim}{X}_k}}^{\infty} r^{-(np-2)}(r^2-\mathrm{tr}\underset{\sim}{\Sigma}_k^{-1}\underset{\sim}{X}'_k\underset{\sim}{X}_k)^{n(p-k)/2-1}\,dF(r)\;,$$

where $\underset{\sim}{\Sigma}_k$ is the first principal minor of order k of $\underset{\sim}{\Sigma}$. By Lemma 13.3.1 of Anderson (1958) again, the density of $\underset{\sim}{W}_k$ is

(5.8) $$\frac{\Gamma(\frac{np}{2})\,|\underset{\sim}{\Sigma}_k|^{-n/2}}{\pi^{k(k-1)/4}\Gamma(\frac{n(p-k)}{2})\prod_{\alpha=1}^{k}\Gamma(\frac{n-\alpha+1}{2})}\,|\underset{\sim}{W}_k|^{(n-k-1)/2}\int_{\sqrt{\mathrm{tr}\underset{\sim}{\Sigma}_k^{-1}\underset{\sim}{W}_k}}^{\infty} r^{-(np-2)}$$
$$(r^2-\mathrm{tr}\underset{\sim}{\Sigma}_k^{-1}\underset{\sim}{W}_k)^{n(p-k)/2-1}\,dF(r)\;.$$

(B) Let

$$\underset{\sim}{X}_{(m)} = \begin{pmatrix} x_{11} & \dots & x_{1p} \\ \vdots & & \vdots \\ x_{m1} & \dots & x_{mp} \end{pmatrix}\;,\; p \le m < n\;,\; \underset{\sim}{W}_{(m)} = \underset{\sim}{X}'_{(m)}\underset{\sim}{X}_{(m)}\;.$$

In a similar way the densities of $\underset{\sim}{X}_{(m)}$ and $\underset{\sim}{W}_{(m)}$ are found as

$$(5.9)\qquad \frac{\Gamma(\frac{np}{2})|\underset{\sim}{\Sigma}|^{-m/2}}{\pi^{mp/2}\Gamma(\frac{(n-m)p}{2})}\int_{\sqrt{\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{X}'_{(m)}\underset{\sim}{X}_{(m)}}}^{\infty} r^{-(np-2)}(r^2-\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{X}'_{(m)}\underset{\sim}{X}_{(m)})^{(n-m)p/2-1}\,dF(r)$$

and

$$(5.10)\qquad \frac{\Gamma(\frac{np}{2})|\underset{\sim}{\Sigma}|^{-m/2}}{\pi^{p(p-1)/4}\Gamma(\frac{(n-m)p}{2})\prod_{\alpha=1}^{p}\Gamma(\frac{m-\alpha+1}{2})}|\underset{\sim}{W}_{(m)}|^{(m-p-1)/2}$$

$$\int_{\sqrt{\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}_{(m)}}}^{\infty} r^{-(np-2)}(r^2-\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}_{(m)})^{(n-m)p/2-1}\,dF(r)\ ,$$

respectively.

Further, partition $\underset{\sim}{X}$ into $k+1$ parts, i.e.

$$\underset{\sim}{X} = \begin{pmatrix}\underset{\sim}{X}_1\\ \underset{\sim}{X}_2\\ \vdots\\ \underset{\sim}{X}_{k+1}\end{pmatrix}\ ,$$

where $\underset{\sim}{X}_1,\underset{\sim}{X}_2,\ldots,\underset{\sim}{X}_{k+1}$ are $n_1\times p,\ n_2\times p,\ldots,n_{k+1}\times p$ matrices, respectively and $p\le n_i<n$, $i=1,\ldots,k$, $n_{k+1}\ge 1$, $\Sigma_1^{k+1} n_i = n$. Let $\underset{\sim}{W}_{(i)} = \underset{\sim}{X}_i'\underset{\sim}{X}_i$, $i=1,\ldots,k$, then the joint density of $\underset{\sim}{W}_{(1)},\ldots,\underset{\sim}{W}_{(k)}$ is

$$(5.11)\qquad \frac{\Gamma(\frac{np}{2})\,|\underset{\sim}{\Sigma}|^{-(n-n_{k+1})/2}}{\pi^{kp(p-1)/4}\,\Gamma(\frac{n_{k+1}p}{2})\prod_{j=1}^{k}\prod_{\alpha=1}^{p}\Gamma(\frac{n_j-\alpha+1}{2})}\prod_{j=1}^{k}|\underset{\sim}{W}_{(j)}|^{(n_j-p-1)/2}$$

$$\int_{(\sum_{j=1}^{k}\operatorname{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}_{(j)})^{\frac{1}{2}}}^{\infty} r^{-(np-2)}\left(r^2-\sum_{j=1}^{k}\operatorname{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}_{(j)}\right)^{pn_{k+1}/2-1} dF(r)\ .$$

The method to obtain (5.11) is the same as to obtain (5.10), but here we need to use Lemma 13.3.1 of Anderson (1958) k times. When $p = 1$, the density (5.11) reduces to $G_{k+1}(n_1/2,\ldots,n_k/2;n_{k+1}/2;\phi)$. We denote the density (5.11) by $MG_{p,k+1}(\underset{\sim}{\Sigma};n_1/2,\ldots,n_k/2;n_{k+1}/2;\phi)$.

Further, if $\underset{\sim}{X}$ has a density (3.13), we rewrite it as

$$|\underset{\sim}{\Sigma}|^{-n/2}\, g\left(\sum_{j=1}^{k+1}\operatorname{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}_{(j)}\right)\ .$$

We use Lemma 13.3.1 of Anderson (1958) $k+1$ times to obtain the density of $\underset{\sim}{W}_{(1)},\ldots,\underset{\sim}{W}_{(k+1)}$ as

$$(5.12)\qquad \frac{\pi^{np/2}|\underset{\sim}{\Sigma}|^{-n/2}}{\pi^{(k+1)p(p-1)/4}}\, g\left(\sum_{j=1}^{k+1}\operatorname{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}_{(j)}\right)\prod_{j=1}^{k+1}\frac{|\underset{\sim}{W}_{(j)}|^{(n_j-p-1)/2}}{\prod_{\alpha=1}^{p}\Gamma(\frac{n_j-\alpha+1}{2})}\ .$$

When $p = 1$, (5.12) reduces to (3.7) with $m = k+1$. Denote (5.12) by $MG_{p,k+1}(\underset{\sim}{\Sigma};n_1/2,\ldots,n_k/2,n_{k+1}/2;\phi)$. In this case as $F(r)$ has a density, we can change (5.11) to the simpler form

$$\frac{2\pi^{np/2-kp(p-1)/4}|\underset{\sim}{\Sigma}|^{-(n-n_{k+1})/2}}{\Gamma(\frac{n_{k+1}p}{2})\prod_{j=1}^{k}\prod_{\alpha=1}^{p}\Gamma(\frac{n_j-\alpha+1}{2})}\prod_{j=1}^{k}|\bar{\underset{\sim}{W}}_{(j)}|^{(n_j-p-1)/2}$$

$$\int_0^\infty r^{n_{k+1}p-1}\, g\left(r^2+\sum_{j=1}^{k}\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{W}_{(j)}\right)dr.$$

(C) Consider the marginal density of $w_{11},\ldots,w_{p-1,p-1},w_{1p},\ldots,w_{1,p-1},$ $w_{2p},\ldots,w_{2,p-1},\ldots,w_{p-2,p-1},w_{1p},\ldots,w_{p-1,p}$ (i.e. the marginal density of $\underset{\sim}{W}$ except w_{pp}); for simplicity we will say the density of $\underset{\sim}{W}^*$.

(1) Let $\underset{\sim}{E} = \underset{\sim}{U}'\underset{\sim}{U}$ where $\underset{\sim}{U}$: $n\times p$ and vec $\underset{\sim}{U} = \underset{\sim}{u}^{(np)}$. Let $\underset{\sim}{Y}$ be defined by (4.3), then

$$\underset{\sim}{E} \overset{d}{=} \underset{\sim}{Y}'\underset{\sim}{Y}/\mathrm{tr}\underset{\sim}{Y}'\underset{\sim}{Y} \overset{d}{=} (e_{ij})\ . \tag{5.13}$$

We want to find the density of $\underset{\sim}{E}^*$ (i.e., the density of $\underset{\sim}{E}$ except e_{pp}). However, the correlation coefficient r_{ij} is

$$r_{ij} = \frac{e_{ij}}{\sqrt{e_{ii}e_{jj}}} = \frac{f_{ij}}{\sqrt{f_{ii}f_{jj}}}\ ,$$

where $(f_{ij}) = \underset{\sim}{F} = \underset{\sim}{Y}'\underset{\sim}{Y}$. As we know, $\{r_{ij},\ i=1,\ldots,j-1, j=2,\ldots,p\}$ are independent of $\{\|\underset{\sim}{y}_i\|,\ i = 1,\ldots,p\}$ and $(\|\underset{\sim}{y}_1\|^2,\ldots,\|\underset{\sim}{y}_{p-1}\|^2)/\mathrm{tr}\underset{\sim}{Y}'\underset{\sim}{Y} \sim$ $D_p(\frac{n}{2},\ldots,\frac{n}{2};\frac{n}{2})$, hence $(e_{11},\ldots,e_{p-1,p-1}) \sim D_p(\frac{n}{2},\ldots,\frac{n}{2};\frac{n}{2})$ and the density of $e_1 = \sqrt{e_{11}},\ldots,e_{p-1} = \sqrt{e_{p-1,p-1}}$ is

$$\frac{2^{p-1}\Gamma\left(\frac{np}{2}\right)}{\Gamma^p\left(\frac{n}{2}\right)}\prod_{i=1}^{p-1} e_i^{n-1}\left(1-\sum_{i=1}^{p-1} e_i^2\right)^{\frac{n}{2}-1} \qquad \text{if } \sum_{1}^{p-1} e_i^2 < 1\ . \tag{5.14}$$

From Corollary 1 of Theorem 4, the joint density of $e_1,\dots,e_{p-1},r_{12},\dots,$ $r_{1p}, r_{23},\dots r_{p-1,p}$ is

$$(5.15)\qquad \frac{2^p\Gamma(\frac{np}{2})}{\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(\frac{n-i+1}{2})}\,|\underset{\sim}{R}|^{\frac{1}{2}(n-p-1)}\prod_{i=1}^{p-1} e_i^{n-1}\Big(1-\sum_{1}^{p-1} e_i^2\Big)^{\frac{n}{2}-1},$$

$$\text{if } \sum_{i=1}^{p-1} e_i^2 < 1 .$$

Consider the following transformation

$$\begin{cases} e_{ii} = e_i^2 & i = 1,\dots,p-1 \\ e_{ij} = e_i e_j r_{ij} & i = 1,\dots,j-1;\ j=1,\dots,p, \end{cases}$$

with $e_p = (1 - \sum_1^{p-1} e_j^2)^{\frac{1}{2}}$. It is easy to see that the matrix of

$$\frac{\partial(e_{11},\dots,e_{p-1,p-1},e_{12},\dots,e_{1,p-1},e_{23},\dots,e_{2,p-1},\dots,e_{p-2,p-1},e_{1p},\dots,e_{p-1,p})}{\partial(e_1,\dots,e_{p-1},r_{12},\dots,r_{1,p-1},r_{23},\dots,r_{2,p-1},\dots,r_{p-2,p-1},r_{1p},\dots,r_{p-1,p})}$$

is lower triangular with diagonal elements $2e_1,\dots,2e_{p-1},e_1e_2,\dots,e_1e_{p-1},e_2e_3,$ $\dots,e_2e_{p-1},\dots,e_{p-2}e_{p-1},e_1e_p,\dots,e_{p-1}e_p$, thus the Jacobian of the transformation is

$$(5.16)\qquad 2^{p-1}\, e_p^{(p-1)}\prod_{i=1}^{p-1} e_i^p = 2^{(p-1)}\Big(1-\sum_1^{p-1} e_i^2\Big)^{(p-1)/2}\prod_{i=1}^{p-1} e_i^p .$$

Combining (5.15) and (5.16), the density of $\underset{\sim}{E}^*$ is

$$(5.17)\qquad \frac{\Gamma(\frac{np}{2})}{\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(\frac{n-i+1}{2})}|\underset{\sim}{R}|^{\frac{1}{2}(n-p-1)}\left(1-\sum_{1}^{p-1}e_{ii}\right)^{\frac{1}{2}(n-p-1)}\cdot\prod_{i=1}^{p-1}e_{ii}^{\frac{1}{2}(n-p-1)}$$

$$=\frac{\Gamma(\frac{np}{2})}{\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(\frac{n-i+1}{2})}|\underset{\sim}{E}|^{\frac{1}{2}(n-p-1)}\ ,$$

with $e_{pp} = 1 - \Sigma_1^{p-1} e_{ii}$.

(2) Let $\underset{\sim}{\Sigma}$ be $p \times p$ a positive definite matrix and $\underset{\sim}{A}$ be an upper triangular matrix such that $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$. Let $\underset{\sim}{V} = \underset{\sim}{A}'\underset{\sim}{E}\underset{\sim}{A} = (v_{ij})$. We want to obtain the density of $\underset{\sim}{V}^*$ (i.e. the density of $\underset{\sim}{V}$ except v_{pp}). Partitioning $\underset{\sim}{V}$, $\underset{\sim}{A}$ and $\underset{\sim}{E}$ as follows

$$\begin{pmatrix} \underset{\sim}{V}_{11} & \underset{\sim}{v}_{(1)} \\ \underset{\sim}{v}'_{(1)} & v_{pp} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{A}'_{11} & \underset{\sim}{0} \\ \underset{\sim}{a}'_{(1)} & a_{pp} \end{pmatrix}\begin{pmatrix} \underset{\sim}{E}_{11} & \underset{\sim}{e}_{(1)} \\ \underset{\sim}{e}'_{(1)} & e_{pp} \end{pmatrix}\begin{pmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{a}_{(1)} \\ \underset{\sim}{0} & a_{pp} \end{pmatrix} ,$$

we have

$$\begin{cases} \underset{\sim}{V}_{11} = \underset{\sim}{A}'_{11}\underset{\sim}{E}_{11}\underset{\sim}{A}_{11} \\ \underset{\sim}{v}_{(1)} = \underset{\sim}{A}'_{11}\underset{\sim}{E}_{11}\underset{\sim}{a}_{(1)} + \underset{\sim}{A}'_{11}\underset{\sim}{e}_{(1)}a_{pp} \\ v_{pp} = a_{pp}^2\, e_{pp} + 2a_{pp}\underset{\sim}{e}'_{(1)}\underset{\sim}{a}_{(1)} + \underset{\sim}{a}'_{(1)}\underset{\sim}{E}_{11}\underset{\sim}{a}_{(1)} \ . \end{cases}$$

As e_{pp} and v_{pp} are not independent variables, the Jacobian of transformation is

$$J(\underset{\sim}{V} \to \underset{\sim}{E}) = J(\underset{\sim}{V}_{11} \to \underset{\sim}{E}_{11}) \cdot J(v_{(1)} \to e_{(1)})$$

$$J(\underset{\sim}{V}_{11} \to \underset{\sim}{E}_{11}) = |\underset{\sim}{A}'_{11}\underset{\sim}{A}_{11}|^{\frac{1}{2}p} = |\underset{\sim}{\Sigma}_{11}|^{p/2} ,$$

where $\underset{\sim}{\Sigma}_{11}$ is the first principal minor of order $(p-1)$ of $\underset{\sim}{\Sigma}$, and

$$J(\underset{\sim}{v}_{(1)} \to \underset{\sim}{e}_{(1)}) = |\underset{\sim}{A}'_{11}|a_{pp}^{p-1} = |\underset{\sim}{\Sigma}_{11}|^{\frac{1}{2}} a_{pp}^{p-1} .$$

Thus

$$(5.18) \qquad J(\underset{\sim}{V} \to \underset{\sim}{E}) = |\underset{\sim}{\Sigma}_{11}|^{\frac{1}{2}(p+1)} a_{pp}^{p-1} = |\underset{\sim}{\Sigma}_{11}|^{\frac{1}{2}(p+1)} \left(\frac{|\underset{\sim}{\Sigma}|}{|\underset{\sim}{\Sigma}_{11}|}\right)^{\frac{1}{2}(p-1)} = |\underset{\sim}{\Sigma}_{11}||\underset{\sim}{\Sigma}|^{\frac{1}{2}(p-1)} .$$

Noting $|\underset{\sim}{V}| = |\underset{\sim}{A}'||\underset{\sim}{E}||\underset{\sim}{A}| = |\underset{\sim}{\Sigma}||\underset{\sim}{E}|$, the density of $\underset{\sim}{V}^*$ is

$$(5.19) \qquad \frac{\Gamma(\frac{np}{2})}{\pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(\frac{n-i+1}{2})} \; |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n+1}|\underset{\sim}{\Sigma}_{11}|^{-1} \; |\underset{\sim}{V}|^{\frac{1}{2}(n-p-1)} .$$

As v_{pp} is not an independent variable, we need to find the relationship between it and the other variables. Since

$$\underset{\sim}{U}'\underset{\sim}{U} = \underset{\sim}{E} = \underset{\sim}{A}'^{-1}\underset{\sim}{V}\underset{\sim}{A}^{-1} ,$$

we have

$$1 = \mathrm{tr}\underset{\sim}{U}'\underset{\sim}{U} = \mathrm{tr}(\underset{\sim}{A}'^{-1}\underset{\sim}{V}\underset{\sim}{A}^{-1}) = \mathrm{tr}\underset{\sim}{V}\underset{\sim}{\Sigma}^{-1} .$$

Denote $\underset{\sim}{\Sigma}^{-1} = (\sigma^{ij})$ and $V = (v_{ij})$; then

(5.20) $v_{pp} = (\sigma^{pp})^{-1}\left(1 - \sum_{i=1}^{p}\sum_{j=1}^{p-1} \sigma^{ij} v_{ij} - \sum_{i=1}^{p-1} \sigma^{ip} v_{ip}\right) \hat{=} (\sigma^{pp})^{-1}(1-v*)$, (say) .

(3) Assume $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and $P(\underset{\sim}{X}=\underset{\sim}{0}) = 0$. From (3.10)

$$\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}\underset{\sim}{A} ,$$

where $\underset{\sim}{U}$: $n \times p$, vec $\underset{\sim}{U} = u^{(np)}$, $\underset{\sim}{A}$ is a $p \times p$ upper triangular matrix and $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$. Then

(5.21) $$\underset{\sim}{W} = \underset{\sim}{X}'\underset{\sim}{X} \overset{d}{=} R^2\underset{\sim}{A}'\underset{\sim}{U}'\underset{\sim}{U}\underset{\sim}{A} = R^2\underset{\sim}{A}'\underset{\sim}{E}\underset{\sim}{A} = R^2\underset{\sim}{V} .$$

We have obtained the density (5.19) of $\underset{\sim}{V}_*$ and it has the form of

$$c|\underset{\sim}{V}|^{\frac{1}{2}(n-p-1)} ,$$

where c is a constant. Thus the cdf of $\underset{\sim}{W}_*$ for $\underset{\sim}{W}$ positive definite is

$$P\{R^2 v_{jj} \le w_{jj},\ j = 1,\dots,p-1;\ R^2 v_{ij} \le w_{ij},\ i < j\}$$

$$= \int_0^\infty P\{v_{jj} \le \frac{w_{jj}}{r^2},\ j = 1,\dots,p-1;\ v_{ij} \le \frac{w_{ij}}{r^2},\ i < j,\ \underset{\sim}{V} \text{ positive definite}\}dF(r).$$

When the probability is written as the definite integral of the density of $\underset{\sim}{V}_*$ differentiation yields as the density of $\underset{\sim}{W}_*$

(5.22) $$c\int_0^\infty r^{-p(p+1)+2}|\underset{\sim}{H}|^{\frac{1}{2}(n-p-1)}\ dF(r) ,$$

where $\underset{\sim}{H} = (h_{ij})$, $h_{ii} = w_{ii}/r^2$, $i = 1,\ldots,p-1$; $h_{ij} = w_{ij}/r^2$, $i < j$ and

$$h_{pp} = (\sigma^{pp})^{-1}(1-w^*/r^2) = (\sigma^{pp}r^2)^{-1}(r^2-w^*) ,$$

where $w^* = \Sigma_{i=1}^{p} \Sigma_{j=1}^{p-1} \sigma^{ij}w_{ij} + \Sigma_{i=1}^{p-1} \sigma^{ip}w_{ip}$. As h_{pp} must be positive, the density of $\underset{\sim}{W}_*$ is

$$(5.23) \qquad c\int_{\sqrt{w^*}}^{\infty} r^{-p(p+1)+2}\, r^{-p(n-p-1)}\,|\underset{\sim}{W}|^{\frac{1}{2}(n-p-1)}\, dF(r)$$

$$= c\int_{\sqrt{w^*}}^{\infty} r^{-(pn-2)}\,|\underset{\sim}{W}|^{\frac{1}{2}(n-p-1)}\, dF(r)$$

$$= \frac{\Gamma(\frac{np}{2})}{\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(\frac{n-i+1}{2})}\,|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n+1}|\underset{\sim}{\Sigma}_{11}|^{-1}\int_{\sqrt{w^*}}^{\infty} r^{-(pn-2)}\,|\underset{\sim}{W}|^{\frac{1}{2}(n-p-1)}\, dF(r),$$

where $\underset{\sim}{W} = (w_{ij})$ with

$$(5.24) \qquad w_{pp} = (\sigma^{pp})^{-1}(r^2-w^*) = (\sigma^{pp})^{-1}\left(r^2 - \sum_{i=1}^{p}\sum_{j=1}^{p-1}\sigma^{ij}w_{ij} - \sum_{i=1}^{p-1}\sigma^{ip}w_{ip}\right) .$$

Now we can summarize the above results as follows:

<u>Theorem 7</u>. Assume that $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and $n > p$; $\underset{\sim}{X}$ is partitioned into $k+1$ parts $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_{k+1}$ which are $n_1 \times p,\ldots,n_{k+1}\times p$ matrices, respectively, $p \leq n_i$, $i = 1,\ldots,k+1$ and $\Sigma_1^{k+1} n_i = n$; $\underset{\sim}{W} = \underset{\sim}{X}'\underset{\sim}{X}$, $\underset{\sim}{W}_{(j)} = \underset{\sim}{X}_j'\underset{\sim}{X}_j$ and $\underset{\sim}{W}_k$ is defined by (5.4).

(1) If $\underset{\sim}{X}$ has a density (5.2), then the density of $\underset{\sim}{W}$ is (5.3) and the joint density of $\underset{\sim}{W}_{(1)},\ldots,\underset{\sim}{W}_{(k+1)}$ is (5.12).

(2) If $P(\underset{\sim}{X}=\underset{\sim}{0}) = 0$, then there exists the marginal density of any (proper) subset of elements of $\underset{\sim}{W}$. In particular, the density of $\underset{\sim}{W}_k$ is (5.8), the density of $\underset{\sim}{W}_*$ (i.e. the density of $\underset{\sim}{W}$ except w_{pp}) is (5.23) and the joint density of $\underset{\sim}{W}_{(1)},\ldots,\underset{\sim}{W}_{(k)}$ is (5.11).

Corollary 1. Assume $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and $n > p$, and $\underset{\sim}{X}$ has a density (5.2). Let $\underset{\sim}{\Psi} = \underset{\sim}{\Sigma}^{-1}$ and $\underset{\sim}{V} = \underset{\sim}{W}^{-1}$, then the density of $\underset{\sim}{V}$ is

$$(5.25) \qquad \frac{\pi^{np/2-p(p-1)/4}}{\prod_{\alpha=1}^{p} \Gamma(\frac{n-\alpha+1}{2})} |\underset{\sim}{\Psi}|^{n/2} |\underset{\sim}{V}|^{-(n+p+1)/2} g(\mathrm{tr}\underset{\sim}{\Psi}\underset{\sim}{V}^{-1}) .$$

Proof. From (5.3) and the fact of the Jacobian of the transformation $\underset{\sim}{W} = \underset{\sim}{V}^{-1}$ is $|\underset{\sim}{V}|^{-(p+1)}$ the corollary follows. Q.E.D.

When $\phi = \exp(-t/2)$, i.e. $\underset{\sim}{X}$ is from a normal population, (5.25) reduces to the inverted Wishart distribution.

Corollary 2. Assume the condition of Corollary 1 holds and $\underset{\sim}{\Sigma} = \underset{\sim}{I}_p$, then the density of the characteristic roots $\lambda_1 > \cdots > \lambda_p$ of $\underset{\sim}{W}$ is

$$(5.26) \qquad \pi^{(n+1)p/2} \left[\prod_{\alpha=1}^{p} \frac{\lambda_\alpha^{(n-p-1)/2}}{\Gamma(\frac{n-\alpha+1}{2})\Gamma(\frac{p-\alpha+1}{2})} \right] \prod_{i<j} (\lambda_i - \lambda_j)\; g\left(\sum_1^p \lambda_i\right) .$$

Proof. From (5.3) and Theorem 13.3.1 of Anderson (1958), the density of $\lambda_1,\ldots,\lambda_p$ is

$$\frac{\pi^{np/2-p(p-1)/4}}{\prod_{\alpha=1}^{p}\Gamma(\frac{n-\alpha+1}{2})}\prod_{i=1}^{p}\lambda_i^{(n-p-1)/2}\, g\left(\sum_1^p \lambda_i\right)\cdot\frac{\pi^{p(p+1)/4}}{\prod_{\alpha=1}^{p}\Gamma(\frac{p-\alpha+1}{2})}\prod_{i<j}(\lambda_i-\lambda_j)$$

and the corollary follows. Q.E.D.

5.2. The distribution of the generalized variance.

Theorem 8. Assume that $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\phi \in \Phi_{np} \leftrightarrow R$, $\underset{\sim}{\Sigma} > 0$, $p(\underset{\sim}{X}=\underset{\sim}{0}) = 0$ and $n > p$. Let $y_1,\ldots,y_{p+1}$ be independently distributed as $\chi^2_n,\chi^2_{n-1},\ldots,\chi^2_{n-p+1},\chi^2_{p(p-1)/2}$, respectively. Then

$$|\underset{\sim}{W}| \overset{d}{=} R^{2p}|\underset{\sim}{\Sigma}|\prod_{i=1}^{p} y_i\Big/\left(\sum_{i=1}^{p+1} y_i\right)^p . \tag{5.27}$$

Proof. From the assumption $\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}\underset{\sim}{A} \overset{d}{=} R\underset{\sim}{Y}\underset{\sim}{A}/\sqrt{\mathrm{tr}\underset{\sim}{Y}'\underset{\sim}{Y}}$, where $\underset{\sim}{Y}$ is defined by (4.3), then

$$|\underset{\sim}{W}| = |\underset{\sim}{X}'\underset{\sim}{X}| \overset{d}{=} R^{2p}|\underset{\sim}{A}'\underset{\sim}{Y}'\underset{\sim}{Y}\underset{\sim}{A}|/(\mathrm{tr}\ \underset{\sim}{Y}'\underset{\sim}{Y})^p = R^{2p}|\underset{\sim}{\Sigma}|\,|\underset{\sim}{Y}'\underset{\sim}{Y}|/(\mathrm{tr}\ \underset{\sim}{Y}'\underset{\sim}{Y})^p . \tag{5.28}$$

Let $\underset{\sim}{T} = (t_{ij})$ be the lower triangular matrix such that $\underset{\sim}{T}\underset{\sim}{T}' = \underset{\sim}{Y}'\underset{\sim}{Y}$. It can be shown that

(a) $t_{ij} \sim N(0,1)$ for any $i > j$;

(b) $t_{ii} \sim \chi_{n-i+1}$ (i.e. $t_{ii}^2 \sim \chi_{n-i+1}^2$), $i = 1,\ldots,p$;

(c) $\{t_{ij}, j \le i\}$ are independent.

(cf. Johnson and Kotz (1972)). Thus

$$|\underset{\sim}{W}| \overset{d}{=} R^{2p}|\underset{\sim}{\Sigma}|\,|\underset{\sim}{T}\underset{\sim}{T}'|/(\mathrm{tr}\ \underset{\sim}{T}\underset{\sim}{T}')^p .$$

As $\underset{\sim}{T}$ is a lower triangular we have

$$|\underset{\sim}{T}\underset{\sim}{T}'| = \prod_{i=1}^{p} t_{ii}^2 \overset{d}{=} \prod_{i=1}^{p} y_i$$

and

$$\mathrm{tr}\ \underset{\sim}{T}\underset{\sim}{T}' = \sum_{i=1}^{p} t_{ii}^2 + \sum_{i>j} t_{ij}^2 \overset{d}{=} \sum_{i=1}^{p} y_i + y_{p+1} = \sum_{i=1}^{p+1} y_i . \quad \text{Q.E.D.}$$

6. A Multivariate Analogue to Cochran's Theorems.

In Section 5 we obtained the distributions of $\underset{\sim}{W}_{(m)}$ and denoted it by $MG_{p,2}(\underset{\sim}{\Sigma}; \frac{m}{2}; \frac{n-m}{2}; \phi)$. Let $\underset{\sim}{D}$ be an $n \times n$ symmetric matrix. We want to know a necessary and sufficient condition for $\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X} \sim MG_{p,2}(\underset{\sim}{\Sigma}; \frac{m}{2}; \frac{n-m}{2}; \phi)$.

<u>Theorem 9</u>. Assume that $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and $P(\underset{\sim}{X}=\underset{\sim}{0}) = 0$, and $\underset{\sim}{D}$ is an $n \times n$ symmetric matrix, then $\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X} \sim MG_{p,2}(\underset{\sim}{\Sigma}; \frac{m}{2}; \frac{n-m}{2}; \phi)$ if and only if $\underset{\sim}{D}^2 = \underset{\sim}{D}$ and $rk(\underset{\sim}{D}) = m$.

<u>Proof</u>. Assume $\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X} \sim MG_{p,2}(\underset{\sim}{\Sigma}; \frac{m}{2}; \frac{n-m}{2}; \phi)$. Since $\underset{\sim}{X} \overset{d}{=} R\underset{\sim}{U}\underset{\sim}{A}$, we have (cf. Theorem 7)

$$R^2\underset{\sim}{A}'\underset{\sim}{U}'\underset{\sim}{D}\underset{\sim}{U}\underset{\sim}{A} \overset{d}{=} \underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X} \overset{d}{=} R^2\underset{\sim}{A}'\underset{\sim}{U}_1'\underset{\sim}{U}_1\underset{\sim}{A} ,$$

where U_1 is an $m \times p$ matrix and $U = (U_1', U_2')'$. The condition $P(X=0) = 0$ implies $P(R^2 > 0) = 1$, i.e., R^2 satisfies the condition (2.6). As R^2 is independent of U, by Lemma 1 of Anderson and Fang (1982) we have

$$A'U'DUA = A'U_1'U_1A \ .$$

We can remove A' and A from both sides because $\Sigma > 0$ and A is non-singular matrix; hence

$$U'DU \stackrel{d}{=} U_1'U_1 \ .$$

Let Y be defined by (4.3), then the above formula becomes

$$Y'DY/\mathrm{tr}\ Y'Y \stackrel{d}{=} Y_1'Y_1/\mathrm{tr}\ YY \ ,$$

where Y is partitioned into Y_1 and Y_2 in the same fashion as U_1 and U_2. As $\mathrm{tr}\ Y'Y$ is independent of $Y/\mathrm{tr}\ Y'Y$, we can multiply by $\mathrm{tr}\ Y'Y$ on both sides (cf. Lemma 1 of Anderson and Fang (1982)) and obtain

$$Y'DY \stackrel{d}{=} Y_1'Y_1 \sim W_p(\Sigma, m) \ .$$

From Cochran's Theorem for the multivariate normal distribution (cf. Anderson (1958)), we have $D^2 = D$ and $\mathrm{rk}(D) = m$.

Assume $D^2 = D$ and $\mathrm{rk}D = m$; there exists an orthogonal matrix Γ such that $\Gamma'D\Gamma = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$. Let $Z = \Gamma X$, then $Z \sim LEC_{n\times p}(0, \Sigma, \phi)$ (cf. Corollary 2 of Lemma 2), and

$$X'DX = Z'\Gamma'D\Gamma Z = Z'\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}Z = Z_1'Z_1 \sim MG_{p,2}(\Sigma; \tfrac{m}{2}; \tfrac{n-m}{2}; \phi).$$

Q.E.D.

There is a close relationship between Cochran's Theorem in the univariate case and one in the multivariate case. When the population is normal this relationship has been established (cf. Anderson (1958), Rao (1973)). We will point out similar results for elliptically contoured distributions.

Corollary 1. Assume $\underset{\sim}{X}$ and $\underset{\sim}{D}$ satisfy the condition of Theorem 9. Then $\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X} \sim MG_{p,2}(\underset{\sim}{\Sigma}; \frac{m}{2}; \frac{n-m}{2}; \phi)$ if and only if $\underset{\sim}{\ell}'\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X}\underset{\sim}{\ell} \sim G_2(\frac{m}{2}; \frac{n-m}{2}; \phi)$ for every $\underset{\sim}{\ell} \in \mathbb{R}^p$ with $\underset{\sim}{\ell}'\underset{\sim}{\Sigma}\underset{\sim}{\ell} = 1$, where $\phi \in \Phi_n \leftrightarrow R^* \stackrel{d}{=} Rb_{n/2,n(p-1)/2}$.

Proof. If $\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X} \sim MG_{p,2}(\underset{\sim}{\Sigma}; \frac{m}{2}; \frac{(n-m)}{2}; \phi)$, then $\underset{\sim}{D}^2 = \underset{\sim}{D}$ and $rk(\underset{\sim}{D}) = m$ by Theorem 9. From Corollary 1 of Lemma 2 $\underset{\sim}{X}\underset{\sim}{\ell} \sim LEC_{n\times 1}(\underset{\sim}{0}, \underset{\sim}{\ell}'\underset{\sim}{\Sigma}\underset{\sim}{\ell}, \phi) = EC_n(\underset{\sim}{0}, \underset{\sim}{I}_n, \phi)$ for all $\underset{\sim}{\ell} \in \mathbb{R}^p$ and $\underset{\sim}{\ell}'\underset{\sim}{\Sigma}\underset{\sim}{\ell} = 1$, where $\phi \in \Phi_n \leftrightarrow R^* = R\cdot b_{n/2,n(p-1)/2}$. Thus $\underset{\sim}{\ell}'\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X}\underset{\sim}{\ell} \sim G_2(\frac{m}{2}; \frac{n-m}{2}; \phi)$ by Theorem 1 of Anderson and Fang (1982).

If $\underset{\sim}{\ell}'\underset{\sim}{X}'\underset{\sim}{D}\underset{\sim}{X}\underset{\sim}{\ell} \sim G_2(\frac{m}{2}; \frac{n-m}{2}; \phi)$ for some $\underset{\sim}{\ell} \in \mathbb{R}^p$ and $\underset{\sim}{\ell}'\underset{\sim}{\Sigma}\underset{\sim}{\ell} = 1$ with $\phi \leftrightarrow R^* = Rb_{n/2,n(p-1)/2}$, then $\underset{\sim}{D}^2 = \underset{\sim}{D}$ and $rk\underset{\sim}{D} = m$ by Theorem 1 of Anderson and Fang (1982). The assertion follows by Theorem 9. Q.E.D.

By using a technique similar to the one for proving Theorem 9, we can obtain the following theorem.

Theorem 10. Assume that $\underset{\sim}{X} \sim LEC_{n\times p}(\underset{\sim}{0}, \underset{\sim}{\Sigma}, \phi)$ with $\underset{\sim}{\Sigma} > 0$ and $P(\underset{\sim}{X}=\underset{\sim}{0}) = 0$, and $\underset{\sim}{D}_1, \ldots, \underset{\sim}{D}_k$ are symmetric matrices; then $(\underset{\sim}{X}'\underset{\sim}{D}_1\underset{\sim}{X}, \ldots, \underset{\sim}{X}'\underset{\sim}{D}_k\underset{\sim}{X}) \sim MG_{p,k+1}(\underset{\sim}{\Sigma}; n_1/2, \ldots, n_k/2; n_{k+1}/2; \phi)$ where $n_i \geq p$, $i = 1, \ldots, k$, $n_{k+1} \geq 1$, $\Sigma_1^{k+1} n_i = n$, if and only if $\underset{\sim}{D}_i^2 = \underset{\sim}{D}_i$, $rk(\underset{\sim}{D}_i) = n_i$, $i = 1, \ldots, k$ and $\underset{\sim}{D}_i\underset{\sim}{D}_j = \underset{\sim}{0}$ for $i \neq j$.

7. Applications.

In this section we apply the theory of multivariate elliptically contoured distributions to the multiple regression model.

We consider the following model

$$(7.1)\qquad \begin{cases} \underset{\sim}{Y}_{n\times p} = \underset{\sim}{X}_{n\times q}\underset{\sim}{B}_{q\times p} + \underset{\sim}{E}_{n\times p}\,, \quad p+q \le n\,, \quad rk\underset{\sim}{X} = q \\ \\ E(Vec\underset{\sim}{E}) = 0\,,\ D(vec\underset{\sim}{E}') = \underset{\sim}{I}_n \otimes \underset{\sim}{\Sigma},\ \underset{\sim}{E} \sim LEC_{n\times p}(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi) \quad \text{with } \underset{\sim}{\Sigma} > 0. \end{cases}$$

Minimizing $|\underset{\sim}{E}'\underset{\sim}{E}|$ or $tr\ \underset{\sim}{E}'\underset{\sim}{E}$ with respect to $\underset{\sim}{B}$ gives the least squares

$$(7.2)\qquad \hat{\underset{\sim}{B}} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Y}\,,$$

since

$$(7.3)\qquad \underset{\sim}{E}'\underset{\sim}{E} = (\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{B})'(\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{B}) = (\underset{\sim}{Y}-\underset{\sim}{X}\hat{\underset{\sim}{B}})'(\underset{\sim}{Y}-\underset{\sim}{X}\hat{\underset{\sim}{B}}) + (\hat{\underset{\sim}{B}}-\underset{\sim}{B})'\underset{\sim}{X}'\underset{\sim}{X}(\hat{\underset{\sim}{B}}-\underset{\sim}{B})\,.$$

By assumption we have

$$(7.4)\qquad \underset{\sim}{E} \overset{d}{=} R\underset{\sim}{U}\underset{\sim}{A}\,,$$

where $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$. Thus

$$(7.5)\qquad \underset{\sim}{Y} \overset{d}{=} \underset{\sim}{X}\underset{\sim}{B} + R\underset{\sim}{U}\underset{\sim}{A}$$

and

$$(7.6)\qquad \hat{\underset{\sim}{B}} \overset{d}{=} \underset{\sim}{B} + R(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{U}\underset{\sim}{A}\,.$$

Theorem 11. Under the assumption in the model (7.1) we have

$$\text{(7.7)} \qquad \mathrm{vec}(\hat{\underset{\sim}{B}}-\underset{\sim}{B}) \sim EC_{pq}(\underset{\sim}{0},\underset{\sim}{\Sigma} \otimes (\underset{\sim}{X}'\underset{\sim}{X})^{-1},\phi) \;,$$

where $\phi \in \Phi_{nq} \leftrightarrow R^* = Rb_{pq/2,(n-q)p/2}$.

Proof. From (7.4) and (7.6) we have $\hat{\underset{\sim}{B}}-\underset{\sim}{B} \overset{d}{=} (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{E}$. Then Theorem 11 follows by Corollary 2 of Lemma 2. Q.E.D.

In general the distribution of $\hat{\underset{\sim}{B}}$ does not belong to the class of the multivariate elliptically contoured distributions unless $(\underset{\sim}{X}'\underset{\sim}{X})^{-1} = c^2\underset{\sim}{I}_q$ where c is a constant in which case

$$\text{(7.8)} \qquad \hat{\underset{\sim}{B}}-\underset{\sim}{B} \sim LEC_{q\times p}(\underset{\sim}{0},c^2\underset{\sim}{\Sigma},\phi) \;,$$

where $\phi \in \Phi_{pq} \leftrightarrow R^*$ being defined by Theorem 11.

Corollary. Denote $\hat{\underset{\sim}{B}} = (\hat{\underset{\sim}{\beta}}_1,\ldots,\hat{\underset{\sim}{\beta}}_p)$. Under the assumption of Theorem 11 we have:

(1) $\hat{\underset{\sim}{\beta}}_j \sim EC_q(\beta_j,\sigma_{jj}(\underset{\sim}{X}'\underset{\sim}{X})^{-1},\phi)$, where $\phi \in \Phi_q \leftrightarrow R^{**} = Rb_{q/2,np-q/2}$.

(2) $\mathrm{cov}(\hat{\underset{\sim}{\beta}}_i,\hat{\underset{\sim}{\beta}}_j) = \text{const.}\ \sigma_{ij}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}$.

Proof. The assertion (1) is a consequence of Lemma 2 and Theorem 11. The assertion (2) follows by Theorem 4 of Cambanis, Huang and Simons (1981). Q.E.D.

Now we consider the distribution of $\hat{E}'\hat{E} \equiv (Y-X\hat{B})'(Y-XB)$. As

$$\hat{E}'\hat{E} = Y'(I_n-X(X'X)^{-1}X')Y = E'(I_n-X(X'X)^{-1}X')E$$

and $(I_n-X(X'X)^{-1}X')$ is a projection matrix with rank $n-q$, if $P(E=0) = 0$, then from Theorem 9,

$$S \equiv \hat{E}'\hat{E} \sim MG_{p,2}(\Sigma;(n-q)/2;q/2;\phi) \ . \tag{7.9}$$

Under the model (7.1) we want to test the following linear hypothesis:

$$H\colon HB = C \ , \quad H\colon t\times q \ , \quad C\colon t\times p \quad \text{and} \quad \mathrm{rk}H = t < p \ . \tag{7.10}$$

Under the condition of $HB = C$ minimizing $|E'E|$ or $\mathrm{tr}\, E'E$ with respect to B gives the *least squares estimator*

$$B_H = \hat{B} - (X'X)^{-1}H'(H(X'X)^{-1}H')^{-1}(H\hat{B}-C) \ , \tag{7.11}$$

where $\hat{B}$ is given by (7.2). Since

$$(Y-XB)'(Y-XB) = (Y-X\hat{B}_H)'(Y-X\hat{B}_H) + (\hat{B}_H-B)'X'X(\hat{B}_H-B) \ .$$

Thus

$$\begin{aligned}\min_{HB=C} |(Y-XB)'(Y-XB)| &= |(Y-X\hat{B}_H)'(Y-X\hat{B}_H)| \\ &= |(Y-X\hat{B})'(Y-X\hat{B}) + (H\hat{B}-C)'(H(X'X)^{-1}H')^{-1}(H\hat{B}-C)| \\ &\equiv |S+T| \quad \text{(say)}.\end{aligned}$$

A statistic for testing the hypothesis H is

$$\text{(7.12)} \qquad \lambda = \frac{\min_{\underset{\sim}{B}} |(\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{B})'(\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{B})|}{\min_{\underset{\sim}{H}\underset{\sim}{B}=\underset{\sim}{C}} |(\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{B})'(\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{B})|} = \frac{|\underset{\sim}{S}|}{|\underset{\sim}{S}+\underset{\sim}{T}|} .$$

Noting $\underset{\sim}{E} \overset{d}{=} R\underset{\sim}{U}\underset{\sim}{A}$ and $\hat{\underset{\sim}{B}} \overset{d}{=} \underset{\sim}{B} + (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{U}\underset{\sim}{A}$, if the hypothesis is true we have

$$\underset{\sim}{S} \overset{d}{=} R^2\underset{\sim}{A}'\underset{\sim}{U}'[\underset{\sim}{I}-\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}']\underset{\sim}{U}\underset{\sim}{A}$$

and

$$\begin{aligned} \underset{\sim}{T} &= (\underset{\sim}{H}\hat{\underset{\sim}{B}}-\underset{\sim}{C})'(\underset{\sim}{H}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{H}')^{-1}(\underset{\sim}{H}\hat{\underset{\sim}{B}}-\underset{\sim}{C}) \\ &= (\hat{\underset{\sim}{B}}-\underset{\sim}{B})'\underset{\sim}{H}'(\underset{\sim}{H}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{H}')^{-1}\underset{\sim}{H}(\hat{\underset{\sim}{B}}-\underset{\sim}{B}) \\ &\overset{d}{=} R^2\underset{\sim}{A}'\underset{\sim}{U}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{H}'(\underset{\sim}{H}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{H}')^{-1}\underset{\sim}{H}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{U}\underset{\sim}{A} , \end{aligned}$$

where $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$ (cf. (7.6)).

Substituting them into (7.12) we see that λ is independent of R^2. By the method applied in Section 4, the distribution of λ is the same as in the normal case; i.e. $U_{p,t,n-q}$ (Wilks' distribution).

Theorem 12. Under the model (7.1) and $P(\underset{\sim}{E}=\underset{\sim}{0}) = 0$, the statistic λ given by (7.12) for testing the linear hypothesis (7.10) is distributed according to $U_{p,t,n-q}$.

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Cochran's theorem, correlation coefficients, generalized variance, Hotelling's T^2, multiple correlation coefficients, multiple regression model, multivariate elliptically contoured distributions, sample covariance matrix.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the multivariate elliptically contoured distributions which generalize the elliptically contoured distribution to the case of a matrix are defined and a special class of multivariate elliptically contoured distributions is studied in detail. For this class we obtain the distributions of the following statistics: correlation coefficients, multiple correlation coefficients, Hotelling's T^2, sample covariance matrix, generalized variance, characteristic roots of the covariance matrix, quadratic forms, etc. Some multivariate statistical applications are discussed.

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