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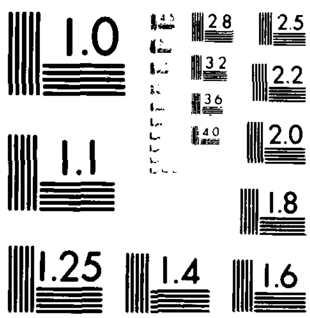

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IMAGE RESTORATION BY THE METHOD OF PROJECTION ONTO CONVEX SETS PART I

Heywood E. Webb

Laboratory Directors' Fund No. LD1215PF

ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
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This report treats the problem of iterative image restoration from projection operators in an infinite dimensional Hilbert Space. A priori information in the form that the solution lies in a convex subset of a closed linear manifold, is assumed. Theoretical and experimental results are included.

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1. Introduction

In a recent paper [11] published by the first author, it was suggested that many problems of image restoration are probably geometric in character and admit the following initial linear formulation: The original f is a vector known a prior to belong to a linear subspace $\mathcal{P}_b$ of a parent Hilbert space $\mathcal{H}$, but all that is available to the observer is its image $P_a f$, the projection of f onto a known linear subspace $\mathcal{P}_a$ (also in $\mathcal{H}$).

1) Find necessary and sufficient conditions under which f is uniquely determined by $P_a f$ and 2) find necessary and sufficient conditions for the stable linear reconstruction of f from $P_a f$ in the face of noise. (In the later case, the reconstruction problem is said to be completely (or well) posed.) The answers turn out to be remarkably simple.

a) f is uniquely determined by $P_a f$ iff $\mathcal{P}_b$ and the orthogonal complement of $\mathcal{P}_a$ have only the zero vector in common.

b) The reconstruction problem is completely posed iff the angle between $\mathcal{P}_b$ and the orthogonal complement of $\mathcal{P}_a$ is greater than zero. (All angles lie in the first quadrant.)

c) *In the absence of noise, there exists in both cases* a) and b) an effective recursive algorithm for the recovery of f employing only the operations of projection onto $\mathcal{P}_b$ and projection onto the orthogonal complement of $\mathcal{P}_a$. These operations define the necessary instrumentation.

This point of view has proved to be very fruitful and can be made to encompass many important applications [12].

However, it appears that a linear formulation is only achievable by discarding information concerning the original f, and as a result the associated restoration problem is often ill-posed [11, Theorem 2]. The entire process of smoothing can be conceptualized as a technique for reintroducing the missing information and therefore plays an essential role in combatting the effects of noise.

As is shown in detail in Part II, a linear image restoration problem transforms, under smoothing, into a nonlinear one of the following kind: The original f is known a priori to belong to the intersection $\mathcal{C}_0$ of m well-defined closed convex sets $C_1, C_2, \ldots, C_m$; i.e.,

$$f \in C_0 = \bigcap_{i=1}^{m} C_i \qquad (1.1)$$

Given only the projection operators P_i onto the individual C_i's, $i = 1 \to m$, restore f, preferably by an iterative scheme. Thus, the P_i's define the necessary instrumentation in an arbitrary Hilbert space setting. We are now in a position to summarize the contents of Part I.

Section 2 provides some basic tutorial material on convex sets and projection operators onto closed convex sets. Section 3 is devoted to an in-depth leisurely study of nonexpansive maps and their fixed points, with special emphasis on iterative methods. Because of the importance of these methods, we have supplied full proofs in order to make the report self-contained. Section 4 applies the theorems in Section 3 to the problem of finding points belonging to the intersection of a finite number of closed convex sets. It is shown that the latter is a particular case of the problem of finding a fixed point of an asymptotically regular nonexpansive operator by iteration. In general, the convergence of this iteration is only weak and the question of strong convergence is examined in some detail. In this regard, Theorem 4.2 is a major result which serves as the starting point for Part II.

Specifically, Section 5 in Part II pursues the consequences of identifying the process of smoothing with that of "opening" up closed convex sets by enlarging them to possess interiors. Section 6 develops algorithms for the realization of several important projection operators onto closed convex sets, and Section 7 describes some numerical results. Lastly, Section 8 concludes with some observations and suggestions for future research.

2. Some Properties of Convex Sets in Hilbert Space

Definition 2.1: A subset $\mathcal{C}$ of $\mathcal{H}$ is said to be convex if together with any x_1 and x_2 it also contains $\mu x_1 + (1-\mu)x_2$ for all μ, $0 \leq \mu \leq 1$.

Theorem 2.1: Let $\mathcal{C}$ denote any closed convex subset of $\mathcal{H}$ and let f be any element of $\mathcal{H}$. Then, there exists a unique $x_0 \in \mathcal{C}$ such that

$$\inf_{x \in \mathcal{C}} \|f-x\| = \|f-x_0\| \quad . \tag{2.1}$$

(Clearly, x_0 is the element in $\mathcal{C}$ closest to f in norm.)

Proof. Let us first demonstrate uniqueness. Suppose that

$$\inf_{x \in \mathcal{C}} \|f-x\| = \delta = \|f-x_0\| = \|f-y_0\| \tag{2.2}$$

where $x_0 \in \mathcal{C}$ and $y_0 \in \mathcal{C}$. Then, by replacing x and y in the identity

$$\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} \tag{2.3}$$

by $f-x_0$ and $f-y_0$, respectively, we obtain

$$\left\|f - \frac{x_0+y_0}{2}\right\|^2 = \delta^2 - \left\|\frac{x_0-y_0}{2}\right\|^2 \leq \delta^2 \quad . \tag{2.4}$$

But, because $\mathcal{C}$ is convex, $\frac{x_0+y_0}{2} \in \mathcal{C}$ so that

$$\left\|f - \frac{x_0+y_0}{2}\right\|^2 \geq \delta^2 \tag{2.5}$$

whence, $\|x_0-y_0\| = 0$ and $x_0 = y_0$.

From the definition of infimum, there exists a sequence $\{x_n\}$ of elements contained in $\mathcal{C}$ such that

$$\lim \|f-x_n\| = \delta \quad . \tag{2.6}$$

By replacing x and y in (2.3) by $f-x_n$ and $f-x_m$ respectively, we get

$$\|x_n-x_m\|^2 = 2(\|f-x_n\|^2 + \|f-x_m\|^2) - 4\left\|f - \frac{x_n+x_m}{2}\right\|^2 \quad . \tag{2.7}$$

Thus, since $\frac{x_n+x_m}{2} \in C$, for $n, m = 1 \to \infty$, (2.5) and (2.7) yield

$$0 \leq \lim_{n,m\to\infty} \|x_n - x_m\|^2 \leq 4\delta^2 - 4\delta^2 = 0 \tag{2.8}$$

which implies that

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0 \quad . \tag{2.9}$$

The sequence $\{x_n\}$ is therefore Cauchy and converges to a limit $x_0 \in C$ because C is closed. From (2.6),

$$\|f - x_0\| = \delta \quad , \tag{2.10}$$

Q.E.D.

Theorem 2.1 is fundamental and leads immediately to the notion of a projection operator: For any $x \in \mathcal{H}$ the projection $P_C x$ of x onto C is the element in C closest to x. As we have seen, if C is closed and convex, $P_C x$ exists and is uniquely determined by x and C from the minimality criterion

$$\|x - P_C x\| = \min_{y \in C} \|x - y\| \quad . \tag{2.11}$$

This rule, which assigns to every $x \in \mathcal{H}$ its nearest neighbor in C, defines the (in general) nonlinear projection operator $P_C : \mathcal{H} \to C$ without ambiguity. Later we shall develop several alternative important characterizations of P_C.

Of course any CLM is automatically convex and closed and the corollary that follows is the classical orthogonal projection theorem.

Corollary 1: Let $\mathcal{G}$ denote any CLM in $\mathcal{H}$. Then, every $x \in \mathcal{H}$ possesses a unique decomposition of the form $x = x_1 + x_2$ where $x_1 = P_{\mathcal{G}} x_1 \in \mathcal{G}$ and $x_2 \in \perp\mathcal{G}$, the orthogonal complement[1] of $\mathcal{G}$.

Proof. Suppose that $x = x_1 + x_2 = x_3 + x_4$ where x_1 and x_3 belong to $\mathcal{G}$ and x_2 and x_4 belong to $\perp\mathcal{G}$. Then, $x_1 - x_3 = x_4 - x_2$ is a member of both $\mathcal{G}$

[1]For any linear manifold $\mathcal{G}$, $y \in \perp\mathcal{G}$ iff $(x, y) = 0$ for all $x \in \mathcal{G}$. It is easily shown that $\perp\mathcal{G}$ is always a CLM and that if $\mathcal{G}$ is a CLM, then $\perp(\perp\mathcal{G}) = \mathcal{G}$. Evidently, $\mathcal{G} \cap (\perp\mathcal{G}) = \{\phi\}$ and $\mathcal{G} \subset \perp(\perp\mathcal{G})$.

and $\perp\mathcal{G}$. Thus, from the previous footnote, $x_1 = x_3$, $x_2 = x_4$ and the decomposition, if it exists, must be unique.

Choose $x_1 = P_{\mathcal{G}}x$ and set $x_2 = x - P_{\mathcal{G}}x$. Then, $x = x_1 + x_2$ and it only remains to show that x_2 is orthogonal to $\mathcal{G}$. That is, $(x_2, y) = 0$ for all $y \in \mathcal{G}$. From the definition of $P_{\mathcal{G}}$ and the convexity of $\mathcal{G}$, for any μ, $0 < \mu < 1$, $\mu y + (1-\mu)P_{\mathcal{G}}x \in \mathcal{G}$ for every $y \in \mathcal{G}$ so that

$$\|x - P_{\mathcal{G}}x\|^2 \leq \|x - \mu y - (1-\mu)P_{\mathcal{G}}x\|^2 \tag{2.12}$$

Or, upon expansion of the right-hand side of (2.12),

$$\|x - P_{\mathcal{G}}x\|^2 \leq \|x - P_{\mathcal{G}}x\|^2 + \mu^2\|y - P_{\mathcal{G}}x\|^2 - 2\mu \mathrm{Re}(x - P_{\mathcal{G}}x, y - P_{\mathcal{G}}x) \quad . \tag{2.13}$$

Hence,

$$2\,\mathrm{Re}(x - P_{\mathcal{G}}x, y - P_{\mathcal{G}}x) \leq \mu\|y - P_{\mathcal{G}}x\|^2 \tag{2.14}$$

and by letting $\mu \to 0$ we find that

$$\mathrm{Re}(x - P_{\mathcal{G}}x, y - P_{\mathcal{G}}x) \leq 0 \tag{2.15}$$

for all $y \in \mathcal{G}$. Now, since $\mathcal{G}$ is also a linear manifold it contains λy for every real λ and (2.14) yields, for λ real,

$$\lambda \mathrm{Re}(x - P_{\mathcal{G}}x, y) \leq (x - P_{\mathcal{G}}x, P_{\mathcal{G}}x) \quad . \tag{2.16}$$

This inequality can only be true if

$$\mathrm{Re}(x - P_{\mathcal{G}}x, y) = 0 \tag{2.17}$$

for all $y \in \mathcal{G}$. If now y in (2.17) is replaced by iy, we get

$$\mathrm{Im}(x - P_{\mathcal{G}}x, y) = 0 \tag{2.18}$$

so that finally

$$(x - P_{\mathcal{G}}x, y) = 0 \ , \quad y \in \mathcal{G} \quad , \tag{2.19}$$

Q.E.D.

Corollary 2: Let $\mathcal{C}$ be a closed convex set. Then, for any $x \in \mathcal{H}$,

$$\mathrm{Re}(x - P_{\mathcal{C}}x, y - P_{\mathcal{C}}x) \leq 0 \tag{2.20}$$

for every $y \in \mathcal{C}$. Conversely, if some $z \in \mathcal{C}$ has the property

$$\mathrm{Re}(x-z, y-z) \leq 0 \tag{2.21}$$

for all $y \in \mathcal{C}$, then $z = P_{\mathcal{C}}x$.

Proof. A review of the proof of corollary 1 reveals that (2.15) is valid for any closed convex set $\mathcal{C}$. Conversely, (2.21) implies that

$$\|x-y\|^2 = \|x-z+z-y\|^2 = \|x-z\|^2 - 2\mathrm{Re}(x-z, y-z) + \|z-y\|^2 \geq \|x-z\|^2 \tag{2.22}$$

for all $y \in \mathcal{C}$. Thus, by Theorem 1, $z = P_{\mathcal{C}}x$, Q.E.D.

Corollary 3: Any closed bounded convex set $\mathcal{C}$ is weakly compact.

Proof. What we have to show is that from every sequence $\{x_n\} \subset \mathcal{C}$ it is possible to extract a subsequence $\{x_{n'}\}$ that converges weakly[2] to some limit f, and that all such weak limits belong to $\mathcal{C}$. By hypothesis, $\mathcal{C}$ is bounded as is therefore every sequence $\{x_n\}$ contained in $\mathcal{C}$. Hence, since Hilbert space $\mathcal{H}$ is weakly compact, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $x_{n'} \rightharpoonup f$, $f \in \mathcal{H}$. Now according to (2.20),

$$\mathrm{Re}(f - P_{\mathcal{C}}f, x_{n'} - P_{\mathcal{C}}f) \leq 0 \tag{2.23}$$

for all n'. Thus, $x_{n'} \rightharpoonup f$ implies that

$$0 \geq \mathrm{Re}(f - P_{\mathcal{C}}f, f - P_{\mathcal{C}}f) = \|f - P_{\mathcal{C}}f\|^2 \geq 0 \tag{2.24}$$

or, $\|f - P_{\mathcal{C}}f\| = 0$. Consequently, $f = P_{\mathcal{C}}f \in \mathcal{C}$, Q.E.D.

The proof of corollary 3 reveals that a closed convex set $\mathcal{C}$, bounded or unbounded, contains all its weak limits and is therefore weakly closed. Now, since a strong limit is automatically weak, it is clear that weak closure of any set implies its strong closure but the converse is in general false. From this point of view, convex sets appear to be quite exceptional.

---

[2] This means that $\lim(g, x_{n'}) = (g, f)$ for every $g \in \mathcal{H}$.

In a real Hilbert space, the inequality (2.20) for closed convex sets $\mathcal{C}$ reads

$$(x-P_{\mathcal{C}}x, y-P_{\mathcal{C}}x) \leq 0 \quad , \quad \text{all } y\in\mathcal{C} \quad . \tag{2.25}$$

In this guise it can be interpreted to mean that the vector $x-P_{\mathcal{C}}x$ is supporting to $\mathcal{C}$ at the point $P_{\mathcal{C}}x\in\mathcal{C}$. As Fig. 2.1 suggests, $x-P_{\mathcal{C}}x$ is "normal" to the "tangent plane" to $\mathcal{C}$ erected at the point $P_{\mathcal{C}}x$. This plane has $\mathcal{C}$ and x on opposite sides and therefore separates one from the other. Note also that the angle ψ between the vectors $x-P_{\mathcal{C}}x$ and $y-P_{\mathcal{C}}x$ is never less than 90°.

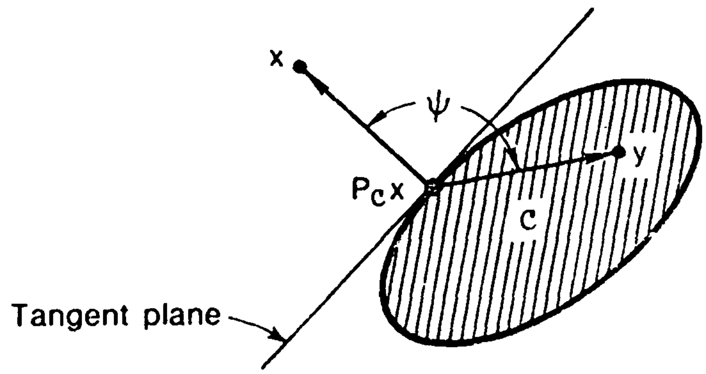


Fig. 2.1. $\mathcal{C}$ is a closed convex set.

Theorem 2.2: Let $\mathcal{C}$ be any closed convex set. Then, for every pair of elements x and y in $\mathcal{H}$,

$$\|P_{\mathcal{C}}x-P_{\mathcal{C}}y\|^2 \leq \mathrm{Re}(x-y, P_{\mathcal{C}}x-P_{\mathcal{C}}y) \quad . \tag{2.26}$$

Proof. Since $P_{\mathcal{C}}x$ and $P_{\mathcal{C}}y$ both belong to $\mathcal{C}$ it follows from (2.20) that

$$\mathrm{Re}(x-P_{\mathcal{C}}x, P_{\mathcal{C}}y-P_{\mathcal{C}}x) \leq 0 \tag{2.27}$$

and

$$\mathrm{Re}(y-P_{\mathcal{C}}y, P_{\mathcal{C}}x-P_{\mathcal{C}}y) \leq 0 \quad . \tag{2.28}$$

Thus, (2.26) is obtained immediately by addition, Q.E.D.

Corollary 1: Projection operators onto closed convex sets $\mathcal{C}$ are nonexpansive and therefore continuous.

Proof. Schwartz's inequality applied to (2.26) yields, for every x and y in $\mathcal{H}$,

$$\|P_{\mathcal{C}}x - P_{\mathcal{C}}y\| \le \|x-y\| \quad . \qquad (2.29)$$

Therefore, under the operator $P_{\mathcal{C}}$, the distance between two images never exceeds the distance between the two originals which is precisely the definition of nonexpansivity. A fortiori, $\|x-y\|$ "small" implies $\|P_{\mathcal{C}}x-P_{\mathcal{C}}y\|$ "small" so that $P_{\mathcal{C}}$ is continuous, Q.E.D.

It is possible to strengthen (2.29) in various directions by imposing one or more constraints on the "curvature" of $\mathcal{C}$. Admittedly, this strengthening appears more natural in a finite-dimensional setting and leads to several technical refinements not fully exploited in this report. Nevertheless, we present the results for possible future applications and because their proofs depend on Theorems 2.1 and 2.2 in an essential way.

Definition 2.2: A convex set $\mathcal{C}$ is said to be strictly convex if $x \in \mathcal{C}$, $y \in \mathcal{C}$ and $x \neq y$ imply that $(x+y)/2$, the midpoint of the "chord" connecting x and y, is an interior point of $\mathcal{C}$.

Definition 2.3: A convex set $\mathcal{C}$ is said to be uniformly convex if there exists a function $\delta(\tau)$ positive for $\tau > 0$, and zero only for $\tau = 0$, such that $x, y \in \mathcal{C}$ and

$$\left\| z - \frac{x+y}{2} \right\| \le \delta(\|x-y\|) \qquad (2.30)$$

imply $z \in \mathcal{C}$.

Definition 2.4: A uniformly convex set $\mathcal{C}$ for which it is possible to choose

$$\delta(\tau) = \mu \tau^2 \quad , \qquad (2.31)$$

μ a positive constant, is said to be strongly convex. (Evidently, strong convexity implies uniform convexity which implies strict convexity.)

Theorem 2.3 [1]: If $\mathcal{C}$ is strictly convex, $y \in \mathcal{C}$, $x \notin \mathcal{C}$, $y \neq P_{\mathcal{C}}x$, we have

$$\mathrm{Re}(x-P_{\mathcal{C}}x, y-P_{\mathcal{C}}x) < 0 \quad . \qquad (2.32)$$

Proof. By definition, if $y \in C$ and $P_C x \neq y$, there exists an $\epsilon > 0$ (which depends in general on the choice of x and y), such that

$$w = z + \frac{P_C x + y}{2} \in C \tag{2.33}$$

for all z that satisfy $\|z\| \leq \epsilon$. Hence, if $x \notin C$ we can choose

$$z = \frac{\epsilon(x - P_C x)}{\|x - P_C x\|} \tag{2.34}$$

and then replace y in (2.20) by w to obtain

$$\mathrm{Re}(x - P_C x, y - P_C x) \leq -2\epsilon \|x - P_C x\| < 0 \quad , \tag{2.35}$$

Q. E. D.

Corollary 3: If C is uniformly convex, $y \in C$ and $x \notin C$, then

$$\mathrm{Re}(x - P_C x, y - P_C x) \leq -2\delta(\|y - P_C x\|) \cdot \|x - P_C x\| \quad . \tag{2.36}$$

If C is strongly convex,

$$\|P_C x - y\| \leq \rho \|x - y\| \tag{2.37}$$

where

$$\rho = (1 + 2\mu \|x - P_C x\|)^{-1} \quad . \tag{2.38}$$

Proof. Observe first that if $y = P_C x$, (2.35) is correct with $\epsilon = 0$. Consequently, in view of the properties of $\delta(\tau)$, (2.36) follows immediately by substituting $\delta(\|y - P_C x\|)$ for ϵ in (2.35). To derive (2.37) we rewrite (2.35) in the form

$$\|P_C x - y\|^2 \leq \mathrm{Re}(x - y, P_C x - y) - 2\epsilon \|x - P_C x\| \quad , \tag{2.39}$$

and then use Schwartz's inequality to obtain

$$\|P_C x - y\|^2 - \|P_C x - y\| \cdot \|x - y\| + 2\epsilon \|x - P_C x\| \leq 0 \quad . \tag{2.40}$$

Or, with ϵ replaced by $\mu \|P_C x - y\|^2$,

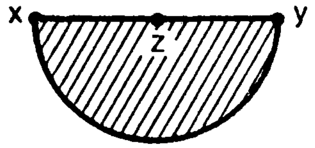
$$\|P_C x - y\| (\|P_C x - y\| + 2\mu \|P_C x - y\| \cdot \|x - P_C x\| - \|x - y\|) \leq 0 \tag{2.41}$$

so that always,

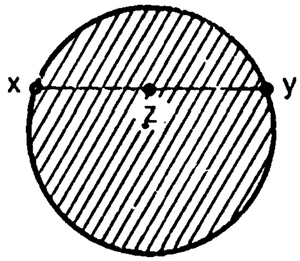
$$\|P_C x - y\| \leq \frac{\|x - y\|}{1 + 2\mu \|x - P_C x\|} , \tag{2.42}$$

Q.E.D.

The closed half-circle C shown in Fig. 2.2a is convex but not strictly convex because the midpoint z is not an interior point. On the other hand, the full closed circle in Fig. 2.2b is the prototype of a strongly convex set.



(a) Closed half-circle.



(b) Closed full-circle.

Figs. 2.2a and 2.2b.

3. Nonexpansive Maps and Their Fixed Points-Basic Theorems

Definition 3.1: A mapping $T: \mathcal{C} \to \mathcal{H}$ is said to be a contraction if there exists a positive constant θ, $0 < \theta < 1$, such that

$$\|Tx - Ty\| \leq \theta \|x - y\| \tag{3.1}$$

for all $x, y \in \mathcal{C}$.

Any contraction has at most one fixed point. For if $Tx_1 = x_1$ and $Tx_2 = x_2$, then

$$\|x_1 - x_2\| = \|Tx_1 - Tx_2\| \leq \theta \|x_1 - x_2\| \tag{3.2}$$

which implies $\|x_1 - x_2\| = 0$ and hence $x_1 = x_2$.

Theorem 3.1: (The contraction principle). If $\mathcal{C}$ is a nonempty closed subset of $\mathcal{H}$, any contraction mapping T of $\mathcal{C}$ into itself possesses a unique fixed point x_∞. Moreover, starting from any element x_0 of $\mathcal{C}$, $T^n x_0 \to x_\infty$ as $n \to \infty$.

Proof. Since we already know that there is at most one fixed point, we need only show that a fixed point exists and is obtainable by iteration. Let x_0 be any member of $\mathcal{C}$ and set

$$x_n = Tx_{n-1} \quad , \quad n = 1 \to \infty \quad . \tag{3.3}$$

Then,

$$\|x_{n+1} - x_n\| \leq \theta \|x_n - x_{n-1}\| \leq \cdots \leq \theta^n \|x_1 - x_0\| \tag{3.4}$$

so that for any $m > n$

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \theta^n (1 + \theta + \cdots + \theta^{m-n-1}) \|x_1 - x_0\| \\ &\leq \theta^n (1 - \theta)^{-1} \|x_1 - x_0\| \quad . \end{aligned} \tag{3.5}$$

Since $0 < \theta < 1$, it follows that $\{x_n\}$ is a Cauchy sequence. Therefore, as $\mathcal{C}$ is closed, it converges to an element $x_\infty \in \mathcal{C}$. But clearly,

$$\|Tx_\infty - x_\infty\| \leq \|Tx_\infty - x_{n+1}\| + \|x_{n+1} - x_\infty\|$$

$$\leq \theta\|x_\infty - x_n\| + \|x_\infty - x_{n+1}\| \qquad (3.6)$$

and the right side tends to zero as $n \to \infty$. Thus, $\|Tx_\infty - x_\infty\| = 0$, i.e., $Tx_\infty = x_\infty$. It should also be noted that (3.5) yields the estimate

$$\|x_\infty - x_n\| \leq \theta^n (1-\theta)^{-1} \|x_1 - x_0\| \quad , \qquad (3.7)$$

Q.E.D.

Corollary: Let T^ℓ, ℓ an integer ≥ 1, be a contraction of a nonempty closed set C into itself. Then, T possesses a unique fixed point x_∞ obtainable as the limit of any sequence $\{T^n x_0\}$, $x_0 \in C$.

Proof. According to Theorem 3.1, there exists a unique $x_\infty \in C$ such that $T^\ell x_\infty = x_\infty$. Hence,

$$Tx_\infty = T(T^\ell x_\infty) = T^\ell (Tx_\infty) \qquad (3.8)$$

and $Tx_\infty \in C$ is also a fixed point of T^ℓ. By uniqueness, $Tx_\infty = x_\infty$ and x_∞ is therefore a fixed point of T. Consider the sequence $\{T^n x_0\}$ as $n \to \infty$ and write $n = q\ell + r$ where q is an integer and $0 \leq r < \ell$. Then, as $n \to \infty$, $q \to \infty$ and

$$\|T^n x_0 - x_\infty\| = \|T^r T^{q\ell} x_0 - T^r x_\infty\| \leq \|T^{q\ell} x_0 - x_\infty\| \to 0 \quad .$$

Consequently, $T^n x_0 \to x_\infty$ as $n \to \infty$, Q.E.D.

Contraction is a very strong requirement to impose on a mapping. A weaker notion is that of "shrinking".

Definition 3.2: A mapping $T: C \to \mathcal{H}$ is said to be shrinking if

$$\|Tx - Ty\| < \|x - y\| \quad , \quad x \neq y \quad , \qquad (3.9)$$

for all $x, y \in C$.

<u>Theorem 3.2</u>: A shrinking map T of a compact[3] set C into itself always possesses a fixed point.

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[3] A subset C of $\mathcal{H}$ is said to be compact if every sequence of elements in C has a subsequence which converges to a limit element in C.

Proof. The numerical function $\|Tx-x\|$ is obviously continuous on $\mathcal{C}$. Hence, since $\mathcal{C}$ is compact there exists an element $x_\infty \in \mathcal{C}$ such that

$$\inf_{x \in \mathcal{C}} \|Tx-x\| = \|Tx_\infty - x_\infty\| \quad . \tag{3.10}$$

But we must have $Tx_\infty = x_\infty$ because otherwise

$$\|T(Tx_\infty) - Tx_\infty\| < \|Tx_\infty - x_\infty\| \quad , \tag{3.11}$$

a contradiction, Q.E.D.

In many applications even shrinking is difficult to achieve and our final weakening of the contraction idea is embodied in the concept of a nonexpansive operator (mapping).

Definition 3.3: A mapping $T: \mathcal{C} \to \mathcal{H}$ is said to be nonexpansive if

$$\|Tx-Ty\| \leq \|x-y\| \tag{3.12}$$

for all $x, y \in \mathcal{C}$.

Theorem 3.3: Let $T: \mathcal{C} \to \mathcal{C}$ be a nonexpansive map whose domain $\mathcal{C}$ is a nonempty closed bounded convex set. Then, T has at least one fixed point.

Proof. Let y_0 be any preselected member of $\mathcal{C}$ and let the set $\mathcal{C}_0 = \{x: x=y-y_0, y \in \mathcal{C}\}$. This translate $\mathcal{C}_0$ is evidently also a closed bounded convex set which moreover contains the zero vector ϕ. Clearly, every $x \in \mathcal{C}_0$ possesses a unique decomposition $x=y-y_0$, $y \in \mathcal{C}$. Let $F: \mathcal{C}_0 \to \mathcal{C}_0$ be defined by

$$Fx = Ty - y_0 \quad . \tag{3.13}$$

This map F is nonexpansive because $x_1 = y_1 - y_0$ and $x_2 = y_2 - y_0$ imply that

$$\|Fx_1 - Fx_2\| = \|Ty_1 - Ty_2\| \leq \|y_1 - y_2\| = \|(y_1-y_0) - (y_2-y_0)\| = \|x_1 - x_2\| \quad . \tag{3.14}$$

For any fixed k, $0 < k < 1$, the map $G = kF$ is a contraction of $\mathcal{C}_0$ into itself. Indeed, for any $x \in \mathcal{C}_0$, $kFx = k(Fx) + (1-k)\phi \in \mathcal{C}_0$ and for all $x_1, x_2 \in \mathcal{C}_0$,

$$\|Gx_1 - Gx_2\| = k\|Fx_1 - Fx_2\| \leq k\|x_1 - x_2\| \quad . \tag{3.15}$$

Hence, invoking Theorem 3.1, there exists for every constant k, $0 < k < 1$, a unique $x_k \in \mathcal{C}_0$ such that

$$x_k = kFx_k \quad . \tag{3.16}$$

If it can now be shown that $x_k \to g$ as $k \to 1$ from below, then by the continuity of F and $g \in \mathcal{C}_0$, it is seen from (3.16) that $g = Fg$. Or, since $g = f - y_0$, $f \in \mathcal{C}$, we obtain $f = Tf$ so that f is a fixed point of T. We shall actually prove that

$$\lim_{k \to 1; 0<k<1} x_k = g \tag{3.17}$$

where g is the (unique) fixed point of F in $\mathcal{C}_0$ of minimum norm.

Assume that $0 < k < \ell \leq 1$, $x_k = kFx_k$, $x_\ell = \ell Fx_\ell$ and let $h = x_\ell - x_k$. Then, since $\|Fx_\ell - Fx_k\| \leq \|x_\ell - x_k\|$, we obtain

$$(\ell^{-1}(x_k + h) - k^{-1}x_\ell \ , \ \ell^{-1}(x_k + h) - k^{-1}x_k) \leq \|h\|^2 \tag{3.18}$$

or,

$$(\ell^{-1} - k^{-1})^2 \|x_k\|^2 + (\ell^{-2} - 1)\|h\|^2 \leq 2\ell\,(k^{-1} - \ell^{-1})\,\mathrm{Re}(x_k, h) \ . \tag{3.19}$$

Thus,

$$\mathrm{Re}(x_k, h) \geq 0 \tag{3.20}$$

which, together with the identity

$$\|x_\ell\|^2 = \|x_k + h\|^2 = \|x_k\|^2 + \|h\|^2 + 2\,\mathrm{Re}(x_k, h) \ , \tag{3.21}$$

leads to the inequality

$$\|x_\ell\|^2 \geq \|x_k\|^2 + \|x_\ell - x_k\|^2 \quad . \tag{3.22}$$

To sum up, for any choice of sequence $0 < k_1 < k_2 < \cdots$ such that $k_i \to 1$, the sequence $\{\|x_{k_i}\|\}$ is monotone nondecreasing and bounded (because $\mathcal{C}_0$ is bounded). It therefore converges and, in particular, from (3.22),

$$\|x_\ell - x_k\|^2 \leq \|x_\ell\|^2 - \|x_k\|^2 \to 0 \tag{3.23}$$

as ℓ, $k \to \infty$. By the completeness of $\mathcal{H}$, $x_{k_i} \to g \in \mathcal{C}_0$ because $\mathcal{C}_0$ is closed. (Of course, the limit g is independent of the particular selection of sequence $k_i \to 1$.)

Lastly, let e be any fixed point of F in $\mathcal{C}_0$. Then, $e = 1 \cdot Fe$ and we can apply (3.22) with $x_\ell = e$, $\ell = 1$, $x_k = x_{k_i}$ and $k = k_i$ for any $i = 1 \to \infty$. As $i \to \infty$, $x_{k_i} \to g$ so that

$$\|e\|^2 \geq \|g\|^2 + \|e-g\|^2 \geq \|g\|^2 \quad . \tag{3.24}$$

Therefore, $\|g\| = \inf \|e\|$, as e ranges over the fixed points of F in $\mathcal{C}_0$, Q.E.D.

Theorem 3.3 is due to Browder [2] but the proof we have presented is, in all its essentials, that given by B. Halpern [3]. Unlike Theorem 3.2 (which it does not subsume), the compactness assumption on $\mathcal{C}$ is dispensed with completely and replaced instead by the much weaker constraints of convexity and boundedness. Nevertheless, in many signal-processing applications even this assumption of boundedness may be questionable because accurate a priori numerical bounds are not always available. However, if the existence of a fixed point is known in advance from say, physical considerations, the boundedness requirement on $\mathcal{C}$ can be dropped. Our immediate objective therefore is to reach Theorem 3.4 (Opial) and to accomplish this we need three important preparatory lemmas, the last two of which we also owe to Opial [4].

Lemma 3.1: The set of fixed points $\mathcal{J}$ of a nonexpansive mapping T with closed convex domain $\mathcal{C}$ and range $\mathcal{H}$ is a closed convex set.

Proof. Let $x_i = Tx_i$, $i = 1 \to \infty$, and suppose that $x_i \to x$. Since $\{x_i\} \subset \mathcal{C}$ which is closed, $x \in \mathcal{C}$ and Tx is well-defined. Thus, invoking nonexpansivity,

$$\|Tx - x\| = \|Tx - Tx_i + x_i - x\| \leq 2\|x - x_i\| \to 0 \tag{3.25}$$

so that $Tx = x$ and $\mathcal{J}$ is closed. To establish convexity we need a very useful inequality.

For any pair $(x, y) \in \mathcal{C}$ we have the easily verified identity

$$\|x-y\|^2 - \|Tx-Ty\|^2 = 4\,\mathrm{Re}(Px-Py, (1-P)x-(1-P)y) \tag{3.26}$$

where 1 is the identity operator and

$$P = \frac{1+T}{2} \quad . \tag{3.27}$$

But T is nonexpansive hence,

$$\mathrm{Re}(Px-Py, (1-P)x-(1-P)y) \geq 0 \tag{3.28}$$

for every $x, y \in C$. Since P and T have precisely the same fixed points it suffices to show that the set of fixed points of P is convex.

Let y be any fixed point of P. Then (3.28) reduces to

$$\mathrm{Re}(Px-y, x-Px) \geq 0 \tag{3.29}$$

for all $x \in C$. Conversely, if some $y \in C$ satisfies (3.29) for every $x \in C$ it satisfies it for $x=y$ which implies that $\|y-Py\| \leq 0$ or $y = Py$. In short, the set of fixed points of T is the set of all $y \in C$ that satisfy (3.29) for all $x \in C$. But this set is obviously convex, Q.E.D.

Corollary: The map T with closed convex domain C is nonexpansive iff $P = (1+T)/2$ satisfies (3.28) for all $x, y \in C$.

Definition 3.4: A map $T: C \to \mathcal{H}$ is said to be demiclosed if from

$$\{x_n\} \subset C \;,\; x_n \rightharpoonup x_o \;,\; x_o \in C \;,\; Tx_n \to y_o \tag{3.30}$$

follows $Tx_0 = y_0$.[4]

Definition 3.5: A map $T: C \to C$ is said to be asymptotically regular if for every $x \in C$, $T^n x - T^{n+1} x \to \phi$.

Lemma 3.2: In a Hilbert space $\mathcal{H}$ let the sequence $\{x_n\}$ converge weakly to x_0. Then, for any $x \neq x_0$,

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[4] In words, T is demiclosed if for any sequence $\{x_n\} \subset C$ which converges weakly to $x_0 \in C$, the strong convergence of the sequence $\{Tx_n\}$ to y_0 implies that $Tx_0 = y_0$.

$$\liminf_{n\to\infty} \|x_n - x\| > \liminf_{n\to\infty} \|x_n - x_0\| \quad . \tag{3.31}$$

Proof. Since a weakly convergent sequence is bounded, both limits in (3.31) are finite. Thus, to prove this inequality it suffices to note that in the estimate

$$\|x_n - x\|^2 = \|x_n - x_0 + x_0 - x\|^2 = \|x_n - x_0\|^2 + \|x_0 - x\|^2 + 2\mathrm{Re}(x_n - x_0, x_0 - x) \tag{3.32}$$

the last term tends to zero as $n \to \infty$, Q.E.D.

Lemma 3.3: Let T be any nonexpansive map with closed convex domain $\mathcal{C} \subset \mathcal{H}$. Then, $1-T$ is demiclosed.

Proof. Let $\{x_n\} \subset \mathcal{C}$ converge weakly to x_0 and let $\{x_n - Tx_n\}$ converge strongly to y_0. Then, since T is nonexpansive,

$$\liminf_{n\to\infty} \|x_n - x_0\| \geq \liminf_{n\to\infty} \|Tx_n - Tx_0\| = \liminf_{n\to\infty} \|Tx_n - x_n + x_n - Tx_0\| =$$

$$= \liminf_{n\to\infty} \|x_n - y_0 - Tx_0\| \geq \liminf_{n\to\infty} \|x_n - x_0\| \tag{3.33}$$

by lemma 3.2. Hence, again by lemma 3.2, $x_0 = y_0 + Tx_0$ or $(1-T)x_0 = y_0$ so that $1-T$ is demiclosed, Q.E.D.

<u>Theorem 3.4</u>: Let $T: \mathcal{C} \to \mathcal{C}$ be an asymptotically regular nonexpansive map with closed convex domain $\mathcal{C} \subset \mathcal{H}$ whose set of fixed points $\mathfrak{F} \subset \mathcal{C}$ is nonempty. Then, for any $x \in \mathcal{C}$, the sequence $\{T^n x\}$ is weakly convergent to an element of $\mathfrak{F}$.

Proof. For every $y \in \mathfrak{F}$, the sequence $\{d_n\} = \{\|T^n x - y\|\}$ is nonincreasing because

$$d_{n+1} = \|T^{n+1}x - y\| = \|T(T^n x) - Ty\| \leq \|T^n x - y\| = d_n \quad . \tag{3.34}$$

Thus, the nonnegative limit

$$d(y) = \lim_{n\to\infty} \|T^n x - y\| \tag{3.35}$$

exists as a finite number for every $y \in \mathfrak{F}$.

According to lemma 3.1, $\mathfrak{T}$ is a closed convex subset of $\mathfrak{C}$ and it follows that for any fixed $\delta \geq 0$, the set

$$\mathfrak{T}_\delta = \{y \in \mathfrak{T} : d(y) \leq \delta\} \tag{3.36}$$

is a closed bounded convex subset of $\mathfrak{T}$ which is nonempty for δ large enough. Indeed, convexity and closure are obvious from (3.35), and boundedness is implied by the inequalities

$$\|y\| = \|y - T^n x + T^n x\| \leq \|T^n x - y\| + \|T^n x\| \tag{3.37}$$

and

$$\|T^n x\| = \|T^n x - y_0 + y_0\| \leq \|T^n x - y_0\| + \|y_0\| \quad , \tag{3.38}$$

where y_0 is any preselected member of $\mathfrak{T}$. Explicitly, for any $y \in \mathfrak{T}_\delta$

$$\|y\| \leq \delta + d(y_0) + \|y_0\| \quad . \tag{3.39}$$

Since bounded closed convex sets are weakly compact (Theorem 2.1, Corollary 3), the intersection of all such nonempty $\mathfrak{T}_\delta$'s is a nonempty closed bounded convex set $\mathfrak{T}_{\delta_0}$. Clearly, δ_0 is the smallest value of δ for which $\mathfrak{T}_\delta$ is nonempty. The set $\mathfrak{T}_{\delta_0}$ can contain only one element say y_0, for if we suppose that it also contains $y_1 \neq y_0$, the identity

$$\left\|T^n x - \frac{y_0 + y_1}{2}\right\|^2 = \frac{1}{2}\left(\|T^n x - y_0\|^2 + \|T^n x - y_1\|^2\right) - \left\|\frac{y_0 - y_1}{2}\right\|^2 \tag{3.40}$$

yields $d(\frac{y_0 + y_1}{2}) < \delta_0$ which contradicts the meaning of δ_0.

The sequence $\{T^n x\}$ converges weakly to y_0. In fact, since this sequence is bounded, it suffices to prove that all possible weak limits of its various subsequences equal y_0.[5] Let $T^{n'} x \rightharpoonup y_1 \neq y_0$. Then, from the

[5] Assume that such is the case but $T^n x \not\rightharpoonup y_0$. Then, for some $f \in \mathfrak{H}$ it is true that the sequence $\{(T^n x, f)\}$ fails to converge to (y_0, f). Hence, there exists a subsequence $\{T^{n'} x\}$ of $\{T^n x\}$ such that $\lim (T^{n'} x, f)$ exists and is unequal to (y_0, f). But by hypothesis, the sequence $\{T^{n'} x\}$ itself contains a subsequence $\{T^{n''} x\}$ converging weakly to y_0; i.e., $(y_0, f) = \lim (T^{n''} x, f) = \lim (T^{n'} x, f) \neq (y_0, f)$, a contradiction.

asymptotic regularity of T,

$$T^{n'}x - T^{n'+1}x = (1-T)T^{n'}x \to \phi \qquad (3.41)$$

and invoking the demiclosed character of 1-T (lemma 3.3), $(1-T)y_1 = \phi$; i.e., y_1 is a fixed point of T. By lemma 3.2,

$$\delta_0 = \lim \| T^{n'}x - y_0 \| > \lim \| T^{n'}x - y_1 \| = d(y_1) \qquad (3.42)$$

which is incompatible with the meaning of δ_0. Thus, for every $x \in C$, the sequence $\{T^n x\}$ is weakly convergent to a fixed point of T, Q.E.D.

Corollary: The sequence $\{T^n x\}$ converges strongly to y_0 iff at least one of its subsequences converges strongly.

Proof. We know that $T^n x \rightharpoonup y_0$, a fixed point of T, for every choice of $x \in C$. Clearly, since the weak and strong limits of a sequence must coincide, the only possible strong limit of any subsequence of $\{T^n x\}$ is y_0. Now, from the equality

$$\| T^n x - y_0 \|^2 = \| T^n x \|^2 - 2\mathrm{Re}(T^n x, y_0) + \| y_0 \|^2 \qquad (3.43)$$

we find that

$$\lim \| T^n x \|^2 = d^2(y_0) + \| y_0 \|^2 \qquad (3.44)$$

as $n \to \infty$. In particular, for any subsequence $\{T^{n'}x\}$ of $\{T^n x\}$,

$$\lim \| T^{n'}x \|^2 = d^2(y_0) + \| y_0 \|^2 \quad . \qquad (3.45)$$

But as we have already remarked, if any subsequence $\{T^{n'}x\}$ converges strongly, it converges strongly to y_0 and therefore $\lim \| T^{n'}x \|^2 = \| y_0 \|^2$ so that necessarily, $d^2(y_0) = 0$ and

$$T^n x \to y_0 \quad , \qquad (3.46)$$

Q.E.D.

This "all or nothing" aspect of strong convergence exhibited by the iterated sequence $\{T^n x\}$ appears to be the rule at the present level of generality. Unfortunately, the lack of strong convergence is a real obstacle from a numerical standpoint, and a more in-depth analysis is undertaken in the next section.

Definition 3.6: A mapping $T: C \to C$ is said to be a reasonable wanderer if for every $x \in C$

$$\sum_{n=0}^{\infty} \|T^n x - T^{n+1} x\|^2 < \infty \quad . \tag{3.47}$$

It should be evident that a reasonable wanderer is automatically asymptotically regular (Definition 3.5).

An example of a nonexpansive operator that is not asymptotically regular is easily constructed. On the real line choose $C = [-1, 1]$ and define T by $Tx = -x$. Clearly, T is nonexpansive, maps C into C and its only fixed point is $x = 0$. However, $T^n x - T^{n+1} x = 2(-1)^n x \not\to 0$ as $n \to \infty$ unless $x = 0$. Nevertheless, Browder and Petryshyn have shown that the convex linear combination

$$T_\alpha = \alpha 1 + (1-\alpha) T \tag{3.48}$$

of T and the identity operator 1 is a reasonable wanderer for all α in the interval $0 < \alpha < 1$ and any nonexpansive T.

Theorem 3.5 [5]: Let $T: C \to C$ be a nonexpansive map with closed convex domain C whose set of fixed points is nonempty. Then, for any fixed α, $0 < \alpha < 1$, $T_\alpha: C \to C$ is a reasonable wanderer and the sequence $\{T^n x\}$ converges weakly to a fixed point of T for every $x \in C$.

Proof. Clearly, T_α maps C into C so that the iterates $x_n = T_\alpha^n x$ are well-defined for all $x \in C$. In addition, for $\alpha \neq 1$ the fixed points of T and T_α coincide and it only remains to prove that T_α is a reasonable wanderer since it is obviously nonexpansive. Let $y \in C$ be a fixed point of T. Then, $y = Ty = T_\alpha y$ and

$$\|x_{n+1} - y\|^2 = \|\alpha x_n + (1-\alpha) T x_n - y\|^2 = \|\alpha(x_n - y) + (1-\alpha)(T x_n - y)\|^2 \tag{3.49}$$

$$= \alpha^2 \|x_n - y\|^2 + 2\alpha(1-\alpha) \mathrm{Re}(x_n - y, T x_n - y) + (1-\alpha)^2 \|T x_n - y\|^2 \quad . \tag{3.50}$$

Similarly,

$$\|x_n-Tx_n\|^2 = \|x_n-y-(Tx_n-y)\|^2 = \|x_n-y\|^2 - 2\mathrm{Re}(x_n-y, Tx_n-y) + \|Tx_n-y\|^2 \tag{3.51}$$

which after multiplication by $\alpha(1-\alpha)$ and addition to (3.51) yields

$$\|x_{n+1}-y\|^2 + \alpha(1-\alpha)\|x_n-Tx_n\|^2 = \alpha\|x_n-y\|^2 + (1-\alpha)\|Tx_n-y\|^2 \leq \|x_n-y\|^2 . \tag{3.52}$$

Hence,

$$\alpha(1-\alpha)\|x_n-Tx_n\|^2 \leq \|x_n-y\|^2 - \|x_{n+1}-y\|^2 \tag{3.53}$$

so that

$$\alpha(1-\alpha)\sum_{n=0}^{N}\|x_n-Tx_n\|^2 = \frac{\alpha}{1-\alpha}\sum_{n=0}^{N}\|T_\alpha^n x - T_\alpha^{n+1}x\|^2 \leq \|x-y\|^2 - \|x_{N+1}-y\|^2 \leq \|x-y\|^2. \tag{3.54}$$

Consequently, since $\alpha \neq 0$,

$$\sum_{n=0}^{\infty}\|T_\alpha^n x - T^{n+1}x\|^2 < \infty \quad , \tag{3.55}$$

Q.E.D.[6]

[6]For $\alpha = 1/2$, $T_\alpha = (1+T)/2$ in which case Theorem 3.5 is easily deduced from the inequality (3.28).

4. Iterative Techniques for Image Restoration in a Hilbert Space Setting

According to the discussion initiated in the introduction, there exists a class of image restoration problems in which the unknown f can be ascertained to lie in an intersection

$$\mathcal{C}_0 = \bigcap_{i=1}^{m} \mathcal{C}_i \tag{4.1}$$

of well-defined closed convex sets $\mathcal{C}_i$, $i = 1 \to m$. Evidently, $\mathcal{C}_0$ is also a closed nonempty convex set containing f and we shall denote the respective projection operators projecting onto $\mathcal{C}_0$ and $\mathcal{C}_i$ by P_0 and P_i, $i = 1 \to m$. Clearly, f is a fixed point of P_0 and all the P_i's. More generally, f is a fixed point of P_0 and of every

$$T_i = 1 + \lambda_i(P_i - 1) \tag{4.2}$$

for any choice of relaxation constants $\lambda_1, \lambda_2, \ldots, \lambda_m$. Hence, under the same conditions, f is a fixed point of P_0 and the composition operator

$$T = T_m T_{m-1} \cdots T_1 \quad . \tag{4.3}$$

Conversely, can we say that any fixed point f of T lies in $\mathcal{C}_0$ and does the sequence $\{T^n x\}$ converge (weakly or strongly) to a point of $\mathcal{C}_0$ as n tends to infinity, irrespective of the initialization $x \in \mathcal{H}$?

Theorem 4.1: Let $\mathcal{C}_0$ be nonempty. Then, for every $x \in \mathcal{H}$ and every choice of constants $\lambda_1, \lambda_2, \ldots, \lambda_m$ in the interval $0 < \lambda < 2$, the sequence $\{T^n x\}$ converges weakly to a point of $\mathcal{C}_0$.

Proof. For $\lambda_i \geq 0$, every

$$T_i = 1 + \lambda_i(P_i - 1) = (1 - \lambda_i)1 + \lambda_i P_i \tag{4.4}$$

is nonexpansive. The assertion is obviously correct for $0 \leq \lambda_i \leq 1$, but if $1 < \lambda_i$ we have $1 - \lambda_i < 0$ and it is necessary to reason differently. With the aid of (2.27) and (2.30), it is found that

$$\|T_i x - T_i y\|^2 = \|(1-\lambda_i)(x-y) + \lambda_i(P_i x - P_i y)\|^2$$

$$= (1-\lambda_i)^2\|x-y\|^2 + 2\lambda_i(1-\lambda_i)\mathrm{Re}(x-y, P_i x - P_i y) + \lambda_i^2\|P_i x - P_i y\|^2$$

$$\leq (1-\lambda_i)^2\|x-y\|^2 + (\lambda_i^2 + 2\lambda_i(1-\lambda_i))\|P_i x - P_i y\|^2$$

$$= (1-\lambda_i)^2\|x-y\|^2 + \lambda_i(2-\lambda_i)\|P_i x - P_i y\|^2$$

$$\leq (\lambda_i(2-\lambda_i) + (1-\lambda_i)^2)\|x-y\|^2 = \|x-y\|^2 \tag{4.6}$$

and nonexpansivity is established. We now show that T is a reasonable wanderer.

For $m=1$, $T=T_1$, $\mathcal{C}_0=\mathcal{C}_1$ and

$$\|x-Tx\|^2 = \lambda_1^2\|x-P_1x\|^2 \quad . \tag{4.7}$$

Moreover, for any $y \in \mathcal{C}_0$, $Ty = P_1 y = y$ and

$$\|Tx-y\|^2 = \|x-y+\lambda_1(P_1x-x)\|^2 \tag{4.8}$$

$$= \|x-y\|^2 + 2\lambda_1 \mathrm{Re}(x-y, P_1x-x) + \lambda_1^2\|x-P_1x\|^2$$

$$= \|x-y\|^2 - \lambda_1(2-\lambda_1)\|x-P_1x\|^2 + 2\lambda_1\mathrm{Re}(x-P_1x, y-P_1x) \tag{4.8a}$$

$$\leq \|x-y\|^2 - \lambda_1(2-\lambda_1)\|x-P_1x\|^2 \tag{4.9}$$

because the last term in (4.8a) is nonpositive. Thus, by combining (4.7) and (4.9) it is found that

$$\|x-Tx\|^2 \leq \frac{\lambda_1}{2-\lambda_1}(\|x-y\|^2 - \|Tx-y\|^2) \tag{4.10}$$

for $0 < \lambda_1 < 2$.

For arbitrary $m \geq 1$, a straightforward induction on m yields the inequality

$$\|x-Tx\|^2 \leq b_m \cdot 2^{m-1}(\|x-y\|^2 - \|Tx-y\|^2) \tag{4.11}$$

where

$$b_m = \sup_{1 \leq i \leq m} \left\{ \frac{\lambda_i}{2-\lambda_i} \right\} . \tag{4.12}$$

(Clearly, (4.10) subsumes the case $m = 1$.) In fact, let $T = T_m K$ where

$$K = T_{m-1} T_{m-2} \cdots T_1 \tag{4.13}$$

and observe that for $m \geq 2$,

$$\begin{aligned} \|x-Tx\|^2 &= \|x-Kx+Kx-Tx\|^2 \\ &\leq (\|x-Kx\| + \|Kx-Tx\|)^2 \\ &\leq 2(\|x-Kx\|^2 + \|Kx-Tx\|^2) \\ &\leq 2(\|x-Kx\|^2 + 2^{m-2}\|Kx-T_mKx\|^2) . \end{aligned} \tag{4.15}$$

Thus, by the induction hypothesis,[7]

$$\|x-Tx\|^2 \leq b_m \cdot 2(2^{m-2}\|x-y\|^2 - 2^{m-2}\|Kx-y\|^2 + 2^{m-2}\|Kx-y\|^2 - 2^{m-2}\|Tx-y\|^2)$$

$$= b_m \cdot 2^{m-1}(\|x-y\|^2 - \|Tx-y\|^2) , \tag{4.16}$$

the desired inequality.

It now follows immediately from (4.16) that

$$\sum_{n=0}^{\infty} \|T^n x - T^{n+1} x\|^2 \leq b_m \, 2^{m-1} \|x-y\|^2 < \infty \tag{4.17}$$

and T is therefore a reasonable wanderer and, a fortiori, asymptotically regular. By Theorem 3.4, the sequence $\{T^n x\}$ converges weakly to a fixed point of T.

---

[7] $b_m \geq \frac{\lambda_m}{2-\lambda_m}$ and $b_m \geq \sup_{1=i \leq m-1} \left\{ \frac{\lambda_i}{2-\lambda_i} \right\}$.

However, the fixed points of T coincide precisely with the points of $\mathcal{C}_0$, the intersection of the $\mathcal{C}_i$'s. Indeed, it is obvious that $x \in \mathcal{C}_0$ implies $x = Tx$ since $x \in \mathcal{C}_i$, $i = 1 \rightarrow m$. Conversely, if $x = Tx$ and $y \in \mathcal{C}_0$,

$$\|x-y\| = \|Tx-Ty\| \leq \|T_1x-T_1y\| = \|T_1x-y\| \leq \|x-y\| \tag{4.18}$$

hence, $\|x-y\| = \|T_1x-y\|$. In view of (4.10) this is only possible if $x = T_1x$ so that $x = T_mT_{m-1} \cdots T_2x$ and a repetition of the argument finally leads to $x \in \mathcal{C}_i$, $i = 1 \rightarrow m$; i.e., $x \in \mathcal{C}_0$ and the proof is complete, Q.E.D.[8]

We are now in a position to address the question of strong convergence. Let $T: \mathcal{C} \rightarrow \mathcal{C}$ denote a nonexpansive asymptotically regular operator with closed convex domain $\mathcal{C} \subset \mathcal{H}$ whose set $\mathfrak{F}$ of fixed points is nonempty. From the corollary to Theorem 3.4, the sequence $\{T^n x\}$ converges strongly to a member of $\mathfrak{F}$ iff at least one of its subsequences converges strongly. Clearly, such is assured if all iterates $T^n x$ lie ultimately in some compact or finite-dimensional subset of $\mathcal{H}$.

The latter possibility exists even if $\mathcal{C}$, the domain of T, is itself not compact as is seen in the case of the composition operator T of Eq. (4.3) with all λ_i's equal to one. Here, $\mathcal{C} = \mathcal{H}$, $T_i = P_i$, $i = 1 \rightarrow m$,

$$T = P_m P_{m-1} \cdots P_1 \tag{4.19}$$

and all iterates $T^n x$, $n \geq 1$, lie in $\mathcal{C}_m$. It suffices therefore that $\mathcal{C}_m$ be compact or finite-dimensional.[9] (Assumptions of this kind appear to be quite natural for those digital restoration problems falling within the scope of Theorem 4.1.)

It is interesting and important to note that if all $\mathcal{C}_i$'s in Theorem 4.1 are CLM's, the sequence of iterates $\{T^n x\}$ not only converges strongly but actually converges to the projection of x onto $\mathcal{C}_0$. This result is an extension of I. Halperin's neat generalization of Von Neumann's celebrated alternating projection theorem [6] from m = 2 to arbitrary [7].

---

[8]That a composition of appropriately relaxed projection operators is automatically a reasonable wanderer is of course directly attributable to the special properties of projections onto closed convex sets.

[9]By this we mean that $\mathcal{C}_m$ is contained in a finite-dimensional linear manifold $\subset \mathcal{H}$.

Corollary 1: Under the constraints of Theorem 4.1, if every $\mathcal{C}_i$ is a CLM, T^n converges strongly to P_0, the orthogonal projection operator onto the CLM $\mathcal{C}_0$.

Proof.[10] Let us first remark that now all P_i's and P_0 are bounded linear self-adjoint operators. Consequently,

$$T_i^* = 1 + \lambda_i(P_i^* - 1) = 1 + \lambda_i(P_i - 1) = T_i \tag{4.20}$$

is self-adjoint and linear, $i = 1 \rightarrow m$, and T^n is also linear for all $n \geq 0$. Moreover, since T is nonexpansive and $T\phi = \phi$,

$$\|Tx\| = \|Tx - T\phi\| \leq \|x - \phi\| = \|x\| \tag{4.21}$$

for all $x \in \mathcal{H}$ so that $\|T\|$ and therefore $\|T^n\| \leq 1$, $n \geq 0$.

As has already been shown in Theorem 4.1, for every $x \in \mathcal{H}$,

$$T^n x - T^{n+1} x = T^n(1-T)x \rightarrow \phi \tag{4.22}$$

hence, $T^n y \rightarrow \phi$ for all y in $R(1-T)$, the range of $1-T$. Now [10] the closure $[R(1-T)]$ of the range of the bounded operator $1-T$ is the orthogonal complement of the null space of

$$(1-T)^* = 1 - (T_m T_{m-1} \ldots T_1)^* = 1 - T_1 T_2 \ldots T_m \ , \tag{4.23}$$

from which it is deduced that x belong to this null space iff

$$x = T_1 T_2 \ldots T_m x \ . \tag{4.24}$$

Thus, in view of Theorem 4.1, this null space is identical with $\mathcal{C}_0$ so that by the projection theorem, $\mathcal{H} = \mathcal{C}_0 \dotplus [R(1-T)]$.

Let $y \in [R(1-T)]$. Then, there exists a sequence $\{y_i\} \subset R(1-T)$ such that $y_i \rightarrow y$. Clearly,

$$\|T^n y\| = \|T^n y_i + T^n y - T^n y_i\| \leq \|T^n y_i\| + \|y - y_i\| \ , \tag{4.25}$$

$n \geq 1$, which immediately gives

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[10] Since $\phi \in \mathcal{C}_i$, $i = 1 \rightarrow m$, $\phi \in \mathcal{C}_0$ which is therefore never empty.

$$\limsup_{n\to\infty} \| T^n y \| \le \| y - y_i \| \tag{4.26}$$

because $T^n y_i \to \phi$ for every fixed $i \geq 1$. But $\|y - y_i\| \to 0$ so that $T^n y \to \phi$ as $n \to \infty$ for every $y \in [R(1-T)]$. Finally, again from the orthogonal projection theorem, every $x \in \mathcal{H}$ admits a unique decomposition $x = z + y$ where $z \in \mathcal{C}_0$ and $y \in [R(1-T)]$. Consequently, $T^n z = z$ for all $n \geq 1$ and

$$T^n x = T^n z + T^n y = z + T^n y \to z = P_0 x \tag{4.27}$$

as $n \to \infty$. Therefore, by definition, T^n converges strongly to the orthogonal projection operator P_0 onto $\mathcal{C}_0$, Q.E.D.

Corollary 2: If each $\mathcal{C}_i$ is a linear variety,[11] $i = 1 \to m$, and if the intersection $\mathcal{C}_0$ of these varieties is nonempty, the sequence $\{T^n x\}$ converges strongly to $P_0 x$ for every $x \in \mathcal{H}$.

Proof. To avoid unnecessary notational complications we assume that $m = 3$. Thus, $\mathcal{C}_i = g_i + \mathcal{S}_i$ where g_i is a fixed element of $\mathcal{H}$ and $\mathcal{S}_i$ is a given CLM, $i = 1 \to 3$. It is easily shown that for each i and any $x \in \mathcal{H}$,

$$P_i x = S_i x + R_i g_i \tag{4.28}$$

where S_i is the orthogonal projection operator onto $\mathcal{S}_i$ and R_i is that onto its orthogonal complement, $\perp\mathcal{S}_i$. Clearly then,

$$T_i x = x + \lambda_i (S_i x + R_i g_i - x) = (1 - \lambda_i) x + \lambda_i S_i x + \lambda_i R_i g_i \tag{4.29}$$

$$= L_i x + \lambda_i R_i g_i \tag{4.30}$$

where

$$L_i = (1 - \lambda_i) 1 + \lambda_i S_i \quad , \quad i = 1 \to 3 \quad . \tag{4.31}$$

Thus, by simple algebra and linearity,

$$Tx = T_3 T_2 T_1 x = \lambda_3 R_3 g_3 + \lambda_2 L_2 R_2 g_2 + \lambda_1 L_3 L_2 R_1 g_1 + L_3 L_2 L_1 x$$

$$= h + Lx \tag{4.32}$$

where

---

[11] A linear variety $\mathcal{C}$ is composed of all vectors x of the form $x = g + y$ where $g \in \mathcal{H}$ is fixed and y ranges throughout some CLM $\mathcal{S}$. For short, $\mathcal{C} = g + \mathcal{S}$. (A linear variety is obviously closed and convex.)

$$h = \lambda_3 R_3 g_3 + \lambda_2 L_2 R_2 g_2 + \lambda_1 L_3 L_2 R_1 g_1 \tag{4.33}$$

and

$$L = L_3 L_2 L_1 \quad . \tag{4.34}$$

It now follows by iteration of (4.32) that

$$T^n x = \sum_{r=0}^{n-1} L^r h + L^n x \quad , \quad n \geq 1 . \tag{4.35}$$

In particular, for $x = f$, f a fixed point of T, $T^n f = f$,

$$f = \sum_{r=0}^{n-1} L^r h + L^n f \tag{4.36}$$

and

$$T^n x - f = L^n (x-f) \quad , \quad n \geq 1 \quad . \tag{4.37}$$

According to corollary 1, $L^n(x-f) \to P_{\mathcal{S}}(x-f)$ hence,

$$T^n x \to f + P_{\mathcal{S}}(x-f) \equiv w \tag{4.38}$$

where $P_{\mathcal{S}}$ is the orthogonal projection operator projecting onto the CLM

$$\mathcal{S} = \bigcap_{i=1}^{3} \mathcal{S}_i \quad . \tag{4.39}$$

Obviously, $w = \lim_{n\to\infty} T^n x$ is independent of the choice of $f \in C_0$.

It is clear from $f \in C_0$ that $w \in C_0$, and invoking the orthogonal projection theorem,

$$x - w = x - f - P_{\mathcal{S}}(x-f) = Q_{\mathcal{S}}(x-f) \tag{4.39a}$$

where $Q_{\mathcal{S}}(x-f)$ is the projection of $x-f$ onto $\perp\mathcal{S}$. Hence

$$\|x-w\| = \|Q_{\mathcal{S}}(x-f)\| \leq \|x-f\| \quad . \tag{4.39b}$$

But f is an arbitrary member of C_0 so that necessarily, $T^n x \to w = P_0 x$, Q.E.D.[12]

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[12] Another proof of corollary 2 (without relaxation) is given by I. Amemiya and T. Ando [8].

In Theorem 4.1 the update rule is given by $x_{n+1} = Tx_n$ with $x_0 = x$ an arbitrary element of $\mathcal{H}$. More generally, we can also work with the single-step cyclic scheme

$$\begin{aligned}
x_1 &= T_1 x \\
x_2 &= T_2 x_1 \\
&\vdots \\
x_m &= T_m x_{m-1} = T_m T_{m-1} \cdots T_1 x = Tx \qquad (4.40) \\
x_{m+1} &= T_1 x_m \\
&\vdots \\
x_{2m} &= T_m x_{2m-1} = T_m T_{m-1} \cdots T_1 Tx = T^2 x \\
x_{2m+1} &= T_1 x_{2m} \\
&\vdots
\end{aligned}$$

Or, put more succinctly,

$$x_{n+1} = T_{\alpha(n)} x_n \tag{4.41}$$

where $x_0 = x$ and[13]

$$\alpha(n) = 1 + n \bmod m \quad , \quad n \geq 0 \quad . \tag{4.42}$$

It is easily shown that the sequences $\{T^n x\}$ and $\{x_n\}$ converge weakly to the same limit $x^* \in C_0$. Write $n = \ell m + r$, $0 \leq r \leq m-1$. Then, for $n \geq 1$

$$x_n = T_r T_{r-1} \cdots T_1 T^{\ell} x \tag{4.43}$$

and $\ell \to \infty$ as $n \to \infty$. Let $T^{\ell} x \rightharpoonup x^*$ and let the subsequence $\{x_{n'}\}$ of $\{x_n\}$ converge weakly to x'.

Then, $n' = m\ell' + r'$, $0 \leq r' \leq m-1$, and $\ell' \to \infty$ as $n' \to \infty$. Moreover, making use of (4.11) with T replaced by $T_{r'} T_{r'-1} \cdots T_1$ and x by $T^{\ell'} x$ we

[13] n mod m is the remainder obtained upon division of n by m and is one of the integers 0, 1, ..., m-1.

obtain, for every $y \in C_0$,

$$\|T^{\ell'}x - x_{n'}\|^2 \leq b_m \cdot 2^{m-1}(\|T^{\ell'}x - y\|^2 - \|x_{n'} - y\|^2) \tag{4.44}$$

$$\leq b_m \cdot 2^{m-1}(\|T^{\ell'}x - y\|^2 - \|T^{\ell'+1}x - y\|^2) \quad . \tag{4.45}$$

Hence, $\|T^{\ell'}x - x_{n'}\| \to 0$ as $n' \to \infty$. But $T^{\ell'}x - x_{n'} \rightharpoonup x^* - x'$ and since[14]

$$\|x^* - x'\| \leq \liminf_{n' \to \infty} \|T^{\ell'}x - x_{n'}\| = 0 \quad , \tag{4.46}$$

$x^* = x'$, as was to be shown. (To the best of our knowledge, the first proof of weak convergence of the iteration (4.41) to a point of C_0 is due to Bregman [9]. However, as we have seen, the underlying theory is the same as that of the full composition operator $T = T_m T_{m-1} \cdots T_1$ and its associated fixed-point recursion, $x_{n+1} = Tx_n$.)

The first lemma of this section is preparatory to a deeper study of the strong convergence of the sequence $\{T^n x\}$.

Lemma 4.1: Let T be any nonexpansive map whose set of fixed points includes a given closed convex set $\mathfrak{J}$, and let $P_{\mathfrak{J}}$ denote the projection operator onto $\mathfrak{J}$. Then, for any $x \in$ domain T,

$$\mathrm{Re}(x - Tx,\ x - P_{\mathfrak{J}}x) \geq 0 \tag{4.47}$$

and

$$\|x - Tx\| \leq 2\|x - P_{\mathfrak{J}}x\| \quad . \tag{4.48}$$

---

[14] Let $y_n \rightharpoonup f$. Then, for every $y \in \mathcal{H}$, $(f, y) = \lim(y_n, y)$ as $n \to \infty$. In particular, $|(f, y)| = \lim |(y_n, y)|$. But from Schwartz's inequality, $|(y_n, y)| \leq \|y\| \cdot \|y_n\|$ so that

$$|(f, y)| \leq \|y\| \cdot \liminf_{n \to \infty} \|y_n\| \quad .$$

Thus, for $y = f$,

$$\|f\| \leq \liminf_{n \to \infty} \|y_n\| \quad .$$

Proof. Clearly, for every x, y in domain T,

$$\|Tx-Ty\|^2-\|x-y\|^2 \leq 0 \quad . \tag{4.49}$$

In particular, if $y = P_{\mathfrak{J}}x$,

$$\|Tx-P_{\mathfrak{J}}x\|^2 \leq \|x-P_{\mathfrak{J}}x\|^2 \tag{4.50}$$

because $P_{\mathfrak{J}}x \in \mathfrak{J}$ and $TP_{\mathfrak{J}}x = P_{\mathfrak{J}}x$, by hypothesis. Thus,

$$\|x-Tx-(x-P_{\mathfrak{J}}x)\|^2 \leq \|x-P_{\mathfrak{J}}x\|^2 \tag{4.51}$$

and an easy simplification yields

$$\|x-Tx\|^2 \leq 2\,\mathrm{Re}(x-Tx, x-P_{\mathfrak{J}}x) \quad , \tag{4.52}$$

$x \in$ domain T. From (4.52) and Schwartz's inequality we now obtain (4.47) and (4.48) immediately, Q.E.D.

Inequality (4.47) simply states that Tx lies on the side of the support plane through x, parallel to that through $P_{\mathfrak{J}}x$, containing $\mathfrak{J}$ (Fig. 4.1).

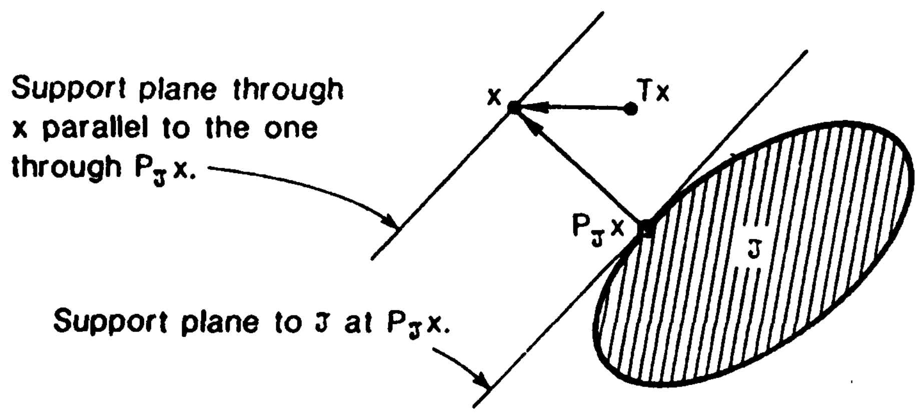

Fig. 4.1

Lemma 4.2: The iterated sequence $\{T^n x\}$ of Theorem 3.4 converges strongly iff

$$\|T^n x-P_{\mathfrak{J}}T^n x\| \to 0 \tag{4.53}$$

as $n \to \infty$.[15]

[15]Clearly, $\|T^n x-P_{\mathfrak{J}}T^n x\| = d(T^n x, \mathfrak{J})$, the distance of $T^n x$ from $\mathfrak{J}$.

Proof. As shown in Theorem 3.4, the sequence $\{T^n x\}$ converges weakly to some point $x^* \in \mathfrak{J}$. Evidently, if the convergence is actually strong,

$$d(T^n x, \mathfrak{J}) = \| T^n x - P_{\mathfrak{J}} T^n x \| \leq \| T^n x - x^* \| \to 0 \qquad (4.54)$$

as $n \to \infty$. Conversely, suppose that $d(T^n x, \mathfrak{J}) \to 0$. Then, since T^k is non-expansive for all integers $k \geq 0$ and includes $\mathfrak{J}$ in its sets of fixed points, (4.48) yields, for all $x \in \mathcal{C}$,

$$\| x - T^k x \| \leq 2 \| x - P_{\mathfrak{J}} x \| \quad , \quad k \geq 0 \quad . \qquad (4.55)$$

In particular, if x is replaced by $T^n x$ we obtain

$$\| T^n x - T^{n+k} x \| \leq 2 \| T^n x - P_{\mathfrak{J}} T^n x \| \to 0 \qquad (4.56)$$

as $n \to \infty$. Thus, the sequence $\{T^n x\}$ is Cauchy and $T^n x \to x^*$, Q.E.D.

According to lemma 4.2, convergence in Theorem 4.1 of the sequence $\{T^n x\}$ to a point of $\mathcal{C}_0$ is strong iff

$$d(T^n x, \mathcal{C}_0) = \| T^n x - P_0 T^n x \| \to 0 \qquad (4.57)$$

as $n \to \infty$. (The set of fixed points of $T = T_m T_{m-1} \cdots T_1$ coincides with $\mathcal{C}_0$.) We can show without the imposition of any additional constraints that for $r = 1 \to m$,

$$d(T^n x, \mathcal{C}_r) = \| T^n x - P_r T^n x \| \to 0 \qquad (4.58)$$

as $n \to \infty$. (The distance of $T^n x$ from every $\mathcal{C}_r$ goes to zero as n goes to infinity.)

To see this, note that as a special case of (4.11),[16]

$$\| T^n x - P_r P_{r-1} \cdots P_1 T^n x \|^2 \leq 2^{r-1} (\| T^n x - y \|^2 - \| P_r P_{r-1} \cdots P_1 T^n x - y \|^2) \qquad (4.59)$$

$$\leq 2^{r-1} (\| T^n x - y \|^2 - \| T^{n+1} x - y \|^2) \to 0 \qquad (4.60)$$

for all $y \in \mathcal{C}_0$ and every $r = 1 \to m$ because,

[16] Identify T with $P_r P_{r-1} \cdots P_1$ and set $m = r$ and $\lambda_i = 1$, $i = 1 \to r$.

$$\|T^{n+1}x-y\| \leq \|P_r P_{r-1} \cdots P_1 T^n x-y\| \tag{4.61}$$

and

$$\lim_{n\to\infty} \|T^n x-y\| \tag{4.62}$$

exists (Theorem 3.4). Thus, since $P_r P_{r-1} \cdots P_1 T^n x \in C_r$,

$$d(T^n x, C_r) \leq \|T^n x-P_r P_{r-1} \cdots P_1 T^n x\| \to 0 \tag{4.63}$$

as $n\to\infty$ and (4.58) is established. Our problem therefore is to find some other restriction, preferably very weak, which together with (4.58) implies (4.57). The next theorem complements Theorem 4.1 and supplies two sufficient conditions for strong convergence, the second of which appeared for the first time in the excellent paper by Gubin, Polyak and Raik [1] already referred to in Theorem 2.3.

Theorem 4.2:[17] Either of the two conditions 1) or 2) stated below suffices to guarantee the strong convergence of the sequence $\{T^n x\}$ of Theorem 4.1 to its weak limit x^*.

1) At least one of the C_i's is uniformly convex and does not contain x^* in its interior.

2) For some α, $1 \leq \alpha \leq m$, the set intersection

$$C_\alpha \cap \Big(\bigcap_{i\neq\alpha}^{m} C_i \Big)^o \tag{4.64}$$

is nonempty (and therefore contains a point of C_0).

In case 2) the strong convergence of $\{T^n x\}$ to x^* is at a geometric rate.

Proof. 1) To be specific, let us suppose that C_1 is uniformly convex and contains x^* on its norm boundary. As we have already shown in (4.58),

$$d(T^n x, C_1) = \|T^n x-P_1 T^n x\| \to 0 \tag{4.65}$$

[17] The interior of any set C is denoted by C^o.

as $n \to \infty$. Clearly,[18] every $y_n = P_1 T^n x$ lies in $\mathcal{C}_1$ and $y_n \to x^*$. Now, if the convergence of the sequence $\{y_n\}$ to x^* is not strong, there exists an $\epsilon > 0$ and a subsequence $\{y_{n'}\}$ such that

$$\|y_{n'} - x^*\| \geq \epsilon \tag{4.66}$$

for all n'.

Thus, since $x^* \in \mathcal{C}_0 \subset \mathcal{C}_1$ and the latter is uniformly convex (Def. 2.3), the point

$$z_{n'} = \frac{y_{n'} + x^*}{2} + h \tag{4.67}$$

is a member of $\mathcal{C}_1$ for all n' and all h such that

$$\|h\| \leq \delta(\epsilon) > 0 \quad . \tag{4.68}$$

But for any such fixed choice of h, $z_{n'} \to x^* + h$ which belongs to $\mathcal{C}_1$ because the latter is weakly closed. Consequently, x^* is the center of a sphere of radius $\delta(\epsilon) > 0$ contained in $\mathcal{C}_1$, and this contradicts the assumption that x^* is not an interior point of $\mathcal{C}_1$. Therefore the convergence of $\{y_n\}$ and hence of $\{T^n x\}$ to x^* must be strong. (Note that $\|T^n x - x^*\| \leq \|T^n x - y_n\| + \|y_n - x^*\| \to 0$ as $n \to \infty$.)

2) Let z belong to the intersection (4.64). Then, $z \in \mathcal{C}_0$ and for some $\delta > 0$ it is true that $h \in \mathcal{C}_i$, $i \neq \alpha$, for all h such that

$$\|h - z\| \leq \delta \quad . \tag{4.69}$$

Let $y = P_\alpha T^n x$. Then, for any $\epsilon > 0$, no matter how small,

$$w = \frac{\epsilon}{\epsilon + \delta} \cdot z + \frac{\delta}{\epsilon + \delta} \cdot y \tag{4.70}$$

is a member of $\mathcal{C}_0$ for all sufficiently large n.

In fact, since $z \in \mathcal{C}_\alpha$ and $y \in \mathcal{C}_\alpha$ and $\mathcal{C}_\alpha$ is convex, $w \in \mathcal{C}_\alpha$. Now observe that

---

[18] For any $z \in \mathcal{H}$,

$$|(T^n x, z) - (y_n, z)| = |(T^n x - y_n, z)| \leq \|T^n x - y_n\| \cdot \|z\| \to 0$$

as $n \to \infty$. Hence, $\lim_{n \to \infty} (y_n, z) = \lim_{n \to \infty} (T^n x, z) = (x^*, z)$ so that $y_n \to x^*$.

$$\|y-P_i y\| = d(y, C_i) \leq \|y-P_i T^n x\| \leq \|T^n x-y\| + \|T^n x-P_i T^n x\| \quad (4.71)$$

$$= d(T^n x, C_\alpha) + d(T^n x, C_i) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (4.72)$$

for n large enough.[19] Thus, for all such n,

$$h = z + \frac{\delta}{\epsilon}(y-P_i y) \quad (4.73)$$

belongs to C_i, $i \neq \alpha$, because $\|h-z\| \leq \delta$. From the identity

$$w = \frac{\epsilon}{\epsilon+\delta}\left(z + \frac{\delta}{\epsilon}(y-P_i y)\right) + \frac{\delta}{\epsilon+\delta} \cdot P_i y = \frac{\epsilon}{\epsilon+\delta} \cdot h + \frac{\delta}{\epsilon+\delta} \cdot P_i y \quad , \quad (4.74)$$

and the convexity of C_1 we also conclude that $w \in C_i$, $i \neq \alpha$; i.e., $w \in C_i$, $i = 1 \to m$, hence $w \in C_0$.

Consequently,

$$d(T^n x, C_0) \leq \|T^n x-w\| \leq \|T^n x-y\| + \|y-w\| \quad (4.75)$$

$$= d(T^n x, C_\alpha) + \frac{\epsilon}{\epsilon+\delta}\|y-z\| \quad (4.76)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon R}{\delta} = \left(1 + \frac{2R}{\delta}\right) \cdot \frac{\epsilon}{2} = \frac{c\epsilon}{2} \quad (4.77)$$

where

$$c = \left(1 + \frac{2R}{\delta}\right) \quad (4.78)$$

and R is an upper bound on $\|y-z\|$.[20] Since ϵ is arbitrary, $d(T^n x, C_0)$ converges to zero as $n \to \infty$ and $\{T^n x\}$ is therefore strongly convergent to x^*. It remains to establish that convergence is at a geometric rate.

Observe first that $\epsilon/2$ in (4.77) is by construction, and for sufficiently large n, an upper bound on the m distances $d(T^n x, C_i)$, $i = 1 \to m$:

$$d(T^n x, C_i) = \|T^n x-P_i T^n x\| \leq \frac{\epsilon}{2} \quad , \quad i = 1 \to m \quad . \quad (4.79)$$

---

[19] Remember that according to (4.58), for $i = 1 \to m$, $\lim_{n\to\infty} d(T^n x, C_i) = 0$.

[20] $\|y-z\| = \|P_\alpha T^n x-z\| \leq \|T^n x-z\|$ for all $z \in C_0$. But the latter is a monotone nonincreasing function of n and is therefore bounded by some finite number R.

Hence, for all large enough n,

$$d(T^n x, \mathcal{C}_0) \le c \cdot \sup_{i=1 \to m} \| T^n x - P_i T^n x \| \quad . \tag{4.80}$$

Choose the positive numbers ϵ_1 and ϵ_2 so that $\epsilon_1 \epsilon_2 / c^2 < 1$ and $0 < \epsilon_1 \le \lambda_i \le 2 - \epsilon_2$, $i = 1 \to m$. Then, from (4.10) and (4.7) with T replaced by T_i, x by $T^n x$ and y by $P_0 T^n x$ we obtain, for every $i = 1 \to m$,

$$d^2(T^n x, \mathcal{C}_0) \le c^2 \cdot d^2(T^n x, \mathcal{C}_i) \le \frac{c^2}{\lambda_i (2-\lambda_i)} (d^2(T^n x, \mathcal{C}_0) - \| T_i T^n x - P_0 T^n x \|^2) \tag{4.81}$$

$$\le \frac{c^2}{\epsilon_1 \epsilon_2} (d^2(T^n x, \mathcal{C}_0) - \| T^{n+1} x - P_0 T^n x \|^2) \tag{4.82}$$

$$\le \frac{c^2}{\epsilon_1 \epsilon_2} (d^2(T^n x, \mathcal{C}_0) - d^2(T^{n+1} x, \mathcal{C}_0)) \quad . \tag{4.83}$$

Therefore,

$$d(T^{n+1} x, \mathcal{C}_0) \le \left(1 - \frac{\epsilon_1 \epsilon_2}{c^2}\right)^{1/2} d(T^n x, \mathcal{C}_0) \tag{4.84}$$

and by iteration,

$$d(T^n x, \mathcal{C}_0) \le \left(1 - \frac{\epsilon_1 \epsilon_2}{c^2}\right)^{n/2} \cdot d(x, \mathcal{C}_0) \tag{4.85}$$

$$\le R\left(1 - \frac{\epsilon_1 \epsilon_2}{c^2}\right)^{n/2} \tag{4.86}$$

The proof can now be brought to a quick conclusion. From (4.56), with $\mathfrak{J} = \mathcal{C}_0$ and $P_{\mathfrak{J}} = P_0$, (4.86) yields, for every integer $k \ge 1$,

$$\| T^n x - T^{n+k} x \| \le 2R\left(1 - \frac{\epsilon_1 \epsilon_2}{c^2}\right)^{n/2} \quad . \tag{4.87}$$

But for fixed n, $T^{n+k} x \to x^*$ as $k \to \infty$ so that a passage to the limit in (4.87) gives

$$\| T^n x - x^* \| \le 2R(1 - \frac{\epsilon_1 \epsilon_2}{c^2})^{n/2} \quad (4.88)$$

i.e., $T^n x$ converges strongly to its limit x* at least as fast as $2Ra^n$ where

$$a = (1 - \frac{\epsilon_1 \epsilon_2}{c^2})^{1/2} < 1 \quad , \quad (4.89)$$

Q.E.D.[21]

The idea behind the proof of 1), Theorem 4.2, is simple but the criterion itself is very effective. It differs fundamentally from 2), and taken in the context of Theorem 4.1, appears to be new. Apart from the important uniform convexity requirement, 1) succeeds because the weak limit x* is on the boundary of at least one C_i, whereas the success of 2) depends entirely on the fact that at least one point of C_0, not necessarily x*, is interior to the set intersection of some m-1 C_i's. In general, without the introduction of smoothing constraints, 2) is of limited applicability because for the class of image-restoration problems we have in mind, the unknown f is usually a boundary point and convex sets with interiors are hard to come by.

For example, suppose that $\mathcal{H} = L_2$, the Hilbert space of all functions f(t) of the real variable t square-integrable over $-\infty < t < \infty$. Let $F(\omega)$ denote its L_2-Fourier transform and let us denote the correspondence by $f(t) \longleftrightarrow F(\omega)$. Then, if $f_i(t) \longleftrightarrow F_i(\omega)$, i = 1, 2,[22]

$$(f_1, f_2) \equiv \int_{-\infty}^{\infty} f_1(t)\overline{f}_2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega)\overline{F}_2(\omega)d\omega \, . \quad (4.90)$$

For any prescribed $b > 0$ let C_b denote the subset of $\mathcal{H}$ composed of all f's bandlimited to b; i.e., $f \in C_b$ iff $F(\omega) = 0$, $|\omega| > b$. It is obvious that C_b is a CLM devoid of any interior points. Indeed, given any $\epsilon > 0$ and any $f \in C_b$ there exists a $g \notin C_b$ such that

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[21] Except for some minor improvements, especially in the part dealing with geometric convergence, our proof of 2), Theorem 4.2, is in all respects the same as the original one given by Gubin et al. [1].

[22] For a scalar x, $\overline{x}$ denotes its complex conjugate, and as usual Re x and Im x are abbreviations for the "real" and "imaginary" parts of x, respectively.

$$\|f-g\|^2 = \int_{-\infty}^{\infty} |f(t)-g(t)|^2 dt \leq \epsilon^2 \quad . \tag{4.91}$$

(Thus, as measured in terms of energy context, any size neighborhood of a bandlimited signal contains signals that are not bandlimited.)

As another illustration, for any prescribed $a > 0$, let $\mathcal{C}_a$ denote the set of all f's whose orthogonal projection $P_a f$ onto the closed interval $-a \leq t \leq a$ is a given function g(t). (Of course, $g(t) = 0$, $|t| > a$.) It is easily seen that $\mathcal{C}_a$ is a linear variety whose interior is also empty. Thus $\mathcal{C}_a$ and $\mathcal{C}_b$ are two practical examples of boundary sets that occur repeatedly in many areas of signal processing.

Let $\mathcal{C}_1$ denote the closed unit sphere in $\mathcal{H}$ and let g be any fixed element of norm equal to one. Clearly, g lies on the boundary of $\mathcal{C}_1$, a closed uniformly convex set.[23] Let $\mathcal{C}_2$ denote a closed half-space whose boundary is tangent to $\mathcal{C}_1$ at g. Obviously, g is the only point common to the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$. To find g we can generate the iterates $T^n x$ where $T = P_2 P_1$ and x is arbitrary. (Figure 4.2 should clarify the mechanics.)

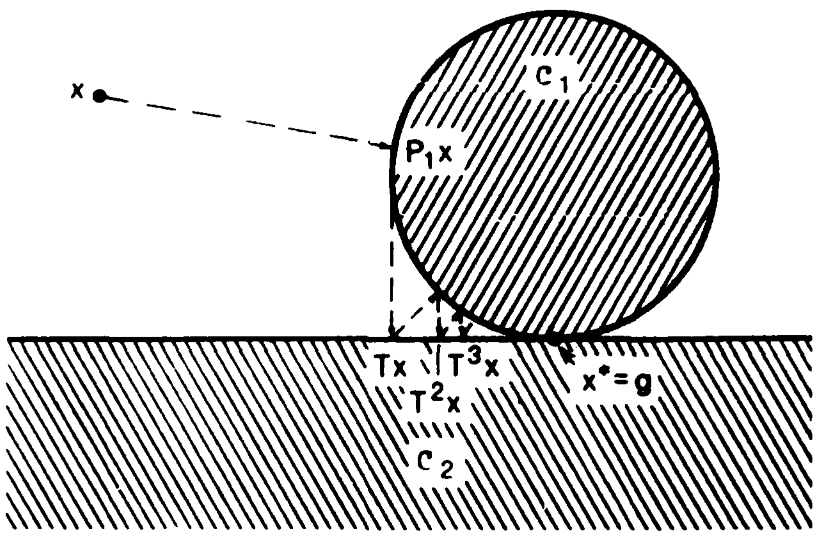

Fig. 4.2

[23] By making use of the identity

$$\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} \quad ,$$

it is easily shown that the closed sphere of radius R is strongly convex (Def. 2.4) with $\delta(\tau) = \mu\tau^2$ and $\mu = 1/8R$.

Since $x^* = g$ lies on the boundary of C_1, criterion 1) guarantees that $T^n x \to g$; however, 2) fails because $C_2 \cap C_1^o$ is empty. Nevertheless, geometric convergence is precluded by the tangency of the two boundaries at the point of contact x^*.

In Fig. 4.3, 2) always succeeds and establishes that $T^n x \to x^*$ at a geometric rate. On the other hand, 1) predicts strong convergence only if C_1 is uniformly convex. Neither 1) nor 2) encompasses corollary 2, Theorem 4.1, in which all C_i's are linear varieties.

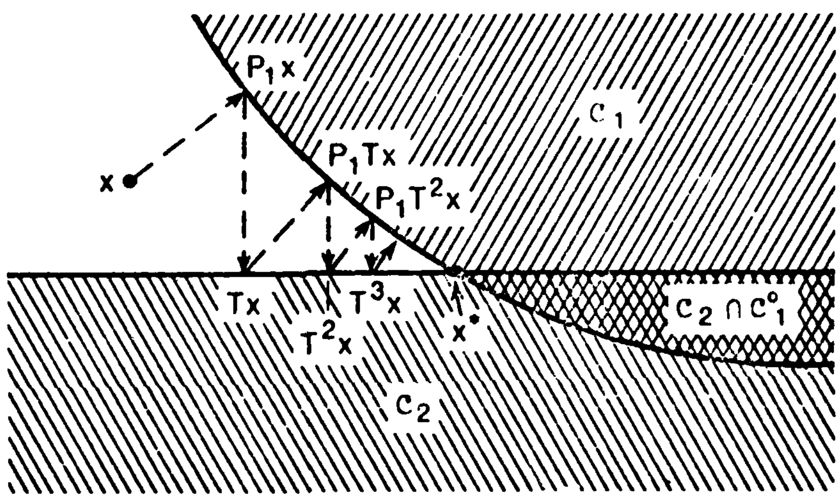

Fig. 4.3

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