MRC Technical Summary Report #2325

ASYMPTOTIC NUMERICAL ANALYSIS FOR
THE NAVIER-STOKES EQUATIONS (I)

C. Foias and R. Temam

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

DTIC
ELECTE
MAY 18 1982
S   D

E.

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

ASYMPTOTIC NUMERICAL ANALYSIS FOR THE NAVIER-STOKES EQUATIONS (I)

C. Foias and R. Temam

ABSTRACT

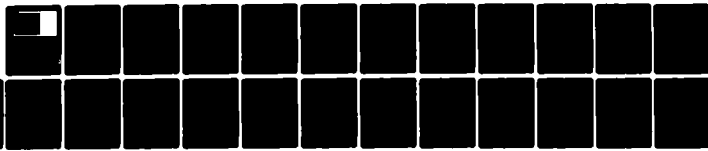Our aim in this work is to show that, in a "permanent regime", the behaviour of a viscous incompressible fluid can be, in principle, determined by the study of a finite number of modes. It is proved that the behaviour for $t \to \infty$ of the solution to the Navier-Stokes equations is completely determined by its projection on appropriate finite dimensional subspaces, corresponding to eigenspaces of the linear operator, or more general subspaces, including finite element subspaces. Some indications on the dimension of such subspaces are given.

SIGNIFICANCE AND EXPLANATION

If a viscous incompressible fluid is driven by time independent forces of sufficient intensity then, after a transient period, the "permanent" regime seems to be totally chaotic and unstructured. The present work is part of a set of articles which, however, tend to show that there may be some structure in such flows (at least in the case where the space dimension is 2), in particular that they are determined by a finite number of parameters.

Accession For
NTIS GRA&I X
DTIC TAB ☐
Unannounced ☐
Justification

By
Distribution/
Availability Codes
Dist Avail and/or
       Special
A

DTIC
COPY
INSPECTED
2

ASYMPTOTIC NUMERICAL ANALYSIS FOR THE NAVIER-STOKES EQUATIONS (I)

C. Foias and R. Temam

Introduction

Up to now the numerical analysis of the Navier-Stokes equations has been limited to
the study of the approximation of time dependent solutions on a finite interval of time or
to the approximation of stationary solutions (cf. among many other references [1,4,8,9]).
In the presence of a turbulent flow driven by a steady excitation a different type of
problem arises naturally: the study of the long time behavior of the solutions.

The present is an essay, the purpose of which is to show that for the 2-D Navier-
Stokes equations and, under some circumstances for the 3-D Navier-Stokes equations, there
is a theoretical basis for determining the qualitative long time behavior of a fluid by the
study of a finite number of adequate modes. A typical result is the following one: Let
W be a finite dimensional subspace of the natural function space V. If W satisfies a
certain condition, then the behavior for $t \to \infty$ of a solution u of the Navier-Stokes
equation is completely determined by the behavior for $t \to \infty$ of its projection on W.

Several results of this type are derived in this article. While this kind of problem
was already discussed in [2], our present interest was aroused and inspired by the
questions, conjectures and ideas due to O. P. Manley and Y. M. Treve [7,12,11], with whom
we acknowledge fruitful discussions and correspondence. In this paper we did not try to
produce the best constants, and we did not try to present the main inequalities in a
nondimensional form. These improvements of the work, and other developments will appear in
a subsequent work [13].

The plan is as follows:

    1. Notations and recapitulation of results.

    2. Approximation in the subspaces V_m.

    3. Approximation in a general subspace.

    4. Time periodic solutions.

    5. Remark on Galerkin approximation.

1. Notations and recapitulation of results

Let Ω be a bounded domain of $\mathbf{R}^{\ell}$, $\ell = 2$ or 3, and let Γ be its boundary. We assume that Γ is a manifold of dimension $\ell - 1$, of class C^4 with a finite number of connected components, and that Ω is locally located on one side of Γ. We shall firstly consider the initial value problem for the Navier-Stokes equations:

$$(1.1) \qquad \frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in} \quad \Omega$$

$$(1.2) \qquad \nabla u = 0 \quad \text{in} \quad \Omega$$

$$(1.3) \qquad u = 0 \quad \text{in} \quad \Gamma$$

$$(1.4) \qquad u\big|_{t=0} = u_0 \ ,$$

where $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2)$ or (u_1, u_2, u_3) is the velocity, a vector-valued function of $x \in \Omega$ and $t \geq 0$, and p is the pressure, $p = p(x,t)$, and f represents the external body force per unit of mass.

All what follows apply to the case where (1.3) is replaced by a nonhomogeneous boundary condition, which corresponds to more realistic physical situations (Couette-Taylor flow, [9]). We will refrain from treating this case to avoid purely technical difficulties.

We denote by $L^2(\Omega)$ the space of square integrable real functions on Ω and by $H_0^1(\Omega)$ the Sobolev space made of the functions which are in $L^2(\Omega)$ together with their first derivative and which vanish on Γ. We set, for u in $L^2(\Omega)$ or $L^2(\Omega)^{\ell}$ (resp. u in $H_0^1(\Omega)$ or $H_0^1(\Omega)^{\ell}$)

$$|u|^2 = \int_{\Omega} |u(x)|^2 dx, \quad \text{resp.} \quad \|u\|^2 = \int_{\Omega} |\nabla u(x)|^2 dx \ .$$

The space $L^2(\Omega)^{\ell}$ admit classically an orthogonal decomposition of the form $H \oplus G$, where

$$G = \{v = \nabla q, \ q \in H^1(\Omega)\}$$

and its orthogonal H satisfies

$$H = \{v \in L^2(\Omega)^{\ell}, \ \text{div } v = 0, \ v \cdot n\big|_{\Gamma} = 0\} \ ,$$

n the unit outward normal of Γ. Let also

$$V = \{v \in H_0^1(\Omega)^{\ell}, \ \text{div } v = 0\} \ .$$

It is clear that H and V are Hilbert spaces for the norms $|\cdot|$ and $\|\cdot\|$, and their corresponding scalar products. While $|v|^2$ is equal to the kinetic energy of the fluid

-2-

with velocity v (the density $\rho = 1$), we recall that for $v \in V$, $\|v\|$ too, reduces to a physical quantity:

$$\|v\|^2 = \int_\Omega |\text{curl } v(x)|^2 dx .$$

Let π denote the orthogonal projection of $L^2(\Omega)^\ell$ onto H and define the operators A and B by

(1.5)
$$Au = -\pi\Delta u \quad \text{for} \quad u \in D(A) = V \cap H^2(\Omega)^\ell ,$$

(1.6)
$$B(u,v) = \pi[(u,\nabla)v], \quad \text{for} \quad u,v \in D(A) .$$

Then A is a self-adjoint operator in H with an orthonormal basis $\{w_m\}_{m \geq 1}$ of eigenvectors, such that

(1.7)
$$Aw_m = \lambda_m w_m, \quad m \geq 1, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots ,$$

$\lambda_m \to +\infty$ as $m \to +\infty$ and

(1.8)
$$V = D(A^{1/2}), \quad \|u\| = |A^{1/2}u| \quad \text{for} \quad u \in V .$$

The operator A is an isomorphism from $D(A)$ onto H and from V onto V' (the dual of V which one can identify to a superspace of H). Concerning B we recall the following fact: B is a compact mapping from $D(A) \times V$ or $V \times D(A)$ into H and from $V \times V$ into V'. Furthermore we have the estimates:

(1.9)
$$\|B(u,v)\|_{V'} \leq \begin{cases} \sqrt{2} \; |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2} & \text{if} \quad \ell = 2 \\ 6|u|^{1/4} \|u\|^{3/4} |v|^{1/4} \|v\|^{3/4} & \text{if} \quad \ell = 3 \end{cases}$$

(1.10)
$$|B(u,v)| \leq c_0 \begin{cases} |u|^{1/2} |Au|^{1/2} \|v\| \\ |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} \end{cases}$$

if $\ell = 2$ and if $\ell = 2$ or 3:

(1.11)
$$|B(u,v)| \leq c_1 \begin{cases} |Au| \|v\| \\ \|u\| |Av| . \end{cases}$$

The constants c_0 and c_1 depend only on Ω but are not easy to determine since they involve the norm of the operator A^{-1}, and Sobolev constants. All the above results can be found for instance in [6], [9].

We can now recall the functional formulation of (1.1)-(1.4): This is the functional differential equation

$$(1.12) \qquad \frac{du}{dt} + \nu Au + B(u,u) = f, \qquad u|_{t=0} = u_0 \ ,$$

where we shall assume for simplicity $u_0 \in V$ and

$$(1.13) \qquad \begin{cases} f \text{ is continuous and bounded from } [0,\infty) \text{ into } H, \\ f' = \dfrac{df}{dt} \text{ belongs to } L^2_{loc}(0,\infty;V') \ . \end{cases}$$

It is well-known that, if $\ell = 2$, there exists a unique function u such that

$$(1.14) \qquad u \in C([0,\infty);H) \cap L^2_{loc}([0,\infty);V)$$

satisfying the equation (1.12) in V' (cf. for instance [9]). Moreover u is actually a continuous function from $(0,\infty)$ into $D(A)$, which is bounded in $D(A)$ as $t \to +\infty$. We set

$$(1.15) \qquad \|u(t)\| \leqslant c_2 = c_2(|u_0|,[f],1/\nu,c_0,\lambda_1) \quad \text{for} \quad t \geqslant 0 \ ,$$

$$(1.16) \qquad |Au(t)| \leqslant c_3 = c_3(|u_0|,[f],1/\nu,c_0,\lambda_1,1/\alpha) \quad \text{for} \quad t \geqslant \alpha > 0$$

where

$$(1.17) \qquad [f] = \sup_{0 < t < \infty} |f(t)| \ .$$

The estimate (1.15) is given in [2]; the estimate (1.16) is more recent and given in [5] (cf. also [10]).

If $\ell = 3$, there exists a weak solution of (1.12) bounded in H. Such a solution may or may not be bounded in V. We will only consider such a solution if it is bounded in V

$$(1.18) \qquad \|u(t)\| \leqslant R \quad \text{for} \quad t \geqslant 0 \ .$$

In this case it follows also from [3], [5], [10], that $u(t)$ belongs to $D(A)$ and is bounded in $D(A)$ as $t \to \infty$:

$$(1.19) \qquad |Au(t)| \leqslant c_4(R,[f],1/\nu,c_1,\lambda_1,1/\alpha) \quad \text{for} \quad t \geqslant \alpha > 0 \ .$$

In the sequel we will let $\ell = 2$ or 3, and consider solutions of (1.12) which are uniformly bounded in V on $[0,\infty)$; the existence of $R < \infty$ is an assumption if $\ell = 3$, and is automatic if $\ell = 2$ ($R = c_2$ cf. (1.15)). See the comment in Remark 2.1.

2. Approximation in the subspaces V_m

2.1. An inequality

For $m \in \mathbb{N}$, we denote by V_m the space spanned by the eigenfunctions $w_1, \ldots, w_m$, of A, and by P_m the projector in V, H or V' onto V_m. Finally $Q_m = I - P_m$. It is easy to see that

$$(2.1) \qquad \|\varphi\| \leq \lambda_m^{1/2} |\varphi|, \qquad \forall \varphi \in V_m$$

$$(2.2) \qquad |\psi| \leq \lambda_{m+1}^{-1/2} \|\psi\|, \qquad \forall \psi \in Q_m V .$$

Let $u(\cdot)$ and $v(\cdot)$ be solutions of the equation

$$(2.3) \qquad u' + \nu Au + B(u,u) = f, \qquad t > 0, \qquad u(0) = u_0 ,$$

$$(2.4) \qquad v' + \nu Av + B(v,v) = g, \qquad t > 0, \qquad v(0) = v_0 ,$$

where $v_0 \in V$ and g satisfies the same assumption (1.13) as f. We set

$$w = u - v, \qquad p_m = P_m w, \qquad q_m = Q_m w, \qquad e = f - g, \qquad e_m = Q_m e .$$

Then

$$q_m' + \nu Aq_m + Q_m B(v,w) + Q_m B(w,u) = e_m$$

and consequently[1],

$$(2.5) \qquad \frac{1}{2} \frac{d}{dt} |q_m|^2 + \nu \|q_m\|^2 = -(B(v,p_m),q_m) - (B(p_m,u),q_m)$$
$$- (B(q_m,u),q_m) + (e_m,q_m) .$$

The right-hand side of (2.5) can be bounded because of (1.11) by[2]:

---

[1] We recall that $(B(\varphi,\psi),\theta) = -(B(\varphi,\theta),\psi)$ for θ, φ, ψ in V.

[2] $c_i, c_i', c_i'', \ldots,$ denote various positive constants. The c_i's are the same all the time, $c_i', c_i'', \ldots,$ may represent different quantities at different places.

$$\lambda_1^{-1/2}|e_m||q_m| + c_1(|Au| + |Av|)|p_m||q_m| + c_1|Au||q_m||q_m|$$

$\leqslant$ (for $\varepsilon > 0$ arbitrary)

$$\leqslant \frac{\varepsilon\nu}{2}|q_m|^2 + \frac{\lambda_1^{-1}}{2\varepsilon\nu}|e_m|^2 + \frac{\varepsilon\nu}{2}|q_m|^2 + \frac{c_1^2}{2\varepsilon\nu}(|Au| + |Av|)^2|p_m|^2 + c_1\lambda_{m+1}^{-1/2}|Au||q_m|^2$$

$\leqslant$ (by (1.16), (1.19))

$$\leqslant \varepsilon\nu|q_m|^2 + \frac{\lambda_1^{-1}}{2\varepsilon\nu}|e_m|^2 + \frac{2c_1^2c_4^2}{\varepsilon\nu}|p_m|^2 + c_1c_4\lambda_{m+1}^{-1/2}|q_m|^2$$

for $t \geqslant \alpha$.

We have thus proved the following

Lemma 2.1

<u>If m is large enough, so that</u>[1]

(2.6)
$$\lambda_{m+1} > \frac{c_1^2c_4^2}{\nu^2},$$

<u>then, for</u> $t \geqslant \alpha > 0$,

(2.7)
$$\frac{d}{dt}|q_m|^2 + \nu'|q_m|^2 \leqslant \frac{\lambda_1^{-1}}{\varepsilon\nu}|e_m|^2 + \frac{4c_1^2c_4^2}{\varepsilon\nu}|p_m|^2,$$

$$\nu' = \nu - c_1c_4\lambda_{m+1}^{-1/2} > 0, \quad \varepsilon = \frac{1}{2} - \frac{c_1c_4\lambda_{m+1}^{-1/2}}{2\nu}.$$

2.2. The main result

We introduce the following weak mode of convergence: we will say that $\varphi(t)$ converges essentially to 0 as $t \to \infty$ (and we write $\varphi(t) \xrightarrow{c.e.} 0$) if:

(2.8)
there exists $\kappa > 0$, such that for every $\delta > 0$, there exists t_δ satisfying

$$\text{meas}\{\tau \in (t, t + 1), |\varphi(t)| \geqslant \delta\} \leqslant \kappa\delta, \quad \text{for every } t > t_\delta.$$

---

[1]We need $\lambda_{m+1} > \frac{36c_1^2}{\nu^2} \cdot (\sup_{t \geqslant \alpha}|Au(t)|^2)$ and actually it is sufficient to have $\lambda_{m+1} > \frac{c_1^2}{\nu^2} \cdot \limsup_{t \to \infty}|Au(t)|^2.$

It is easy to see that if $\varphi(t) \to 0$ for $t \to \infty$, then $\varphi(t) \xrightarrow{c.e.} 0$, for $t \to \infty$. Also if $\varphi \in L^1(0,\infty)$ then $\varphi(t) \xrightarrow{c.e.} 0$ as $t \to \infty$.

Theorem 2.1

We assume that $\ell = 2$ or that $\ell = 3$ and that u and v are solutions of (2.3), (2.4) uniformly bounded in V. We assume also that (2.6) is satisfied. Then:

i) If $|p_m(u(t) - v(t)| \to 0$, $|(I - P_m)(f(t) - g(t))| \to 0$, for $t \to \infty$, then

$$(2.9) \qquad |(I - P_m)(u(t) - v(t))| \to 0 \quad \text{for} \quad t \to \infty$$

$$(2.10) \qquad |u(t) - v(t)| \to 0 \quad \text{for} \quad t \to \infty \; .$$

ii) If $|p_m(u(t) - v(t))|^2 \xrightarrow{c.e.} 0$, $|(I - P_m)(f(t) - g(t))|^2 \xrightarrow{c.e.} 0$, for $t \to \infty$, then (2.9) still holds but instead of (2.10),

$$(2.11) \qquad |u(t) - v(t)|^2 \xrightarrow{c.e.} 0 \quad \text{for} \quad t \to \infty \; .$$

Proof

We infer from (2.2) and (2.7)

$$\frac{d}{dt} |q_m|^2 + \nu'\lambda_{m+1}^{1/2}|q_m|^2 < \frac{\lambda_1^{-1}}{\varepsilon\nu} |e_m|^2 + \frac{4c_1^2 c_4^2}{\varepsilon\nu} |p_m| \; .$$

Whence for $t > t_0 > \alpha$:

$$(2.12) \qquad |q_m(t)|^2 < |q_m(t_0)|^2 e^{-\nu'\lambda_{m+1}^{1/2}(t-t_0)} +$$

$$+ \int_{t_0}^t [\frac{\lambda_1^{-1}}{\varepsilon\nu} |e_m(\tau)|^2 + \frac{4c_1^2 c_4^2}{\varepsilon\nu} |p_m(\tau)|^2] e^{-\nu'\lambda_{m+1}^{1/2}(t-\tau)} \, d\tau \; .$$

There exists $\kappa, \kappa' > 0$ such that for every $\delta > 0$, there exists t_δ which satisfies for $t > t_\delta$:

$$\text{meas}\{\tau \in (t, t + 1), \; |P_m(u(\tau) - v(\tau))|^2 > \delta\} < \kappa\delta$$

$$\text{meas}\{\tau \in (t, t + 1), \; |P_m(f(\tau) - g(\tau))|^2 > \delta\} < \kappa'\delta \; .$$

For a fixed integer M we take $t > t_0 + M$, $t_0 > \max(t_\delta, \alpha)$. Then (2.12) implies

$$|q_m(t)|^2 \le c(u,v)e^{-\nu'\lambda_{m+1}^{1/2}(t-t_0)} + \delta\Big(\frac{\lambda_1^{-1}}{\varepsilon\nu} + \frac{4c_1^2c_4^2}{\nu}\Big)\int_{t-M}^{t} e^{-\nu'\lambda_{m+1}^{1/2}(t-\tau)}\,d\tau$$

$$+ \Big[\frac{\lambda_1^{-1}}{\varepsilon\nu}c(f,g) + \frac{4c_1^2c_4^2}{\varepsilon\nu}c(u,v)\Big]\cdot\Big(\int_{t_0}^{t-M} e^{-\nu'\lambda_{m+1}^{1/2}(t-\tau)}\,d\tau\Big)$$

$$+ M\delta\Big(\kappa'\frac{\lambda_1^{-1}}{\varepsilon\nu}c(f,g) + \kappa\frac{4c_1^2c_4^2}{\nu}c(u,v)\Big)$$

where

$$c(u,v) = \sup_{t\geqslant 0}|u(t) - v(t)|, \quad c(f,g) = \sup_{t\geqslant 0}|f(t) - g(t)| .$$

Therefore, as $t \to \infty$,

$$\limsup_{t\to\infty}|q_m(t)|^2 \le \delta\Big(\frac{\lambda_1^{-1}}{\varepsilon\nu} + \frac{4c_1^2c_4^2}{\varepsilon\nu}\Big)\Big(\frac{1 - e^{-\nu'\lambda_{m+1}^{1/2}M}}{\nu'\lambda_{m+1}^{1/2}}\Big)$$

$$\Big[\frac{\lambda_1^{-1}}{\varepsilon\nu}c(f,g) + \frac{4c_1^2c_4^2}{\varepsilon\nu}c(u,v)\Big]\Big(\frac{e^{-\nu'\lambda_{m+1}^{1/2}M}}{\nu'\lambda_{m+1}^{1/2}}\Big)$$

$$+ M\delta\Big[\kappa'\frac{\lambda_1^{-1}}{\varepsilon\nu}c(f,g) + \kappa\frac{4c_1^2c_4^2}{\nu}c(u,v)\Big] .$$

We let $\delta \to 0$ and then $M \to \infty$, and we obtain (2.9).

Remark 2.1

This result which is contained in a slightly weaker form in [2], when the dimension of space is $\ell = 2$, is reproduced here for the convenience of the reader.

3. Approximation in a general subspace

In the applications the utilization of the basis $\{w_m\}_{m \geqslant 1}$ formed by the eigenvectors of A is not practical since these functions are not easy to determine. Therefore we shall now show how the preceding theorem can be extended when V_m is replaced by a general finite dimensional subspace W of V.

3.1. Assumptions

Let W be a finite dimensional subspace of V and let $P(W)$ be the projector in H onto $W, Q(W) = I - P(W)$. Since $P(W)$ is not a projector in V, it may happen that $((\varphi, \psi)) \neq 0$ if $\varphi \in W$ and $\psi \in V$, $P(W)\psi = 0$. However, one can show (cf. Lemma 3.2 below) that there exists $\rho(W)$, $0 \leqslant \rho(W) < 1$ such that

$$(3.1) \qquad |((\varphi, \psi))| \leqslant \rho(W) \|\varphi\| \|\psi\|, \; \forall \varphi \in W, \; \forall \psi \in V, \; P(W)\psi = 0.$$

We associate also to W the two numbers $\lambda(W), \mu(W)$

$$\lambda(W) = \mathrm{Inf}\{\|\varphi\|^2, \varphi \in V, \; P(W)\varphi = 0, \; |\varphi| = 1\}$$

$$\mu(W) = \mathrm{Sup}\{\|\psi\|^2, \; \psi \in W, \; |\psi| = 1\},$$

so that

$$(3.2) \qquad |\varphi| \leqslant \lambda(W)^{-1/2} \|\varphi\|, \; \forall \varphi \in V, \; P(W)\varphi = 0$$

$$(3.3) \qquad \|\psi\| \leqslant \mu(W)^{1/2} |\psi|, \; \forall \psi \in W.$$

When it is not necessary to mention the dependence on W, we will write simply P, Q, ρ, λ, μ, instead of $P(W), \ldots$.

3.2. An inequality

We consider as in Section 2, the solutions $u(\cdot), v(\cdot)$ of (2.3), (2.4) and we set

$$w = u - v, \; p = Pw, \; q = Qw, \; e = f - g.$$

We have

$$q' + \nu QAw + QB(v,w) + QB(w,u) = Qe$$

and taking the scalar product in H with q, we get

$$\frac{1}{2} \frac{d}{dt} |q|^2 + \nu \|q\|^2 = -(B(v,p),q) - (B(p,u),q),$$

$$-(B(q,u),q) + (Qe,q) - \nu((p,q)) .$$

Using (1.11) and (3.1)-(3.3) we find that the right-hand side of this inequality is less than

$$\lambda_1 |\varrho e| |q| + \nu \rho \mu^{1/2} |p| |q| + c_1 (|Au| + |Av|) |p| |q| + c_1 |Au| |q| |q|$$

$$\leqslant \quad \text{(by (1.16), (1.19))}$$

$$\leqslant \frac{\varepsilon \nu}{3} |q|^2 + \frac{3\lambda_1^{-1}}{4\varepsilon \nu} |\varrho e|^2 + \frac{\varepsilon \nu}{3} |q|^2 + \frac{3\nu}{\varepsilon} \rho^2 \mu |p|^2$$

$$+ \frac{\varepsilon \nu}{3} |q|^2 + \frac{3c_1^2}{4\varepsilon \nu} (|Au| + |Av|)^2 |p|^2 + c_1 c_4 \lambda^{-1/2} |q|^2$$

$$\leqslant \varepsilon \nu |q|^2 + \frac{3\lambda_1^{-1}}{4\varepsilon \nu} |\varrho e|^2 + \frac{3}{\varepsilon} \left(\frac{c_1^2 c_4^2}{\nu} + \nu \rho^2 \mu \right) |p|^2 + c_1 c_4 \lambda^{-1/2} |q| \quad \text{for} \quad t \geqslant \alpha > 0 .$$

If

$$(3.4) \qquad \qquad \qquad \lambda(W) > \frac{c_1^2 c_4^2}{\nu^2}$$

then we set

$$(3.5) \qquad \qquad \nu' = \nu - c_1 c_4 \lambda(W)^{-1/2} > 0, \quad \varepsilon = \frac{1}{2} - \frac{c_1 c_4 \lambda(W)^{-1/2}}{2\nu}$$

and we have established:

Lemma 3.1

If (3.1), (3.4) hold, then for $t \geqslant \alpha > 0$,

$$(3.6) \qquad \qquad \frac{d}{dt} |q|^2 + \nu' |q|^2 \leqslant \frac{3\lambda_1^{-1}}{2\varepsilon \nu} |\varrho e|^2 + \frac{6}{\varepsilon} \left(\frac{c_1^2 c_4^2}{\nu} + \nu \rho^2 \mu \right) |p|^2 ,$$

ν', ε as in (3.5).

3.3. Statement of the result

As in Section 2.2, we have

Theorem 3.1

We assume that $\ell = 2$ or that $\ell = 3$ and that u and v are solution of (2.3), (2.4) uniformly bounded in V. Let W be a finite dimensional subspace of V such that (3.4) is satisfied. Then

1) If $|P(u(t) - v(t))| \to 0$, $|(I - P)(f(t) - g(t))| \to 0$ for $t \to \infty$, then

$$(3.7) \qquad \qquad |(I - P)(u(t) - v(t))| \to 0 \quad \text{for} \quad t \to \infty ,$$

$$(3.8) \qquad \qquad |u(t) - v(t)| \to 0 \quad \text{for} \quad t \to \infty .$$

ii) If $|P(u(t) - v(t))|^2 \xrightarrow{c.e.} 0$, $|(I - P)(f(t) - g(t))|^2 \xrightarrow{c.e.} 0$ for $t \to \infty$, then

(3.7) holds and, instead of (3.8):

$$(3.9) \qquad\qquad |u(t) - v(t)| \xrightarrow{\;c.e.\;} 0, \quad \text{for} \quad t \to \infty .$$

The proof, starting from (3.6), is essentially the same as that of Theorem 2.1.

In the rest of this section we give examples and show that (3.1) is always satisfied, while assumption (3.4) is satisfied if W is "sufficiently large".

3.4. Assumption (3.1).

Lemma 3.2

Under the assumptions of Section 3.1, there exists $\rho = \rho(W)$, $0 \leq \rho < 1$ such that (3.1) holds.[1]

Proof

If (3.1) was not true, we could find two sequences $\{\varphi_j\}_{j \geq 1}$, $\{\psi_j\}_{j \geq 1}$, $\varphi_j \in W$, $\psi_j \in V$, $P\psi_j = 0$, such that

$$|\varphi_j|\,|\psi_j| \geq |((\varphi_j , \psi_j))| \geq (1 - \tfrac{1}{j})|\varphi_j|\,|\psi_j| .$$

Setting $\varphi_j' = \dfrac{\varphi_j}{|\varphi_j|}$, $\psi_j' = \dfrac{\psi_j}{|\psi_j|}$, we find

$$(3.10) \qquad\qquad 1 \geq |((\varphi_j' , \psi_j'))| \geq \left(1 - \tfrac{1}{j}\right) .$$

We can extract a subsequence (still denoted j) such that φ_j' converges to some limit φ, $|\varphi| = 1$, $\varphi \in W$ (W has finite dimension), and ψ_j' converge weakly in V to ψ, $\psi \in V$, $|\psi| \leq 1$, $P\psi = 0$. At the limit, (3.10) gives

$$|((\varphi , \psi))| = 1, \quad |\varphi| = 1, \quad |\psi| \leq 1 ,$$

so that $|\psi| = 1$, $\psi = \kappa\varphi \neq 0$, by contradiction with $P\psi = 0$.

3.5. Assumption (3.4) - Example

We consider the following situation which is classical in the numerical analysis of partial differential equations and in particular of Navier-Stokes equations (cf. [9], Chap. I, §4):

---

[1] Actually the fact that $\rho < 1$ which is important in other developments, did not play any role in the proof above. The inequality (3.1) with $\rho(W) = 1$ is trivial.

We are given a family $\{W_h\}_{h\in H}$, of finite dimensional subspaces of V. The set of indices H is arbitrary but is equiped with a concept of limit[1], which we denote for simplicity $\lim_{h\to 0}$. For example, in the Galerkin method, $H = 1/\mathbb{N}$, $h = 1/m$, and we pass to the limit $m \to \infty$, $h \to 0$. For finite element methods (cf. [9]), H is a family of regular triangulations of the domain Ω), and we let the diameter of the largest triangle go to 0.

The main assumption on the spaces W_h is the following one

(3.11)
$$\forall \varphi \in V, \quad \operatorname*{Inf}_{\psi \in W_h} \|\varphi - \psi\| \to 0 \quad \text{as} \quad h \to 0 .$$

In the case of an increasing sequence of subspaces W_m of V (Galerkin method), assumption (3.11) means simply that

(3.12)
$$\bigcup_{m\in\mathbb{N}} W_m \quad \text{is dense in } V .$$

Then we see that assumption (3.4) is satisfied for h "sufficiently small".

<u>Lemma 3.3</u>

<u>Under assumption</u> (3.11),

(3.13)
$$\lim_{h\to 0} \lambda(W_h) = +\infty ,$$

<u>and</u> (3.4) <u>is satisfied for</u> h <u>sufficiently small.</u>

<u>Proof</u>

The proof consists in showing the more precise following statement

(3.14)   For every integer m, there exists h_m and, for $h \leqslant h_m$, $\lambda(W_h) \geqslant \lambda_m - 1$.

For given m and $\delta > 0$, the assumption (3.11) written with $\varphi = w_j$, $j = 1,\dots,m$, shows that there exists h_m such that

$$\operatorname*{Inf}_{\psi \in W_h} \|w_j - \psi\| \leqslant \delta, \quad \text{for} \quad j = 1,\dots,m \text{ and for every } h \leqslant h_m .$$

---

[1] A filter F with a denumerable basis; $\lim_{h\to 0}$ means roughly speaking $\lim_F$.

Thus for every $h \leqslant h_m$, there exists $\tilde{w}_1, \ldots, \tilde{w}_m$ in W_h, with $\|w_j - \tilde{w}_j\| \leqslant \delta$. Therefore if $\varphi \in V$, $(I - P(W_h))\varphi = 0$, we have

$$\|\varphi\|^2 = \|P_m \varphi\|^2 + \|(I - P_m)\varphi\|^2$$

$$\geqslant \lambda_{m+1} |(I - P_m)\varphi|^2 + \lambda_1 |P_m \varphi|^2$$

$$\geqslant \lambda_{m+1} |\varphi|^2 - (\lambda_{m+1} - \lambda_1)|P_m \varphi|^2$$

$$= \lambda_{m+1} |\varphi|^2 - (\lambda_{m+1} - \lambda_1) \sum_{j=1}^{m} (\varphi, w_j - \tilde{w}_j)^2$$

$$\geqslant (\lambda_{m+1} - (\lambda_{m+1} - \lambda_1)\lambda_1 m \delta^2)|\varphi|^2 .$$

This implies

$$\lambda(W_h) \geqslant \lambda_{m+1} - (\lambda_{m+1} - \lambda_1)\lambda_1 m \delta^2 ,$$

and the result follows by taking δ small enough. ∎

It is also useful for later purposes to establish

Lemma 3.4

Under assumption (3.11),

(3.15)
$$\lim_{h \to 0} \mu(W_h) = +\infty .$$

Proof

Due to (3.11), for every $\psi \in V$ and for every h, there exists $\psi_h \in W_h$, with

$$\lim_{h \to 0} \|\psi - \psi_h\| = 0 .$$

Due to (3.3), $\|\psi_h\| \leqslant \mu(W_h)^{1/2}|\psi_h|$, and if the family $\mu(W_h)$ does not converge to $+\infty$ for $h \to 0$, $\liminf_{h \to 0} \mu(W_h) \leqslant K < \infty$, we would have at the limit: $\|\psi\| \leqslant K|\psi|$, $\forall \psi \in V$, and this is impossible.

4. Time periodic solutions

The notations are the same as in Section 3. If the assumption (3.4) is satisfied, we infer from Lemma 3.1 that there exist two positive constants, $c_5 > 0$, $\eta > 0$, independent of u and v such that

$$(4.1) \qquad |(I - P)(u(t) - v(t))|^2 \leqslant c_5 \{e^{-\eta(t-t_0)}$$

$$+ \int_{t_0}^{t} e^{-\eta(t-\tau)} [|P(u(\tau) - v(\tau)|^2 + |(I - P)(f(\tau) - g(\tau))|^2] d\tau \},$$

for $t \geqslant t_0 \geqslant \alpha > 0$.

We can now prove the following:

Theorem 4.1

We assume that $\ell = 2$ or that $\ell = 3$ and that u is a solution of (2.3) uniformly bounded in V. Let W be a finite dimensional subspace of V such that (3.4) is satisfied. Assume moreover that there exist periodic functions $f_\infty(\cdot)$, $p_\infty(\cdot)$ with value in H and W and period $T > 0$, such that

$$(4.2) \qquad \lim_{t \to \infty} |f(t) - f_\infty(t)| = \lim_{t \to \infty} |Pu(t) - p_\infty(t)| = 0 .$$

Then there exists a periodic solution u_∞ with period T of the equation

$$(4.3) \qquad u_\infty' + \nu A u_\infty + B(u_\infty, u_\infty) = f_\infty ,$$

such that

$$(4.4) \qquad \lim_{t \to \infty} |u(t) - u_\infty(t)| = 0 .$$

Proof

Apply (4.1) to u and $v, v(t) = u(t + jT)$. Then for $\ell T \geqslant t_0$, we obtain at time $t + \ell T$:

(4.5) $\qquad |(I - P)(u(t + \ell T) - u(t + (j + \ell)T))|^2 \leqslant$

$$\leqslant c_5 \left\{ e^{-\eta(t+\ell T-t_0)} + \int_{t_0}^{t+\ell T} e^{-\eta(t+\ell T-\tau)} \cdot \right.$$

$$\left. \cdot \left[|P(u(\tau) - u(\tau + jT))|^2 + |(I - P)(f(\tau) - f(\tau + jT))|^2 \right] d\tau \right\} .$$

For $\varepsilon > 0$ given, let $t_\varepsilon \geqslant t_0$ be such that for $t \geqslant t_\varepsilon$

$$|Pu(t) - p_\infty(t)| \leqslant \varepsilon, \qquad |f(t) - f_\infty(t)| \leqslant \varepsilon .$$

Then from (4.5) we obtain that

$$|(I - P)(u(t + \ell T) - u(t + (j + \ell)T))|^2 \leqslant c_5 e^{-\eta(t+\ell T-t_0)}$$

$$+ c_5 c(u,f) \int_{t_0}^{t_\varepsilon} e^{-\eta(t+\ell T-\tau)} d\tau + 8c_5 \varepsilon^2 \int_{t_\varepsilon}^{t+\ell T} e^{-\eta(t+\ell T-\tau)} d\tau$$

so that

$$\sup_{t \geqslant t_0} |(I - P)(u(t + \ell T) - u(t + (\ell + j)T))|^2 \leqslant c_\varepsilon e^{-\eta\ell T} + \frac{c_5 \varepsilon^2}{\eta} ,$$

for every $j \geqslant 0$ and $\ell \geqslant \dfrac{t_0}{T}$. Therefore $\{u(t + \ell T)\}$ is a Cauchy sequence in the space of continuous founded functions from $[t_0,\infty)$ into $(I - P)W$. Thus there exists a continuous bounded function u_∞ from $[t_0,\infty)$ into H, such that

(4.6) $\qquad u(t + \ell T) \to u_\infty(t)$ as $\ell \to \infty$, in H, uniformly in t on $[t_0,\infty)$.

Since $|Au(t + \ell T)| \leqslant c_4$ for all $t \geqslant \alpha$, $\ell \geqslant 1$, we see that $|Au_\infty(t)| \leqslant c_4$ for all $t \geqslant \alpha$, and

(4.7) $\qquad u(t + \ell T) \to u_\infty(t)$ in V, $\forall$, $t \geqslant \alpha$.

It is then easy to see that u_∞ is a solution, bounded in V, of (4.3), and u_∞ is periodic of period T, just because of (4.6). The convergence

(4.8) $\qquad \lim_{t\to\infty} |u(t) - u_\infty(t)| = 0 ,$

follows immediately from (4.6) as well as the T periodicity of $u_\infty(\cdot)$. The convergence (4.4) follows from (4.8) and the fact that $|Au(t) - Au_\infty|$ remains bounded (by $2c_4$) as $t \to \infty$. ∎

We then deduce the following result for stationary solutions

Theorem 4.2

The assumptions are similar to that of Theorem 4.1. We assume that there exist $f_\infty \in H$ and $p_\infty \in W$ such that

(4.9) $$|f(t) - f_\infty| \to 0, \quad |Pu(t) - p_\infty| \to 0, \quad \text{for} \quad t \to \infty .$$

It follows that there exists $u_\infty \in D(A)$ such that

(4.10) $$\|u(t) - u_\infty\| \to 0 \quad \text{for} \quad t \to \infty ,$$

where u_∞ is a (stationary) solution of the Navier-Stokes equation

(4.11) $$vAu_\infty + B(u_\infty, u_\infty) = f_\infty .$$

Proof

We apply Theorem 4.1 with $T > 0$ fixed, arbitrary, and we obtain (4.3), (4.4). Now $T > 0$ can be chosen arbitrarily small and since u_∞ must be independent of T in (4.4), we conclude that u_∞ has period 0, i.e. u_∞ is independent of t. ∎

5. Remark on Galerkin approximation

We now assume that the dimension $\ell = 2$.

For simplicity we restrict ourself to a Galerkin approximation of Navier-Stokes equations based on the spaces V_m, i.e. the family w_j of eigenfunctions of the Stokes problem. We will show that if m is sufficiently large, the behavior as $t \to \infty$ of the Galerkin approximation u_m, is completely determined by the behavior as $t \to \infty$ of a certain number m_* of its modes, i.e. of $P_{m_*} u_m$, $m_* < m$. This number m_* does not depend on m.

5.1. Galerkin approximation

For fixed m, the Galerkin approximation u_m of the solution u of (2.3) is defined by:

$$(5.1) \qquad \begin{cases} u_m' + \nu A u_m + P_m B(u_m, u_m) = P_m f, & t > 0 , \\ u_m(0) = P_m u_0 . \end{cases}$$

It is classical to derive a priori estimates independent on m on u_m: for example, for every $t \geqslant s \geqslant 0$:

$$(5.2) \qquad |u_m(t)|^2 + \nu \int_s^t \|u_m(\sigma)\|^2 d\sigma \leqslant |u_m(s)|^2 + \frac{\lambda_1(t-s)}{\nu} [f]$$

and

$$(5.3) \qquad |u_m(t)|^2 \leqslant |u_0|^2 e^{-\nu\lambda_1 t} + \frac{(1 - e^{-\nu\lambda_1 t})}{\nu^2 \lambda_1} [f]^2 .$$

The following a priori estimate is verified by u_m:

Lemma 5.1

$\|u_m(t)\|$ is bounded independently of m and t for $t \geqslant \alpha > 0$, $m \geqslant 0$.

Proof

Taking the scalar product of (5.1) with $A u_m$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_m\|^2 + \nu |A u_m|^2 = (f, A u_m) - (B(u_m, u_m), A u_m .$$

Because of (1.10), (1.17), the right-hand side of this expression is majorized by

-17-

$$[f]|Au_m| + c_0|u_m|^{1/2}\|u_m\| |Au_m|^{3/2} \leq \frac{\nu}{4}|Au_m|^2 + \frac{1}{\nu}[f]^2 + \frac{\nu}{4}|Au_m|^2 + c'|u_m|^2\|u_m\|^4 .$$

Therefore, with (5.3)

$$(5.4) \qquad \frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 \leq c''(1 + \|u_m\|^2)^2$$

and for $0 \leq s \leq t$, we can show by integration that

$$(1 + \|u_m(t)\|^2) \leq (1 + \|u_m(s)\|^2)e^{\displaystyle c''\int_s^t (1+\|u_m(\sigma)\|^2)d\sigma} .$$

If $t \geq \alpha > 0$, we integrate in s from $t - \alpha$ to t and we find

$$c''\alpha(1 + \|u_m(t)\|^2) \leq \left[e^{\displaystyle c''\int_{t-\alpha}^t (1+\|u_m(\sigma)\|^2)d\sigma} - 1 \right] .$$

Using (5.2), we see that the right-hand side of this inequation is bounded by a constant depending on α but independent of t and m and the Lemma follows. ∎

5.2. Behavior as $t \to \infty$ of the Galerkin approximation

Let v_m be the Galerkin approximation of the solution v of (2.4)

$$(5.5) \qquad \begin{cases} v_m' + \nu A v_m + P_m B(v_m, v_m) = P_m g, & t > 0 , \\ v_m(0) = P_m v_0 , \end{cases}$$

and, as before, u_m denotes the Galerkin approximation of the solution u of (2.3).

We set for $m_* \leq m$,

$$w_m = u_m - v_m, \quad p_{m_*} = P_{m_*}w_m, \quad q_{m_*} = Q_{m_*}w_m, \quad e = f - g, \quad e_{m_*} = Q_{m_*}e .$$

Then

$$\frac{dq_{m_*}}{dt} + \nu A q_{m_*} + Q_{m_*}P_m B(v_m, w_m) + Q_{m_*}P_m B(w_m, u_m) = Q_{m_*}P_m e$$

and consequently,

$$\frac{1}{2}\frac{d}{dt}|q_{m_*}|^2 + \nu\|q_{m_*}\|^2 = -(B(v_m, w_m), P_m q_{m_*}) - (B(w_m, u_m), P_m q_{m_*}) + (Q_{m_*}e, P_m q_{m_*}) .$$

The right-hand side is equal to

$$(Q_{m_*}e, P_m q_{m_*}) - (B(v_m, p_{m_*}), P_m q_{m_*}) - (B(v_m, (I - P_m)q_{m_*}), P_m q_{m_*})$$

$$- (B(p_{m_*}, u_m), P_m q_{m_*}) - (B(q_{m_*}, u_m), P_m q_{m_*}) ,$$

and because of (1.10) and Lemma 5.1, this quantity is bounded by

$$|Q_{m_*} e| |q_{m_*}| + c' |v_m|^{1/2} \|v_m\|^{1/2} |p_{m_*}|^{1/2} \|q_{m_*}\|^{1/2} |q_{m_*}|$$

$$+ c' |v_m|^{1/2} \|v_m\|^{1/2} |q_{m_*}|^{1/2} \|q_{m_*}\|^{3/2} + c' |p_{m_*}|^{1/2} \|p_{m_*}\|^{1/2} \|u_m\| |q_{m_*}|$$

$$+ c' |q_{m_*}|^{1/2} \|q_{m_*}\|^{3/2} \|u_m\|$$

$$\leq \lambda_1^{1/2} |Q_{m_*} e| \|q_{m_*}\| + c'' \lambda_{m_*+1}^{1/4} |p_{m_*}| \|q_{m_*}\| + c'' \lambda_{m_*+1}^{-1/4} \|q_{m_*}\|^2 .$$

If

$$(5.6) \qquad\qquad \lambda_{m_*+1} > \left(\frac{c''}{\nu}\right)^4$$

we set

$$(5.7) \qquad\qquad \nu' = 2(\nu - c'' \lambda_{m_*+1}^{-1/4}) > 0, \quad \varepsilon = \frac{1}{2} - \frac{c'' \lambda_{m_*+1}^{-1/4}}{2\nu} > 0 ,$$

and we bound the *last quantity above* (5.6) by

$$\frac{\varepsilon \nu}{2} \|q_{m_*}\|^2 + \frac{\lambda_1^{-1}}{2\varepsilon\nu} |Q_{m_*} e|^2 + \frac{\varepsilon\nu}{2} \|q_{m_*}\|^2 + \frac{(c'')^2 \lambda_{m_*+1}^{1/4}}{2\varepsilon\nu} |p_{m_*}|^2 + c'' \lambda_{m_*+1}^{-1/4} \|q_{m_*}\|^2$$

and we find

$$(5.8) \qquad \frac{d}{dt} |q_{m_*}|^2 + \nu' \|q_{m_*}\|^2 \leq \frac{\lambda_1}{\varepsilon\nu} |Q_{m_*} e|^2 + \frac{c''' \lambda_{m_*+1}^{1/2}}{\varepsilon\nu} |p_{m_*}| .$$

As for Theorem 2.1 we obtain

Theorem 5.1

We assume that $\ell = 2$ and that $m \geq m_*$, m_* sufficiently large so that (5.6) is verified. Then:

-19-

i) <u>If</u>

$$|P_{m_*}(u_m(t) - v_m(t))| \to 0, \quad |(I - P_{m_*})(f(t) - g(t))| \to 0$$

for $t \to \infty$; <u>then</u>

(5.9)
$$|(I - P_{m_*})(u_m(t) - v_m(t))| \to 0 \quad \text{for} \quad t \to \infty,$$

(5.10)
$$|u_m(t) - v_m(t)| \to 0 \quad \text{for} \quad t \to \infty.$$

ii) <u>If</u>

$$|P_{m_*}(u_m(t) - v_m(t))|^2 \xrightarrow{\text{c.e.}} 0, \quad |(I - P_{m_*})(f(t) - g(t))|^2 \xrightarrow{\text{c.e.}} 0$$

<u>for</u> $t \to \infty$, <u>then</u> (5.9) <u>holds and instead of</u> (5.10),

(5.11)
$$|u_m(t) - v_m(t)|^2 \xrightarrow{\text{c.e.}} 0, \quad \text{for} \quad t \to \infty.$$

<u>Remark 5.1</u>

This theorem will take its full interest if we can relate the behavior for $t \to \infty$ of the Galerkin approximation u_m of u, to the behavior for $t \to \infty$ of u itself. This question will be considered in a subsequent work.

REFERENCES

[1] A. J. Chorin, Lectures on Turbulence Theory, Publish or Perish Inc., Boston, 1975 and
its bibliography.

[2] C. Foias and G. Prodi, Sur le comportement global des solutions nonstationnaires
des équations de Navier-Stokes en dimension 2, Rend. Sem. Mat. Univ. Padova, 39
(1967), p. 1-34.

[3] C. Foias and R. Temam, Some analytic and geometric properties of the evolution
Navier-Stokes equations, J. Math. Pures Appl., 58 (1979), p. 339-368.

[4] V. Girault and P. A. Raviart, Finite element approximation of the Navier-Stokes
equations, Lecture Notes in Math., vol. 749, Springer-Verlag, 1979.

[5] C. Guillope, Comportement a l'infini des solutions des équations de Navier-Stokes
et propriétés des ensembles fonctionnels invariants (on attracteurs), Annales de
l'Institut Fourier, to appear.

[6] J. L. Lions, Quelques méthodes de resolution des problèmes aux limites nonlineaires,
Dunod, Paris 1969.

[7] O. P. Manley and Y. M. Treve, Minimum number of modes in approximate solutions to
equations of hydrodynamics, to appear.

[8] S. Orszag, Lectures on the Statistical Theory of Turbulence, in Fluid Dynamics, Les
Houches 1973, R. Balian and J. L. Peube eds., Gordon and Breach, New York, 1974, p.
235-374.

[9] R. Temam, Navier-Stokes Equations, Theory and Numerical Analysis, North-Holland,
Amsterdam 1979 (2nd edition).

[10] R. Temam, Behaviour at time $t = 0$ of the solutions of semilinear evolution
equations, J. Differential Equations, to appear.

[11] Y. M. Treve, A numerical test of the reliability of Galerkin Approximations to the
solutions of the Navier-Stokes equations, to appear.

[12] Y. M. Treve and O. P. Manley, A possible strange attractor in MHD flow, in Intrinsic
Stochasticity in Plasmas, G. Laval, D. Gresillon eds., Editions de Physique, Orsay
1979.

[13] C. Foias, O. P. Manley, R. Temam, and Y. Treve, <u>Asymptotic numerical analysis for the</u> <u>Navier-Stokes equations</u> (II), to appear.

CF:RT:scr

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2325	2. GOVT ACCESSION NO. AT-A11419	3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle) ASYMPTOTIC NUMERICAL ANALYSIS FOR THE NAVIER-STOKES EQUATIONS (I)	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s) C. Foias and R. Temam	8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis

11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709	12. REPORT DATE January 1982
	13. NUMBER OF PAGES 22

14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Navier-Stokes equations
Galerkin method
Finite elements
Asymptotic numerical analysis
Behaviour for $t \to \infty$

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Our aim in this work is to show that, in a "permanent regime", the behaviour of a viscous incompressible fluid can be, in principle, determined by the study of a finite number of modes. It is proved that the behaviour for $t \to \infty$ of the solution to the Navier-Stokes equations is completely determined by its projection on appropriate finite dimensional subspaces, corresponding to eigenspaces of the linear operator, or more general subspaces, including finite element subspaces. Some indications on the dimension of such subspaces are given.

DD $^{\text{FORM}}_{\text{1 JAN 73}}$ 1473   EDITION OF 1 NOV 65 IS OBSOLETE