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APPROXIMATION BY SMOOTH MULTIVARIATE SPLINES.(U)
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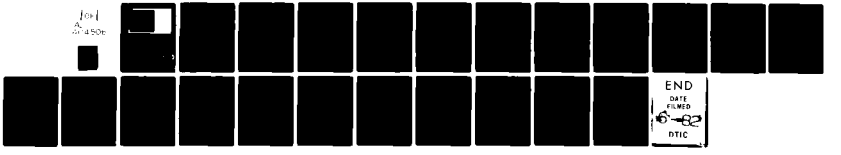

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APPROXIMATION BY SMOOTH MULTIVARIATE SPLINES

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THE UNIVERSITY OF WISCONSIN - MADISON
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APPROXIMATION BY SMOOTH MULTIVARIATE SPLINES

C. de Boor[1] and R. DeVore[2]

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ABSTRACT

The degree of approximation achievable by piecewise polynomial functions on certain regular grids in the plane is shown to be adversely affected by smoothness requirements, - in stark contrast to the univariate situation. For a rectangular grid, and for the triangular grid derived from it by adding all northeast diagonals, the maximum degree of approximation (as the grid size $1/n$ goes to zero) to a suitably smooth function is shown to be $O(n^{-\rho-2})$ in case we insist that the approximating functions are in C^{ρ}. This only holds as long as $\rho \leq \frac{r-3}{2}$ and $\rho \leq \frac{2r-4}{3}$, respectively, with r the total order of the polynomial pieces. In the contrary case, some smooth functions are not approximable at all. In the discussion of the second mesh, a new and promising kind of multivariate B-spline is introduced.

AMS (MOS) Subject Classification: 41A15, 41A63, 41A25

Key words: multivariate, splines, piecewise polynomial, smoothness, degree of approximation, B-splines

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SIGNIFICANCE AND EXPLANATION

One of the important properties of univariate splines is that in most senses smooth splines approximate just as well as do piecewise polynomials on the same mesh. This report shows this to be untrue in the multivariate setting. In particular, it details the cost in approximating power one may have to pay for the luxury of a smooth piecewise polynomial approximant. In an extreme case, piecewise polynomials of total degree $< r$ on a rectangular grid with all derivatives of order $\leq \rho$ continuous will fail to approximate certain smooth functions at all (as the grid goes to zero) unless ρ is kept below $(r-3)/2$.

During the analysis of approximation on a certain regular triangular grid, a novel kind of bivariate B-spline is introduced. This B-spline, in contrast to the established multivariate B-spline derived from a simplex, can be made to have all its breaklines in a given regular grid. This makes it a prime candidate for use in the construction of smooth multivariate piecewise polynomial approximation, and its properties will be explored further.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

APPROXIMATION BY SMOOTH MULTIVARIATE SPLINES

C. de Boor[1] and R. DeVore[2]

1. Introduction. One of the important properties of univariate splines is that in most senses smooth splines approximate just as well as piecewise polynomials. As we shall see, this is no longer the case for multivariate splines where both the smoothness of the spline and the geometry of the partition can have a limiting effect on the order of approximation. This type of limitation has already been recognized in certain cases [9], such as low order splines on a rectangular grid. For example, there is no effective approximation by $C^{(1)}$-cubics on rectangular grids. The purpose of this paper is to give a systematic study of this and related questions. We restrict our inquiry to bivariate approximation but it will be clear that our techniques extend to higher dimensions.

Let $\Pi = \{\pi_i\}$ be partition of $\mathbb{R}^2$ into triangles and/or rectangles. Denote by $\Pi_n := \{\pi_i(n)\}$ the corresponding scaled partition, with $\pi_i(n) := \pi_i/n$, all i. Denote by $S_{r,\rho}(\Pi_n)$ the space of splines of order r and smoothness ρ on Π_n, i.e., $s \in S_{r,\rho}(\Pi_n)$ if and only if $s \in C^\rho(\mathbb{R}^2)$ and, on each $\pi_i(n)$, s is a polynomial of <u>total</u> order r (i.e. total degree $< r$). We are interested in when $\bigcup_{n=1}^{\infty} S_{r,\rho}(\Pi_n)$ is dense in $C_0(\mathbb{R}^2)$. We study this problem in detail for two particular partitions: $\Sigma := \{\sigma_{ij}\}$, consisting of the squares $\sigma_{ij} := [i,i+1]\times[j,j+1]$, and the partition Δ which results when each square of Σ is divided into two triangles by introducing the northeast diagonal. The technique developed for these two cases can be used for more general partitions as well.

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In Section 2, we study approximation on Σ_n. This modest example already has the salient features of the general problem. We show that $S_{r,\rho}(\Sigma_n)$ is effective if and only if $\rho \leq \frac{r-3}{2}$. Thus, roughly speaking, only smoothness up to one-half the order of the polynomial pieces is allowable in this case. It turns out that smoothness also affects the rate of approximation in that certain C_0^∞-functions can only be approximated to within $O(n^{-r+\rho+1})$. Thus any smoothness condition reduces the achievable order of approximation. This should be compared with the univariate case where the full order of approximation is achievable regardless of smoothness.

We study approximation on Δ_n in Section 3. Here, the role of the geometry of the partition becomes more apparent. The splines $S_{r,\rho}(\Delta_n)$ are effective if and only if $\rho \leq \frac{2r-4}{3}$, hence there is a gain over the case Σ_n from roughly $r/2$ to $2r/3$. It is clear from our techniques that this is due to the fact that the partition Δ has three pairwise independent directions, viz. (1,0), (0,1) and (1,1), whereas Σ has only two, viz. (1,0) and (0,1). More generally, if the partition Π is generated by m pairwise independent vectors (what we mean by this is made precise in Section 4), then $S_{r,\rho}(\Pi_n)$ is effective if and only if $\rho \leq r-1-\left\lceil \frac{r+1}{m} \right\rceil$. Thus the more directions, the higher the allowable smoothness, but of course at the expense of a more complicated partition.

The results just described have two components. First they say that $\cup_n S_{r,\rho}(\Pi_n)$ is not dense in $C_0(\mathbb{R}^2)$ if ρ is too large. This rests on the fact that $S_{r,\rho}(\Pi_n)$ will not contain splines of finite support when ρ is too large. Our approach for this part of the problem is more or less the same in both cases $\Pi = \Sigma$ and $\Pi = \Delta$. The second half of the analysis is to show that $\cup_n S_{r,\rho}(\Pi_n)$ is dense when ρ is suitably restricted. This requires the construction of appropriate approximation methods. We develop different methods for constructing such approximations in the two cases.

For Σ it is easy to see that $S_n := S_{r,\rho}(\Sigma_n)$ is effective when $\rho \leq \frac{r-3}{2}$ since S_n then contains the tensor products of univariate splines of order $\rho+2$ and smoothness ρ. The more difficult problem is to show that S approximates any C_0^∞-function f to within $O(n^{-r+\rho+1})$. This is done by approximating the derivative $D^{(1,1)}f$ by splines of

lower order and smoothness, integrating and making local corrections for the approximation to f.

Our construction of approximants from $S := S_{r,\rho}(\Delta_n)$ has a completely different flavor, being based on extensions of the idea of multivariate B-splines. These ideas extend readily to more general partitions. Recall that multivariate B-splines are defined by certain cross sectional volumes of simplices. If the simplex is replaced by a more general polyhedron, the resulting function is still a piecewise polynomial, but now perhaps with fewer lines of discontinuity. In particular, by a suitable choice of the polyhedron, we can force the discontinuities of the resulting spline to be contained in the mesh lines of Δ_n. As mentioned in Section 4, other choices for the polyhedra will handle other partitions Π of $\mathbb{R}^2$. We feel that this viewpoint for constructing multivariate splines may prove to be very useful. For example, some of the standard finite elements can be described this way [3].

Here are some notational conventions used. $C_0^r(A)$ stands for the set of r times continuously differentiable functions on the set A with compact support in A. $\|\cdot\|$ is the L_∞-norm on $\mathbb{R}^2$, and $\|\cdot\|(A)$ is the L_∞-norm over the set A. Further,

$$\|f\|_r := \|f\|_{W_\infty^r} := \max_{\alpha+\beta\le r} \|D^{(\alpha,\beta)}f\| \quad \text{and} \quad |f|_r := \max_{\alpha+\beta=r} \|D^{(\alpha,\beta)}f\| .$$

$\lfloor a \rfloor$ denotes the largest integer no bigger than a, i.e., the "floor" function, and $\lceil a \rceil$ denotes its companion, the "ceiling" function, which gives the smallest integer no smaller than a. Further,

$$\underline{m} := \{1,2,\dots,m\} .$$

We also need the difference operator Δ_h^r defined on the function g by the rule

$$\Delta_h^r g : (x,y) \mapsto \sum_{p,q=0}^{r} (-1)^{p+q} \binom{r}{p}\binom{r}{q} g(x+ph,y+qh) .$$

This is the tensor product of the univariate r-th order difference operators in the coordinate directions. Explicitly, $\Delta_h^r = \delta_{(h,0)}^r \cdot \delta_{(0,h)}^r$, with

$$\delta_{(a,b)}^r g : (x,y) \mapsto \sum_{p=0}^{r} (-1)^{r-p} \binom{r}{p} g(x+pa,y+pb) .$$

2. Approximation on rectangular grids. Let $\Sigma := \{\sigma_{ij}\}$ with $\sigma_{ij} := [x_i,x_{i+1}]\times[y_j,y_{j+1}]$ and $x_i := y_i := i$. We are interested in the restriction of $S := S_{r,\rho}(\Sigma)$ to some compact set.

Lemma 1. *Every* $s \in S = S_{r,\rho}(\Sigma)$ *can be represented on*

$$\mathbb{R}_+^2 := \{(x,y) : x,y \geq 0\}$$

as a linear combination of the truncated powers in

$$T := T_{r,\rho}(\Sigma) := \{(x-x_i)_+^p(y-y_j)_+^q : i,j,p,q \geq 0;\ p+q < r\ ;\ \frac{p}{q} > \rho \text{ if } \frac{i}{j} > 0\}\ .$$

Proof. This is proved by induction, similar to the proof that truncated power functions are a basis for univariate splines. |||

Recall that Σ_n denotes the partition Σ scaled by $1/n$. Let $x_i(n) := y_i(n) := i/n$. Set $S_n := S_{r,\rho}(\Sigma_n)$.

Theorem 1. $\overset{\infty}{\underset{n=1}{\cup}} S_n$ *is dense in* $C_0(\mathbb{R}^2)$ *if and only if* $\rho \leq \frac{r-3}{2}$.

Proof. If $\rho \leq \frac{r-3}{2}$, then S_n contains all the truncated powers $(x,y) \longmapsto (x-x_i(n))_+^{\rho+1}(y-y_j(n))_+^{\rho+1}$ and hence the space $\tilde{S}_{\rho+2,\rho}(\Sigma_n)$ of splines of coordinate order $\rho+2$ and smoothness ρ . It is known that $\overset{\infty}{\underset{n}{\cup}}\tilde{S}_{\rho+2,\rho}(\Sigma_n)$ is dense in $C_0(\mathbb{R}^2)$; see for example [5] .

Suppose that $\rho > \frac{r-3}{2}$ and that $f \in C_0^\infty(\mathbb{R}^2)$ with $\|f - s_n\| = o(1)$ as $n \to \infty$, for some splines $s_n \in S_n$. Suppose without loss of generality that f has its support in $\mathbb{R}_+^2$. Since $\rho > \frac{r-3}{2}$, the only truncated power functions in T (as introduced in Lemma 1) are of the form $t(x,y) = x^p(y-y_j)_+^q$ or $t(x,y) = (x-x_i)_+^p y^q$ with $p + q < r$. Since $\Delta_h^r t = 0$ for such t , we have from Lemma 1 that $\Delta_h^r s = 0$ on $\mathbb{R}_+^2$ for all $s \in S$. This implies that we also have $\Delta_h^r s_n = 0$ on $\mathbb{R}_+^2$ for $n=1,2,\ldots$, from which it follows that $\Delta_h^r f = 0$. If we divide by h^{2r} and take the limit as $h \to 0$, we find that $D^{(r,r)}f = 0$. Since there are C_0^∞-functions for which $D^{(r,r)}f \neq 0$, we have proved the theorem. |||

Theorem 1 shows that only splines with smoothness less than about one-half the order r will be effective for approximation. It turns out that even when $\rho \leq \frac{r-3}{2}$, the order of approximation is affected negatively by smoothness. More precisely, we now show that the optimal order of approximation achievable with splines of order r and smoothness ρ is $n^{-r+\rho+1}$.

Theorem 2. *Let* $\rho \leq \frac{r-3}{2}$ *and* $k := r-\rho-1$. *Then*

(i) there are functions $f \in C_0^\infty(\mathbb{R}^2)$ *for which* $\operatorname{dist}(f,S_n) \neq o(n^{-k})$, $n \to \infty$;

(ii) for each $f \in C_0^\infty(\mathbb{R}^2)$, $\operatorname{dist}(f,S_n) = O(n^{-k})$.

Proof. We will show that if $f \in C_0^\infty(\mathbb{R}^2)$ and $\operatorname{dist}(f,S_n) = o(n^{-k})$, then $D^{(r+k,r)}f$ vanishes at 0, and this shows (i). Actually, with a finer analysis, we could show that all k-th order derivatives of $D^{(r,r)}f$ vanish on all of $\mathbb{R}^2$.

Suppose that $f \in C_0^\infty(\mathbb{R}^2)$ and that there are functions $s_n \in S_n$ such that

$$\|f - s_n\| = o(n^{-k}) \quad \text{as} \quad n \to \infty. \tag{2.1}$$

Let m be a positive integer and set $h := 1/m$. If $n = bm$ with b an integer, then $u_n := \Delta_h^r s_n$ is in S_n and, with $g := \Delta_h^r f$, we have

$$\|g - u_n\| = o(n^{-k}). \tag{2.2}$$

Now any truncated power $t(x,y) = (x - x_i)_+^p y^q$ or $t(x,y) = x^p(y - y_j)_+^q$ in $T = T_{r,\rho}(\Sigma)$ is annihilated by Δ_h^r. Hence, on $\mathbb{R}_+^2$, u_n is a linear combination of the splines $\Delta_h^r t$, with $t(x,y) = (x-x_i(n))_+^p(y-y_j(n))_+^q$ in S_n and $i,j > 0$. Such t has $\rho < p,q$ and $p+q < r$, and therefore $p,q < k$. Thus u_n is a polynomial of coordinate order k on each $\sigma_{ij}(n) \in \Sigma_n$ with $i,j > 0$.

For given $\eta > 0$, choose b so that $1/((b+1)m) \leq k\eta < 1/(bm)$. Then the points $(i\eta,0)$, $0 \leq i \leq k$, are in $\sigma_{00}(bm)$ and so $\left(\delta^k_{(\eta,0)}u_{bm}\right)(0) = 0$. Using (2.1), we have

$$\left|\left(\delta^k_{(\eta,0)}g\right)(0)\right| = \left|\left(\delta^k_{(\eta,0)}(g-u_{bm})\right)(0)\right| \leq \text{const}\,\|g-u_{bm}\|_\infty = o\left((bm)^{-k}\right) = o(\eta^k),$$

therefore $D^{(k,0)}g(0) = 0$. If we now let $m \to \infty$ and recall that $h = 1/m$ and $g = \Delta_h^r f$, we find $D^{(r+k,r)}f(0) = 0$, as desired.

We now prove (ii). Without loss of generality we assume from now on that f is supported in the unit cube $Q := [0,1]^2$. If $\rho = \lfloor \frac{r-3}{2} \rfloor$, then (ii) follows from the fact that S_n contains the tensor product splines of order $k = r-\rho-1$. For general ρ, the argument is more involved. We need a certain subspace of S_n, the space $\tilde{S}_{r,\rho}(\Sigma_n)$ of those $s \in S_n$ for which $s_{|\sigma_{ij}(n)}$ agrees with a polynomial in the span of $\{x^p y^q : p+q < r, p,q < k\}$, all i,j. We will prove by induction on ρ the following

Claim. *For any ρ and any r with $\rho \leq \frac{r-3}{2}$ there is const so that, for any $f \in C_0^r(\mathbb{R}^2)$ supported in Q and any n, there is an $s \in \tilde{S}_{r,\rho}(\Sigma_n)$ supported in the square $(\rho+1)^2 Q$ for which*

$$\|f - s\| \leq \text{const} \, |f|_r \, n^{-r+\rho+1} . \tag{2.3}$$

which in turn gives (ii).

Certainly the claim is true when $\rho = -1$, the case of piecewise polynomial approximation (see, e.g., [5]). Suppose then that the claim has been established for all $\rho < \rho_0$ and consider $\rho = \rho_0$. Take r so that $\rho \leq \frac{r-3}{2}$. Take $f \in C_0^r(\mathbb{R}^2)$ supported in Q, and let $g := D^{(1,1)} f$. Then

$$f(x,y) = \int_0^x \int_0^y g(\xi,\eta) \, d\xi d\eta \quad \text{for all} \quad (x,y) \in \mathbb{R}_+^2 .$$

By induction hypothesis, we may choose $u \in \tilde{S}_{r-2,\rho-1}(\Sigma_n)$ so that the support of u is in $\rho^2 Q$ and

$$\|g - u\| \leq \text{const} \, |g|_{r-2} \, n^{-(r-2)+(\rho-1)+1} \leq \text{const} \, |f|_r \, n^{-k+1} . \tag{2.4}$$

Our approximation s to f is gotten by integrating u and making local corrections using B-splines.

For $p = (p_1,p_2)$, let $M_p(x,y) := M_{p_1}(x) M_{p_2}(y)$, with M_j the univariate B-spline with knots $x_j(n), \ldots, x_{j+\rho+1}(n)$. Then M_j is of order $\rho+1$ and smoothness $\rho-1$, and is supported on the interval $[x_j(n), x_{j+\rho+1}(n)]$. It follows that M_p is in $\tilde{S}_{r-2,\rho-1}(\Sigma_n)$. We assume M_j to be normalized to have integral 1.

Consider the spline $N_p := M_{(\rho+1)p}$. It is supported on the square

$$Q_p := [x_{(\rho+1)p_1}(n), \, x_{(\rho+1)(p_1+1)}(n)] \times [y_{(\rho+1)p_2}(n), \, y_{(\rho+1)(p_2+1)}(n)]$$

and therefore the N_p's have disjoint support.

For each p, define $a_p := \int_{Q_p} (g - u)$. From (2.4), it follows that

$$|a_p| \leq |Q_p| \text{ const } \|g\|_{r-2} n^{-k+1} \leq \text{const } \|f\|_r n^{-k-1}. \tag{2.5}$$

Consider now the spline

$$v := u + \sum_p a_p N_p .$$

For each p, we have

$$\int_{Q_p} (g - v) = 0 . \tag{2.6}$$

Hence, if we set

$$s(x,y) := \int_0^x \int_0^y v(\xi,\eta)\, d\xi d\eta ,$$

then s will be in $S_{r,\rho}(\Sigma_n)$ and if $(x,y) \in Q_p$ then

$$f(x,y) - s(x,y) = \int_0^x\int_0^y (g - v)(\xi,\eta)\, d\xi d\eta = \int_E (g - v)$$

with

$$E := \left([0,x]\times[y_{(\rho+1)p_2}(n),y]\right) \left([x_{(\rho+1)p_1}(n),x]\times[0,y]\right) .$$

Now, for any $(x,y) \in (\rho+1)Q$, we have $|E| \leq \text{const } n^{-1}$. Further, $\int_E N_p$ is nonzero for at most $O(n)$ values of p. Therefore, (2.4), (2.5) give

$$\begin{aligned} |f(x,y) - s(x,y)| = \left|\int_E \Big(g - u - \sum_p a_p N_p\Big)\right| &\leq \|g - u\|\, |E| + \max_p |a_p| \sum_p \int_E N_p \\ &\leq \text{const } \|f\|_r n^{-k} . \end{aligned} \tag{2.7}$$

This proves (2.3) for the approximation just constructed. Also, using (2.6) and the fact that f and u have support in $\rho^2 Q$, it follows that $s = 0$ outside of $[\rho^2 + (\rho+1)]Q \subseteq (\rho+1)^2 Q$, as desired. |||

3. Approximation on triangular grids. We now show how the results of the last section can be extended to triangular grids. We focus on the partition Δ which is gotten by dividing each square of Σ into two triangles by adding the northeast diagonal. It will be clear, however, that our techniques apply to more general partitions, a point made in Section 4. We begin by developing methods for constructing smooth spline approximants. This turns out to be the more significant part of the problem. Our construction will be based on some variants of the ideas of multivariate B-splines.

Let $p < m$ and let P denote the projection of $\mathbb{R}^m$ onto $\mathbb{R}^p$, i.e., $z = (Pz, y)$ for $z \in \mathbb{R}^m$. If A is a simplex in $\mathbb{R}^m$ with vertices $v_0, \ldots, v_m$, then the function

$$M(x) := \mathrm{vol}_{m-p}\{z \in A : Pz = x\}, \quad x \in \mathbb{R}^p, \tag{3.1}$$

is a piecewise polynomial [4], [8] of order $m-p+1$ which is, up to a constant (viz., $\mathrm{vol}_m(A)$), completely determined by the points $Pv_0, \ldots, Pv_m$ in $\mathbb{R}^p$ and has its support in their convex hull.

When $p = 2$, the piecewise polynomial M has any segment connecting any two of the projected vertices $Pv_0, \ldots, Pv_m$ as a mesh line, and is a polynomial on any connected set not intersected by such a mesh line. For many purposes (including ours), this results in too complicated a grid in the plane. This can be avoided if we replace the simplex A in (3.1) by an appropriate m-dimensional convex polyhedron B and so define

$$M_B(x) := \mathrm{vol}_{m-p}\{z \in B : Pz = x\}, \quad x \in \mathbb{R}^p. \tag{3.2}$$

The function M_B so defined is again piecewise polynomial of total order $m-p+1$ because B can be decomposed into disjoint simplices and therefore M_B is a sum of (simplicial) B-splines. It has its support in $P(B)$.

A more useful analytic description of M_B is given by the identity (used first in the context of simplicial B-splines by Micchelli, see [8])

$$\int_{\mathbb{R}^p} \phi(x)\, M_B(x)\, dx = \int_B \phi(Pz)\, dz \tag{3.3}$$

which holds for any test function ($\phi \in C_0^\infty$). The right hand side defines M_B as a distribution on $\mathbb{R}^p$. One of the advantages of (3.3) is that it can be used to define M_B when B is a convex polyhedron of dimension $q < m$. The right hand side is then interpreted as a q-dimensional surface integral. This definition also makes sense when $P(B)$ has dimension $< p$. In this case, M_B is defined only as a distribution.

Suppose then that B is a convex polyhedron of dimension $q \leq m$. As is pointed out in [2], it is easy to check the smoothness of M_B by using the differentiation formula

$$D_{Pz}\, M_B = -\Sigma\, (z \cdot n_i) M_{B_i} \tag{3.4}$$

proved there. Here, the B_i are the facets of B, i.e., the faces of dimension $q-1$,

which make up the boundary of B, and n_i are their respective normals. Repeated application of (3.4) shows that $D^{(\alpha,\beta)}M_B$ can be written as a linear combination of M_F's where each F is a face of B of dimension $q - \alpha - \beta$. In particular, if dim $P(F) = p$ for all faces, then $D^{(\alpha,\beta)}M_B$ is again a piecewise polynomial. Thus M_B is in C^{q-d-2}, with d the largest integer for which there is a face F of B of dimension d with $P(F)$ of dimension $< p$. Note also that the discontinuities of M_B occur across the $(p-1)$-dimensional sets $P(F)$, with F a face of B.

We now turn to the construction of spline approximants on the triangular partition Δ of $\mathbb{R}^2$. Let $e_i := (\delta_{ij})_{j=1}^m$ be the unit coordinate vectors in $\mathbb{R}^m$ and define

$$v_1 := e_1, \quad v_2 := e_2,$$

$$v_j := \left\{ \begin{array}{ll} e_1 + e_j, & \text{if } j = 1 (\text{mod } 3) \\ e_2 + e_j, & \text{if } j = 2 (\text{mod } 3) \\ e_1 + e_2 + e_j, & \text{if } j = 3 (\text{mod } 3) \end{array} \right\}, \quad j = 3,4,\ldots,m.$$

The vectors $v_1, \ldots, v_m$ determine a parallelepiped B with vertices $\sum_1^m \varepsilon_i v_i$, where $\varepsilon_i \in \{0,1\}$, all i. The translated parallelepipeds $B_{ij} := ie_1 + je_2 + B$ have pairwise disjoint interiors and fill out the slab $\mathbb{R}^2 \times [0,1]^{m-2}$, hence form a partition for that slab.

Consider the functions $M := M_B$ and $M_{ij}(x,y) := M(x-i,y-j) = M_{B_{ij}}(x,y)$ defined by (3.3) (or, equivalently, by (3.2)) with $p = 2$.

Lemma 2. (i) $\sum M_{ij} = 1$ on $\mathbb{R}^2$.

(ii) $M_{ij} \in S_{r,\rho}(\Delta)$ for $r := m-1$ and $\rho := \left\lfloor \frac{2r-4}{3} \right\rfloor$.

Proof. The identity (i) follows from the representation (3.2) and the fact that $\cup B_{ij} = \mathbb{R}^2 \times [0,1]^{m-2}$, with the B_{ij} having pairwise disjoint interiors.

We have already noted that M is a piecewise polynomial of total order $m-1 = r$. To check the smoothness of M, note that any face F of B is of the form

$$F = \{v + \sum_{i \in I} \alpha_i v_i : 0 < \alpha_i < 1\}$$

with v some vertex of B and $I \subset \underset{\sim}{m}$. If $P(F)$ is a segment, i.e., $P(F)$ has dimension

1, then all $i \in I$ are congruent modulo 3, hence $|I| \le \lceil \frac{m+2}{3} \rceil$. Therefore, M has smoothness $\rho := m - \lceil \frac{m+2}{3} \rceil - 2 = \lfloor \frac{2r-4}{3} \rfloor$. Such a segment $P(F)$ must lie on the line

$$P(v) + xP(v_j), \quad -\infty < x < \infty,$$

with $j = i \pmod 3$, all $i \in I$, and $j \in \underline{3}$. This is one of the mesh lines which form the partition Δ. Since M is discontinuous only across such segments $P(F)$, it follows that M is in $S_{r,\rho}(\Delta)$, and therefore so are all the M_{ij}. |||

Lemma 3. *If* $\rho > \frac{2r-4}{3}$, *then every* $s \in S_{r,\rho}(\Delta)$ *can be represented on* R_+^2 *as a linear combination of the truncated powers in*

$$T := T_{r,\rho}(\Delta) := \{x^p y^q,\ (x-x_i)_+^p y^q,\ x^p (y-y_j)_+^q,\ (x-y-x_i)_+^p (x+y)^q : p,q \ge 0;\ p+q<r;$$
$$p,q > \rho \text{ whenever they exponentiate a truncated function}\}.$$

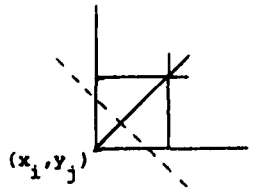
Proof. Denote by σ_{ij}^- and σ_{ij}^+ the left, respectively right, triangle in Δ which make up the square σ_{ij}. Let $s \in S$ and let t_0 be the polynomial of order r which agrees with s on σ_{00}^-. Then $s - t_0 = 0$ on the line $y = x$. Therefore $s - t_0 = u$ on σ_{00}^+, with

$$u(x,y) = \sum_{\substack{p \ge \rho \\ p+q<r}} c_{pq}(x-y)^p (x+y)^q.$$

Set $t_1 := t_0 + \sum c_{pq}(x-y)_+^p (x+y)^q$. Then $s = t_1$ on σ_{00}. We now continue in this manner, moving to the right, using the truncated powers $(x-x_i)_+^p y^q$ and $(x-y-x_i)_+^p (x+y)^q$ to construct a function t in the span of T which agrees with s on σ_{i0}, $i=0,1,2,\ldots$. On the first column $(\sigma_{0i})_{i=1}^{\infty}$ of squares, this t is just the polynomial t_0. Proceeding now up that column, we further modify t by the addition of a suitable linear combination of the truncated powers $x^p (y-y_i)_+^q$ and $(x+y)^p (x-y-y_j)_+^q$ (note that $(x-y-y_j)_+^q = (x-y-x_i)_+^q$ for suitable x_i) and so obtain a new t in the span of T which agrees with s on the first row and first columnn of squares, i.e., on

$$\bigcup_{i=0}^{\infty} \sigma_{i0} \cup \sigma_{0i}.$$

We claim that now $s = t$ on all of R_+^2. We prove this by showing that $s = t$ on σ_{ij}, $i,j > 0$, using lexicographic ordering and induction.



Assume that $s = t$ on $\sigma_{\mu\nu}$ for all $\mu < i$ and for all (μ,ν) with $\mu = i$ and $\nu < j$ (as is the case for $(i,j) = (1,1)$). Then $u := s-t$ is a piecewise polynomial function of total order r and smoothness ρ which vanishes for $0 < x < x_i$ and for $0 < y < y_j$. If now u were nonzero somewhere in σ_{ij}, then, on restricting u to some line $x+y = c$, with $x_i+y_j < c < x_{i+1}+y_j$, we would obtain a <u>univariate</u> piecewise polynomial function v of order r and smoothness ρ which vanishes outside some interval $[a,b]$ and has just three knot locations, viz. the points a, $(a+b)/2$, and b. Further, v would be nonzero somewhere in $[a,b]$. This would imply that the sum of the multiplicities of the knots a, $(a+b)/2$, and b is at least $r+1$, which would imply that at least one knot has multiplicity $\leq \frac{r+1}{3}$, therefore $\rho \leq r-1 - \lceil \frac{r+1}{3} \rceil = \lfloor \frac{2r-4}{3} \rfloor$, a contradiction. This advances the induction hypothesis and so finishes the proof. |||

Theorem 3. $\bigcup_n S_{r,\rho}(\Delta_n)$ <u>is dense in</u> $C_0(\mathbb{R}^2)$ <u>if and only if</u> $\rho \leq \frac{2r-4}{3}$.

Proof. By Lemma 2, $S_{r,\rho}(\Delta)$ contains the local and positive partition of unity (M_{ij}) as long as $\rho \leq \frac{2r-4}{3}$. This implies the density of $\bigcup_n S_{r,\rho}(\Delta_n)$ by the following standard argument: The simple approximation map

$$Tf := \sum_{i,j} f(x_i,y_j)M_{ij}$$

carries f into $S_{r,\rho}(\Delta)$ and satisfies

$$|(f-Tf)(x,y)| = \Big| \sum_{i,j} \big(f(x,y)-f(x_i,y_j)\big)M_{ij}(x,y) \Big|$$

$$\leq \max_{M_{ij}(x,y)\neq 0} |f(x,y) - f(x_i,y_j)|$$

Consequently, $\|f - Tf\| \leq \omega(f;\ \mathrm{diam\ supp\ } M)$. Now scale.

For the converse, assume that $\rho > \frac{2r-4}{3}$. Consider the difference operator

$\Delta_h := \Delta_h^r \cdot \delta_{(h,h)}^r$. Since Δ_h^r and $\delta_{(h,h)}^r$ commute, we have $\Delta_h t = 0$ for all t in the span of $T_{r,\rho}(\Delta)$ defined in Lemma 3 . Therefore, by Lemma 3, $\Delta_h s = 0$ on $\mathbb{R}_+^2$ for every $s \in S_{r,\rho}(\Delta)$.

Suppose now that $f \in C_0^\infty(\mathbb{R}^2)$ and that $\|f - s_n\| \to 0$ (as $n \to \infty$) for some $s_n \in S_{r,\rho}(\Delta_n)$, $n=1,2,\ldots$. Assume without loss of generality that f has its support in $\mathbb{R}_+^2$. Since Δ_n is obtained from Δ by scaling, it follows that $\Delta_h s_n = 0$ on $\mathbb{R}_+^2$, therefore $\Delta_h f = 0$. Dividing by h^{3r} and taking the limit as $h \to 0$ shows that

$$(D_{e_1} D_{e_2} D_{e_1+e_2})^r f = 0 ,$$

which shows that f is not an arbitrary function in $C_0^\infty(\mathbb{R}^2)$. |||

Next we consider the approximation from $S_{r,\rho}(\Delta_n)$ to smooth functions. For this, we need to consider the B-splines associated with faces of B . Any $I \subseteq \underline{\mathbf{m}}$ is associated with a face, viz. the face

$$F := F_I := \{ \sum_{i \in I} \alpha_i v_i : 0 \le \alpha_i \le 1 \} .$$

This is a face of dimension $|I|$, or, a $|I|$-face, for short.

Denote the corresponding B-spline M_{F_I} by M^I . We are particularly interested in faces for which

$$\underline{2} \subseteq I \quad \text{and} \quad |I| > 2 .$$

For such a face, set

$$Q := \{ \sum_{i \notin I} \alpha_i v_i : 0 \le \alpha_i \le 1 \} .$$

Then, for any test function ϕ ,

$$\int_{\mathbb{R}^2} \phi M^I = \int_F \phi \circ P = \int_{F+Q} \phi \circ P ,$$

using the facts that $\phi(Pz) = \phi(Pz_0)$ in case $z \in z_0 + Q$, and that the $(m-|I|)$-dimensional volume of Q is 1 . This shows that, for such a face,

(3.5) $$M^I(x,y) = \mathrm{vol}_{m-2}\{z \in F+Q : Pz = (x,y)\} = \mathrm{vol}_{|I|-2}\{z_0 : Pz_0 = (x,y)\} .$$

Now define

$$M_{ij}^I := M_{F_{ij}} ,$$

with $F_{ij} := (i,j) + F$ the face F translated. The F_{ij} have pairwise disjoint interiors and form the set $\mathbb{R}^2 \times C$ with $C := \{ \sum_{i \in I \setminus 2} \alpha_i v_i : 0 \leq \alpha_i \leq 1 \}$. Hence it follows from (3.5) that

$$\sum_{i,j} M^I_{ij} = \mathrm{vol}_{|I|-2}(C) = 1 . \tag{3.6}$$

Our next lemma is a special case of a result in [3].

Lemma 4. Let $r := m-1$ and $k := \left\lfloor \frac{2r+2}{3} \right\rfloor$. Then, for $\alpha+\beta < k$, there are polynomials $Q_{\alpha\beta}$ of the form $Q_{\alpha\beta}(x,y) = \frac{x^\alpha y^\beta}{\alpha!\beta!} + R_{\alpha\beta}(x,y)$ with $R_{\alpha\beta}$ of coordinate order (α,β) such that

$$\sum_{i,j} Q_{\alpha\beta}(i,j)\, M_{ij}(x,y) = \frac{x^\alpha y^\beta}{\alpha!\beta!} . \tag{3.7}$$

Proof. The proof is an adaptation of the argument in an early draft of [3] to our particular context. It is essentially an inductive proof on the number $|I|$ of the following

Claim. Let $\mathbb{Z}^2 \subseteq I \subseteq \mathbb{Z}^2$. Let $h := h_I := |I| - d$, with d the largest integer for which there is a d-dimensional face G of $F = F_I := \{\sum_{i \in I} \alpha_i v_i : 0 < \alpha_i < 1\}$ with $\dim P(G) < 2$. Then, for any $\alpha+\beta < h$, the function $\phi_{\alpha\beta}$, given by

$$\phi_{\alpha\beta} := \sum_{i,j} i^\alpha j^\beta M^I_{ij},$$

is a polynomial of coordinate degree (α,β) with leading coefficient 1, i.e.,

$$\phi_{\alpha\beta}(x,y) = x^\alpha y^\beta + \text{terms of order } (\alpha,\beta) . \tag{3.8}$$

To prove this claim, consider first the case $|I| = 3$. Let $I = \{1,2,u\}$. If $u = 1$ or $2 \pmod 3$, then $d = 2$, hence $h = 1$ and the claim is just (3.6). If $u = 3 \pmod 3$, then M^I vanishes at all mesh points (i,j) except that $M^I(1,1) = 1$, while $\Sigma_{ij} \alpha_{ij} M^I_{ij}$ is continuous and piecewise linear. Therefore

$$\sum_{i,j} f(i+1,j+1)\, M^I_{ij} = f$$

for all linear functions f, and the claim follows also for this case.

Let now $s > 3$, assume the claim proved for all $|I| < s$, and consider some I with $|I| = s$.

If $d = |I| - 1$, then $h = 1$ and the claim reduces once again to (3.5). Otherwise, $d < |I| - 1$. In this case, consider $\mu \in I \setminus \mathcal{Z}$ and set $u := v_\mu$. We calculate $D_{Pu}\phi_{\alpha\beta}$ for some $\alpha+\beta < h$. By (3.4),

$$D_{Pu}M^I = -\Sigma(u\cdot n_i)\, M_{F_i} \tag{3.9}$$

where the F_i are the $(|I|-1)$-faces of F_I and the n_i are the corresponding outward normals. This implies that $(u\cdot n_i) = 0$ for any face F_i parallel to u. There are only two faces which are not parallel to u, viz. the faces F_J and F_J+u, with

$$J := I \setminus \{u\} .$$

Since their normals sum to zero and $|(u\cdot n_i)| = 1$, (3.8) becomes simply

$$D_{Pu}M^I = -M^J + M^J_{Pu} .$$

This implies that

$$D_{Pu}\phi_{\alpha\beta} = \sum_{i,j} f_{\alpha\beta}(i,j)\, M^J_{ij} \tag{3.10}$$

with $f_{\alpha\beta}(x,y) := x^\alpha y^\beta - \xi^\alpha \eta^\beta$ and $(\xi,\eta) := (x,y) - Pu$. More explicitly,

$$\text{if } Pu = \begin{Bmatrix} e_1 \\ e_2 \\ e_1+e_2 \end{Bmatrix} \text{ then } f_{\alpha\beta}(i,j) = \begin{Bmatrix} \alpha i^{\alpha-1} j^\beta \\ \beta i^\alpha j^{\beta-1} \\ \alpha i^{\alpha-1} j^\beta + \beta i^\alpha j^{\beta-1} \end{Bmatrix} .$$

Since $h_J \geq h - 1$ and $|J| < |I|$ and $f_{\alpha\beta}$ is of coordinate degree $< (\alpha,\beta)$, we may apply the induction hypothesis to conclude that $D_{Pu}\phi_{\alpha\beta}$ is a polynomial; more precisely, that

$$D_{Pu}\phi_{\alpha\beta}(x,y) = \begin{Bmatrix} \alpha x^{\alpha-1} y^\beta \\ \beta x^\alpha y^{\beta-1} \\ \alpha x^{\alpha-1} y^\beta + \beta x^\alpha y^{\beta-1} \end{Bmatrix} + \text{ lower order terms, if } Pu = \begin{Bmatrix} e_1 \\ e_2 \\ e_1+e_2 \end{Bmatrix} .$$

Integrating back up, we find that

$$\phi_{\alpha\beta}(x,y) = x^\alpha y^\beta + \text{lower order terms} + g_{\alpha\beta}(z_1 x + z_2 y)$$

with g a univariate function and $z \in \mathbb{R}^2$ perpendicular to Pu.

But now, since $|I| > 3$ and $d < |I| - 1$, it is possible to choose u in different ways, say $u = v$ and $u = w$, so that Pv and Pw are linearly independent. This allows the conclusion that actually (3.8) holds and so advances the induction hypothesis.

This finishes the proof of the Claim. Taking now, in particular, $I = \mathfrak{m}$, we find that $d = \lceil \frac{m}{3} \rceil$, hence $h_I = m - \lceil \frac{m}{3} \rceil = \lfloor \frac{2m}{3} \rfloor = k$, and we conclude that (3.8) holds for all $\alpha+\beta < k$. Repeated application of this fact gives us, for each $\alpha+\beta < k$, a polynomial $Q_{\alpha\beta}$ of coordinate degree (α,β) with leading coefficient $1/(\alpha!\beta!)$ such that

$$\sum_{i,j} Q_{\alpha\beta}(i,j)\, M_{ij}(x,y) = \frac{x^\alpha y^\beta}{\alpha!\beta!} . \quad |||$$

We conclude that, for any r, s, and any $\alpha+\beta < k$,

$$\frac{(x-r)^\alpha (y-s)^\beta}{\alpha!\ \beta!} = \sum_{i,j} Q_{\alpha\beta}(i-r,j-s)\, M_{i-r,j-s}(x-r,y-s)$$

$$= \sum_{i,j} Q_{\alpha\beta}(i-r,j-s)\, M_{ij}(x,y) .$$

Consequently, for any $p \in P_k$,

$$p(x,y) = \sum_{\alpha+\beta<k} \frac{(x-r)^\alpha (y-s)^\beta}{\alpha!\ \beta!} \left(D^{(\alpha,\beta)}p\right)(r,s) = \sum_{i,j} q_{rs}(i,j)\, M_{ij}(x,y) \tag{3.11a}$$

with

$$q_{rs}(x,y) := \sum_{\alpha+\beta<k} Q_{\alpha\beta}(x-r,y-s)\left(D^{(\alpha,\beta)}p\right)(r,s) . \tag{3.11b}$$

The next lemma implies that q_{rs} is, in fact, independent of r and s.

Lemma 5. *The map* $P_k \longrightarrow \operatorname{span}(M_{ij}) : q \longmapsto \sum_{i,j} q(i,j)\, M_{ij}$ *is one-one.*

Proof. Suppose $q \in P_k \backslash \{0\}$. Then, for any $r > 0$, there exists $c \in \mathbb{R}^2$ so that q is of one strict sign on the ball $B_r(c)$ of radius r and center c. Choose $r = \operatorname{diam} \operatorname{supp}(M)$. Since

$$(i,j) \in \operatorname{supp}(M_{ij}) = (i,j) + \operatorname{supp}(M) ,$$

it follows that $\{q(i,j)M_{ij}(c) : M_{ij}(c) \neq 0\}$ are all of one strict sign, hence

$$\sum_{i,j} q(i,j)M_{ij} \neq 0 . \quad |||$$

We conclude that q_{rs} in (3.11) does not depend on r, s. In particular,

$$q_{rs}(i,j) = q_{ij}(i,j) = \sum_{\alpha+\beta<k} Q_{\alpha\beta}(0,0)(D^{(\alpha,\beta)}p)(i,j) .$$

Corollary. *For any* $p \in \mathbb{P}_k$, $p = \sum_{i,j} \lambda(p(\cdot+i,\cdot+j))\, M_{ij}$ *with*

$$\lambda f := \sum_{\alpha+\beta<k} Q_{\alpha\beta}(0,0)\, (D^{(\alpha,\beta)}f)(0,0) .$$

We use the linear functional λ to define an approximation from $S_{r,\rho}(\Delta)$ in the now standard quasi-interpolant fashion. First, we modify λ so as to make it applicable to any $f \in C_0(\mathbb{R}^2)$. For this, let μ be a bounded extension of λ from $\mathbb{P}_k$ to $C(\sigma_{00}^+)$. Then μ can be taken to be a bounded linear functional on all of $C_0(\mathbb{R}^2)$. In this way, we obtain a linear map

$$Lf := \sum_{i,j} \mu(f(\cdot+i,\cdot+j))\, M_{ij}$$

on $C_0(\mathbb{R}^2)$ to $S_{r,\rho}(\Delta)$ which is local, reproduces $\mathbb{P}_k$, and is bounded by $\|\mu\|$ since $\Sigma_{ij} M_{ij} = 1$. This implies that

$$\|f - Lf\|(A) \le (\|\mu\| + 1)\, \mathrm{dist}_{N(A)}(f, \mathbb{P}_k) \tag{3.12}$$

with $N(A) := \cup\{\mathrm{supp}(M_{ij}) : \mathrm{supp}(M_{ij}) \cap A \neq \emptyset\}$. Scaling by $1/n$ to get to the partition Δ_n gives the map L_n to $S_{r,\rho}(\Delta_n)$ for which we have the following

Theorem 4. *Set* $m := r-1$, $\rho := \lfloor \frac{2r-4}{3} \rfloor$ *and* $k := \lfloor \frac{2r+2}{3} \rfloor = \rho+2$. *If* $f \in C_0(\mathbb{R}^2)$, *then* $L_n f \in S_{r,\rho}(\Delta_n)$ *and*

$$\|f - L_n f\| \le \mathrm{const}_k\, \omega_k(f, 1/n) , \quad n=1,2,\ldots \tag{3.13}$$

with ω_k *the* k*-th order modulus of smoothness of* f. *In particular, if* $f \in C_0^k(\mathbb{R}^2)$, *then*

$$\|f - L_n f\| \le \mathrm{const}_k\, |f|_k\, n^{-k} . \tag{3.14}$$

Proof. Let $A = \sigma_{ij}(n)$ and, correspondingly

$$N(A) := \cup\{\mathrm{supp}(M_{pqn}) : \mathrm{supp}(M_{pqn}) \cap A \neq \emptyset\} .$$

By (3.12),

$$\|f - L_n f\|(A) \le (\mathrm{const}_r + 1)\, \mathrm{dist}_{N(A)}(f, \mathbb{P}_k) , \tag{3.15}$$

while, e.g., from [5],

$$\operatorname{dist}_{N(A)}(f,\mathbb{P}_k) \leq \text{const}_k\, \varepsilon$$

with $\varepsilon = \omega_k(f, 1/n)$ or $\varepsilon = |f|_k n^{-k}$ since diam $N(A) = O(1/n)$. |||

4. Concluding remarks. The construction of smooth multivariate spline interpolants developed in Section 3 can be extended to more general partitions (see also [3]). Suppose that Π is a partition of $\mathbb{R}^2$ which can be obtained as follows. Starting with a regular partition Π_0 , associated with the two independent directions d_1 and d_2 , we add mesh lines through all the vertices of Π_0 in the directions of the vectors $d_3, \ldots, d_p$, with these d_i's nonzero vertices of Π_0 other than d_1 or d_2 , and pairwise linearly independent.

Thus Σ is associated with the directions e_1 and e_2 , Δ is associated with the directions e_1, e_2 and e_1+e_2 , while the four directions e_1, e_2, e_1+e_2, e_1-e_2 are associated with a partition in which both diagonals are drawn into every square.

Given such a partition Π and $m \geq p$, define vectors $v_1, \ldots, v_m$ in $\mathbb{R}^m$ by $v_1 := d_1$, $v_2 := d_2$, and , for $i > 2$, $v_i := d_j + e_i$, with $j \in \underset{\sim}{p}$ and $j = i \pmod p$. Set $B := \{\sum_1^m \alpha_i v_i : 0 \leq \alpha_i \leq 1\}$. Then the corresponding B-spline M_B given by (3.3) is of order $r := m-1$ and smoothness $\rho := r-1-\left\lceil \frac{r+1}{p} \right\rceil$. If (x_i,y_i) are the vertices of Π_0 , then the *translated splines* M_i *given by* $M_i(x,y) := M_B(x-x_i,y-y_i)$, all i , form a local partition of unity. Using arguments like those for Lemma 4 and Theorem 4 , they can be used to construct a local and bounded quasiinterpolant on $C_0(\mathbb{R}^2)$ into $S_{r,\rho}(\Pi_n)$ which reproduces polynomials of total order $k := \rho+2$, hence approximates $C_0^k(\mathbb{R}^2)$-functions to within $O(n^{-k})$.

The particular choice $d_j := \left(\cos \frac{\pi j}{3}, \sin \frac{\pi j}{3}\right)$, j=1,2,3, results in a partition of $\mathbb{R}^2$ into equilateral triangles. We have recently learned that P. Frederickson [6], [7] has studied spline approximation on this partition and has shown the existence of spline interpolants from $S_{r,\rho}(\Pi_n)$ in the case $\rho = \frac{2r-4}{3}$ and $r = 2 \pmod 3$. This is accomplished by using a partition of unity given as translates of a fixed finite support spline obtained by a certain convolution.

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The degree of approximation achievable by piecewise polynomial functions on certain regular grids in the plane is shown to be adversely affected by smoothness requirements, - in stark contrast to the univariate situation. For a rectangular grid, and for the triangular grid derived from it by adding all northeast diagonals, the maximum degree of approximation (as the grid size $1/n$ goes to zero) to a suitably smooth function is shown to be $O(n^{-\rho-2})$ in case we insist that the approximating functions are in C^{ρ}. This only holds as long as

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20. Abstract (continued)

$\rho \leq \frac{r-3}{2}$ and $\rho \leq \frac{2r-4}{3}$, respectively, with r the total order of the polynomial pieces. In the contrary case, some smooth functions are not approximable at all. In the discussion of the second mesh, a new and promising kind of multivariate B-spline is introduced.