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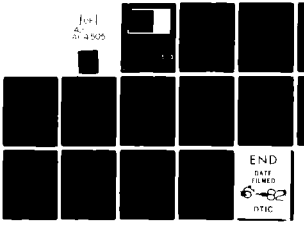

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Tai-Ping Liu[1] and Michel Pierre[1,2]

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ABSTRACT

We study the uniqueness of the solutions to the scalar conservation law $u_t + \varphi(u)_x = 0$ when the initial datum is a finite measure. The case of a Dirac mass is particularly emphasized: it is shown how it provides a description of the asymptotic behavior of the solutions initiated by an arbitrary integrable function. This behavior is proved to depend on one parameter in the case when ϕ is odd while it depends on two when ϕ is convex.

AMS (MOS) Subject Classifications: Primary 35L65, 35G25; Secondary 35B40

Key Words: initial-value problem, conservation law, measures, asymptotic behavior

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

It is well-known that the solutions of the linear heat equation

$$(1) \qquad u_t - u_{xx} = 0 \qquad (t,x) \in (0,\infty) \times \mathbf{R}$$

behave for large time in a way that depends only on the total mass $\int_{\mathbf{R}} u(0,x)dx$ of the initial data. This property is also shared by the solutions of the nonlinear equation modelling the filtration of a gas through a porous medium

$$(2) \qquad u_t - (u^2)_{xx} = 0 \qquad (t,x) \in (0,\infty) \times \mathbf{R} .$$

Although it may seem paradoxical, this property turns out to be strongly related to the uniqueness of the so-called "fundamental solution" of (1) - or (2) - when the initial datum is the unit Dirac mass at the origin.

Here we study the same kind of relationship for the scalar conservation law

$$(3) \qquad u_t + \varphi(u)_x = 0 \qquad (t,x) \in (0,\infty) \times \mathbf{R} ,$$

where $\varphi : \mathbf{R} \to \mathbf{R}$ is a given constitutive function. We prove that the above properties are also valid for (3) when φ is odd. However, when φ is convex it is known that the asymptotic behavior of the solutions of (3) depends on two parameters (rather than just one, namely $\int_{\mathbf{R}} u(0,x)dx$). This corresponds to the existence of a one-parameter family of fundamental solutions of (3). The corresponding initial-value problem has then to be understood in a stronger sense.

In addition to the above motivation and some obvious applications to physical situations involving initial masses concentrated at some points, this study also represents a contribution to the theoretical question concerning the extension of nonlinear semigroups in L^1 to the space of measures.

SOURCE-SOLUTIONS AND ASYMPTOTIC BEHAVIOR IN CONSERVATION LAWS

Tai-Ping Liu[1] and Michel Pierre[1,2]

INTRODUCTION

The main goal of this paper is the study of the uniqueness of the solutions to the scalar conservation law

$$u_t + \varphi(u)_x = 0 \tag{0.1}$$

when the initial data is a given finite measure μ and its application to the description of the asymptotic behavior of the solutions to (0.1) with integrable initial data.

Solutions to (0.1) are understood in Kružkov's sense [10] which includes the usual "entropy" conditions. The initial data is understood in the sense of measures, namely

$$\lim_{t \downarrow 0} \int_{\mathbf{R}} \psi(x)u(t,x)dx = \int_{\mathbf{R}} \psi(x)d\mu(x) , \tag{0.2}$$

for all continuous, bounded functions ψ on $\mathbf{R}$.

The results turn out to be quite different whether one assumes the nonnegativity of the solutions or not and whether φ is odd or convex. Namely, given a nonnegative finite measure μ, we prove that there exists a unique nonnegative solution to (0.1), (0.2). But the problem is not well-posed in general if the assumption of nonnegativity of the solutions is dropped even if the initial measure is nonnegative. Explicit counterexamples are known in the literature when φ is a strictly convex function.

However, one can prove that the problem (0.1), (0.2) remains well-posed if φ is assumed to be odd.

This emphasizes a difference in nature between the two typical cases $\varphi(u) = u^2$ and $\varphi(u) = u^3$ which does not appear as long as the initial data is a bounded integrable function. Actually, this difference corresponds to a fundamental difference in the

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asymptotic behavior of the solutions to (0.1). This is one of the motivations of our study.

Indeed, at least when φ homogeneous, there is a strong relationship between the asymptotic behavior of the solutions to (0.1) (with $u(0) \in L^1(\mathbb{R})$) and the "fundamental solutions" of (0.1), that is the solutions whose initial data are concentrated at the origin. Using similarity transformations, we can prove that, when t is large, the solutions of

$$u_t + (|u|^m)_x = 0 \quad \text{or} \quad u_t + (\text{sign } u|u|^m)_x = 0 \ ,$$

are "closed" to a solution of the same problem with the initial datum "$\delta \cdot \int_{\mathbb{R}} u(0,x)dx$" where δ is the Dirac mass at the origin. In the case when φ is odd, this solution is unique: this proves that the asymptotic behavior depends only on one parameter (namely $\int_{\mathbb{R}} u_0$). In the case when φ is convex, the lack of uniqueness corresponds to the fact that the asymptotic behavior depends on two parameters as it is well-known for $\varphi(u) = u^{2k}$ (see [11], [6], [12], [5] and the references in them). In the latter case, it is necessary to put more information in the initial data.

The use of similarity solutions to study asymptotic behaviors has already been applied to parabolic problems like the heat equation $u_t - \Delta u = 0$, or more generally the so-called "porous media equation" $u_t - \Delta u^m = 0$ (see [9], [7]). Related uniqueness questions for initial data measures have also been treated in [8], [13]. Here the same main ideas are used, but the hyperbolic structure leads to specific difficulties. In particular, it is interesting to mention how the proof of the uniqueness involves the resolution of a "dual" linear problem

$$(\alpha\psi)_t + (\beta\psi)_x = 0 \ ,$$

when α, β are discontinuous coefficients. The built-in property $\alpha_t + \beta_x \leq 0$, coming from the "entropy" condition, insures the existence of solutions to this problem.

Besides the above motivations and some obvious applications to "physical" situations involving initial masses concentrated at some point, let us mention that this study is also a contribution to a theoretical question concerning the extension of nonlinear semigroups on L^1 to the space of measures. It shows that the answer depends strongly on φ in the

case of the semi-groups generated by the operators $Au = "\varphi(u)_x"$ (see [3] for a precise definition) although they are m-accretive in $L^1(\mathbf{R})$ for most continuous φ (see [3]).

Some Notations. For ω open set in $\mathbf{R}^n$, $C_0^\infty(\omega)$ denotes the space of infinitely differentiable functions with compact support in ω and $\mathcal{D}'(\omega)$ the space of distributions on ω.

For simplicity, we will write

$$\int_{\mathbf{R}} f \quad \text{for} \quad \int_{\mathbf{R}} f(x)dx, \qquad \int_0^\infty \int_{\mathbf{R}} f \quad \text{for} \quad \int_0^\infty \int_{\mathbf{R}} f(t,x)dtdx ,$$

$$\int_{\mathbf{R}} f d\mu \quad \text{for} \quad \int_{\mathbf{R}} f(x)d\mu(x) \quad \text{if} \quad \mu \text{ is a measure on } \mathbf{R} .$$

Given a function f, we set

$$f^+(\xi) = \begin{cases} f(\xi) & \text{if } f(\xi) \geq 0 \\ 0 & \text{if } f(\xi) \leq 0 \end{cases}, \qquad f^-(\xi) = \begin{cases} 0 & \text{if } f(\xi) \geq 0 \\ -f(\xi) & \text{if } f(\xi) \leq 0 \end{cases}.$$

In the same way, μ^+ and μ^- are the positive and negative part of a given Radon measure μ on $\mathbf{R}$.

We will indifferently use f_x or $\frac{\partial f}{\partial x}$ for partial derivatives of a function f.

We set

$$\operatorname{sign} r = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ 1 & \text{if } r > 0 . \end{cases}$$

Some facts about measures. Given a sequence of signed Radon measures $(\mu_n)_{n\geq 0}$ on $\mathbf{R}$, we will say that μ_n converges narrowly to μ if, for all f continuous, bounded on $\mathbf{R}$

$$\lim_{n\to\infty} \int_{\mathbf{R}} f d\mu_n = \int_{\mathbf{R}} f d\mu .$$

We will use that a sequence μ_n is relatively compact for the narrow convergence if and only if (see [14])

(i) the total mass $\int_{\mathbf{R}} d|\mu_n|$ is uniformly bounded

(ii) $\lim_{R\to\infty} \int_{|x|>R} d|\mu_n| = 0$ uniformly in n.

SECTION 1: UNIQUENESS

Throughout this section, we are given

(1.1) $\varphi : \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous with $\varphi(0) = 0$.

By solution on $(0,T)$ of

(1.2) $$u_t + \varphi(u)_x = 0 ,$$

we mean a function $u(t,x)$ satisfying (see [10])

(1.3) $$u \in L^\infty(0,T; L^1(\mathbb{R})) \cap L^\infty((\tau,T) \times \mathbb{R}) \quad \forall \tau \in (0,T)$$

(1.4)
$$\begin{cases} \forall k \in \mathbb{R}, \quad \forall \psi \in C_0^\infty((0,T) \times \mathbb{R}), \quad \psi \geq 0 , \\ \\ \int_0^T \int_{\mathbb{R}} |u - k| \psi_t + \operatorname{sign}(u - k)(\varphi(u) - \varphi(k))\psi_x \geq 0 . \end{cases}$$

Theorem 1.1 (Nonnegative solutions). Assume $\varphi([0,\infty)) \subset [0,\infty)$. Then, for any nonnegative finite measure μ on $\mathbb{R}$, there exists at most one nonnegative solution u to (1.3), (1.4) such that

(1.5) $$\lim_{t \downarrow 0} \text{ess } u(t) = \mu \quad \text{narrowly in } \mathbb{R} .$$

Remark 1.1. Since $u(t)$ (and $\hat{u}(t)$) are not a priori defined for all $t \in (0,T)$, by "lim ess" we mean limit outside a set of zero measure. Actually, using Kružkov's uniqueness result (applied to $u(t)$ for all $t \in (0,T)$), we know that any solution to (1.3), (1.4) belongs to $C((0,T); L^1(\mathbb{R}))$ the space of continuous functions from $(0,T)$ into $L^1(\mathbb{R})$.

Remark 1.2. The uniqueness result stated in Theorem 1 fails if one drops the assumption of nonnegativity for the solutions. Indeed when $\varphi(u) = u^2$, there exists a 1-parameter family of "source-solutions" to (1.3), (1.4) satisfying

$$u(t) \to \delta \quad \text{(Dirac mass at the origin)} ,$$

for the narrow convergence on $\mathbb{R}$ (see [11], [6], [5] and Section 2). However, this does not happen if $\varphi(u) = \operatorname{sign} u|u|^m$ $(m > 1)$ and it is the purpose of our next result.

Theorem 1.2. Assume φ is odd and $\varphi(r) \cdot r \geq 0$ for all $r \in \mathbb{R}$. Then, for any nonnegative finite measure μ on $\mathbb{R}$, there exists at most one solution u to (1.3),

(1.4) such that

$$\lim_{t \downarrow 0} \operatorname{ess}\ u(t) = \mu \quad \text{narrowly in } \mathbf{R}\ .$$

Remark 1.3. By symmetry, the same result holds for nonpositive finite measures μ. By combining existence results and the ideas used in the proof of Theorem 1.2, one could also prove the same result for any finite signed measure.

In order to have uniqueness for more general φ, it is necessary to understand the initial data in a stronger sense. To illustrate this, let us state one more result which can be applied to $\varphi(u) = |u|^m$, $m > 1$ and which also directly gives informations about the asymptotic behavior of the solutions (see Section 3).

Theorem 1.3. Assume $\varphi \geq 0$ on $\mathbf{R}$. Let μ be a finite signed measure on $\mathbf{R}$ and $v: \mathbf{R} \to \mathbf{R}$ of bounded variation such that $\frac{d}{dx} v(x) = \mu$ and $v(-\infty) = 0$. Then there exists at most one solution u to (1.3), (1.4) such that

$$(1.6) \qquad \begin{cases} \lim_{t \downarrow 0} \operatorname{ess}\ u(t) = \mu \quad \text{narrowly in } \mathbf{R} \\ \\ \forall x \in \mathbf{R},\ \lim_{t \downarrow 0} \operatorname{ess} \int_{-\infty}^{x} u(t,\xi)d\xi = v(x)\ . \end{cases}$$

Remark 1.4. These results show that the solution is determined by v rather than by μ. If μ has a mass at x_0, then there are infinitely many choices for $v(x_0)$. They are not always arbitrary. Indeed we will see that v is necessarily lower-semicontinuous when $\varphi \geq 0$. Under the assumptions of Theorems 1.1, 1.2, v is exactly equal to $\int_{(-\infty,x]} d\mu$. But if $\varphi(u) = u^2$, other nontrivial choices of v do occur.

In the proof of the theorems, we will need the following consequences of Kružkov's definition of solution (1.4).

Lemma 1.1. Let u be a solution to (1.2), (1.4). Then $u \in C((0,T); L^1(\mathbf{R}))$ and

$$\forall k \in \mathbf{R}, \qquad \forall \psi \in C_0^\infty((0,T) \times \mathbf{R}), \quad \psi \geq 0\ .$$

$$(1.7) \qquad \int_0^T \int_{\mathbf{R}} (u^+ - k)^+ \psi_t + \operatorname{sign}^+(u^+ - k)(\varphi(u^+) - \varphi(k))\psi_x \geq 0$$

$$(1.8)\qquad \int_0^T \int_{\mathbf{R}} (u^- - k)^+ \psi_t + \operatorname{sign}^+(u^- - k)(\varphi(-k) - \varphi(-u^-))\psi_x \geq 0$$

$$(1.9)\qquad u_t + \varphi(u)_x = 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathbf{R})$$

$$(1.10)\qquad u_t^+ + \varphi(u^+)_x \leq 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathbf{R})$$

$$(1.11)\qquad u_t^- - \varphi(-u^-)_x \leq 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathbf{R}) .$$

Proof. We use the fact proved in [10] that any solution of (1.3), (1.4) is in $C([\tau,T); L^1(\mathbf{R}))$ for all $\tau \in (0,T)$ and satisfies: for any H convex on $\mathbf{R}$.

$$(1.12)\qquad \int_0^T \int_{\mathbf{R}} H(u)\psi_t + \Big[\int_k^u \varphi'(s)H'(s)ds\Big]\psi_x \geq 0 ,$$

for all $k \in \mathbf{R}$ and nonnegative $\psi \in C_0^\infty((0,T) \times \mathbf{R})$. Applying this successively with $H(r) = (r^+ - k)^+$ and $H(r) = (r^- - k)^+$ gives (1.7) and (1.8).

To obtain (1.9), since $u \in L^\infty((\tau,T) \times \mathbf{R})$ for all $\tau \in (0,T)$, one can apply (1.4) with ψ supported in $(\tau,T) \times \mathbf{R}$ and successively $k > \max\{u(t,x);\ t \in (\tau,T), x \in \mathbf{R}\}$, $k < \min\{u(t,x);\ t \in (\tau,T), x \in \mathbf{R}\}$.

For (1.10) and (1.11), one uses (1.7) and (1.8) with $k = 0$.

For convenience, we will call subsolution of (1.2) a function u satisfying (1.3) and

$$(1.15)\qquad \begin{cases} \forall k \in \mathbf{R}, \quad \forall \psi \in C_0^\infty((0,T) \times \mathbf{R}), \quad \psi \geq 0 \\ \\ \displaystyle\int_0^T \int_{\mathbf{R}} (u - k)^+ \psi_t + \operatorname{sign}^+(u - k)(\varphi(u) - \varphi(k))\psi_x \geq 0 . \end{cases}$$

Remark 1.5. By (1.12) applied with $H(r) = (r - k)^+$, a solution of (1.2) is also a subsolution. Note that Lemma 1.1 says that, if u is a solution of (1.2), then u^+ is a subsolution of (1.2) and u^- is a subsolution of $u_t - \varphi(-u)_x = 0$.

The first main step in the proof of the theorems is the following lemma due to Kružkov (see [10] for a proof).

Lemma 1.2. Let $u, \hat{u}$ be solutions of (1.2). Then

$$\frac{\partial}{\partial t}|u - \hat{u}| + \frac{\partial}{\partial x}\operatorname{sign}(u - \hat{u})(\varphi(u) - \varphi(\hat{u})) \leq 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}) . \tag{1.16}$$

Remark 1.6. The relation (1.15) is the main ingredient in Kružkov's uniqueness proof of the solutions to (1.2) belonging to $C([0,T]; L^1(\mathbb{R})) \cap L^\infty((0,T) \times \mathbb{R})$. Indeed, from (1.15) one can prove

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} |u(t) - \hat{u}(t)| \leq 0 ,$$

by choosing suitable test-functions (see [10]). Then $u = \hat{u}$ follows if $u, \hat{u} \in C([0,T]; L^1(\mathbb{R}))$ and $u(0) = \hat{u}(0)$. This method does not apply when the initial data is only a measure. A more sophisticated analysis is then needed which is partly contained in the next lemma.

Lemma 1.3. Let $u, \hat{u}$ be two solutions to (1.2). For all $t \in (0,T)$, we set

$$v(t,x) = \int_{-\infty}^{x} u(t,\xi)d\xi, \quad \hat{v}(t,x) = \int_{-\infty}^{x} \hat{u}(t,\xi)d\xi . \tag{1.17}$$

Then, for all $\theta \in C_0^\infty(\mathbb{R})$, $\theta \geq 0$, and all $\tau \in (0,T)$, there exists a nonnegative $\psi \in L^\infty((0,\tau) \times \mathbb{R})$ such that

$$\text{(1.18)} \quad \text{a.e. } s \in (0,\tau), \int_{\mathbb{R}} |u(\tau) - \hat{u}(\tau)|(v(\tau) - \hat{v}(\tau))\theta = \int_{\mathbb{R}} |u(s) - \hat{u}(s)|(v(s) - \hat{v}(s))\psi(s) .$$

Proof. Let us denote

$$g(t,x) = v(t,x) - \hat{v}(t,x), \quad \alpha(t,x) = |u(t,x) - \hat{u}(t,x)|$$
$$\beta(t,x) = \operatorname{sign}(u(t,x) - \hat{u}(t,x))[\varphi(u(t,x) - \varphi(\hat{u}(t,x))] .$$

Integrating with respect to x the relations

$$u_t + \varphi(u)_x = 0, \quad \hat{u}_t + \varphi(\hat{u})_x = 0$$

gives

$$(v - \hat{v})_t + \varphi(u) - \varphi(\hat{u}) = 0 .$$

By multiplying this by α, we obtain

$$\alpha g_t + \beta g_x = 0 . \tag{1.19}$$

Now, ψ will be constructed as the solution to the dual problem

$$(\alpha\psi)_t + (\beta\psi)_x = 0, \qquad \psi(\tau) = \theta \,, \tag{1.20}$$

where $\theta \in C_0^\infty(\mathbb{R})$, $\theta \geq 0$ and $\tau \in (0,T)$ are given. Since α and β are not continuous, it is not a priori clear that (1.20) has a solution. But using (1.16), which says that

$$\alpha_t + \beta_x \leq 0 \,, \tag{1.21}$$

we are going to prove that (1.20) can be solved.

For this, let us regularize α and β: set $\alpha_n = \rho_n * (\alpha + \frac{1}{n}\eta)$, $\beta_n = \rho_n * \beta$ where ρ_n is a sequence of mollifiers in $\mathbb{R} \times \mathbb{R}$ and $\eta = \eta(x)$ is a positive function of $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then α_n and β_n are defined on some $(\varepsilon_n,\tau) \times \mathbb{R}$ with $\varepsilon_n \to 0$ when $n \to \infty$ and $\alpha_n > 0$. There exists $\psi_n \in C^\infty((\varepsilon_n,\tau) \times \mathbb{R})$ solution of

$$(\alpha_n\psi_n)_t + (\beta_n\psi_n)_x = 0, \quad \psi_n(\tau) = \theta \,. \tag{1.22}$$

(Note that, by setting $\varphi_n = \alpha_n\psi_n$, this equation can be rewritten $\varphi_{nt} + \left(\frac{\beta_n}{\alpha_n}\varphi_n\right)_x = 0$, $\varphi_n(\tau) = \alpha_n(\tau)\theta$.

First $\theta \geq 0 \Longrightarrow \psi_n \geq 0$.

Then if we expand (1.22), we obtain

$$(\alpha_{nt} + \beta_{nx})\psi_n + \alpha_n\psi_{nt} + \beta_n\psi_{nx} = 0 \,.$$

By (1.21), this implies

$$\psi_{nt} + \frac{\beta_n}{\alpha_n}\psi_{nx} \geq 0 \quad \text{on} \quad (\varepsilon_n,\tau) \,.$$

By maximum principle, we then have

$$\forall t \in (\varepsilon_n,\tau), \ \|\psi_n(t)\|_\infty \leq \|\psi_n(\tau)\|_\infty = \|\theta\|_\infty \,. \tag{1.23}$$

As a consequence, there exists a subsequence of ψ_n (still denoted by ψ_n) converging to $\psi \in L^\infty((0,\tau) \times \mathbb{R})$ in $L^{\infty *}((s,\tau) \times \mathbb{R})$ for all $s \in (0,\tau)$, i.e.

$$\forall s \in (0,\tau), \ \forall f \in L^1((0,\tau) \times \mathbb{R}), \ \int_s^\tau \int_{\mathbb{R}} \psi_n f \to \int_s^\tau \int_{\mathbb{R}} \psi f \,.$$

One could show that ψ is a solution of (1.20). But since our purpose is to prove (1.18), let us rather pass to the limit in the following equality obtained by multiplying

(1.22) by q and integrating by parts:

$$\forall s \in (0,\tau),\ \int_{\mathbf{R}} \alpha_n(\tau)g(\tau)\theta = \int_{\mathbf{R}} \alpha_n(s)g(s)\psi_n(s) + \int_s^\tau \int_{\mathbf{R}} \psi_n[(\alpha_n - \alpha)g_t + (\beta_n - \beta)g_x] .$$

But g_t, g_x are in $L^1((s,\tau) \times \mathbf{R})$, α_n, β_n are uniformly bounded on $(s,\tau) \times \mathbf{R}$ and converge pointwise to α,β. Hence the last integral above converges to 0 for all $s \in (0,\tau)$. Now integrating this equality from σ to ρ, $0 < \sigma < \rho < \tau$, we have at the limit

$$(\rho - \sigma) \int_{\mathbf{R}} \alpha(\tau)g(\tau)\theta = \lim_{n\to\infty} \int_\sigma^\rho ds \int_{\mathbf{R}} \alpha_n(s)g(s)\psi_n(s) .$$

Since α_n converges to α in $L^1((\sigma,\rho) \times \mathbf{R})$ and q is bounded on (σ,ρ), this limit is $\int_\sigma^\rho ds \int_{\mathbf{R}} \alpha(s)g(s)\psi(s)$. Dividing by $\rho - \sigma$ and letting ρ tend to σ yield the relation (1.18).

Remark 1.7. Next, we will have to let s tend to 0 in (1.18). For this, we need some information about the behavior of $v(s)$ and $\hat{v}(s)$ when $s \downarrow 0$. For this, let us prove another lemma.

Lemma 1.4. Assume $\varphi([0,\infty)) \subset [0,\infty)$. Let $u \in C((0,T); L^1(\mathbf{R}))$ be a nonnegative subsolution of (1.2) such that $\{u(t);\ t \in (0,T)\}$ is narrowly relatively compact. Then

(i) There exists a unique nonnegative measure μ on $\mathbf{R}$ such that

$$\mu = \lim_{t\downarrow 0} u(t) \quad \text{narrowly on } \mathbf{R} .$$

(ii) When t decreases to 0, $v(t,x) = \int_{-\infty}^x u(t,\xi)d\xi$ increases pointwise to the lower-semi-continuous (l.s.c.) function $v(0,x) = \int_{(-\infty,x]} d\mu$.

Proof of Lemma 1.4. Integrating $u_t + \varphi(u)_x \le 0$ leads to $v_t + \varphi(u) \le 0$. Since $\varphi(u) \ge 0$ here, it proves that $t \to v(t)$ is nonincreasing. When t decreases to 0, $v(t,\cdot)$ is then a nondecreasing family of continuous, nondecreasing functions whose total variation is uniformly bounded (since $u \in L^\infty(0,T; L^1(\mathbf{R}))$. Hence $v(t,\cdot)$ converges pointwise to a

l.s.c. and nondecreasing function. Moreover, by compactness, $u(t,\cdot)$ converges narrowly to $\mu = \frac{d}{dx} v(0,x)$. The narrow convergence of $u(t)$ implies (see [14])

$$\lim_{x \to -\infty} v(t,x) = 0 \quad \text{uniformly for} \quad t \in (0,T) .$$

In particular $v(0,-\infty) = 0$ and

$$\text{(1.24)} \qquad \text{a.e. } x \in \mathbf{R} \qquad v(0,x) = \int_{(-\infty,x)} d\mu .$$

But $v(0,x)$ is left-continuous, since it is l.s.c. and nondecreasing. Therefore (1.24) holds for all $x \in \mathbf{R}$.

We now come to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $u, \hat{u}$ be two solutions to (1.2) with

$$\lim_{t \downarrow 0} \operatorname{ess} u(t) = \lim_{t \downarrow 0} \operatorname{ess} \hat{u}(t) = \mu \quad \text{narrowly} .$$

Let $h \in (0,T)$ be fixed. By Lemma 1.3 applied to $u(\cdot + h)$ and $\hat{u}$, one has for all $\tau \in (0, T - h)$ and $\theta \in C_0^\infty(\mathbf{R})$, $\theta \geq 0$:

$$\text{a.e. } s \in (0,\tau) \quad \int_{\mathbf{R}} \alpha(\tau)(v(\tau + h) - \hat{v}(\tau))\theta \leq \int_{\mathbf{R}} \alpha(s)(v(s + h) - \hat{v}(s))\psi(s)$$

where $\alpha(t) = |u(t + h) - u(t)|$, $\|\psi\|_{L^\infty((0,\tau)\times\mathbf{R})} = M < +\infty$, $\psi \geq 0$ and

$v(t + h,x) = \int_{-\infty}^{x} u(t + h,\xi)d\xi$, $\hat{v}(t,x) = \int_{-\infty}^{x} \hat{u}(t,\xi)d\xi$. By the monotonicity proved in Lemma 1.4, for a.e. s $0 < s < s_0$,

$$\text{(1.26)} \qquad \int_{\mathbf{R}} \alpha(\tau)(v(\tau + h) - \hat{v}(\tau))\theta \leq \int_{\mathbf{R}} \alpha(s)\psi(s)(v(h) - \hat{v}(s_0)) .$$

If $\rho(s) = \alpha(s)\psi(s)$, $\int_{\mathbf{R}} \rho(s) \leq M \int_{\mathbf{R}} u(s + h) + \hat{u}(s)$. Since $u(t), \hat{u}(t)$ are uniformly bounded in $L^1(\mathbf{R})$, so is $\rho(s)$. Moreover, for all $R > 0$

$$\int_{|x|>R} \rho(s) \leq M \int_{|x|>R} u(s + h) + \hat{u}(s) .$$

By narrow compactness, the right-hand side tends to 0 uniformly in $s \in (0,\tau)$ when R tends to ∞. These two estimates on $\rho(s)$ prove that $\{\rho(s);\ s \in (0,\tau)\}$ is narrowly

compact (see [14]). Therefore, one can let s tend to 0 (according to a suitable subsequence) to find the existence of a nonnegative finite measure ρ such that

$$\forall 0 < s < s_0, \int_{\mathbf{R}} \alpha(\tau)(v(\tau + h) - \hat{v}(\tau))\theta \leq \int_{\mathbf{R}} (v(h) - \hat{v}(s_0))d\rho \tag{1.27}$$

(remember that $v(h)$ and $\hat{v}(s_0)$ are continuous and bounded). Now we let s_0 decrease to 0 in (1.27). By monotonicity

$$\int_{\mathbf{R}} \alpha(\tau)(v(\tau + h) - \hat{v}(\tau))\theta \leq \int_{\mathbf{R}} (v(h) - \hat{v}(0))d\rho .$$

By Lemma 1.4, $v(h,x) \leq \int_{(-\infty,x]} d\mu = \int_{(-\infty,x]} d\hat{\mu} = \hat{v}(0,x)$ for all $x \in \mathbf{R}$. Since θ is arbitrary, we obtain

$$\alpha(\tau)(v(\tau + h) - \hat{v}(\tau)) \leq 0$$

and by letting h go to 0

$$|u(\tau) - \hat{u}(\tau)|(v(\tau) - \hat{v}(\tau)) \leq 0 .$$

By symmetry we obtain

$$0 = (u(\tau) - \hat{u}(\tau))(v(\tau) - \hat{v}(\tau)) = \frac{\partial}{\partial x} \frac{1}{2} (v(\tau) - \hat{v}(\tau))^2 .$$

This proves $v = \hat{v}$ and $u = \hat{u}$.

Proof of Theorem 1.2. Let u be a solution to (1.2) with $\mu = \lim_{t \downarrow 0} \operatorname{ess}\, u(t)$ narrowly in $\mathbf{R}$ ($\mu \geq 0$). By Lemma 1.1 and since φ is odd, $u^+(t)$ and $u^-(t)$ are subsolutions to $u_t + \varphi(u)_x = 0$. Since $u(t)$ is narrowly compact, so are $u^+(t)$ and $u^-(t)$ (see [14]). By Lemma 1.4, there exists μ_+ and μ_- nonnegative finite measures such that

$$\mu_+ = \lim_{t \downarrow 0} u^+(t), \qquad \mu_- = \lim_{t \downarrow 0} u^-(t) \quad \text{narrowly} .$$

Moreover, $v_+(t,x) = \int_{-\infty}^{x} u^+(t,\xi)d\xi$ and $v_-(t,x) = \int_{-\infty}^{x} u^-(t,\xi)d\xi$ converge monotonically to $v_+(0,x) = \int_{(-\infty,x]} d\mu_+$, $v_-(0,x) = \int_{(-\infty,x]} d\mu_-$. One has $\mu_+ - \mu_- = \mu$. Since the mapping $\mu \to \mu^+$ is not continuous for the narrow convergence, we a priori do not know that

$$\mu_+ = \mu, \quad \mu_- = 0 . \tag{1.28}$$

We are going to prove it by using Lemma 1.3 again. From this, it will follow that $v_- = 0$ and $u^- = 0$. The uniqueness result of Theorem 1.2 will then be a consequence of Theorem 1.1.

Let us introduce $f_h(t,x)$, $g_k(t,x)$ the solutions to (1.2) with the initial data $f_h(0) = u^+(h)$, $g_k(0) = u^-(k) + \mu_k$, where $\mu_k \in L^1(\mathbf{R})$ is nonnegative and such that μ_k converges narrowly to μ and $\int_{(-\infty,x]} d\mu_k$ increases to $\int_{(-\infty,x]} d\mu$ when k decreases to 0. By the results in [10], we know that f_h, g_k exist and are unique (in particular, they are nonnegative). Moreover, for all $t \in (0,T-h) \cap (0,T-k)$

$$u(t+h) \le f_h(t), \qquad -u(t+k) \le g_k(t),$$

since it holds for $t = 0$ (see [10] or [3]). Taking the positive part, we obtain

$$u^+(t+h) \le f_h(t), \qquad u^-(t+h) \le g_k(t). \tag{1.29}$$

Let us prove that $f_h(t)$ and $g_k(t)$ converge in $\mathcal{D}'(\mathbf{R})$ to the <u>same</u> measure $f(t)$ for all $t \in (0,T)$ when h and k tend to 0. For this, set

$$F_h(t,x) = \int_{-\infty}^{x} f_h(t,\xi)d\xi, \qquad G_k(t,x) = \int_{-\infty}^{x} g_k(t,\xi)d\xi .$$

Since $F_h(0) = v_+(h)$ increases when h decreases to 0, so does $F_h(t)$ for all $t \in (0,T)$ (one can use Lemma 1.3) for instance. Also $G_k(t)$ increases when $k \downarrow 0$. Moreover, by Lemma 1.4

$$F_h(t,x) \le F_h(0,x) = v_+(h,x) \le \int_{(-\infty,x]} d\mu_+ , \tag{1.30}$$

$$G_k(t,x) \le G_k(0,x) = v_-(k,x) + \int_{(-\infty,x]} d\mu_k \le \int_{(-\infty,x]} d\mu_- + d\mu = \int_{(-\infty,x]} d\mu_+ . \tag{1.31}$$

Hence $F_h(t,x)$ converges pointwise to some bounded function $F(t,x)$ when $h \downarrow 0$ and $G_k(t,x)$ converges to some bounded function $G(t,x)$. Moreover, $f_h(t)$, $g_k(t)$ converge narrowly to $f(t) = \frac{\partial}{\partial x} F(t,\cdot)$, $g(t) = \frac{\partial}{\partial x} G(t,\cdot)$. Let us prove $F = G$ (so that $f = g$). Applying Lemma 1.3 to f_h and g_k gives (with obvious notations)

$$\int_{\mathbf{R}} |f_h(\tau) - g_k(\tau)|(F_h(\tau) - G_k(\tau))\theta = \int_{\mathbf{R}} |f_h(s) - g_k(s)|(F_h(s) - G_k(s))\psi(s) .$$

Using monotonicity properties of Lemma 1.4, one has with $\rho_k(s) = \psi(s)|f_h(s) - g_k(s)|$

$$(1.32)\qquad \begin{cases} \int_{\mathbf{R}} \theta \frac{\partial}{\partial x} \frac{1}{2} (F_h(\tau) - G_k(\tau))^{+2} \leq \int_{\mathbf{R}} (F_h(s) - G_{k_0}(s_0))\rho_k(s) \\ \qquad + \int_{\mathbf{R}} |f_h(\tau) - g_k(\tau)|(G_k(\tau) - F_h(\tau))^{+}\theta \ , \end{cases}$$

for all $s \in (0,s_0)$, $k \in (0,k_0)$. To get rid of the last term, we choose θ supported in the set

$$\omega = \{ x \in \mathbf{R};\quad \bar{G}(\tau) - F_h(\tau) < 0\} \ ,$$

where $\bar{G}(\tau,x) = G(\tau,x)$ a.e. x and $x \mapsto \bar{G}(\tau,x)$ is right-continuous (note that $\bar{G}(\tau)$ dominates $G_k(\tau)$). Since $\bar{G}(\tau)$ is upper-semi-continuous in x and $F_h(\tau)$ continuous in x, ω is open.

As in the proof of Theorem 1.1, we check that $\rho_k(s)$ is narrowly compact for $s \in (0,s_0)$, $k \in (0,k_0)$. So that we can let s tend to 0 in (1.32) to obtain the existence of a nonnegative measure ρ such that:

$$\langle \theta, \frac{\partial}{\partial x} \frac{1}{2} (F_h(\tau) - G(\tau))^{+2} \rangle_{C_0^\infty(\omega)\times\mathcal{D}'(\omega)} \leq \int_{\mathbf{R}} (F_h(s) - G_{k_0}(s_0))d\rho \ .$$

We let s_0, k_0 tend to 0. Since $\lim_{k_0 \downarrow 0} G_{k_0}(0,x) = \int_{(-\infty,x]} d\mu_- + d\mu = \int_{(-\infty,x]} d\mu_+$, using (1.30) we obtain

$$\langle \theta, \frac{\partial}{\partial x} \frac{1}{2} (F_h(\tau) - G(\tau))^{+2} \rangle_{C_0^\infty(\omega)\times\mathcal{D}'(\omega)} \leq 0$$

or $\frac{\partial}{\partial x} \frac{1}{2} (F_h(\tau) - G(\tau))^{+2} \leq 0$ in ω. Hence if $U = \frac{1}{2} (F_h(\tau) - \bar{G}(\tau))^{+2}$, $x \to U(x)$ is right-continuous, lower-semi-continuous and $\frac{d}{dx} U(x) \leq 0$ on $[U > 0]$. This implies $U = 0$ since $U(-\infty) = 0$. Therefore $F_h(\tau) \leq G(\tau)$ (a.e. x). Since h is arbitrary and by symmetry $F = G$.

We finish the proof by passing to the limit in (1.29) to obtain:

$$(1.33)\qquad u^+(t) \leq f(t), \quad u^-(t) \leq f(t) \quad \text{in} \quad \mathcal{D}'(\mathbf{R}) \ .$$

Since $u^+ \cdot u^- = 0$, this implies $u^+(t) + u^-(t) \leq f(t)$. Integrating this and letting $t \downarrow 0$ leads to

$$(1.34)\qquad \int_{\mathbf{R}} d\mu_+ + d\mu_- \leq \lim_{t\downarrow 0} \int_{\mathbf{R}} f(t) \leq \int_{\mathbf{R}} d\mu_+ \ .$$

This proves $\mu_- = 0$.

Remark 1.8. The main point in the proof of $\mu_- = 0$ is to reach (1.33). The idea behind it is that $f(t)$ is a "solution" of (1.2) with initial value $\mu_+ = \mu + \mu_-$ and $u^+(t)$, $u^-(t)$ are subsolutions of (1.2) with initial value respectively μ_+ and $\mu_- \leq \mu + \mu_-$. But we do not know in general if $f(t)$ is a solution in the sense (1.3), (1.4). A priori $f(t)$ is only a measure. This leads to a more complicated analysis than in Theorem 1 although we deal with nonnegative functions.

Proof of Theorem 1.3. Let $u, \hat{u}$ be two solutions to (1.3), (1.4) satisfying (1.6). Set $v(t,x) = \int_{-\infty}^{x} u(t,\xi)d\xi$, $\hat{v}(t,x) = \int_{-\infty}^{x} \hat{u}(t,\xi)d\xi$. By Lemma 1.1, $v_t + \varphi(u) \leq 0$, $\hat{v}_t + \varphi(\hat{u}) \leq 0$. Since $\varphi \geq 0$, $t \to v(t,x)$ and $t \to \hat{v}(t,x)$ are monotone (increasing when t decreases to 0). By Lemma 1.3 applied to $u(t + h)$ and $\hat{u}(t)$, one has

$$\int_{\mathbf{R}} \alpha(\tau)(v(\tau + h) - \hat{v}(\tau))\theta \leq \int_{\mathbf{R}} \alpha(s)(v(s + h) - \hat{v}(s))\psi(s) ,$$

where $\alpha(t) = |u(t + h) - u(t)|$ and θ, τ, ψ are specified as in the lemma. By monotonicity, for all $s \in (0,s_0)$

$$\int_{\mathbf{R}} \alpha(\tau)(v(\tau + h) - \hat{v}(\tau))\theta \leq \int_{\mathbf{R}} (v(h) - \hat{v}(s_0))\rho(s) ,$$

where $\rho(s) = \alpha(s)\psi(s)$ is narrowly compact for $s \in (0,s_0)$ (see the proof of Theorem 1.1). We let s tend to 0 to obtain a nonnegative finite measure ρ such that

$$\int_{\mathbf{R}} \alpha(\tau)(v(\tau + h) - \hat{v}(\tau))\theta \leq \int_{\mathbf{R}} (v(h) - \hat{v}(s_0))d\rho .$$

Finally we let s_0 decrease to 0. Since $\hat{v}(s_0,x)$ increases to $v(x)$ (given by assumption), the right-hand side converges to $\int_{\mathbf{R}} (v(h) - v)d\rho$ which is nonpositive. Then we finish as in Theorem 1.1 to obtain $v(\tau) = \hat{v}(\tau)$ and $u = \hat{u}$.

SECTION 2: EXISTENCE

We easily check that for $\varphi(u) = |u|^m$, $m > 1$, the following functions are solutions of (1.3), (1.4) with the initial data $(q - p)\delta$ where δ is the Dirac mass in 0, $0 \leq p,q$.

$$(2.1) \qquad w_{p,q}(t,x) = \begin{cases} \operatorname{sign} x \; |x|^{\frac{1}{m-1}} (mt)^{\frac{1}{1-m}} & -\eta(t) \leq x \leq \xi(t) \\ 0 & x < -\eta(t) \text{ or } x > \xi(t) , \end{cases}$$

with $\xi(t) = m\left(\frac{q}{m-1}\right)^{\frac{m-1}{m}} t^{\frac{1}{m}}$, $\eta(t) = m\left(\frac{p}{m-1}\right)^{\frac{m-1}{m}} t^{\frac{1}{m}}$. When t tends to 0, $v(t,x) = \int_{-\infty}^{x} w(t,\xi)d\xi$ converges to

$$v(x) = \begin{cases} 0 & \text{if } x < 0 \\ -p & \text{if } x = 0 \\ q - p & \text{if } x > 0 . \end{cases}$$

If $\varphi(u) = \operatorname{sign} u|u|^m$, the unique fundamental solution $(u(0) = \delta)$ is the one corresponding to $p = 0$, $q = 1$.

These explicit solutions are known as "N-waves" and were already known as describing the asymptotic behavior of any bounded solution with compact support in the case $\varphi(u) = \frac{1}{2k} u^{2k}$ (see [5] and the references in it).

In order to obtain solutions to (1.2) with initial data a signed measure μ, a natural way is to approximate μ by $\mu_n \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for which the problem (1.2) has a solution u_n in the sense (1.3), (1.4). In order to pass to the limit, it is necessary to have a uniform estimate of $\|u_n(t)\|_\infty$ in terms of $\|\mu_n\|_{L^1(\mathbb{R})}$ and some compactness property for $\varphi(u_n)$. This is realized as soon as an estimate

$$\int_{\mathbb{R}} |u_t| \leq C(t, \int |u_0|) ,$$

is valid for the solution to (1.2). For instance, using the results proved in [4], one can prove the following.

Proposition 2.1. Let $\varphi:[0,\infty) \to [0,\infty)$ be increasing, $\varphi(0) = 0$, and such that

$$(2.2) \qquad \exists\, \alpha \in (0,1), \quad r \to \varphi^{1-\alpha}(r) \text{ is convex on } (0,\infty) .$$

Then, for any nonnegative finite measure μ on $\mathbf{R}$, there exists a unique solution u to (1.3), (1.4) with $T = +\infty$ such that

$$\lim_{t \downarrow 0} u(t) = \mu \quad \text{narrowly in } \mathbf{R} . \tag{2.3}$$

Remark. The condition (2.2) is obviously satisfied if $\varphi(r) = r^m$, $m > 1$, with $1 - \alpha = \frac{1}{m}$.

Proof. Let $\mu_n \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $\mu_n \geq 0$ converging narrowly to μ. It is well-known that there exist a unique $u_n \in C([0,\infty); L^1(\mathbf{R}))$, nonnegative, solution of (1.3), (1.4) (see [10]). Thanks to (2.2), we have as proved in [4]

$$\int_{\mathbf{R}} \frac{|u_n(t + h) - u_n(t)|}{h} \leq \frac{C(\alpha)}{t} \int_{\mathbf{R}} \mu_n . \tag{2.4}$$

(The results in [3] can be used to prove that our particular case falls under the scope of the abstract setting in [4]). From the estimate (2.4) together with (1.3), (1.4), one easily obtains

$$\int_{\mathbf{R}} |\varphi(u_n(t,x + h)) - \varphi(u_n(t,x))| \leq h \frac{C(\alpha)}{t} \int_{\mathbf{R}} \mu_n \tag{2.5}$$

$$|\varphi(u_n(t,x))| \leq \frac{C(\alpha)}{t} \int_{\mathbf{R}} \mu_n . \tag{2.6}$$

The condition (2.2) implies that $\lim_{r \to +\infty} \varphi(r) = +\infty$. Since φ is strictly monotone, (2.6) can be rewritten

$$\|u_n(t)\|_{L^\infty(\mathbf{R})} \leq \varphi^{-1}\left(\frac{C(\alpha)}{t} \int_{\mathbf{R}} \mu_n\right) = h(t) . \tag{2.7}$$

Finally

$$\int_{\mathbf{R}} |\varphi(u_n(t + h,x)) - \varphi(u_n(t,x))| \leq M(t) \int_{\mathbf{R}} |u_n(t + h,x) - u_n(t,x)| , \tag{2.8}$$

where $M(t) = \sup\{\varphi'(r);\ 0 \leq r \leq h(t)\}$.

By (2.5), (2.8), $\varphi(u_n(t,x))$ is relatively compact in $L^1_{loc}((0,\infty) \times \mathbf{R})$. There exists a subsequence of $\varphi(u_n)$, still denoted by $\varphi(u_n)$, converging in $L^1_{loc}((0,\infty) \times \mathbf{R})$ and

a.e. $(x,t) \in (0,\infty) \times \mathbf{R}$. Since φ is strictly monotone and u_n is locally bounded on $(0,\infty) \times \mathbf{R}$, we can also assume that u_n converges in $L^1_{loc}((0,\infty) \times \mathbf{R})$ to u. Hence we can pass to the limit in (1.4) to say that u is a solution of (1.2).

It remains to check that $\mu = \lim_{t \downarrow 0} u(t)$ narrowly. For this let us assume we have chosen a sequence μ_n such that $\int_{-\infty}^{x} \mu_n(\xi)d\xi$ increases to $\int_{(-\infty,x]} d\mu$. Then we easily verify (for instance by using Lemma 1.3) that $v_n(t,x) = \int_{-\infty}^{x} u_n(t,\xi)d\xi$ increases with n to some $v(t,x)$. On the other hand, by monotonicity we have (see Lemma 1.4)

$$v_n(t,x) \leq v_n(0,x) = \int_{-\infty}^{x} \mu_n \leq \int_{(-\infty,x]} d\mu \ .$$

In particular, this implies that $v(t,x) = \int_{-\infty}^{x} u(t,\xi)d\xi$ (since $v(-\infty) = 0$), $\int_{\mathbf{R}} u(t) = \lim_{n \to \infty} \int_{\mathbf{R}} u_n(t) = \int_{\mathbf{R}} d\mu$ and

$$v(t,x) \leq \int_{(-\infty,x]} d\mu \ . \tag{2.9}$$

Using the monotonicity of $v(t)$ in t, the fact that $u(t)$ is uniformly bounded in $L^1(\mathbf{R})$ and (2.9), we conclude that $\{u(t);\ t \in (0,t_0)\}$ is narrowly relatively compact. By Lemma 1.4, $\hat{\mu} = \lim_{t \downarrow 0} u(t)$ exists and by (2.9)

$$\int_{(-\infty,x]} d\hat{\mu} \leq \int_{(-\infty,x]} d\mu \ . \tag{2.10}$$

But also, for all $t > 0$

$$\int_{(-\infty,x]} u_n(t) \leq \int_{(-\infty,x]} u(t) \leq \int_{(-\infty,x]} d\hat{\mu} \Longrightarrow \int_{(-\infty,x]} \mu_n = \int_{(-\infty,x]} u_n(0) \leq \int_{(-\infty,x]} d\hat{\mu}.$$

This together with (2.10) proves $\mu = \hat{\mu}$.

SECTION 3: APPLICATION TO THE ASYMPTOTIC BEHAVIOR

We state here two results, obtained as a consequence of the Section 1, which say that the asymptotic behavior of the solutions to (1.2) depends only on

$$\int_{\mathbf{R}} u_0 \quad \text{if} \quad \varphi(r) = \operatorname{sign} r|r|^m, \qquad m > 1$$

$$p = -\min_{x\in\mathbf{R}} \int_{-\infty}^{x} u_0(\xi)d\xi \quad \text{and} \quad q = \max_{x\in\mathbf{R}} \int_{x}^{+\infty} u_0(\xi)d\xi \ ,$$

if $\varphi(r) = |r|^m$, $m > 1$.

The fact that the invariants p,q describe the asymptotic behavior in the case when φ is convex is not new (see [11], [6], [12], [5] - see also [2] for other results). Here the method is different: it directly applies to L^1-functions and treats as well the non-convex case.

We set $\|u\|_r = \left(\int_{\mathbf{R}} |u(x)|^r dr\right)^{\frac{1}{r}}$.

Theorem 3.1. Let $\varphi(r) = \operatorname{sign} r|r|^m$, $m > 1$ and $u_0 \in L^1(\mathbf{R})$. Then the solution u to (1.3), (1.4) with $u(0) = u_0$ satisfies

$$\lim_{t\to+\infty} t^{\frac{r-1}{mr}} \|u(t) - w(t)\|_r = 0 \qquad \forall \ 1 \le r < \infty \ , \tag{3.1}$$

where $w = w_{0,\int u_0}$ if $\int_{\mathbf{R}} u_0 > 0$ and $w = w_{-\int u_0,0}$ if $\int_{\mathbf{R}} u_0 < 0$ as given by (2.1).

Theorem 3.2. Let $\varphi(r) = |r|^m$, $m > 1$ and $u_0 \in L^1(\mathbf{R})$. Then the solution u to (1.3), (1.4) with $u(0) = u_0$ satisfies

$$\lim_{t\to\infty} t^{\frac{r-1}{mr}} \|u(t) - w(t)\|_r = 0 \qquad \forall \ 1 > r < \infty \ , \tag{3.2}$$

where $w = w_{p,q}$ with $q = \max_{x\in\mathbf{R}} \int_{x}^{+\infty} u_0(\xi)d\xi$, $-p = \min_{x\in\mathbf{R}} \int_{-\infty}^{x} u_0(\xi)d\xi$.

As an application of our general uniqueness result of Theorem 1.2, one can in fact study the asymptotic behavior for more general φ that behave like a power near the origin. Here we borrow from [9] where the same kind of assumptions (but weaker) are introduced to study the behavior in the large of the solutions to $u_t - \varphi(u)_{xx} = 0$.

Theorem 3.3. Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and let $\varphi \in C^2(\mathbb{R})$ be increasing, odd and satisfying

$$\exists\, 0 < \alpha < \beta, \quad \alpha \leq \frac{\varphi(r)\varphi''(r)}{\varphi'(r)^2} \leq \beta \qquad \forall r \in [-\|u_0\|_\infty, \|u_0\|_\infty] , \tag{3.3}$$

$$\lim_{r \to 0} \frac{\varphi'(r)}{m r^{m-1}} = 1, \qquad m > 1 . \tag{3.4}$$

Then (3.1) holds.

Remark. As we saw in Section 2 (see (2.2)), the condition (3.3) insures the existence of solutions to (1.2) originated from a Dirac mass. Here we will use again the estimates of type (2.4) established in [4] to provide some compactness argument. Note that, if $u_0 \geq 0$, *only the existence of* α is needed. The condition (3.3) is then equivalent to the convexity of $|\varphi^{1-\alpha}(r)|$ on $(-\|u_0\|_\infty, \|u_0\|_\infty)$ (see [4]).

Note that (3.4) implies

$$\varphi(r) = (\operatorname{sign} r)|r|^m(1 + \varepsilon(r)), \quad \lim_{r \to 0} \varepsilon(r) = 0 , \tag{3.5}$$

$$\exists\, a,b > 0, \ \forall r \in (-\|u_0\|_\infty, \|u_0\|_\infty), \quad a|r|^m \leq |\varphi(r)| \leq b|r|^m . \tag{3.6}$$

Proof of Theorem 3.3. For $\lambda > 1$, let us consider

$$u_\lambda(t,x) = \lambda u(\lambda^m t, \lambda x)$$

where u is the given function. We are going to prove that u_λ converges to w defined by (3.1) when λ tends to $+\infty$. Going back to (1.4), one easily checks that u_λ is a solution (in the sense (1.3), (1.4)) to

$$u_{\lambda_t} + \varphi_\lambda(u_\lambda)_x = 0 ,$$

with $\varphi_\lambda(r) = \lambda^m \varphi(\frac{r}{\lambda})$. This new function φ_λ also satisfies

(3.7) $\forall r \in [-\lambda\|u_0\|_\infty, \lambda\|u_0\|_\infty]$, $\alpha \leq \dfrac{\varphi_\lambda(r)\varphi_\lambda''(r)}{\varphi_\lambda'(r)^2} \leq \beta$, $a|r|^m \leq |\varphi_\lambda(r)| \leq b|r|^m$.

By the results in [4], we have

(3.8) $$\int_{\mathbf{R}} \frac{|u_\lambda(t+h) - u_\lambda(t)|}{h} \leq \frac{C}{t} \int_{\mathbf{R}} |u_\lambda(0)| \quad \text{for some} \quad C = C(\alpha,\beta) .$$

Since $\int_{\mathbf{R}} |u_\lambda(0)| = \int_{\mathbf{R}} |u_0|$, this gives an estimate uniform in λ. As in Section 2, we then obtain

(3.9) $$\int_{\mathbf{R}} |\varphi_\lambda(u_\lambda(t,x+h)) - \varphi_\lambda(u_\lambda(t,x))| \leq C\frac{h}{t} \int_{\mathbf{R}} |u_0| ,$$

(3.10) $$|\varphi_\lambda(u_\lambda(t,x))| \leq \frac{C}{t} \int_{\mathbf{R}} |u_0| .$$

This together with (3.7) proves that

(3.11) $$|u_\lambda(t,x)| \leq t^{-1/m} C(\int_{\mathbf{R}} |u_0|) = C(t) .$$

We then obtain

(3.12) $$\int_{\mathbf{R}} |\varphi_\lambda(u_\lambda(t+h)) - \varphi_\lambda(u_\lambda(t))| \leq M(t) \int_{\mathbf{R}} |u_\lambda(t+h) - u_\lambda(t)| ,$$

where $M(t) = \sup\{\varphi_\lambda'(r);\ |r| \leq C(t),\ \lambda \geq 1\}$ is finite for all $t > 0$. Indeed $\varphi_\lambda'(r) = \lambda^{m-1}\varphi'(\frac{r}{\lambda})$ and, by (3.4), for s small enough

$$0 \leq \varphi'(s) \leq \frac{3}{2} m|s|^{m-1} .$$

The estimates (3.8), (3.9), (3.12) imply that $\varphi_\lambda(u_\lambda)$ is precompact in $L^1_{loc}((0,\infty) \times \mathbf{R})$ for $\lambda \geq 1$. There exists a subsequence $\lambda_n \to \infty$ such that $\varphi_{\lambda_n}(u_{\lambda_n})$ converges in $L^1_{loc}((0,\infty) \times \mathbf{R})$ and a.e. (x,t) to some $\varphi^*(x,t)$. But, by (3.5)

(3.13) $$\varphi_\lambda(u_\lambda) = (\text{sign } u_\lambda)|u_\lambda|^m(1 + \varepsilon(\frac{u_\lambda}{\lambda})) .$$

This implies that $u_{\lambda_n}^m(x,t)$ converges for a.e. (x,t). Let $w(x,t) = \lim_{\lambda_n \to \infty} u_{\lambda_n}(x,t)$. Then (3.13) says that $\varphi^*(x,t) = (\text{sign } w)|w|^m$.

Now, using the dominated convergence theorem, one can pass to the limit in (1.4) (applied with $u_\lambda, \varphi_\lambda$) to obtain that w is a solution of (1.3), (1.4) with $\varphi(w) = (\text{sign } w)|w|^m$.

Let us now identify the initial data of the limit w. For all $\lambda \geq 1$ and $t \geq 0$

$$u_\lambda(t) - u_\lambda(0) + \frac{\partial}{\partial x}\int_0^t \varphi_\lambda(u_\lambda) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbf{R}). \tag{3.14}$$

In order to pass to the limit in (3.14), let us remark that

$$\int_{\mathbf{R}}\int_0^t |\varphi_\lambda(u_\lambda)| \leq b \int_{\mathbf{R}}\int_0^t |u_\lambda(s)|^m ds$$

and, by (3.11)

$$\int_{\mathbf{R}} |u_\lambda(s)|^m ds \leq \Big(\int_{\mathbf{R}} |u_\lambda(s)|\Big) \frac{C}{s^{\frac{m-1}{m}}} \leq \frac{C}{s^{\frac{m-1}{m}}} \int_{\mathbf{R}} |u_0| \ .$$

Hence

$$\int_{\mathbf{R}}\int_0^t |\varphi_\lambda(u_\lambda)| \leq C\Big(\int_{\mathbf{R}} |u_0|\Big) t^{\frac{1}{m}} \ . \tag{3.15}$$

Thanks to (3.15) and to the fact that $\varphi_{\lambda_n}(u_{\lambda_n})$ converges to $\varphi(w)$ in $L^1_{loc}((0,\infty) \times \mathbf{R}^n)$, we obtain that $\int_0^t \varphi_\lambda(u_\lambda(\sigma))d\sigma$ converges in $L^1_{loc}(\mathbf{R})$ to $\int_0^t \varphi(w(\sigma))d\sigma$.

Now we can pass to the limit in (3.14). Using that $u_\lambda(0) = \lambda u_0(\lambda x)$ converges in $\mathcal{D}'(\mathbf{R})$ to $\delta \int_{\mathbf{R}} u_0$ we have

$$w(t) - \delta \cdot \int_{\mathbf{R}} u_0 + \frac{\partial}{\partial x}\int_0^t \varphi(w(\sigma))d\sigma = 0 \quad \text{in} \quad \mathcal{D}'(\mathbf{R}) \ . \tag{3.16}$$

Let us prove that $w(t)$ converges narrowly to $\delta \cdot \int_{\mathbf{R}} u_0$ when $t \to 0$. Assume first $u_0 \geq 0$, so that $u_\lambda(t) \geq 0$ and $w(t) \geq 0$. Set

$$v(t,x) = \int_{-\infty}^x w(t,\xi)d\xi, \qquad v_0(x) = \Big(\int_{\mathbf{R}} u_0\Big) \cdot \int_{(-\infty,x]} \delta \ .$$

Integrating (3.16) gives

$$v(t) - v_0 + \int_0^t \varphi(w(\sigma))d\sigma = 0 \ . \tag{3.17}$$

Using the estimate (3.15) (also true for $\varphi(w)$) we deduce from (3.17) that $v(t)$ increases to v_0 and that $w(t) = \frac{\partial}{\partial x} v(t)$ converges narrowly to $\delta \cdot \int_R u_0 = \frac{\partial}{\partial x} v_0$ (we use that by (3.16) $\int_R w(t) = \int_R u_0$).

For general u_0, we use the fact that $u_-(t) \leq u(t) \leq u_+(t)$, where $u_+(t)$, $u_-(t)$ are the solutions to (1.2) with the initial data u_0^+ and $-u_0^-$. Since the corresponding $w_+(t)$, $w_-(t)$ are narrowly convergent when $t \neq 0$ by the previous proof, $w(t)$ is narrowly compact. The only possible limit is $\delta \cdot \int_R u_0$ by (3.16).

We now can use our uniqueness Theorem 1.2 to say that w is the function defined by (3.1). (Note that we a priori do not know the sign of w.)

The last step is to prove that $u_\lambda(t)$ converges in $L^1(R)$ (rather than in $L^1_{loc}(\mathbb{R})$) to $w(t)$.

If $u_0 \geq 0$ ($\Longrightarrow u_\lambda(t) \geq 0$) this is obvious since

$$\forall \lambda > 1, \ \int_R u_\lambda(t) = \int_R u_0 \ ,$$

and by Fatou lemma and (3.19):

$$\int_R w_0 = \int_R w(t) \leq \lim_{\lambda \to 0} \int_R u_\lambda(t) = \int_R u_0 \ .$$

For general u_0, we use again that

$$u_{-\lambda}(t) \leq u_\lambda(t) \leq u_{+\lambda}(t)$$

where u_+, u_- correspond to the initial data u_0^+ and $-u_0^-$.

So, in any case, for all $t > 0$

$$\lim_{\lambda \to \infty} \int_R |\lambda u(\lambda^m t, \lambda x) - w(t,x)|dx = 0 \ . \tag{3.18}$$

Using the invariance of w by the similarity transformations ($\lambda w(\lambda^m t, \lambda x) = w(t,x)$),

choosing $t = 1$, and $\tau = \lambda^m$ in (3.21) yields

$$\lim_{\tau \to \infty} \int_{\mathbf{R}} |u(\tau) - w(\tau)| = 0 ,$$

which is (3.1) for $r = 1$. For $r > 1$, one has:

$$\int_{\mathbf{R}} |u(\tau) - w(\tau)|^r \leq \|u(\tau) - w(\tau)\|_{L^\infty(\mathbf{R})}^{r-1} \int_{\mathbf{R}} |u(\tau) - w(\tau)| . \tag{3.19}$$

But by (3.11) applied to u and the definition of w

$$\|u(\tau) - w(\tau)\|_{L_\infty(\mathbf{R})} \leq \frac{C}{\frac{1}{\tau^m}} , \tag{3.20}$$

where C depends on m and $\int |u_0|$. Then (3.19) and (3.20) give (3.1).

Proof of Theorem 3.1. Note that if $\varphi(r) = \text{sign } r|r|^m$, (3.3) holds on $\mathbf{R}$ with $\alpha = \beta = \frac{m-1}{m}$. Theorem 3.1 is then a consequence of Theorem 3.3. The assumption $u_0 \in L^\infty(\mathbf{R})$ can be easily dropped.

Proof of Theorem 3.2. Except for the uniqueness part, the proof is exactly the same as for Theorem 3.3. The fundamental estimate (3.8) is also true in this case thanks to the homogeneity of $\varphi(u) = |u|^m$ as proved in [1]. (Again one can use [3] to verify that our case falls under the scope of the abstract setting in [1].) The only extra work is to identify the limit when $t \downarrow 0$ of

$$v(t,x) = \int_{-\infty}^{x} w(t,\xi)d\xi$$

in order to apply Theorem 1.3. Since $w(t)$ converges narrowly to $\delta \int_{\mathbf{R}} u_0$, we have

$$\forall x < 0, \quad \lim_{t \downarrow 0} v(t,x) = 0 \tag{3.21}$$

$$\forall x > 0, \quad \lim_{t \downarrow 0} v(t,x) = \int_{\mathbf{R}} u_0 . \tag{3.22}$$

It remains to determine $\lim_{t\downarrow 0} v(t,0)$. Note that, since $\varphi(u) = |u|^m \geq 0$, $t \to v(t,0)$ is increasing when $t \downarrow 0$ so that $-\ell = \lim_{t\downarrow 0} v(t,0)$ exists.

Thanks to (3.21), (3.22), (2.1) and the uniqueness Theorem 1.3, we necessarily have $w = w_{\ell,\ell+\int u_0}$. To finish the proof, it is sufficient to prove $\ell = p$. But ℓ is characterized by

$$-\ell = \min_{x\in\mathbb{R}} \int_{-\infty}^{x} w_{\ell,\ell+\int u_0}(t,\xi)d\xi \ .$$

So, we are reduced to prove

$$-p = \min_{x\in\mathbb{R}} \int_{-\infty}^{x} w(t,\xi)d\xi \ . \tag{3.23}$$

Since $u_\lambda(t)$ converges to $w(t)$ in $L^1(\mathbb{R})$, $\int_{-\infty}^{x} u_\lambda(t,\xi)d\xi$ converges uniformly for $x \in \mathbb{R}$ to $\int_{-\infty}^{x} w(t,\xi)d\xi$ when $\lambda \uparrow \infty$. In particular

$$\min_{x\in\mathbb{R}} \int_{-\infty}^{x} w(t,\xi)d\xi = \lim_{\lambda\uparrow\infty} \min_{x\in\mathbb{R}} \int_{-\infty}^{x} u_\lambda(t,\xi)d\xi = \lim_{\lambda\to\infty} \min_{x\in\mathbb{R}} \int_{-\infty}^{x} u(\lambda^m t,\xi)d\xi \ . \tag{3.24}$$

But, since $\varphi(u) = |u|^m$ is convex (see [5] and references)

$$\min_{x\in\mathbb{R}} \int_{-\infty}^{x} u(\lambda^m t,\xi)d\xi = \min_{x\in\mathbb{R}} \int_{-\infty}^{x} u(0,\xi)d\xi = -p \ . \tag{3.25}$$

The relations (3.24), (3,25) yield (3.23).

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We study the uniqueness of the solutions to the scalar conservation law $u_t + \varphi(u)_x = 0$ when the initial datum is a finite measure. The case of a Dirac mass is particularly emphasized: it is shown how it provides a description of the asymptotic behavior of the solutions initiated by an arbitrary integrable function. This behavior is proved to depend on one parameter in the case when φ is odd while it depends on two when φ is convex.

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