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B-SPLINES FROM PARALLELEPIPEDS

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B-SPLINES FROM PARALLELEPIPEDS

C. de Boor[1] and K. Höllig[1,2]

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ABSTRACT

We study multivariate B-splines M obtained as "shadows" of parallelepipeds, and spaces spanned by their translates $S = \operatorname{span}_{j \in \mathbb{Z}^m} M(\cdot - j)$. Recurrence relations for M are obtained and a necessary condition for the stability of the B-spline basis is given. We further determine the polynomials contained in S and the optimal degree of approximation from S.

AMS (MOS) Subject Classifications: 41A15, 41A63, 41A25

Key Words: B-splines, multivariate, spline functions, degree of approximation

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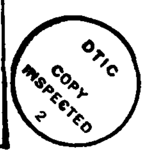
SIGNIFICANCE AND EXPLANATION

Local support bases for piecewise polynomial spaces are important for applications such as finite element methods, data fitting etc. In [BH_1] a general construction principle for such "B-splines" was described. A special case are the so called box-splines. They have a particularly regular discontinuity pattern and coincide in special cases with standard finite elements.

It is hoped that using translates of box-splines will lead, at least in two variables, to a unified theory for piecewise polynomial functions on regular meshes.

This note is a first attempt in this direction and deals with basic approximation properties of translates of one box-spline such as stability, degree of approximation etc.



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B-SPLINES FROM PARALLELEPIPEDS

C. de Boor[1] and K. Höllig[1,2]

0. Introduction. Following $[BH_1]$, we define the **B-spline** M_B as the **m-shadow** of the polyhedral convex body $B \subseteq \mathbb{R}^n$, i.e., as the distribution on $\mathbb{R}^m$ given by the rule

(1) $$M_B\phi := \int_B \phi \circ P, \quad \text{all } \phi \in D(\mathbb{R}^m) .$$

Here, $P:\mathbb{R}^n \longrightarrow \mathbb{R}^m : x \longmapsto (x(i))_1^m$ is the canonical projection and $\int_K$ denotes the k-dimensional integral over K in case $\dim K = k$, i.e., K spans a k-dimensional flat.

It is obvious that M_B is nonnegative, with $\operatorname{supp} M_B \subseteq P[B]$. It is easy to see that M_B is a piecewise polynomial function of degree $\le n-m$ once one knows the recurrence relation $[BH_1]$

(2) $$D_{Pz}M_B = -\Sigma_i \langle z|n_i\rangle M_{B_i}, \quad \text{all } z \in \mathbb{R}^n .$$

Here,

$$D_y f := \Sigma\, y(i)\, D_i f ,$$

with $D_i f$ the partial derivative of f with respect to its i-th argument. Further, B_i denotes the typical (n-1)-dimensional polyhedron of which the boundary of B consists, and n_i denotes the corresponding outward normal. Finally, $\langle\cdot|\cdot\rangle$ denotes the scalar product.

In principle, M_B can be evaluated with the aid of the stable recurrence $[BH_1]$

(3) $$(n-m)M_B(Pz) = \Sigma_i \langle b_i - z|n_i\rangle M_{B_i}(Pz), \quad \text{all } z \in \mathbb{R}^n ,$$

with b_i an arbitrary point in the flat spanned by B_i.

Cases of particular interest are:

(i) the simplex spline, obtained when B is a simplex. These B-splines were introduced in [B] following up on [S] and have already been studied intensively, mostly by W. Dahmen and C. A. Micchelli $[M_{1-2}]$, $[D_{1-4}]$, $[DM_{1-3}]$, but also by Goodman & Lee [GL], Hakopian $[Hk_{1-3}]$, and by Höllig $[Hl_{1-2}]$.

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(ii) the truncated powers or cone splines, obtained when B is a proper polyhedral cone spanned by some basis for $\mathbb{R}^n$. These were introduced by Dahmen [D_2] and further studied by Dahmen and Micchelli in [DM_3]. For example, they show that, near an extreme point of its support, a simplex spline coincides with a truncated power.

(iii) the box spline, obtained when B is a parallelepiped. These splines were introduced in [BD] and are the object of study of the present note.

To be precise, a box spline is, by a definition slightly more general than the one in [BD], a distribution M_Ξ on $\mathbb{R}^m$ given by the rule

$$M_\Xi : \phi \longmapsto \int_{[0,1]^r} \phi\Big(\sum_{i=1}^{r} \lambda(i)\xi_i\Big)\, d\lambda \tag{4}$$

for some sequence $\Xi := (\xi_i)_1^r$. If $\dim \langle\Xi\rangle = m$, then

$$M_\Xi = M_B / \mathrm{vol}_r B$$

with

$$B := \{\sum_{i=1}^{r} \lambda(i)\hat{\xi}_i : \lambda \in [0,1]^r\}$$

the parallelepiped spanned by some linearly independent sequence $(\hat{\xi}_i)_1^r$ in $\mathbb{R}^n$ for which $P\hat{\xi}_i = \xi_i$, all i. We would like, though, to consider M_Ξ also in case $\dim \langle\Xi\rangle < m$.

For this, we find it convenient to enlarge the above definition of the B-spline M_B by allowing P in (1) to be an arbitrary linear map on B into $\mathbb{R}^m$. Then (1) defines M_B as the P-shadow of B. One checks that this leaves the recurrence relations (2) and (3) unchanged (see Sect.1).

In these terms, the box spline M_Ξ defined by (4) is the P-shadow of the box $[0,1]^r$, with P the linear map given by

$$P\lambda := \sum_{i=1}^{r} \lambda(i)\xi_i .$$

Here is an outline of the paper. We discuss P-shadows in Section 1. In Section 2, we give some basic information about the box spline M_Ξ, such as its recurrence relations, its Fourier transform, and its relationship to the difference operator Δ_Ξ and to the truncated powers. We show in Section 3 that it is usually possible to make a partition of unity out of the box spline and certain of its translates in many ways. We use this fact in Section 4 to show that the box spline and its translates are usually globally linearly dependent, thus destroying all hopes for stability or the existence of a set of dual linear functionals for such sets except in special circumstances. One such is discussed in [BH_2].

In the remaining sections, we consider the space

$$S_\Xi \;:=\; \{\, \Sigma_V\, a(j)M_j \;:\; a \in \mathbb{R}^V \}$$

with

$$M_j \;:=\; M_\Xi(\cdot - j)\;, \quad \text{all } \; j \in V := \mathbb{Z}^m\;,$$

and under the assumption that $\Xi \subseteq V$ and that $\langle \Xi \rangle = \mathbb{R}^m$. In Section 5, we determine all polynomials in S_Ξ as well as the largest k for which all polynomials of (total) degree k or less are contained in S_Ξ. We use this information in Section 6 to construct a quasi-interpolant from S_Ξ and thereby to obtain statements about the degree of approximation obtainable from $S_{\Xi,h} := \{x \longmapsto f(x/h) : f \in S_\Xi\}$ as $h \longrightarrow 0$.

We could have obtained our results concerning S_Ξ with the aid of the general theory of spaces spanned by translates of a fixed function developped by Fix and Strang [FS], particularly if we had been content to discuss only L_2. We chose to derive our results directly since it seems no more effort to do this than it is to verify that the general conditions given in [FS] are satisfied for our specific examples.

We point out in Section 2 that $S_\Xi \subseteq L_\infty^{(d)}$, with

$$d \;:=\; \max\{\, r : \langle \Xi \backslash Z \rangle = \mathbb{R}^m \; \text{ for all } \; Z \subseteq \Xi \; \text{ with } \; |Z| = r \,\}$$

(see Section 2 for how we treat the sequence Ξ as a set). This raises the question of the relationship of S_Ξ to the space of all pp functions in $L_\infty^{(d)}$ on the same mesh and of degree $\le |\Xi| - m$. We study this difficult question in [BH_2] just for $m = 2$ and mainly only for the 3-direction mesh, i.e., for $\operatorname{ran} \Xi = \{e_1, e_2, e_1+e_2\}$.

Notation. With $A \subseteq \mathbb{R}^m$, we denote by $[A]$ the convex hull of A and by $\langle A \rangle$ its linear span. We use $x(r)$ for the r-th entry of the vector x. For $x \in \mathbb{R}^m$ and $j \in \mathbb{Z}_+^m$, the number x^j is computed as

$$x^j \;:=\; x(1)^{j(1)} \ldots x(m)^{j(m)},$$

as usual. We denote by π the class of all polynomials (on $\mathbb{R}^m$), and by π_k its subspace made up of those of total degree no larger than k. Thus

$$\pi_k \;:=\; \{x \longmapsto \sum_{|j| \le k} a(j)x^j\}$$

with $|j| := j(1) + \ldots + j(m)$. We also use $D^j := D_1^{j(1)} \ldots D_m^{j(m)}$ and, more generally, $p(D) := \Sigma_k\, a(k)D^k$ in case $p:x \longmapsto \Sigma_k a(k)x^k$. Here, we use again the notation $D_i f$ for the partial derivative with respect to its i-th argument of the function f with domain in $\mathbb{R}^m$. We also use the notation $D_y := \Sigma_1^m y(i)D_i f$. For a sequence of vectors in $\mathbb{R}^m$, such as $\Xi = (\xi_1,\ldots,\xi_r)$, we use

$$D_\Xi := D_{\xi_1} \ldots D_{\xi_r} .$$

We also use $\Delta_\Xi := \Delta_{\xi_1} \ldots \Delta_{\xi_r}$ and $\nabla_\Xi := \nabla_{\xi_1} \ldots \nabla_{\xi_r}$, with

$$\Delta_y f := f(\cdot+y) - f , \quad \nabla_y f := f - f(\cdot-y) .$$

Finally, we denote by $\mathbf{D}(\mathbb{R}^m)$ the space of tempered distributions on $\mathbb{R}^m$.

1. P-shadows. As defined in the introduction, the P-shadow of a convex polyhedron B in $\mathbb{R}^n$ is the distribution M on $\mathbb{R}^m$ given by the rule

$$M : \phi \longmapsto \int_B \phi \circ P\ , \quad \text{all} \quad \phi \in D(\mathbb{R}^m)\ ,$$

with $P:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ an affine map.

We claim that the recurrence relations for B-splines established in [BH$_1$;Theorem 2] remain valid for these more general B-splines and state this in the following theorem, for the record. For this, we make the assumption that P is a linear map, i.e., $P0 = 0$. This can always be achieved by a translation in $\mathbb{R}^n$. Further, we assume that B is proper, i.e., n-dimensional. If $r := \dim B < n$, then this can be achieved by restricting P to the affine hull of B and identifying this hull with $\mathbb{R}^r$. Given that B is a proper convex polyhedron, its boundary is made up of (n-1)-dimensional convex polyhedra B_i, with corresponding outward normals n_i, and b_i denotes an arbitrary point in the affine hull of B_i. Further, D stands for the first order differential operator given by the rule

$$Df := \sum_{i=1}^{r} \phi_i D_i f$$

in case f has its domain in $\mathbb{R}^r$, with

$$(\phi_i f)(x) := x(i)f(x)\ .$$

Thus $(Df)(x) = (D_x f)(x)$ and the adjoint of D is $-\Sigma\, D_i \phi_i$.

Theorem 1. <u>Let</u> B <u>be a proper convex polyhedron in</u> $\mathbb{R}^n$ <u>and let</u> M <u>be its P-shadow in</u> $\mathbb{R}^m$ <u>under the linear map</u> $P:\mathbb{R}^n \longrightarrow \mathbb{R}^m$. <u>Then</u>

<u>(i)</u> $D_{Pz}M = -\Sigma_i \langle z|n_i\rangle M_i$, <u>all</u> $z \in \mathbb{R}^n$.

<u>(ii)</u> $(n-m)M(Pz) = \Sigma_i \langle b_i - z|n_i\rangle M_i(Pz)$, <u>all</u> $z \in \mathbb{R}^n$.

<u>(iii)</u> $DM = (n-m)M - \Sigma_i \langle b_i|n_i\rangle M_i$.

Proof. The proof is a slight extension of the arguments for Theorem 2 of [BH$_1$]. The proof of (i) only needs the additional observation that

(1) $$D_y(\phi \circ P) = (D_{Py}\phi)\circ P\ .$$

This also implies that

(2) $$(D\phi)(Px) = (D_{Px}\phi)(Px) = (D_x(\phi \circ P))(x) = (D(\phi \circ P))(x)\ ,$$

of use in a moment. The recurrence relation (ii) follows from (i) and (iii). As to (iii), observe that

$$D_j\Phi_j = 1 + \Phi_j D_j ,$$

therefore

$$-(DM)\phi = \int_B \Big(\sum_{j=1}^{m} D_j\Phi_j\phi\Big)\circ P = mM\phi + \int_B (D\phi)\circ P$$

and

$$\int_B \sum_{i=1}^{n} D_i\Phi_i(\phi\circ P) = nM\phi + \int_B D(\phi\circ P) .$$

Here, the last integral in the first line equals the last integral in the second, by (2).

Thus

$$(DM)\phi = (n-m)M\phi - \sum_{i=1}^{n}\int_B D_i\Phi_i(\phi\circ P) ,$$

and the argument now finishes as in [BH_1]. |||

2. Basic properties of the box spline. The box spline M_Ξ defined in (0.4) (with $r = n$) is a <u>symmetric</u> function of the sequence $\Xi = (\xi_i)_1^n$. In other words, $M_{\Xi'} = M_\Xi$ for any rearrangement Ξ' of the sequence Ξ. For this reason, we find it excusable and, in any case, convenient to treat Ξ in the sequel as if it were a <u>set, of cardinality</u> n, rather than a sequence, even though the set $\{\xi_i : i=1,\ldots,n\}$ may well have fewer than n elements. Thus, we write

$$\sum_{\xi\in\Xi} \lambda(\xi)\xi \quad \text{instead of} \quad \sum_{i=1}^{n} \lambda(i)\xi_i$$

or

$$\Xi\setminus\xi \quad \text{instead of} \quad (\xi_{s(i)})_1^{n-1}$$

for the appropriate subsequence $s(1), \ldots, s(n-1)$ of $1, \ldots, n$. In the latter example, this abuse of notation stresses the fact that it doesn't matter which one of the possibly several occurrences of the vector ξ in the sequence Ξ is being omitted.

It is clear that M_Ξ is nonnegative and that

$$\text{(1)} \qquad \operatorname{supp} M_\Xi = \{ \Sigma\lambda(\xi)\xi : \lambda \in [0,1]^\Xi \} .$$

Further, from (0.4),

$$\text{(2)} \qquad \|M_\Xi\| = 1$$

as a linear functional on $C(\mathbb{R}^m)$. Also,

$$\text{(3)} \qquad M_\Xi \in L_\infty \quad \text{iff} \quad \langle\Xi\rangle = \mathbb{R}^m .$$

The **recurrence relations** of Theorem 1 for general B-splines simplify for the box spline as follows.

Proposition 2.1. <u>If</u> $z = \Sigma_\Xi \lambda(\xi)\xi$, <u>then</u>

$$\text{(4)} \qquad D_z M_\Xi = \sum_{\xi\in\Xi} \lambda(\xi)\big(M_{\Xi\setminus\xi} - M_{\Xi\setminus\xi}(\cdot-\xi)\big) ,$$

$$\text{(5)} \qquad (n-m)M_\Xi(z) = \sum_{\xi\in\Xi} \big(\lambda(\xi)M_{\Xi\setminus\xi}(z) + (1-\lambda(\xi))M_{\Xi\setminus\xi}(z-\xi)\big) .$$

Proof. The typical facet (i.e., (n-1)-dimensional face) of $B = [0,1]^n$ has the form

$$B_\xi := \{\mu \in [0,1]^\Xi : \mu(\xi) = 0\}$$

or else the form $e_\xi + B_\xi$, for some $\xi \in \Xi$. Further, B_ξ and $e_\xi + B_\xi$ have the outward normal $-e_\xi$ and e_ξ, respectively. Thus

$$\langle\mu|n_i\rangle = \begin{cases} -\mu(\xi)\ , & B_i = B_\xi \\ \mu(\xi)\ , & B_i = e_\xi + B_\xi \end{cases} ,$$

and (4) and (5) now follow from Theorem 1.(i) and (ii) . |||

Smoothness. We associate with Ξ the number

(6) $$d := \max \{ r : \langle \Xi \setminus Z \rangle = \mathbb{R}^m \text{ for all } Z \subseteq \Xi \text{ with } |Z| = r \}$$

and say for short that Ξ is **d-spanning.** (We take $d = -1$ in case $\langle\Xi\rangle \neq \mathbb{R}^m$). The number d is of interest here since it follows from (4) and (3) that all r-th order derivatives of M_Ξ are in L_∞ as long as $\langle\Xi\setminus Z\rangle = \mathbb{R}^m$ for all $Z \subseteq \Xi$ with $|Z| = r$. Thus

$$M_\Xi \in L_\infty^{(d)} \subseteq C^{(d-1)} .$$

Obviously, d cannot be bigger than $|\Xi| - m$ which is the total degree of the polynomial pieces of which M_Ξ consists. Precisely, on each connected component of the complement of

$$\{ [\Xi\setminus Z] + \sum_H n : H \subseteq Z ,\ \langle\Xi\setminus Z\rangle \neq \mathbb{R}^m \} ,$$

M_Ξ agrees with some polynomial of degree $\leq |\Xi| - m$.

Examples. (i) For $m = 1$ and $\xi = e_1$, all $\xi \in \Xi$, M_Ξ is just the forward cardinal B-spline, i.e., $M_\Xi = M(\cdot;0,1,\ldots,n)$. For $m > 1$ and Ξ containing only $e_1, \ldots, e_m$ (each at least once), M_Ξ is the tensor product of such univariate B-splines.

(ii) For $m = 2$ and $|\Xi| = n = 3$, with $d = 1$, we obtain a standard linear finite element.

(iii) For $m = 2$ and $\Xi = (e_1, e_2, e_1+e_2, e_1-e_2)$, M_Ξ is a piecewise quadratic function first studied by Zwart [Z] and independently derived by Powell [P] and Sabin [PS]. Its support is shown in Figure 1 together with its "mesh", i.e., its lines of transition from one polynomial piece to a neighboring piece. The dotted mesh lines occur in the above references. Our construction makes clear that they do not appear in actuality since they do not lie in some one-dimensional image $P[F]$ of some face F of $[0,1]^n$.

(iv) Further examples for Ξ containing only e_1, e_2, e_1+e_2 and/or e_1-e_2 can be found in Sablonniere's study [S1] of smooth finitely supported pp functions on regular

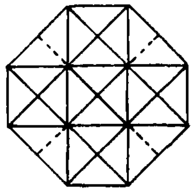

Figure 1. Support and meshlines for a C^1-quadratic box spline

meshes. The *"generalized triangular splines" of Frederickson* $[F_{1-2}]$ *can now be recognized* as spanned by the box spline M_Ξ, with Ξ containing each of the three vectors e_1, e_2, and e_3 the same number of times.

Associated difference operator. It follows from (4) that

$$D_\xi M_\Xi = M_{\Xi\setminus\xi} - M_{\Xi\setminus\xi}(\cdot-\xi) = \nabla_\xi M_{\Xi\setminus\xi},$$

therefore

(7) $$D_Z M_\Xi = \nabla_Z M_{\Xi\setminus Z}, \quad \text{for} \quad Z \subseteq \Xi .$$

In particular,

$$D_\Xi M_\Xi = \nabla_\Xi \delta ,$$

since $M_\emptyset = \delta :=$ point evaluation at 0. Therefore

(8) $$\int M_\Xi D_\Xi \phi = (\Delta_\Xi \phi)(0), \quad \text{for all} \quad \phi \in C^{|\Xi|}(\mathbb{R}^m) .$$

This close association between the box spline and the forward difference operator brings to mind the well known association

$$\int M(x;t_0,\ldots,t_n)\, \phi^{(n)}(x)/n!\, dx = [t_0,\ldots,t_n]\phi$$

between the univariate B-spline and the divided difference.

The **Fourier transform** of M_Ξ is quite simple,

(9) $$\hat{M}_\Xi(x) = \prod_{\xi\in\Xi} \frac{1 - e^{-i\xi\cdot x}}{i\xi\cdot x} .$$

From this we see that

(10) $$M_\Xi * M_Z = M_{\Xi\cup Z} .$$

Symmetries and local structure. We pointed out earlier that M_Ξ does not depend on the order of the vectors in the sequence Ξ. This is due to the fact that any linear map in $\mathbb{R}^n$ which permutes the unit vectors leaves the box $[0,1]^n$ invariant. Multiplication of some of the unit vectors by -1 will change $[0,1]^n$, but $[0,1]^n$ can be restored by a subsequent shift. Therefore

(11) $$M_{\Xi_\sigma} = M_\Xi(\cdot - \sum_{\xi\in\Xi} \frac{\sigma(\xi)-1}{2}\xi)$$

in case Ξ_σ is obtained from Ξ by multiplying each $\xi \in \Xi$ by $\sigma(\xi) \in \{-1,1\}$. A symmetry of a different sort occurs when Ξ' is the image of Ξ under some invertible linear map Q on $\mathbb{R}^m$. Precisely,

(12) $$M_\Xi = |\det Q|\, M_{Q\Xi} \circ Q .$$

This implies certain symmetries for M_Ξ in case $\Xi = Q\Xi$.

The box spline is particularly simple near an extreme point of its support.

Proposition 2.2. *If e is an extreme point of $\operatorname{supp} M_\Xi$, then, in a neighborhood of e , M_Ξ agrees with some truncated power of degree $|\Xi| - m$. In particular, $M_\Xi(\cdot + e)$ is homogeneous of degree $|\Xi| - m$ near 0 .*

Proof. We pointed out earlier that

$$\operatorname{supp} M_\Xi = \{ \sum_{\xi\in\Xi} \lambda_\xi \xi : \lambda\in[0,1]^\Xi \} .$$

Thus any extreme point e of the (closed) support is necessarily of the form

$$e = \sum_{\zeta\in Z} \zeta$$

for some $Z \subseteq \Xi$ for which, further, there exists $\eta \in \mathbb{R}^m$ so that $\langle\eta|\xi\rangle > 0$ for all $\xi \in \Xi_\sigma$ with

$$\sigma(\xi) := \begin{cases} -1 , & \xi\in Z \\ 1 , & \xi\in\Xi\setminus Z \end{cases} .$$

Therefore, from (11),

$$M_\Xi(\cdot+e) = M_{\Xi_\sigma} ,$$

showing that it is sufficient to consider the case that $e = 0$ and, for some $\eta \in \mathbb{R}^m$ and all $\xi \in \Xi$, $\langle\eta|\xi\rangle > 0$.

In this case,

$$\varepsilon := \operatorname{dist}(0, [\Xi]) > 0 .$$

Therefore, for all test functions ϕ with $\operatorname{supp}\phi \subseteq B_\varepsilon(0)$:= ball of radius ε and center 0 ,

$$M_\Xi\phi = \int_{[0,1]^n} \phi\big(\sum \lambda(\xi)\xi\big)\, d\lambda = \int_{\mathbb{R}_+^n} \phi\big(\sum \lambda(\xi)\xi\big)\, d\lambda \quad . |||$$

3. Partition of unity. In this section, we show that appropriate translates of an appropriately scaled version of the box spline

$$M := M_{\Xi}$$

form a partition of unity. By a standard argument, this implies that the space spanned by these translates can, at least, approximate continuous functions (as the mesh size is reduced by scaling).

Proposition 3. Suppose Ξ contains the basis Z (for $\mathbb{R}^m$). Then

(1)
$$\sum_{j\in\mathbb{Z}^Z} M\Big(\cdot - \sum_{\zeta\in Z} j(\zeta)\zeta\Big) = 1/|\det Z| .$$

Proof. Since $\mathbb{R}^m$ is the essentially disjoint union of the sets

$$\Big\{ \sum_{\zeta\in Z} \big(\lambda(\zeta)-j(\zeta)\big)\zeta : \lambda \in [0,1]^Z \Big\} , \quad j \in \mathbb{Z}^Z ,$$

we find that

$$\sum_{j\in\mathbb{Z}^Z} M\big(\cdot - \Sigma_Z j(\zeta)\zeta\big)\phi = \int_{[0,1]^{n-m}} \Big(\int_{\mathbb{R}^m} \phi\big(\Sigma_Z\lambda(\zeta)\zeta + \Sigma_{\Xi\setminus Z}\mu(\xi)\xi\big)\, d\lambda\Big)\, d\mu .$$

The change of variables $\lambda \longmapsto \Sigma_Z\lambda(\zeta)\zeta$ carries this to

$$\int_{[0,1]^{n-m}} \Big(\int_{\mathbb{R}^m} \phi\big(x + \Sigma_{\Xi\setminus Z}\mu(\zeta)\zeta \big)\, dx/|\det Z| \Big)\, d\mu$$

and this equals

$$\int_{\mathbb{R}^m} \phi \ \mathrm{vol}_{n-m}[0,1]^{n-m}/|\det Z| \quad . \ |||$$

Corollary. If $\Xi \subseteq \mathbb{Z}^m$ and $\langle\Xi\rangle = \mathbb{R}^m$, then $\sum_{j\in\mathbb{Z}^m} M(\cdot-j) = 1$.

Proof. Let $Z \subseteq \Xi$ be a basis (for $\mathbb{R}^m$). Then

$$A := \{\Sigma_Z\, j(\zeta)\zeta : j \in \mathbb{Z}^Z\}$$

is a subgroup of $\mathbb{Z}^m$ and its factor group $G := \mathbb{Z}^m/A$ has (finite) order $|\det Z|$. Therefore

$$\sum_{j\in\mathbb{Z}^m} M(\cdot-j) = \sum_{g\in G} \sum_{j\in A} M(\cdot-g-j) = |G|/|\det Z| = 1 \quad . \ |||$$

4. Linear independence of translates. For any particular subset V of $\mathbb{R}^m$, we consider the collection of translates $M_v := M(\cdot - v)$, $v \in V$, of the box spline

$$M := M_{\Xi} .$$

Such a collection is always (algebraically) linearly independent: Indeed, if $f := \Sigma_V a(v) M_v$ with $W := \operatorname{supp} a$ a finite nonempty set, then

$$(\operatorname{supp} M_w) \setminus \bigcup_{v \in W \setminus w} (\operatorname{supp} M_v) \neq \emptyset$$

for some $w \in W$, hence $f \neq 0$.

We are interested in considering nontrivial sums of infinitely many translates. For this, we make the assumption that V has no finite limit points. Then only finitely many of the translates have any particular point in their support and thus, for $a \in \mathbb{R}^V$ with suitably controlled growth at infinity,

$$f := \Sigma_V a(v) M_v$$

defines a distribution on $\mathbb{R}^m$.

Assume that M is a function, i.e., that Ξ contains a basis (for $\mathbb{R}^m$). Then

$$S_{\Xi,V} := \operatorname{span} (M_v)_V := \{\Sigma_V a(v) M_v : a \in \mathbb{R}^V\}$$

is a space of piecewise polynomial functions, possibly quite smooth, and it becomes of interest to find out to what an extent $(M_v)_V$ is a basis for this space or one of its subspaces. We call $(M_v)_V$ **(globally) linearly independent** if the linear map

$$a \longmapsto \Sigma_V a(v) M_v \tag{1}$$

is 1-1 on $\mathbb{R}^V$. Such linear independence is a first necessary condition for other properties of interest to hold. One such property is **stability** of the basis $(M_v)_V$ for $S_{\Xi,V}$, i.e., the property that the map (1) is bounded and bounded below on $\ell_\infty(V)$ into $L_\infty(\mathbb{R}^m)$. Another is the possibility of interpolation from $S_{\Xi,V}$, i.e., the existence of points p_v , typically with $M_v(p_v) \neq 0$, all $v \in V$, so that, for any function f in some class K , there exists one and only one $s \in S_{\Xi,V} \cap K$ which agrees with f at $(p_v)_V$. We make clear below that this (global) linear independence is usually not present. Yet, as is pointed out in [BD], if $\Sigma_V a(v) M_v = 0$ and $a \neq 0$, then there exists $r > 0$ so that, for all $v \in V$, a changes sign on $V \cap B_r(v)$. This implies that the map (1) is 1-1 on $\pi_{|V}$.

The space $S_{\Xi,V} = \text{span}\,(M_v)_V$ becomes interesting when V is related to Ξ. By assumption, M is a function, i.e., Ξ contains a basis Z (for $\mathbb{R}^m$). Therefore, according to Proposition 3.1, the collection M_v, $v \in V_Z := \{\ \Sigma_Z\ j(\zeta)\zeta\ :\ j \in \mathbb{Z}^Z\}$ forms a partition of the constant $1/|\det Z|$. This suggests consideration of V of the form V_Z for some basis Z in Ξ. We go one step further, though, and consider from now on only the following <u>normalized</u> situation:

$$\Xi \subseteq V = \mathbb{Z}^m . \tag{2}$$

This is the same, up to an affine transformation, as the assumption that $V_Z \subseteq V$ for all bases Z in Ξ. We abbreviate

$$S_\Xi := S_{\Xi,\mathbb{Z}^m} .$$

Proposition 4. <u>Under the assumption (2),</u> $(M_j)_V$ <u>is linearly dependent unless</u>

$$|\det Z| = 1 \quad \underline{\text{for all bases}} \quad Z \subseteq \Xi .$$

Proof. By assumption, Ξ contains a basis Z for $\mathbb{R}^m$, therefore $(|\det Z|M_j)_{V_Z}$ provides a partition of unity as does $(M_j)_V$, by Proposition 3 and its corollary. If now $|\det Z| \neq 1$ for some basis Z in Ξ, then $V_Z \neq V$, yet

$$\Sigma_{V_Z} |\det Z|\ M_j = 1 = \Sigma_V\ M_j . \ |||$$

Remark. It would be nice to know whether the converse of this proposition holds.

5. The polynomials in S_Ξ. In this section, we determine $S_\Xi \cap \pi$. This information is important in the discussion of the degree of approximation to smooth functions attainable from $S_{\Xi,h}$. We continue to use the abbreviations and assumptions introduced in Section 4.

Lemma 5.1. $\pi \cap \ker D_\Xi = \pi \cap \ker \Delta_\Xi$.

Proof. Recall from (2.8) that

$$(\Delta_\Xi f)(x) = \int M \, D_\Xi f(\cdot + x) .$$

Therefore $\pi \cap \ker \Delta_\Xi \supseteq \pi \cap \ker D_\Xi$. For the converse, observe that $\Delta_\Xi f = 0$ implies $\int M(\cdot - x) \, D_\Xi f = 0$ for all x. Since M is nonnegative and of compact support, this cannot hold for a polynomial f unless the polynomial $D_\Xi f$ vanishes identically. |||

We also note that (2.7) together with summation by parts gives

$$D_Z\bigl(\Sigma\, a(j)\, M_j\bigr) = \Sigma\, \bigl(\nabla_Z a\bigr)(j)\, M_{\Xi \setminus Z}(\cdot - j) , \quad \text{all } Z \subseteq \Xi . \tag{1}$$

Theorem 5. Let $K := \bigcap_{Z \in \Xi^*} \ker D_Z$ with $\Xi^* := \{Z \subseteq \Xi : \langle \Xi \setminus Z \rangle \neq \mathbb{R}^m\}$. Then

$$\pi \cap S_\Xi = \pi \cap K . \tag{2}$$

Proof. Let

$$\Sigma\, a(j)\, M_j =: p \in \pi_k \cap S_\Xi .$$

If $Z \subseteq \Xi$, then, by (1), the polynomial $D_Z p$ can be written

$$D_Z p = \Sigma\, \bigl(\nabla_Z a\bigr)(j)\, M_{\Xi \setminus Z}(\cdot - j) .$$

If also $\langle \Xi \setminus Z \rangle \neq \mathbb{R}^m$, then $\operatorname{supp} M_{\Xi \setminus Z}$ has zero measure, hence the polynomial $D_Z p$ must vanish identically.

For the converse statement, we prove by induction on k that

$$S_\Xi \supseteq \pi_k \cap K , \tag{3}$$

it being trivially true for $k = -1$. For the induction step, we show now that

$$p \in \pi_k \cap K \quad \text{implies} \quad q := p - \Sigma\, p(j) M_j \in \pi_{k-1} \cap K . \tag{4}$$

Since p belongs to K, so does $\Sigma\, p(j) M_j$, by Lemma 5.1 and (1) (making use of the fact that $\ker \Delta_\Xi = \ker \nabla_\Xi$). Thus it remains to show that $q \in \pi_{k-1}$. This is established once we show that, for any $y \in \mathbb{R}^m$,

$$(D_y)^k q = 0 .$$

For this, we note that, whenever $Z \not\in \Xi^*$, then

$$(D_y)^s D_Z = (D_y)^{s-1} \sum_{\xi \in \Xi \setminus Z} a(\xi)\, D_{Z\setminus\xi}$$

(with $y = \Sigma_{\xi \in \Xi\setminus Z} a(\xi)\xi$). Repeated application of this formula justifies the claim that

$$(D_y)^k = \sum_{Z \in \Xi^*, |Z| < k} a(Z) (D_y)^{k-|Z|} D_Z + \sum_{Z \in \Xi, Z \not\in \Xi^*, |Z| = k} a(Z) D_Z .$$

It follows that

$$(D_y)^k q = \sum_{Z \in \Xi, Z \not\in \Xi^*, |Z| = k} a(Z) \big(D_Z p - \Sigma (\nabla_Z p)(j) M_{\Xi\setminus Z}(\cdot - j) \big)$$

and this is zero since, for each $Z \subseteq \Xi$ with $Z \not\in \Xi^*$, we have $\Sigma\, M_{\Xi\setminus Z}(\cdot - j) = 1$ by the corollary to Proposition 3, while $|Z| = k$ implies that $\nabla_Z p = D_Z p$ is some constant, since $p \in \pi_k$.

It is now easy to complete the induction step. If $p \in \pi_k \cap K$, then, by (4), $p \in S_\Xi + \pi_{k-1} \cap K$, *hence* $p \in S_\Xi$ *by induction hypothesis.* |||

Corollary 1. <u>For each</u> k, <u>the map</u> $T : p \longmapsto \Sigma\, p(j)\, M_j$ <u>carries</u> $\pi_k \cap S_\Xi$ <u>1-1 onto itself.</u>

Proof. We mentioned already in Section 4 that T is 1-1 on π. Thus it is sufficient to show that T carries $\pi_k \cap S_\Xi$ into itself. But that is obvious since, by (4), even $(1-T)[\pi_k \cap S_\Xi] \subseteq \pi_{k-1} \cap S_\Xi$. |||

Corollary 2. <u>As in (2.6), let</u>

$$d := \max\{ r : \langle \Xi \setminus Z \rangle = \mathbb{R}^m \text{ for all } Z \subseteq \Xi \text{ with } |Z| = r \} .$$

<u>Then</u> $\pi_k \subseteq S_\Xi$ <u>if and only if</u> $k \leq d$.

Proof. From the theorem

$$\pi_k \subseteq S_\Xi \quad \text{iff} \quad \pi_k \subseteq K := \bigcap_{Z \in \Xi^*} \ker D_Z .$$

Further, the differential operator D_Z decreases the total degree of any polynomial by at least $|Z|$ and, for some polynomials, by exactly $|Z|$. Finally,

$$d+1 = \min_{Z \in \Xi^*} |Z| .$$

This shows that $\pi_d \subseteq \ker D_Z$ for all $Z \in \Xi^*$. It also shows that, for some $Z \in \Xi^*$, $|Z| = d+1$, hence $D_Z p \neq 0$ for some $p \in \pi_{d+1}$. |||

6. Degree of approximation from S_Ξ **.** In this last section, we discuss the degree of approximation from $S_{\Xi,h}$ to a sufficiently smooth function, as $h \longrightarrow 0$. Here, h indicates a scaling of the mesh, i.e.,

$$S_{\Xi,h} := \{x \longmapsto f(x/h) : f \in S_\Xi\} .$$

Theorem 6. <u>If</u> $k \le d$ (<u>with</u> d <u>given by (2.6)), then there exists a linear functional</u> λ <u>on</u> π_k <u>so that</u> $p = \Sigma_j \lambda p(\cdot+j) M_j$ <u>for all</u> $p \in \pi_k$.

Proof. By Corollary 2 of Theorem 5, $\pi_k \subseteq S_\Xi$, while, by Corollary 1 of Theorem 5, the map $T{:}p \longmapsto \Sigma\, p(j)\, M_j$ is 1-1 onto π_k . Thus

$$p \;=\; \Sigma_j (T^{-1}p)(j)\, M_j \;, \quad \text{all} \quad p \in \pi_k \;,$$

with $T^{-1} := \left(T_{|\pi_k}\right)^{-1}$. Further, with T_i the shift by the vector i , i.e.,

$$(T_i p)(x) \;:=\; p(x+i) \;, \quad \text{all } x \;,$$

we have

$$\Sigma\, (T_i p)(j) M_j \;=\; \Sigma\, p(j+i) M_j \;=\; \Sigma\, p(j) M_j(\cdot+i) \;=\; T_i\big(\Sigma\, p(j) M_j\big) \;,$$

showing that T commutes with T_i , hence so does T^{-1} . This proves the theorem, with $\lambda p := (T^{-1}p)(0)$. |||

The theorem implies statements about degree of approximation to smooth functions from S_Ξ in the now standard quasi-interpolant fashion: Let K be the class of functions which belong locally to some function space K_0 , e.g., to L_1 . Extend the linear functional λ of the theorem to a continuous linear functional μ on K_0 and with support in supp M . The quasi-interpolant scheme

$$Q{:}K \longrightarrow S_\Xi{:}\; f \longmapsto \Sigma\, \mu f(\cdot+j) M_j$$

then reproduces π_k and is local. This implies that

$$f - Qf \;=\; (f-p) - Q(f-p) \;, \text{ for all } p \in \pi_k$$

and that

$$|(Qg)(x)| \;=\; |\Sigma \mu g(\cdot+j) M_j(x)| \;\le\; \|\mu\| \max\{\, \|g_{|\mathrm{supp} M_j}\| : M_j(x) \ne 0\} \;.$$

Therefore

$$|(f - Qf)(x)| \leq \mathrm{dist}_\rho(f, \pi_k)$$

with

$$\rho(g) := |g(x)| + \|\mu\| \max\{\|g_{|\mathrm{supp} M_j}\| : M_j(x) \neq 0\} .$$

This shows that Qf approximates to f locally as well as local polynomial approximation. Here is a particular result along these lines.

Corollary. Let μ be an extension of λ to a continuous linear functional on $L_\infty(\mathrm{supp}\ M)$ and let $Q_h := S_h Q S_{1/h}$ with $Q: f \longmapsto \Sigma\ \mu f(\cdot+j) M_j$ and $(S_h f)(x) := f(xh)$. Then, for $f \in L_\infty^{(k+1)}$, $\|f - Q_h f\|_\infty = O(h^{k+1})$.

Sharpness. The order $O(h^{d+1})$ is, in general, best possible for the approximation from S_Ξ to smooth functions. To see this, choose $Z \in \Xi^*$ (cf. Theorem 5) with $|Z| = d+1$ and a polynomial $p \in \pi_{d+1}$ with $D_Z p = 1$. If, for some approximating sequence $s_h \in S_{\Xi,h}$, we have

$$\|s_h - p\|_{L_1[0,1]^m} = o(h^{d+1}) ,$$

it follows from the standard Markov inequality for piecewise polynomials that

$$\|D_{Z'} s_h - D_{Z'} p\|_{L_1[0,1]^m} = o(h) \tag{1}$$

for any $Z' \subseteq Z$ with $|Z'| = d$. Set $z = Z \setminus Z'$. Since the support of $D_z(D_{Z'} s_h)$ is contained in hyperplanes, $(D_{Z'} s_h)$ is piecewise constant on lines in the direction z. On the other hand, $D_{Z'} p$ restricted to these lines is of the form $(D_{Z'} p)(x + tz) = (D_{Z'} p)(x) + t$ which contradicts (1). |||

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We study multivariate B-splines M obtained as "shadows" of parallelepipeds, and spaces spanned by their translates $S = \operatorname{span}_{j\in\mathbb{Z}^m} M(\cdot - j)$. Recurrence relations for M are obtained and a necessary condition for the stability of the B-spline basis is given. We further determine the polynomials contained in S and the optimal degree of approximation from S.

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