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A NEW APPROACH TO EULER SPLINES II

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UNIVERSITY OF WISCONSIN - MADISON
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ABSTRACT

Let $t \in C$ be a constant which is not real and negative, $t \neq 0$, $t \neq 1$. In [4] it was shown that the exponential Euler splines $S_n(x) = S_n(x;t)$, as introduced by the author in [2], can be obtained by the following recursive procedure: We define $S_1(x)$ as the cardinal linear spline interpolating the biinfinite sequence (t^ν) $(\nu \in Z)$ and then defined recursively

$$S_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} S_{n-1}(u)du \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{n-1}(u)du \quad (n = 2,3,\ldots) .$$

In [4] this was used to derive all the known properties of these splines and also some new ones. In the present short note, written for Euler's bicentenial of 1983, we just show that the resulting $S_n(x)$ are identical with the splines defined in [2].

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SIGNIFICANCE AND EXPLANATION

In the first paper [4] on the subject the new recursive approach to the exponential Euler splines was used to derive the known properties, and some new ones, of these most attractive among cardinal splines. In the present short note, written for Euler's bicentenial of 1983, we just show that the new recursive construction leads precisely to the exponential Euler splines as introduced in [2].

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NEW APPROACH TO EULER SPLINES II

I. J. Schoenberg

For Leonard Euler's bicentenial of 1983.

1. Introduction. Recently I have written the article [4] on this subject, but the present note can be read independently of it. Its aim is to show briefly that the new approach of §3 leads to the exponential Euler splines introduced in [2]. We need a few definitions.

Let $\mathcal{S}_n = \{S(x)\}$ $(n \geqq 1)$ denote the class of cardinal spline $S(x)$ of degree n. This means that in each unit interval $(\nu, \nu+1)$ $(\nu \in \mathbb{Z}) S(x)$ is a polynomial of degree $\leqq n$, with the strong restriction that $S(x) \in C^{n-1}(\mathbb{R})$. We also need the class of midpoint cardinal splines $\mathcal{S}_n^* = \{S(x);\ S(x+1/2) \in \mathcal{S}_n\}$. In this case the junction points (or knots) between the polynomial components of $S(x)$ are at $\nu + 1/2$ $(\nu \in \mathbb{Z})$. Actually the action will take place within the class

$$(1.1) \qquad \tilde{\mathcal{S}}_n = \begin{cases} \mathcal{S}_n & \text{if } n \text{ is odd ,} \\ \mathcal{S}_n^* & \text{if } n \text{ is even .} \end{cases}$$

A convenient basis for the class $\tilde{\mathcal{S}}_n$ is furnished by the central B-splines which are described as follows (see [3, Lecture 2, §1]). We define

$$(1.2) \qquad M_1(x) = \begin{cases} 1 & \text{if } -1/2 \leqq x \leqq 1/2 \ , \\ 0 & \text{elsewhere ,} \end{cases}$$

and convolute it with itself to obtain

$$(1.3) \qquad M_2(x) = M_1 * M_1(x) = \begin{cases} 1 - |x| & \text{if } -1 \leqq x \leqq 1 \ , \\ 0 & \text{elsewhere .} \end{cases}$$

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Early in this century it was realized that $S(x) = \sum_{-\infty}^{\infty} c_\nu M_2(x-\nu)$ is a unique representation of every element of $S_1 = \tilde{S}_1$. This extends to $\tilde{S}_n$ as follows: We form the convolution

$$(1.4)\qquad M_{n+1}(x) = \overbrace{M_1 * M_1 *\ldots* M_1}^{n+1 \text{ factors}}(x) = \frac{1}{n!}\sum_{0}^{n+1} (-1)^\nu \binom{n+1}{\nu}(x - \nu + \tfrac{1}{2}n + \tfrac{1}{2})_+^n ,$$

where $x_+ = \max(0,x)$. This an even bell-shaped function which is positive in its support $(-\tfrac{1}{2}(n+1), \tfrac{1}{2}(n+1))$. Moreover $M_{n+1}(x) \in \tilde{S}_n$, and <u>every</u> $S(x) \in \tilde{S}_n$ <u>admits a unique representation</u> $S(x) = \sum_{-\infty}^{\infty} c_\nu M_{n+1}(x-\nu)$ <u>and conversely</u> [3, Lecture 2, §2].

Let t be a constant such that

$$(1.5)\qquad t = |t|e^{i\alpha},\ -\pi < \alpha < \pi,\ t \neq 0,\ t \neq 1 ,$$

the objective being to find all solutions $S(x) \in \tilde{S}_n$ of the functional equation

$$(1.6)\qquad S(x+1) = tS(x), \quad (x \in R) .$$

From [2, Part I, §1] we need the very simple result

I. <u>Setting</u>

$$(1.7)\qquad G_n(x) = G_n(x;t) = \sum_{-\infty}^{\infty} t^\nu M_{n+1}(x-\nu) ,$$

<u>the most general</u> $S(x) \in \tilde{S}_n$ <u>satisfying</u> (1.6) <u>is of the form</u> $S(x) = C\cdot G_n(x)$.

From [3, Lecture 2, §§4 and 5] we also need

II. <u>If</u> t <u>satisfies</u> (1.5), <u>then</u>

$$(1.8)\qquad G_n(x) = \sum_{-\infty}^{\infty} t^\nu M_{n+1}(x-\nu) \neq 0 \text{ \underline{for all real} } x .$$

This is evident if $t > 0$ because $G_n(x) > 0$ for all x. For complex-valued t (1.8) follows from the fact that the curve of the complex plane

$$z = G_n(x), \quad (-\infty < x < \infty)$$

spirals "convexly" around the origin O without ever reaching it.

2. Euler's generating function. Our actual discussion starts with Euler's generating function [1, Chap. VII, §178]

$$\frac{t-1}{t-e^z} e^{xz} = \sum_0^\infty \frac{A_n(x;t)}{n!} z^n \tag{2.1}$$

which defines the Appell sequence of monic polynomials $A_n(x;t) = A_n(x)$. These are the Eulerian polynomials. If we differentiate (2.1) with respect to x ν times $(\nu \geq 0)$ we obtain from $A_\nu^{(\nu)}(x;t) = \nu!$ that

$$\frac{t-1}{t-e^z} e^{xz} z^\nu = \sum_{n=\nu}^\infty \frac{A_n^{(\nu)}(x;t)}{n!} z^n = z^\nu + \sum_{n=\nu+1}^\infty \frac{A_n^{(\nu)}(x;t)}{n!} z^n .$$

Canceling the factor z^ν and setting successively $x = 0$ and $x = 1$ we obtain

$$\frac{t-1}{t-e^z} - 1 = \sum_{n=\nu+1}^\infty \frac{A_n^{(\nu)}(0;t)}{n!} z^{n-\nu}, \quad \frac{t-1}{t-e^z} e^z - 1 = \sum_{n=\nu+1}^\infty \frac{A_n^{(\nu)}(1;t)}{n!} z^{n-\nu} .$$

Multiplying the first equation by t, we find that the two left sides become identically equal. Therefore the right sides are identical and show that

$$A_n^{(\nu)}(1;t) = tA_n^{(\nu)}(0;t) \quad (\nu = 0,1,\ldots,n-1) . \tag{2.2}$$

This is a characteristic property of the Eulerian polynomial $A_n(x;t)$.

The definition of the exponential Euler spline $s_n(x)$ as I defined it in [2] is now immediate: We set

$$s_n(x) = A_n(x;t)/A_n(0;t) \quad \underline{\text{if}} \quad 0 \leq x < 1 , \tag{2.3}$$

and extend its definition to all real x by the functional equation

$$s_n(x+1) = ts_n(x) \quad (x \in \mathbb{R}) . \tag{2.4}$$

From (2.4) and the boundary property (2.2) we find that $s_n(x) \in C^{n-1}(\mathbb{R})$ and therefore

$$s_n(x) \in \mathcal{S}_n . \tag{2.5}$$

From (2.3) and (2.4) we obtain that

$$s_n(\nu) = t^\nu \quad (\nu \in \mathbb{Z}) . \tag{2.6}$$

The exponential Euler spline $s_n^*(x)$ of S_n^* is now obtained as

$$s_n^*(x) = s_n(x + \tfrac{1}{2})/s_n(\tfrac{1}{2}) \quad (x \in \mathbb{R}) \quad . \tag{2.7}$$

It satisfies

$$s_n^*(x+1) = ts_n^*(x) \quad (x \in \mathbb{R}) \tag{2.8}$$

and therefore

$$s_n^*(\nu) = t^\nu \quad (\nu \in \mathbb{Z}) \quad . \tag{2.9}$$

Is $A_n(0;t) \neq 0$ in (2.3)? It is, for if $A_n(0;t) = 0$ we could define a cardinal spline $\hat{s}(x)$ by setting $\hat{s}(x) = A_n(x;t)$ in $[0,1)$ and extend it to satisfy $\hat{s}(x+1) = t\hat{s}(x)$. By I. it follows that $\hat{s}(x) = C \cdot G_n(x)$ $(C \neq 0)$, and now $s(0) = A_n(0;t) = 0$ would contradict II.

3. The new approach to exponential Euler splines. As we are going to alternate between the classes $\mathcal{S}_n$ and $\mathcal{S}_n^*$ we find it convenient to define

$$\tilde{\mathcal{S}}_n = \begin{cases} \mathcal{S}_n & \text{if } n \text{ is odd} \; , \\ \mathcal{S}_n^* & \text{if } n \text{ is even} \; . \end{cases} \tag{3.1}$$

The corresponding splines we denote by

$$\tilde{s}_n(x) = \begin{cases} s_n(x) & \text{if } n \text{ is odd} \; , \\ s_n^*(x) & \text{if } n \text{ is even} \; . \end{cases} \tag{3.2}$$

We now define a new sequence of functions

$$S_n(x) = S_n(x;t) \quad (n = 1,2,\ldots) \quad , \tag{3.3}$$

by first setting

$$S_1(x) = \tilde{s}_1(x) = s_1(x) = \sum_{-\infty}^{\infty} t^\nu M_2(x-\nu) \quad , \tag{3.4}$$

and determine the functions (3.3) by the recurrence relation

$$S_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} S_{n-1}(u)du \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{n-1}(u)du \quad (n = 2,3,\ldots) \quad . \tag{3.5}$$

Our main result is

<u>Theorem 1.</u> <u>The sequence of functions</u> (3.3) <u>defined by</u> (3.4) <u>and</u> (3.5) <u>is identical with the exponential Euler splines of the classes</u> $\tilde{S}_n$, <u>hence</u>

$$S_n(x) = \tilde{s}_n(x), \quad (n = 1,2,\dots) \quad . \tag{3.6}$$

Let us first establish for the functions (1.7) the relation

$$G_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} G_{n-1}(u)du \quad (n = 2,3,\dots) \quad . \tag{3.7}$$

Indeed, from (1.4) we have $M_{n+1}(x) = M_1 * M_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} M_n(u)du$, whence

$$\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} M_n(u-\nu)du = M_{n+1}(x-\nu) \quad .$$

Now (1.7) implies

$$\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} G_{n-1}(u)du = \sum_{\nu} t^{\nu} \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} M_n(u-\nu)du = \sum_{\nu} t^{\nu} M_{n+1}(x-\nu) = G_n(u) \quad ,$$

proving (3.7).

We observe next that $M_{n+1}(x) \in \tilde{S}_n$; also that by I. and II. we have

$$\tilde{s}_n(x) = G_n(x)/G_n(0) \quad . \tag{3.8}$$

However, from (3.7) we obtain

$$G_n(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} G_{n-1}(u)du \quad , \tag{3.9}$$

and therefore by (3.7) and (3.9)

$$\tilde{s}_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} G_{n-1}(u)du \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} G_{n-1}(u)du \quad .$$

If we define here the terms of the fraction on the right by $G_{n-1}(0)$ we find from $G_{n-1}(u)/G_{n-1}(0) = \tilde{s}_{n-1}(u)$ that

$$(3.10) \qquad \tilde{s}_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \tilde{s}_{n-1}(u)du \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{s}_{n-1}(u)du \quad (n = 2,3,\ldots) .$$

Since $\tilde{s}_1(x) = S_1(x)$, the relations (3.10) clearly imply that $S_n(x) = \tilde{s}_n(x)$ and Theorem 1 is established.

Remarks. 1. Notice that (3.4) and (3.5) have produced only one half of the exponential Euler splines, i.e. only those described by (3.2). If we retain the relations (3.5), but start from $S_1(x) = s_1^*(x)$, we would get the other half: $S_n(x) = s_n^*(x)$ if n is odd and $S_n(x) = s_n(x)$ if n is even.

2. If we retain (3.4), but modify (3.5) to

$$S_n(x) = \int_x^{x+1} S_{n-1}(u)du \Big/ \int_0^1 S_{n-1}(u)du, \quad (n = 2,3,\ldots) ,$$

we would obtain that $S_n(x) = s_n(x)$ for all n.

3. So fare we have excluded the case when t is real and negative. Actually, the case $t < 0$ leads to the so-called eigensplines of the classes $\mathcal{S}_n$ and $\mathcal{S}_n^*$, which are fundamental for the problem of cardinal spline interpolation (see [3, Lecture 4]). Exceptional is the case of the Euler splines which arise if $t = -1$. In this case our Theorem 1 again holds, as shown in [4, §6].

4. In [2, §11] and again in [4,§5] I have recommended the use of the exponential Euler splines for the programming of the exponential function on a computer. Especially for $t = 2$ this would produce, by a simple algorithm, very smooth and close approximations to 2^x.

5. We hope to have shown in this note the fundamental nature of Leonard Euler's contribution to cardinal spline interpolation.

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Let $t \in C$ be a constant which is not real and negative, $t \neq 0$, $t \neq 1$. In [4] it was shown that the exponential Euler splines $S_n(x) = S_n(x;t)$, as introduced by the author in [2], can be obtained by the following recursive procedure: We define $S_1(x)$ as the cardinal linear spline interpolating the biinfinite sequence (t^ν) $(\nu \in \mathbb{Z})$ and then defined recursively

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ABSTRACT (continued)

$$S_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} S_{n-1}(u)du / \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{n-1}(u)du \quad (n = 2,3,\ldots) \quad .$$

In [4] this was used to derive all the known properties of these splines and also some new ones. In the present short note; written for Euler's bicentenial of 1983, we just show that the resulting $S_n(x)$ are identical with the splines defined in [2].

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