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"ON THE STEADY-STATE CONTINUOUS CASTING STEFAN PROBLEM WITH NON-LINEAR COOLING"

By

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ABSTRACT - A steady-state one phase Stefan problem corresponding to the solidification process of an ingot of pure metal by continuous casting with a non-linear lateral cooling is considered via the weak formulation introduced in [BKS] for the dam problem. Two existence results are obtained for a general non-linear flux and for a maximal monotone flux. Comparison results and the regularity of the free boundary are discussed. An uniqueness theorem is given for the monotone case.

CONTENTS

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0. INTRODUCTION

In this paper we study the one phase model of the solidification of a pure metal in continuous casting submitted to a non-linear lateral cooling.

In the liquid phase we assume that the metal is at the melting temperature, which is zero after a normalization. In the solid phase the temperature Θ satisfies the heat equation. The ingot is extracted with constant velocity b, and the liquid - - solid interface (the free boundary) is unknown but steady with respect to a fixed system of coordinates of $\mathbb{R}^3$, in which our problem will be studied. Assuming that the free boundary Φ is representable by a surface $z=\phi(x,y)$, the steady Stefan condition is

$$(0.1) \qquad \Theta_z-\Theta_x\phi_x - \Theta_y\phi_y = \lambda b, \text{ for } z=\phi(x,y)$$

where λ is a positive constant representing the heat of melting.

In the lateral boundary one specifies a non-linear flux condition

$$(0.2) \qquad - \partial\Theta/\partial n = G(\Theta)$$

which expresses the law of cooling, and may be quite general. Namely, we shall consider a maximal monotone graph G, which may include a cooling process with climatization as in Chapter 1 of the book of Duvaut and Lions [DL].

This model has been considered in a particular case by Rubinstein [Ru] and, with a linear flux condition of Newton type, by Brière [Br] and Rodrigues [R], via variational inequalities after a transformation of Baiocchi's type. However this approach doesn't work with a non linear cooling.

Since this problem has some similarities with the dam problem, we formulate it in section 1 using the method of Brēzis, Kinderlehrer and Stampacchia [BKS]. In sections 2 and 3 we prove the existence theorems, first using compactness arguments and next combining compacity and monotonicity tecniques for the maximal monotone case.

In section 4 we discuss comparison properties which show that when the extraction velocity b is small the ingot solidifies immediatly and there is no free boundary. For some type of cooling and for a high enough velocity b one can show the existence of a free boundary. In this case it is shown, in section 5, that the free boundary is an analytic surface and a weak solution is also a classic one, as in the linear case of [R].

To conclude this paper we give an uniqueness theorem for the monotone case in section 6, using the technique of Carrillo-Chipot [CC].

1. MATHEMATICAL FORMULATION

Let Ω denote a cylindric domain in $\mathbb{R}^3$, in the form $\Omega=\Gamma\times]0,H[$, where $\Gamma\subset\mathbb{R}^2$ is a bounded domain with lipschitz boundary $\partial\Gamma$ representing a section of the ingot and $H>0$ its height. We denote $\Gamma_i=\Gamma\times\{i\}$, for $i=0,H$, the bottom and the top of the ingot respectively, and by $\Gamma_1=\partial\Gamma\times]0,H[$ its lateral boundary. We have $\partial\Omega=\Gamma_0\cup\overline{\Gamma}_1\cup\Gamma_H$.

Considering $\vec{z}$ the direction of extraction, we can formulate our problem in its classical form:

PROBLEM (C) : Find a couple (Θ,ϕ), such that

(1.1) $\Theta\geq 0$ in Ω and $\Theta=0$ for $0\leq z\leq\phi(x,y)<H$

(1.2) $\Delta\Theta=b\,\Theta_z$ for $0\leq\phi(x,y)<z<H$

(1.3) $\Theta=0$ on Γ_0 , $\Theta=h(x,y)>0$ on Γ_H

(1.4) $-\partial\Theta/\partial n=g(\Theta)$ on Γ_1

(1.5) $\Theta_z-\Theta_x\phi_x-\Theta_y\phi_y = \lambda b$, if $z=\phi(x,y) > 0$

(1.5') $\Theta_z \geq \lambda b$, if $z=\phi(x,y)=0$.

In this formulation b and λ are positive constants, h is a given function, and g will be specified in the next two sections. The reader will note that the condition (1.5') is a degeneration of the Stefan condition (1.5) in the case when the free boundary Φ can touch the known boundary Γ_0, where the melting condition $\Theta=0$ is assumed by (1.3).

Let us remark that by the maximum principle it must be $\Theta>0$ for $z>\phi(x,y)$. Denoting by χ^+ the characteristic function of the set $\Omega_+=\{\Theta>0\}$ and integrating formaly by parts, for every regular function ζ, such that $\zeta=0$ on Γ_H and $\zeta\geq 0$ on Γ_0, from Problem (C) one has

$$\int_\Omega(\nabla\Theta\cdot\nabla\zeta+b\Theta_z\zeta-\lambda b\chi^+\zeta_z) = \int_{\Omega_+}(\nabla\Theta\cdot\nabla\zeta+b\Theta_z\zeta-\lambda b\zeta_z)$$

$$= \int_{\Omega_+}(-\Delta\Theta+b\Theta_z)\zeta + \int_{\Gamma_1\cup\Phi\cup\Gamma_0}\frac{\partial\Theta}{\partial n}\zeta+\lambda b\int_{\Phi\cup\Gamma_0}\ell\zeta$$

$$= -\int_{\Gamma_1}g(\Theta)\zeta+\int_{\Gamma_0}\ell\zeta(\lambda b-\Theta_z)+\int_\Phi\ell\zeta(\Theta_x\phi_x+\Theta_y\phi_y-\Theta_z+\lambda b)$$

$$\leq -\int_{\Gamma_1}g(\Theta)\zeta,$$

where $\ell^{-2}=\phi_x^2+\phi_y^2+1$. Therefore, following [BKS], we introduce the weak formulation of Problem (C) :

PROBLEM (P) : Find a couple $(\Theta,\chi) \in H^1(\Omega)\times L^\infty(\Omega)$, such that,

(1.6) $\Theta \geq 0$ a.e. in Ω, $\Theta=0$ on Γ_0 and $\Theta=h$ on Γ_H;

(1.7) $0 \leq \chi \leq 1$ a.e. in Ω and $\chi=1$ where $\Theta>0$;

(1.8) $\int_\Omega (\nabla\Theta\cdot\nabla\zeta + b\Theta_z\zeta - \lambda b\chi\zeta_z) + \int_{\Gamma_1} g(\Theta)\zeta \leq 0$, for every $\zeta \in H^1(\Omega)$, such that $\zeta \geq 0$ on Γ_0 and $\zeta=0$ on Γ_H.

If we consider a more restrictive class of test functions one can introduce a more general formulation, which we call Problem (P'), if we replace (1.8) by

(1.9) $\int_\Omega (\nabla\Theta\cdot\nabla\zeta + b\Theta_z\zeta - \lambda b\chi\zeta_z) + \int_{\Gamma_1} g(\Theta)\zeta = 0,\ \forall\zeta\in H^1(\Omega): \zeta=0$ on $\Gamma_0 \cup \Gamma_H$.

It is clear that every solution of Problem (P) verifies (1.9), but the Problem (P') has more solutions than Problem (P). In particular, if

PROBLEM (P_1) : Find Θ verifying (1.6) and

(1.10) $\int_\Omega (\nabla\Theta\cdot\nabla\zeta + b\Theta_z\zeta) + \int_{\Gamma_1} g(\Theta)\zeta = 0,\ \forall\zeta\in H^1(\Omega);\ \zeta=0$ on $\Gamma_0 \cup \Gamma_H$

has a solution $\Theta \geq 0$, by the maximum principle, one has $\Theta>0$ in Ω and $(\Theta,1)$ is a solution to Problem (P'), which may not satisfy (1.5')(see Proposition 4).

2. EXISTENCE OF A WEAK SOLUTION

In this section we assume the lateral cooling given by

$$(2.1) \qquad -\frac{\partial\Theta}{\partial n}(X) = g(X,\rho(X),\Theta(X)), \; X \in \Gamma_1$$

where $\rho \geq 0$ is a given function representing the cooling temperature, and

(2.2) $g(X,\rho,\theta)$ is a bounded Carathéodory function, i.e., is continuous in $\theta \in \mathbb{R}$, a.e. $(X,\rho) \in \Gamma_1 \times \mathbb{R}_+$, measurable in (X,ρ) for all θ, and maps bounded sets of $\Gamma_1 \times \mathbb{R}_+ \times \mathbb{R}$ in bounded sets of $\mathbb{R}$.

Since the cooling process is determined by ρ, we shall assume that

$$(2.3) \qquad g(X,\rho,\theta) \leq 0, \quad \text{a.e.} \quad (X,\rho,\theta) \in \Gamma_1 \times \mathbb{R}_+ \times \mathbb{R}$$

$$(2.4) \qquad g(X,\rho,\theta) = 0 \quad \text{for} \quad |\theta| \geq \rho, \; \text{a.e. } X \in \Gamma_1.$$

Consider a parameterized family of functions $\chi_\varepsilon \in C^\infty(\mathbb{R})$ such that

$$(2.5) \qquad \chi_\varepsilon(t) = \begin{cases} 0 & , \text{ for } t \leq 0 \\ 0 \leq \chi_\varepsilon(t) \leq 1 & , \text{ for } 0 \leq t \leq \varepsilon \\ 1 & , \text{ for } t \geq \varepsilon \end{cases}$$

and so it approaches the Heaviside function when $\varepsilon \searrow 0$.
Introduce now the following penalized problem, where for the sake of simplicity we denote $g(X,\rho(X),\Theta(X))$ by $g(\Theta)$:

<u>PROBLEM (P_ε)</u> Find $\Theta^\varepsilon \in H^1(\Omega) \cap C^0(\overline{\Omega})$, such that,

$$(2.6) \qquad \Theta^\varepsilon = 0 \;\text{ on }\; \Gamma_0, \quad \Theta^\varepsilon = h \;\text{ on }\; \Gamma_H,$$

(2.7) $\int_\Omega [\nabla\Theta^\varepsilon\cdot\nabla\zeta + b\Theta_z^\varepsilon\zeta - \lambda b\,\chi_\varepsilon(\Theta^\varepsilon)\zeta_z] + \int_{\Gamma_1} g(\Theta^\varepsilon)\zeta = 0, \ \forall\zeta\varepsilon H^1(\Omega): \zeta=0 \text{ on } \Gamma_o \cup \Gamma_H.$

Assuming the functions h and ρ verify

(2.8) $0 < h(x,y) \leq M, \quad$ a.e. $(x,y)\ \varepsilon\ \Gamma_H$,

(2.9) $0 \leq \rho(X) \leq M, \quad$ a.e. $X\ \varepsilon\ \Gamma_1$,

one can prove the following "a priori" estimate:

<u>LEMMA 1</u> If Θ^ε is a solution to Problem (P_ε) with assumptions (2.2-4) and (2.8-9), one has

(2.10) $0 \leq \Theta^\varepsilon(X) \leq M$, for all $X\ \varepsilon\ \overline{\Omega}$ and $0<\varepsilon\leq M$.

Proof : Let $\zeta = [\Theta^\varepsilon]^-$ in (2.7). One has

$$0 = \int_\Omega \{\nabla\Theta^\varepsilon\cdot\nabla[\Theta^\varepsilon]^- + b\Theta_z^\varepsilon[\Theta^\varepsilon]^- - \lambda b\ \chi_\varepsilon(\Theta^\varepsilon)[\Theta^\varepsilon]_z^-\} + \int_\Gamma g(\Theta^\varepsilon)[\Theta^\varepsilon]^-$$

$$\leq -\int_\Omega \{|\nabla[\Theta^\varepsilon]^-|^2 + b[\Theta^\varepsilon]_z^-[\Theta^\varepsilon]^-\} = -\int_\Omega |\nabla[\Theta^\varepsilon]^-|^2$$

from which it follows $[\Theta^\varepsilon]^- = 0$ and $\Theta^\varepsilon \geq 0$.

From (2.4) (2.9) and (2.5), one has respectively

$g(\Theta^\varepsilon)[\Theta^\varepsilon - M]^+ = 0$ and $\chi_\varepsilon(\Theta^\varepsilon)[\Theta^\varepsilon - M]_z^+ = [\Theta^\varepsilon - M]_z^+$ for $0<\varepsilon\leq M$.

Then $\zeta = [\Theta^\varepsilon - M]^+$ in (2.7) implies

$$0 = \int_\Omega \{\nabla\Theta^\varepsilon\cdot\nabla[\Theta^\varepsilon - M]^+ + b\ \Theta_z^\varepsilon[\Theta^\varepsilon - M]^+ - \lambda b[\Theta^\varepsilon - M]_z^+\}$$

$$= \int_\Omega |\nabla[\Theta^\varepsilon - M]^+|^2,$$

and therefore $[\Theta^\varepsilon - M]^+ = 0$. The lemma is proved. ∎

We shall need the L^∞ and the Hölder estimates due to Stampacchia [S] for the following eliptic problem with mixed boundary conditions:

$$- \Delta u + b u_z = f \quad \text{in } \Omega\ , \quad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_1 \text{ and } u = h \quad \text{on } \Gamma_0 \cup \Gamma_H . \tag{2.11}$$

<u>LEMMA 2</u> [S] The unique solution of (2.11) verifies

$$\|u\|_{L^\infty(\Omega)} \le C_1 (\|f\|_{W^{-1,p}(\Omega)} + \|g\|_{L^q(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0 \cup \Gamma_H)}) \tag{2.12}$$

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \le C_2 (\|f\|_{W^{-1,p}(\Omega)} + \|g\|_{L^q(\Gamma_1)} + \|h\|_{C^{0,1}(\overline{\Gamma}_0 \cup \overline{\Gamma}_H)}) \tag{2.13}$$

for all $p>3$ and $q>2$, and for some constants $C_1, C_2 > 0$ and $0<\alpha<1$ which are independent of f,g,h and u.

Proof : See the results of §5 of [S] or a more explicit result extended to variational inequalities in Section 2 of [MS] ∎

Now we can state an existence result for the penalized problem, from which we shall construct a sequence of functions converging to a solution of Problem (P).

<u>PROPOSITION 1</u> Under assumptions of Lemma 1, and if

$$h \in C^{0,1}(\overline{\Gamma}_H) \tag{2.14}$$

then there exists a solution Θ^ε to Problem (P_ε) for all $0<\varepsilon\le M$ satisfying the estimate

$$\|\Theta^\varepsilon\|_{H^1(\Omega)} + \|\Theta^\varepsilon\|_{C^{0,\alpha}(\overline{\Omega})} \le C, \tag{2.15}$$

where the constants $C>0$ and $0<\alpha<1$ are independent of ε.

Proof : For $\tau \in B_R=\{\tau \in C^0(\bar{\Omega}) : \|\tau\|_{C^0(\bar{\Omega})}\leq R\}$, $(R>0)$, define

$$\Theta = S_\varepsilon(\tau)$$

as the unique solution of the following mixed linear problem

$$\Theta = 0 \quad \text{on} \quad \Gamma_0 \quad , \quad \Theta=h \quad \text{on} \quad \Gamma_H$$

$$\int_\Omega (\nabla\Theta\cdot\nabla\zeta+b\Theta_z\zeta)=\lambda b\int_\Omega \chi_\varepsilon(\tau)\zeta_z - \int_{\Gamma_1} g(\tau)\zeta, \quad \forall\zeta\varepsilon H^1(\Omega):\zeta=0 \text{ on } \Gamma_0 U\Gamma_H$$

Since, by definition, $0 \leq \chi_\varepsilon \leq 1$ and g is bounded independently of τ (for $|\tau(X)|\geq M\geq \rho(X)$ one has $g(X,\rho(X),\tau(X))=0)$ by (2.4)) one can apply Stampacchia's estimate (2.13). Therefore, there exists $C > 0$ and $0<\alpha<1$, independent of τ and ε such that

$$\|\Theta\|_{C^{0,\alpha}(\bar{\Omega})}\leq C_2(\lambda b+\|g\|_{L^\infty} + \|h\|_{C^{0,1}}) \leq C$$

and for $R\geq C$ one has $S_\varepsilon(B_R) \subset B_R$.

From the compactness of the imbedding $C^{0,\alpha}(\bar{\Omega})\hookrightarrow C^0(\bar{\Omega})$ one finds that S_ε is a continuous and compact mapping of B_R into itself. By the Schauder fixed point theorem there exists a function $\Theta^\varepsilon \in B_R$ satisfying $\Theta^\varepsilon=S_\varepsilon(\Theta^\varepsilon)$, which is clearly a solution to Problem (P_ε).

The estimate in $H^1(\Omega)$ is classical, since χ^ε and $g(C^\varepsilon)$ are bounded independently of ε. ∎

<u>THEOREM 1</u> Assuming (2.2,3,4) and (2.8,9,14) there exists a solution $(\Theta,\chi)\varepsilon[H^1(\Omega)\cap C^{0,\alpha}(\bar{\Omega})]\times L^\infty(\Omega)$ to Problem (P).

Proof : By (2.15) one can consider a sequence of solutions Θ^ε of Problem (P_ε), such that, when $\varepsilon \downarrow 0$

(2.16) $$\Theta^\varepsilon \rightharpoonup \Theta \quad \text{in } H^1(\Omega)\text{-weak}$$

(2.17) $$\Theta^\varepsilon(X) \to \Theta(X) \text{ uniformly in } X=(x,y,z) \in \bar{\Omega}$$

(2.18) $$\chi_\varepsilon(\Theta^\varepsilon) \rightharpoonup \chi \quad \text{in } L^\infty(\Omega)\text{-weak } *,$$

where Θ is some function belonging to $H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ satisfying (2.10) and $0 \leq \chi \leq 1$. Moreover in the open set $\{\Theta>0\}$ one has $\chi_\varepsilon(\Theta^\varepsilon) \to 1$ and therefore $\chi=1$ a.e. in $\{\Theta>0\}$.

Let $\zeta \in H^1(\Omega)$, $\zeta \geq 0$ on Γ_0 and $\zeta=0$ on Γ_H.

By the Green's formula and since $\partial\Theta^\varepsilon/\partial n \leq 0$ on Γ_0, one has

$$\int_\Omega [\nabla\Theta^\varepsilon \cdot \nabla\zeta + b\Theta_z^\varepsilon \zeta - \lambda b \chi_\varepsilon(\Theta^\varepsilon)\zeta_z] + \int_{\Gamma_1} g(\Theta^\varepsilon)\zeta = \int_{\Gamma_0} \frac{\partial\Theta^\varepsilon}{\partial n} \zeta \leq 0$$

and in the limit we obtain (1.8). The proof is complete. ∎

3. THE CASE OF A MAXIMAL MONOTONE COOLING

In this section we consider the existence of a weak solution with a lateral cooling

(3.1) $$-\frac{\partial\Theta}{\partial n} \in G(\Theta) \quad \text{on } \Gamma_1,$$

where G denotes a maximal monotone graph , that is, G is a multivalued function which graph is a continuous monotone increasing curve in $\mathbb{R}^2$ (see [B]). We shall assume

(3.2) $$G(0) \subset]-\infty,0]$$

(3.3) $$[0,+\infty[\subset \text{Dom } (G) \equiv \{x \in \mathbb{R} \mid G(x) \neq \emptyset\} .$$

The weak formulation of the corresponding problem takes now the following form:

PROBLEM ($\tilde{P}$) Find $(\Theta,\chi,g)\varepsilon\ H^1(\Omega)xL^\infty(\Omega)xL^2(\Gamma_1)$, such that

(3.4) $\Theta\geq 0$ a.e. in Ω , $\Theta=0$ on Γ_0 and $\Theta=h$ on Γ_H;

(3.5) $0\leq\chi\leq 1$ a.e.in Ω, $\chi=1$ if $\Theta>0$;

(3.6) $\int_\Omega(\nabla\Theta\cdot\nabla\zeta+b\Theta_z\zeta-\lambda b\chi\zeta_z)+\int_{\Gamma_1} g\zeta\leq 0$, $\forall\zeta\varepsilon H^1(\Omega):\zeta\geq 0$ on $\Gamma_0,\zeta=0$ on Γ_H;

(3.7) $g(X)\ \varepsilon\ G(\Theta(X))$ a.e. $X\ \varepsilon\ \Gamma_1$.

We shall obtain a solution to Problem ($\tilde{P}$) as the limit of a sequence of solutions to Problem (P) with a non-linear cooling given by a function g satisfying :

(3.8) g is monotone increasing, lipschitz and such that $g(0)\leq 0$.

THEOREM 2 Assume (3.8) and let $h\ \varepsilon\ H^{1/2}(\Gamma_H)$, $h>0$. Then Problem (P) has a solution.

Proof : The proof follows the lines of the one in theorem 1. by considering the penalized problem (P_ε) with g satisfying (3.8). The fixed point is now constructed in $L^2(\Omega)$ by means of the mapping

$$L^2(\Omega)\ni\tau\mapsto\xi=T_\varepsilon(\tau)\ \varepsilon\ V.$$

where $V=\{v\varepsilon H^1(\Omega):\ v=0\ \text{ on }\Gamma_0\}$ and ξ is the unique solution of the following problem

$$(3.9)\quad \begin{cases} \xi \in V\,, \quad \xi = h \quad \text{on } \Gamma_H \\ \int_\Omega (\nabla\xi\cdot\nabla\zeta + b\xi_z\zeta) + \int_{\Gamma_1} g(\xi)\zeta = \lambda b \int_\Omega \chi_\varepsilon(\tau)\zeta_z\,, \ \forall\zeta\in V: \zeta=0 \quad \text{on } \Gamma_H. \end{cases}$$

which is a coercive and (strictly) monotone problem in V by assumption (3.8) (see [L]). Denoting by $\tilde{h}$ some function in V, which trace on Γ_H is h, and letting $\zeta=\xi-\tilde{h}$ in (3.9) one easily finds

$$\|\xi\|_{H^1(\Omega)} \leq C = C(\tilde{h}),$$

where C is a constant independent of τ and ε.

Since the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the Schauder fixed point Theorem assures the existence of a solution Θ^ε to Problem (P_ε). As in Lemma 1 one finds that $\Theta^\varepsilon \geq 0$, since g is monotone increasing and $g(0)\leq 0$, and therefore one has $g(\Theta^\varepsilon)\cdot [\Theta^\varepsilon]^- \leq 0$.

The passage to the limit as $\varepsilon \downarrow 0$ is straightforward since $\Theta^\varepsilon \longrightarrow \Theta$ in $H^1(\Omega)$-weak and g is a lipschitz function. ∎

<u>REMARK 1</u> Since g is lipschitz, by Sobolev imbeddings one has $g(\Theta) \in H^{1/2}(\Gamma_1) \hookrightarrow L^4(\Gamma_1)$ (see [A, p. 218]) and therefore applying Lemma 2, it follows that

i) if $h \in L^\infty(\Gamma_H)$, then $\Theta \in L^\infty(\Omega)$; and

ii) if $h \in C^{0,1}(\overline{\Gamma}_H)$, then $\Theta \in C^{0,\alpha}(\overline{\Omega})$, for some $0<\alpha<1$. ∎

Since G is a maximal monotone operator one can introduce the Yosida regularization, defined by

$$g_\delta = \frac{1}{\delta}(I - J_\delta) \quad , \text{ for } \quad \delta > 0,$$

where $J_\delta=(I+\delta G)^{-1}$ is the resolvent of G. Consider $\tau=J_\delta(0)$, that is $0\in(I+\delta G)(\tau)$. From the monotonicity of $I+\delta G$ and using assumption (3.2) one finds $\tau\geq 0$. Therefore $g_\delta(0)=-J_\delta(0)/\delta\leq 0$, which means that, for each $\delta>0$, the Yosida regularization g_δ satisfies the condition (3.8) (see [B]). So we may apply Theorem 2 to conclude the existence of a solution $(\Theta^\delta,\chi^\delta)\in H^1(\Omega)xL^\infty(\Omega)$ to Problem (P) with lateral cooling given by g_δ. We shall obtain a solution to Problem $(\tilde{P})$ by considering a subsequence $\delta\downarrow 0$.

THEOREM 3 The Problem $(\tilde{P})$ with a maximal monotone graph G satisfying (3.2) and (3.3), and with $h\in H^{1/2}(\Gamma_H)\cap L^\infty(\Gamma_H)$ has a solution $(\Theta,\chi,g)\in[H^1(\Omega)\cap L^\infty(\Omega)]xL^\infty(\Omega)xL^\infty(\Gamma_1)$.

Moreover, if $h\in C^{0,1}(\overline{\Gamma}_H)$ one has $\Theta\in C^{0,\alpha}(\overline{\Omega})$, for some $0<\alpha<1$.

Proof : Consider the (unique) solution Θ^0 of the following mixed problem.

$$(3.10)\quad \begin{cases} \Theta^0\in H^1(\Omega),\quad \Theta^0=0 \text{ on } \Gamma_0,\quad \Theta^0=h \text{ on } \Gamma_H \\ \int_\Omega(\nabla\Theta^0\cdot\nabla\zeta+b\Theta^0_z\zeta)+\int_{\Gamma_1}g^0(0)\zeta=0,\quad \forall\zeta\in H^1(\Omega),\zeta=0 \text{ on } \Gamma_0\cup\Gamma_H. \end{cases}$$

where $g^0(t)=\mathrm{Proj}_{G(t)}0$ is the smallest (in norm) number of G(t). Since $g^0(0)\leq 0$ it is easy to show that $\Theta^0\geq 0$. Since $h\in L^\infty(\Gamma_H)$ one has $\Theta^0\in L^\infty(\Omega)$ by (2.12), and we assume that $\Theta^0\leq M^0=M^0(h,g^0(0))$.

Then, for every solution Θ^δ to Problem (P) with g_δ, we have

$$(3.11)\qquad 0\leq\Theta^\delta\leq\Theta^0\leq M^0.$$

Indeed (3.11) follows by a comparison argument: take $\zeta=[\Theta^\delta-\Theta^0]^+$ in $(1\text{-}8)_\delta$ and in (3.10); one has

$$(3.12) \qquad \int_\Omega |\nabla[\Theta^\delta-\Theta^o]^+|^2 - \lambda b\int_\Omega \chi^\delta[\Theta^\delta-\Theta^o]_z^+ + \int_{\Gamma_1}[g_\delta(\Theta^\delta)-g^o(0)][\Theta^\delta-\Theta^o]^+ \le 0;$$

Since $\Theta^o \ge 0$ and $\chi^\delta=1$ in $\{\Theta^\delta>0\}$, the middle term in (3.12) vanishes; using $g_\delta(0)\le 0$, together with

$$(3.13) \qquad |g_\delta(t)| \le |g^o(t)| \qquad \text{(see [B], p.28)}$$

in order to deduce the chain

$$g_\delta(\Theta^\delta) \ge g_\delta(\Theta^o) \ge g_\delta(0) \ge g^o(0),$$

one finds that the last term in (3.12) is non-negative, which proves (3.11).

Using again (3.13), by (3.11) one has

$$(3.14) \qquad |g_\delta(\Theta^\delta)| \le |g^o(\Theta^\delta)| \le \max\,[|g^o(0)|,|g^o(M^o)|] \equiv \ell,$$

from where we easily conclude

$$\|\Theta^\delta\|_{H^1(\Omega)} \le C(\text{=const.independ. of } \delta).$$

It follows that there exists a subsequence $\delta \downarrow 0$ such that

$$(3.15) \qquad \Theta^\delta \rightharpoonup \Theta \quad \text{in } H^1(\Omega)\text{-weak, and } 0\le\Theta\le M^o$$

$$(3.16) \qquad \chi^\delta \rightharpoonup \chi \quad \text{in } L^\infty(\Omega)\text{-weak *, } 0\le\chi\le 1$$

$$(3.17) \qquad g_\delta(\Theta^\delta) \rightharpoonup g \text{ in } L^\infty(\Gamma_1)\text{-weak *, with } \|g\|_{L^\infty} \le \ell\ .$$

Since one can also consider $\Theta^\delta \to \Theta$ uniformly in each compact subset $K\subset\Omega$, one has $\chi=1$ in the open set $\{\Theta>0\}$.

Using the compactness of the trace mapping, one can consider $\Theta^\delta\to\Theta$ in $L^2(\Gamma_1)$-strong and from (3.3) $J_\delta(\Theta^\delta)\to\Theta$ in $L^2(\Gamma_1)$. Since $g_\delta(\Theta^\delta)\in G(J_\delta(\Theta^\delta))$, it follows, by a classical argument

([B],p.27), that $g \in G(\Omega)$.

If we assume $h \in C^{0,1}(\bar{\Gamma}_H)$, by Lemma 2 one easily concludes that $\Theta \in C^{0,\alpha}(\bar{\Omega})$ for some $0<\alpha<1$. The proof is complete. ∎

REMARK 2 Assuming that there exists some $\nu \geq 0$ such that $0 \in G(\nu)$, one can find a more simple estimate in $L^\infty(\Omega)$ for every solution Θ to Problem $(\tilde{P})$:

$$\Theta \leq M = \max\left(\nu, \|h\|_{L^\infty(\Gamma_H)}\right) .$$

Indeed, it is sufficient to consider $\zeta=[\Theta-M]^+$ in (3.6) and to recall that the monotonicity of G implies $g \geq 0$ if $\Theta > M$. ∎

REMARK 3 The results of this section can be easily extended to the case of a lateral boundary condition

$$-\frac{\partial\Theta}{\partial n}(X) \in G(z,\Theta(X)) \ , \text{ for } X=(x,y,z) \in \Gamma_1 ,$$

where , for each $z \in]0,H[$, $G(z,\cdot)$ denotes a maximal monotone graph satisfying (3.2),(3.3) and ℓ in (3.14) being uniformly bounded in z.

An interesting case could be a lateral boundary submitted to N differents cooling zones, that is, when, for $i=1,\dots,N$,

$$G(z,\cdot)=G_i(\cdot),\ 0=z_0<\dots<z_{i-1}<z<z_i<\dots z_N=H. \ ∎$$

4. COMPARISON RESULTS

If the cooling is given by a monotone function one can adapt the technique of [BKS] to prove the

<u>PROPOSITION 2</u> Let Θ^{ε} (resp. $\bar{\Theta}^{\varepsilon}$) a solution to Problem (P_{ε}) and corresponding to g and h (resp. $\bar{g}$ and $\bar{h}$), where g and $\bar{g}$ are monotone functions satisfying (3.8) . Then if $\bar{h}\geq h$ and $\bar{g}\leq g$ it follows that $\bar{\Theta}^{\varepsilon}\geq\Theta^{\varepsilon}$.

Proof :

Set $f_{\delta}(t)=[1-\delta/t]^{+}$, $t \in \mathbb{R}$ and $\delta>0$.

From (2.7) and denoting $\eta=\Theta^{\varepsilon}-\bar{\Theta}^{\varepsilon}$, one has

$$\int_{\Omega}\nabla\eta\cdot\nabla\zeta=b\int_{\Omega}\{\eta+\lambda[\chi_{\varepsilon}(\Theta^{\varepsilon})-\chi_{\varepsilon}(\bar{\Theta}^{\varepsilon})]\}\zeta_{z}-\int_{\Gamma_1}[g(\Theta^{\varepsilon})-\bar{g}(\bar{\Theta}^{\varepsilon})]\zeta$$

for every $\zeta \in H^{1}(\Omega)$, $\zeta=0$ on $\Gamma_0 \cup \Gamma_H$. *In particular, for* $\zeta=f_{\delta}(\eta)$, which is different from zero if $\Theta^{\varepsilon}\geq\bar{\Theta}^{\varepsilon}$ where $g(\Theta^{\varepsilon})\geq g(\bar{\Theta}^{\varepsilon})\geq\bar{g}(\bar{\Theta}^{\varepsilon})$, it follows

$$(4.1) \qquad \left|\int_{\Omega}\nabla\eta\cdot\nabla f_{\delta}(\eta)\right| \leq b\, L_{\varepsilon}\int_{\Omega}|\eta|\cdot|[f_{\delta}(\eta)]_{z}|,$$

being L_{ε} the Lipschiz constant of $t \mapsto t+\lambda\,\chi_{\varepsilon}(t)$.

As in [BKS], (4.1) implies, for any $\delta>0$,

$$\int_{\Omega}\left|\log\left(1+\frac{[\eta-\delta]^{+}}{\delta}\right)\right|^{2} \leq C(=\text{const.independ.of } \delta)$$

from which it follows $\Theta^{\varepsilon}-\bar{\Theta}^{\varepsilon}=\eta\leq 0$. ∎

<u>REMARK 4</u>. This argument also proves the uniqueness of the solution of the Problem (P_{ε}) when g is monotone. Of course if Θ(resp.$\bar{\Theta}$) is a solution of (P) which is the limit of the subsequence $\Theta^{\varepsilon'}$ (resp.$\bar{\Theta}^{\varepsilon'}$) the above proposition implies that $\bar{\Theta}\geq\Theta$.

∎

Next we shall prove comparison results with respect to the extraction velocity b.

PROPOSITION 3 Assume that there exists constants μ, M such that

(4.2) $0 < \mu \leq h(x,y) \leq M$, a.a.$(x,y) \in \Gamma_H$.

and that the function g verifies (3.8) with

(4.3) $\{t : g(t)=0\} \subset [M,+\infty[$,

or else that g verifies (2.2,3,4,9). Then if $b \leq \frac{1}{H}\log(1+\frac{\mu}{\lambda})$ a solution Θ to Problem (P_1) is also a solution to Problem (P) with $\chi=1$.

Proof : If g satisfies (3.8), then the Problem (P_1) has a unique solution (let $\chi_\varepsilon \equiv 0$ in (3.9)). Moreover by (4.3) one has $g(\Theta) \leq 0$ (see Lemma 1).

Under assumptions (2.2,3,4,9) the existence of Θ may be shown essentialy as in Proposition 1, being also $g(\Theta) \leq 0$, by hypothesis.

Consider now the function $\Theta_\mu(z)=\mu(e^{bz}-1)(e^{bH}-1)^{-1}$. Taking $\zeta=(\Theta_\mu-\Theta)^+$ in (1.10) and since $g(\Theta) \leq 0$ in both cases, one easily finds that $\Theta \geq \Theta_\mu$. Therefore, if follows

$$\frac{\partial\Theta}{\partial n} \leq \frac{\partial\Theta_\mu}{\partial n} = -\mu b(e^{bH}-1)^{-1} \text{ on } \Gamma_o.$$

Using the Green's formula with a smooth function ζ such that $\zeta \geq 0$ on Γ_o and $\zeta=0$ on Γ_H, one has

$$\int_\Omega (\nabla\Theta\cdot\nabla\zeta + b\,\Theta_z\zeta - \lambda b\zeta_z) + \int_{\Gamma_1} g(\Theta)\zeta = \int_{\Gamma_o} \left(\frac{\partial\Theta}{\partial n} + \lambda b\right)\zeta \leq 0$$

for $\lambda b \leq \mu b(e^{bH}-1)^{-1}$. This means that, for all $bH \leq \log(1+\mu/\lambda)$, $(\Theta,1)$ is also a solution to Problem (P). ∎

This proposition suggests that, for small velocities b, the whole region Ω is occupied by solid metal, since if the Problem (P) admits only one solution Θ, one has $\Theta>0$ in Ω for $0<b\leq\frac{1}{H}\log(1+\mu/\lambda)$. Conversely the next proposition suggests that for big velocities the free boundary exists, since we will show that the volume of the set $\{\Theta>0\}$ vanishes when $b\uparrow\infty$.

PROPOSITION 4. Under assumptions of the Theorem 1 or Theorem 3

and denoting by $|\Omega_+|$ the Lebesque measure of the set $\Omega_+=\{X|\Theta(X)>0\}$, one has

$$(4.4) \qquad |\Omega_+| \leq \frac{C}{\lambda b},$$

where C is a positive constant independent of λ and b. Moreover, for b big enough, one has $\chi\not\equiv 1$.

Proof. Let $\zeta=H-z$ in (1.8) and in (3.6). One has

$$(4.5) \qquad -\int_\Omega \Theta_z + b\int_\Omega \Theta_z(H-z) + \lambda b\int_\Omega \chi + \int_{\Gamma_1} g(H-z) \leq 0,$$

where $g=g(\Theta)$ and $g\in G(\Theta)$, respectively. In the first case, g is a bounded function and from $0\leq\Theta\leq M$ (see Theorem 1 and Lemma 1), we may assume $-\ell_1\leq g\leq 0$, with ℓ_1 independent of b and λ. In the second one, by (3.17) and (3.14) we have $\|g\|_{L^\infty}\leq\ell$ and ℓ is also independent of b and λ.

Denoting $L=\max(\ell,\ell_1)$ from (4.5) it follows that

$$\lambda b \int_\Omega \chi \leq \int_{\Gamma_H} h + L \int_{\Gamma_1} (H-z),$$

since one has

$$\int_\Omega \Theta_z = \int_{\Gamma_H} h \quad \text{and} \quad \int_\Omega \Theta_z (H-z) = \int_\Omega \Theta \geq 0 .$$

Recalling that $0 \leq \chi \leq 1$ and $\chi = 1$ in Ω_+, one has

$$|\Omega_+| \leq \int_\Omega \chi \leq |\Gamma| \; (M + LH^2/2) \; / \lambda b,$$

which completes the proof of the proposition . ∎

Now we assume the existence of d, $0<d<H$, such that

$$(4.6) \qquad g(X,\rho,\theta) = 0 \quad \text{for} \quad 0<z<d, \; \forall (X,\rho,\theta) \in \Gamma_1 \times \mathbb{R}_+ \times \mathbb{R}$$

or, for the monotone case (see Remark 3),

$$(4.7) \qquad G(z,.) \equiv 0 \quad \text{for } 0<z<d<H .$$

<u>THEOREM 4</u> . Let (Θ,χ) (resp. (Θ,χ,g)) a solution to Problem (P) (resp. $(\tilde{P})$) under assumptions of Theorem 1 with (4.6) (resp. Theorem 3 with (4.7)). Then there exists δ, $0<\delta<d$, such that

$$(4.8) \qquad \Theta(x,y,z) \leq \lambda b [z-\delta]^+, \quad \forall (x,y,z) \in \bar{\Omega}$$

$$(4.9) \qquad \Theta = \chi = 0 \quad \text{for} \quad 0<z<\delta,$$

for all $b > M/\lambda d$, where $M \geq \|\Theta\|_{L^\infty}$ is a constant independent of b (see (2.10) and (3.15)).
The proof of this theorem uses the following lemma.

LEMMA 3. Under assumptions of Theorem 4, one has

$$\int_{Z_\delta} \chi(\lambda b\chi-\Theta_z) \leq \int_{Z_\delta} (b\Theta+\lambda b\chi-\Theta_z) \leq 0 \tag{4.10}$$

for $0<\delta\leq d$ and $Z_\delta=\{(x,y,z)\in \Omega \mid 0<z<\delta\}$.

Proof : Let $\zeta=[\delta-z]^+$ in (1.8) or in (3.6) . One has

$$\int_{Z_\delta} [-\Theta_z+b\Theta_z(\delta-z)+\lambda b\chi] \leq 0,$$

because (4.6) or (4.7) imply $g[\delta-z]^+=0$. Since

$$\int_{Z_\delta} \Theta_z(\delta-z) = \int_{Z_\delta} \Theta \geq 0 \quad \text{and} \quad 0\leq\chi\leq 1$$

it follows

$$\int_{Z_\delta} \chi(\lambda b\chi-\Theta_z) \leq \int_{Z_\delta} (\lambda b\chi-\Theta_z) \leq \int_{Z_\delta} (b\Theta+\lambda b\chi-\Theta_z) \leq 0. \quad \blacksquare$$

PROOF OF THEOREM 4. ; Consider $\mu=\mu(z)=\lambda b[z-\delta]^+$ with δ fixed such that $0<\delta\leq d-M/\lambda b$. The function $\zeta=[\Theta-\mu]^+$ vanishes on $z=0$ and for $z\geq d$. Therefore $g[\Theta-\mu]^+=0$ and from (1.8) or from (3.6) , one has

$$\int_\Omega \nabla\Theta\cdot\nabla[\Theta-\mu]^+ + b\int_\Omega \Theta_z[\Theta-\mu]^+ - \lambda b \int_\Omega \chi[\Theta-\mu]^+_z \leq 0$$

or

$$\int_{Z_\delta} (|\nabla\Theta|^2-\lambda b\chi\Theta_z) + \int_{(\Omega\setminus Z_\delta)\cap\{\Theta>0\}} \{\nabla\Theta\cdot\nabla[\Theta-\mu]^+ - \lambda b[\Theta-\mu]^+_z\} + b\int_\Omega \Theta_z[\Theta-\mu]^+ \leq 0 .$$

Adding the quantity

$$\lambda b\int_{Z_\delta}\chi(\lambda b\chi-\Theta_z) - b\int_{\Omega\setminus Z_\delta}\lambda b[\Theta-\mu]^+$$

which is non-positive by Lemma 3, one obtains

$$\int_{Z_\delta}\{\Theta_x^2+\Theta_y^2+(\Theta_z-\lambda b\chi)^2\}+\int_{\Omega\setminus Z_\delta}|\nabla[\Theta-\mu]^+|^2+b\int_\Omega(\Theta-\mu)_z[\Theta-\mu]^+\leq 0.$$

Since the last term is zero, if follows that $\Theta\leq\mu$ in $\Omega\setminus Z_\delta=\{z\geq\delta\}$ and $\Theta_x=\Theta_y=0$, $\Theta_z=\lambda b\chi$ in $Z_\delta=\{z<\delta\}$. Since $\Theta=0$ for $z=0$ and $z=\delta$, one has $\Theta=0$ for $z\leq\delta$ and consequently also $\chi=0$ for $z\leq\delta$. ∎

5. REGULARITY OF THE FREE BOUNDARY

The goal of Theorem 4 is to provide sufficient conditions in order to assume the global existence of a free boundary. In this case we shall prove that the free boundary is an analytic surface.

We begin with the following

PROPOSITION 5. A solution (Θ,χ) (resp. (Θ,χ,g)) to Problem (P) (resp. $(\tilde{P})$) satisfies

(5.1) $$-\Delta\Theta+b\Theta_z+\lambda b\,\chi_z = 0 \quad \text{in} \quad \mathcal{D}'(\Omega),$$

(5.2) $$\chi_z \geq 0 \quad \text{in} \quad \Omega.$$

Proof: The equation (5.1) follows immediatly by taking $\zeta \in \mathcal{D}(\Omega)$ in (1.8) or in (3.6)

Choosing as a test function in (1.8) or in (3.6) $\zeta = \min(\Theta, \varepsilon\eta)$, where $\varepsilon > 0$ and $\eta \in \mathcal{D}(\Omega), \eta \geq 0$ one has

$$I = \int_\Omega \nabla\Theta \cdot \nabla \min(\Theta,\varepsilon\eta) + b\int_\Omega \Theta_z \min(\Theta,\varepsilon\eta) - \lambda b \int_\Omega [\min(\Theta,\varepsilon\eta)]_z \leq 0$$

since $\chi = 1$ in $\{\Theta > 0\}$. Since $\min(\Theta,\varepsilon\eta) = 0$ on $\partial\Omega$, the last integral is zero and it follows

$$I = \int_{\{\Theta \leq \varepsilon\eta\}} |\nabla\Theta|^2 + \varepsilon \int_{\{\Theta > \varepsilon\eta\}} \nabla\Theta \cdot \nabla\eta + b\int_\Omega \{\varepsilon\eta\Theta_z + \Theta_z\,[\min(\Theta,\varepsilon\eta) - \varepsilon\eta]\}$$

$$\geq \varepsilon \int_{\{\Theta > \varepsilon\eta\}} \nabla\Theta \cdot \nabla\eta + \varepsilon b \int_\Omega \Theta_z \eta - b \int_\Omega \Theta_z [\varepsilon\eta - \Theta]^+ ,$$

from which one concludes

$$\int_\Omega \chi_{\{\Theta > \varepsilon\eta\}} \nabla\Theta \cdot \nabla\eta + b\int_\Omega \Theta_z \eta \leq b \int_\Omega \Theta_z [\eta - \tfrac{\Theta}{\varepsilon}]^+ .$$

Passing to the limit $\varepsilon \searrow 0$, one obtains

$$\int_\Omega (\nabla\Theta \cdot \nabla\eta + b\Theta_z \eta) \leq 0, \qquad \forall\ \eta \in \mathcal{D}(\Omega) : \eta \geq 0$$

and using (5.1), one deduces (5.2). ∎

From (5.1) it follows that the function Θ is locally

Hölder continuous. Therefore the set

$$(5.3) \qquad \Omega_+ \equiv \{X \in \Omega \mid \Theta(X)>0\}$$

is an open set. Since χ is monotonous increasing in the z-coordinate one can introduce

$$(5.4) \qquad \phi(x,y) = \inf \{z : \Theta(x,y,z)>0, \ (x,y,z)\in\Omega\}$$

where ϕ is an upper semi-continuous function, by the continuity of Θ. Then we can state.

<u>THEOREM 5.</u> For any solution of Problem (P) or ($\tilde{P}$) one has

$$(5.5) \qquad \Omega_+ \equiv \{\Theta>0\} = \{X\in\Omega : z>\phi(x,y)\}$$

where ϕ is an upper semi-continuous function given by (5.4) ∎

<u>COROLLARY 1.</u> Under conditions of Theorem 4, for all $b>M/\lambda d$, one has

$H > \phi(x,y) \geq d-M/\lambda d > 0$, for all $(x,y)\in \Gamma$, which, in particular, assures the existence of a free boundary. ∎

Consider now the function

$$(5.6) \qquad u(x,y,z) = \int_0^z \Theta(x,y,t)\,dt, \text{ for } (x,y,z) \in \overline{\Omega},$$

which is a Baiocchi type transformation (see [BC] for instance).

<u>THEOREM 6.</u> Let (Θ,χ) (resp. (Θ,χ,g)) be a solution to Problem (P) (resp. $(\tilde{P})$) under the assumptions of Theorem 4. Then the function u defined by (5.6) satisfies the following variational inequality in Ω

$$u \geq 0, \quad (-\Delta u+bu_z+\lambda b) \geq 0, \quad u\cdot(-\Delta u+bu_z+\lambda b)=0, \tag{5.7}$$

and χ is a characteristic function, being

$$\chi=\chi(\Theta)=\chi(u) \quad \text{a.e. in } \Omega \tag{5.8}$$

where $\chi(v)$ denotes the characteristic function of the set $\{v>0\}$.

Proof : From definition (5.6) and recalling $\Theta \geq 0$ it is obvious that $u \geq 0$. Since $\Theta=u_z$ and Θ satisfies (5.1) one has

$$(-\Delta u+bu_z+\lambda b\chi)_z = -\Delta\Theta+b\Theta_z+\lambda b\chi_z = 0$$

which, together with (4.9) and $0 \leq \chi \leq 1$, imply

$$0= -\Delta u+bu_z+\lambda b\chi \leq -\Delta u+bu_z+\lambda b. \tag{5.9}$$

Recalling (5.5) it is clear that

$$\{\Theta>0\} = \{u>0\} \tag{5.10}$$

from which one deduces $\chi=1$ if $u>0$, and the third condition of (5.7) follows by (5.9).

From the classical regularity to solutions of variational inequalities one has

(5.11) $\qquad u \in W^{2,\infty}_{loc}(\Omega)$ (see [KS], for instance) and (5.8)

follows easily from (5.9) and (5.10). ∎

REMARK 5 If one considers a linear flux

$$(5.12) \quad g(X,\rho(X),\theta(X)) = \alpha(z)\ [\theta(X)-\rho(X)]$$

with $\rho\geq 0$ and $\alpha(z)=0$ for $0< z <d$ and $\alpha(z)=\alpha=\text{const.}>0$ for $d< z <H$, then we have that u is the unique solution of the following variational inequality with mixed boundary conditions (see [Br] and [R]):

$$u \in \mathbb{K} = \{v\in H^1(\Omega) \mid \ v\geq 0 \text{ in } \Omega,\ v=0 \text{ on } \Gamma_0\}$$

$$\int_\Omega \nabla u\cdot\nabla(v-u)+b\int_\Omega u_z(v-u)+\alpha\int_{\Gamma_1} u(v-u)\geq\int_{\Gamma_H} h(v-u)-\lambda b\int_\Omega (v-u)+\alpha\int_{\Gamma_1}\tilde{\rho}(v-u),$$

$$\forall\ v \in \mathbb{K},$$

where $\tilde{\rho}(z)=\int_d^z \rho(t)dt$ for $z\geq d$.

In particular, this implies the uniqueness of the solution of Problem (P) for a linear cooling given by (5.12). ∎

The transformation (5.6) and its consequence (5.8) allow us to include the study of the free boundary

$$\Phi = \Omega \cap \partial\Omega_+$$

in the known results of Caffarelli [C] Kinderlehrer and Nirenberg [KN]. In order to apply these results we must show that Φ has not singular points. This may be done by using a technique due to Alt [Aℓ] for the dam problem.

LEMMA 4. Let $X_0\in\Phi$ and $B_r(X_0)\subset \Omega$. Then there is a cone $\Lambda_r\subset\{X\in\mathbb{R}^3 \mid z<0\}$. such that

$$(5.13) \quad \frac{\partial u}{\partial \eta}(X)=\nabla u(X).\eta\leq 0 \quad \text{for } X\in B_{r/2}(X_0), \text{ whenever } \eta\in\Lambda_r\cap S^2.$$

Proof: Recalling (5.11) and that $u_z=\Theta\geq 0$ in Ω, the proof of this lemma is a simple adaptation of Lemma 6.9 of [KS], page 255, and therefore we omit it. ∎

THEOREM 7. Let (Θ,χ) (resp. (Θ,χ,g)) be a solution to Problem (P) (resp. $(\tilde{P})$) under conditions of Theorem 4. Then the free boundary Φ is an analytic surface given by

$$\Phi:\ z=\phi(x,y) \text{ for } (x,y)\in\Gamma,$$

and Θ is also a classical solution of Problem (C).

Proof: By (5.13) the function ϕ defined by (5.4) is a lipschitz function in Γ and we can apply Theorem 3 of [C] to conclude that (5.14) ϕ is C^1 and $u\in C^2(\Omega_+ \cup \Phi)$. Therefore from equation (5.1) and Green's formula one finds that condition (1.5) is verified in every point of the free boundary $z=\phi(x,y)$, for all $(x,y)\in\Gamma$, by Corollary 1.

To conclude that Φ is an analytic surface it is sufficient to apply Theorem 1 of [KN], using (5.14) and recalling that the equation satisfied by u in Ω_+ has constant coefficients. ∎

6. UNICITY IN THE MONOTONE CASE

In Remark 5 we have already stated the uniqueness of the solution of Problem (P) with a particular linear cooling.

Adapting to our problem the technique of Carrillo and Chipot ([CC]) we shall prove an uniqueness result for the maximal monotone case assuming that χ is a characteristic function, that is, assuming

$$\chi=\chi(\Theta), \tag{6.1}$$

to which we have already stated sufficient conditions in Theorems

4 and 6.

Denote by (Θ_i,χ_i,g_i), with $\chi_i=\chi(\Theta_i)$ and $g_i \in G(\Theta_i)$, for i=1,2, two solutions of the Problem $(\tilde{P})$ and set

$$\Theta_0=\min(\Theta_1,\Theta_2),\ \chi_0=\min(\chi_1,\chi_2),\ \phi_0=\sup(\phi_1,\phi_2).$$

<u>LEMMA 5.</u> Assuming (6.1), one has

$$(6.2)\qquad \int_\Omega \{\nabla(\Theta_i-\Theta_0).\nabla\eta+b(\Theta_i-\Theta_0)_z\eta-\lambda b(\chi_i-\chi_0)\eta_z\}dx\,dy\,dz$$

$$\leq \lambda b\int_{D_i}\eta(x,y,\phi_i(x,y))dx\,dy$$

for any $\eta \in H^1(\Omega)\cap C^0(\bar{\Omega})$, $\eta\geq 0$, where

$$D_i=\{(x,y)\in\Gamma \mid \phi_i(x,y)<\phi_0(x,y)\}\quad ,\ i=1,2.$$

Proof: Choosing the test functions $\pm\zeta=\pm\min(\Theta_i-\Theta_0,\varepsilon\eta)$, $\varepsilon>0$, from (3.6) one obtains for $i\neq j$ $(i,j=1,2)$

$$\int_\Omega\{\nabla(\Theta_i-\Theta_j)\cdot\nabla\zeta+b(\Theta_i-\Theta_j)_z\zeta-\lambda b\,(\chi_i-\chi_j)\zeta_z\}+\int_{\Gamma_1}(g_i-g_j)\zeta=0.$$

By the monotonicity of G, one has

$$\int_{\Gamma_1}(g_i-g_j)\min(\Theta_i-\Theta_0,\varepsilon\eta)\geq 0$$

since it is sufficient to integrate in $\{\Theta_i>\Theta_0\}$ where $\Theta_j=\Theta_0$.

Then it follows

$$\int_\Omega\{\nabla(\Theta_i-\Theta_0).\nabla\min(\Theta_i-\Theta_0,\varepsilon\eta)+b(\Theta_i-\Theta_0)_z\min(\Theta_i-\Theta_0,\varepsilon\eta)$$

$$-\lambda b(\chi_i-\chi_0)\,[\min(\Theta_i-\Theta_0,\varepsilon\eta)]_z\}\leq 0$$

or, using $\min(u,v)=v-[v-u]^+$,

$$\int_{\{\Theta_i-\Theta_o>\varepsilon\eta\}} \nabla(\Theta_i-\Theta_o)\cdot\nabla\eta + b\int_\Omega (\Theta_i-\Theta_o)_z\eta + \lambda b(\chi_i-\chi_o)\,\eta_z$$

$$\leq b\int_\Omega \{(\Theta_i-\Theta_o)_z\left[\eta-\frac{\Theta_i-\Theta_o}{\varepsilon}\right]^+ - \lambda(\chi_i-\chi_o)\left[\eta-\frac{\Theta_i-\Theta_o}{\varepsilon}\right]^+_z\}\ .$$

Since the χ_i are characteristic functions, integrating in z, one has

$$-\int_\Omega (\chi_i-\chi_o)\left[\eta-\frac{\Theta_i-\Theta_o}{\varepsilon}\right]^+_z = -\int_{\{\phi_i<z<\phi_o\}}\left[\eta-\frac{\Theta_i-\Theta_o}{\varepsilon}\right]^+_z \leq \int_{D_i}\left[\eta-\frac{\Theta_i}{\varepsilon}\right]^+(x,y,\phi_i)\leq\int_{D_i}\eta\,(x,y,\phi_i)$$

and (6.2) follows by passing to the limite $\varepsilon\searrow 0$ in

$$\int_{\{\Theta_i-\Theta_o>\varepsilon\eta\}}\nabla(\Theta_i-\Theta_o)\,.\nabla\eta+b\int_\Omega[(\Theta_i-\Theta_o)_z\eta-\lambda(\chi_i-\chi_o)\eta_z] \leq$$

$$\leq b\int_\Omega(\Theta_i-\Theta_o)_z\left[\eta-\frac{\Theta_i-\Theta_o}{\varepsilon}\right]^+ + \lambda b\int_{D_i}\eta(x,y,\phi_i)\,.\ \blacksquare$$

<u>THEOREM 8.</u> Assuming (6.1) , the Problem $(\tilde{P})$ has at most one solution.

Proof : For $\varepsilon>0$, consider a smooth function α_ε, such that, $0\leq\alpha_\varepsilon\leq 1$, and

$$\alpha_\varepsilon=1 \ \text{ in } \ A_o=\{\Theta_o>0\}\cup\Gamma_1 \text{ and } \alpha_\varepsilon(X)=0 \text{ if } d(X,A_o)>\varepsilon.$$

Since $1-\alpha_\varepsilon=0$ on $\{\Theta_o>0\}$, for all $\eta\in H^1(\Omega)$, one has

$$\int_\Omega\{\nabla\Theta_o\cdot\nabla(1-\alpha_\varepsilon)\eta+b\Theta_{oz}(1-\alpha_\varepsilon)\eta-\lambda b\chi_o\left[(1-\alpha_\varepsilon)\eta\right]_z\} = 0.$$

For $\eta \in H^1(\Omega)\cap C^0(\bar{\Omega})$, $\eta \geq 0$, $\zeta=(1-\alpha_\varepsilon)\eta$ is a test function in (3.6), and it follows (since $1-\alpha_\varepsilon=0$ on Γ_1)

$$\int_\Omega \{\nabla(\Theta_i-\Theta_0).\nabla(1-\alpha_\varepsilon)\eta+b(\Theta_i-\Theta_0)_z\ (1-\alpha_\varepsilon)\eta-\lambda b(\chi_i-\chi_0)[(1-\alpha_\varepsilon)\eta]_z\}$$

$$\leq 0 \quad (i=1,2).$$

Using (6.2), we obtain

$$\int_\Omega \{\nabla(\Theta_i-\Theta_0).\nabla\eta+b(\Theta_i-\Theta_0)_z\eta-\lambda b(\chi_i-\chi_0)\eta_z\} \leq \lim_{\varepsilon\downarrow 0} \lambda b\int_{D_i}(\alpha_\varepsilon\eta)(x,y,\phi(x,y))=0.$$

Choosing in this inequality $\eta=z$ and $\eta=H-z$, after a simples calculation one obtains

$$\int_\Omega(\Theta_i-\Theta_0) + \lambda\int_\Omega(\chi_i-\chi_0) = 0,$$

from where one deduces $\Theta_i=\Theta_0$ and $\chi_i=\chi_0$, for i=1,2, which proves the uniqueness of the solution. ∎

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A steady-state one phase Stefan problem corresponding to the solidification process of an ingot of pure metal by continuous casting with a non-linear lateral cooling is considered via the weak formulation introduced in (BKS) for the dam problem. Two existence results are obtained for a general non-linear flux and for a maximal monotone flux. Comparison results and the regularity of the free boundary are discussed. An uniqueness theorem is given for the monotone case.

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