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ANOTHER STRATEGY FOR FAST 'POISSON' SOLVING WITH NON-CONSTANT C--ETC(U)
APR 82 J P BORIS
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The key to fast Poisson solving in an extended domain is nonlocal residual error dispersal. The fastest, most stable iterations are obtained when "most" of the nonconstant-coefficient elliptic operator can be inverted implicitly. Although local inversion approaches (ICCG, SOR, etc.) are workable, there are good incentives to seek computationally inexpensive nonlocal techniques whose worst case convergence is expected to be far better than for local inverse techniques. This note presents one such approach for a useful class of variable coefficient problems.

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ANOTHER STRATEGY FOR FAST "POISSON" SOLVING WITH NON-CONSTANT COEFFICIENTS

I. The Introduction

The key to fast Poisson solving in an extended domain is nonlocal residual error dispersal. On a two-dimensional domain characterized by N grid points in one direction (taken as x below) and M grid points in another (taken as y), the general finite difference equation could be solved in $\sim (NM)^3$ operations by matrix inversion. Because the matrix is so sparse, however, faster algorithms are sought. For a number of useful cases, usually characterized by cyclic reduction, Fourier transforms, or similar folding techniques, fast NM $\log_2$NM algorithms exist and have been in use for some time. These direct solution methods are non-iterative and disperse the errors globally and correctly in one application because the complete elliptic operator can be inverted.

When complex stretched grids are used or diffusion/dielectric coefficients vary generally in space, the nature of the physical solutions does not change but the variable nature of the finite-difference coefficients in the resulting elliptic equations obviates the use of direct methods. Although local inversion approaches such as ICCG, SOR, etc. are often useful, there are still good reasons to seek computationally inexpensive nonlocal techniques whose worst case convergence is expected to be far better than for local inverse techniques.

A nonlocal technique spreads residual error at each point all over the mesh in a single iteration cycle. The analytic formulation for this involves

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the manipulation of the Green's function. Numerically, the use of Green's function in a general case can be prohibitively expensive, so cheaply manipulated approximate Green's functions are sought. It is not necessary for an algorithm to be local, to be fast.

II. The Idea

This short note contains an idea for fast cheap "Poisson" solving on a rectangular domain. Consider a finite-difference formulation of the equation

$$\underset{\sim}{\nabla}\cdot\tilde{\nu}(x,y)\,\underset{\sim}{\nabla}\tilde{P}(x,y) - \tilde{\omega}(x,y)\,\tilde{P}(x,y) = \tilde{S}(x,y) \qquad (1)$$

where $\tilde{\nu}(x,y) > 0$, $\tilde{\omega}(x,y) \geq 0$ and (x,y) in the domain $(0 \leq x \leq x_{max}, 0 \leq y \leq y_{max})$. Heat transport, special treatments of fluid dynamics, and electromagnetic propagation in varying media all satisfy similar equations. For the scalar field $\{P_{ij}\}$ $(1 \leq i \leq N,\ 1 \leq j \leq M)$, the following finite difference equation plus well posed boundary conditions approximates (1),

$$\begin{aligned} &\nu^x_{i+1/2j}(P_{i+1j} - P_{ij}) - \nu^x_{i-1/2j}(P_{ij} - P_{i-1j}) - \omega_{ij}P_{ij} + \\ &\nu^y_{ij+1/2}(P_{ij+1} - P_{ij}) - \nu^y_{ij-1/2}(P_{ij} - P_{ij-1}) = S_{ij}. \end{aligned} \qquad (2)$$

Suppose $\{P_{ij}\}$ is unknown with $\{\omega_{ij}\}$, $\{S_{ij}\}$, $\{\nu^x_{i+1/2j}\}$, and $\{\nu^y_{ij+1/2}\}$ given. A new set of unknowns $\{\phi_{ij}\}$ can be defined by the substitution in Eq. (2),

$$P_{ij} = \Psi_{ij}\phi_{ij},\ (i = 1, \ldots, N;\ j = 1, \ldots, M). \qquad (3)$$

The purpose of the transformation is to provide a set of adjustable coefficients $\{\Psi_{ij}\}$ which will be optimized with reference to the coefficients $\{\nu^x_{i+1/2j}\}$, $\{\nu^y_{ij+1/2}\}$, $\{\omega_{ij}\}$ but not with respect to $\{S_{ij}\}$ which may change from one timestep to the next.

By appropiate choice of $\{\psi_{ij}\}$, the coefficients in the derived Poisson-like elliptic equation for $\{\phi_{ij}\}$ can be made as nearly constant as possible. Since constant coefficient problems can be solved using fast algorithms, the non-constant portion of the coefficients are minimized row by row and added as a residual to the source term on the right hand side. The object equation derived from (2) by substituting (3) is

$$\nu^x_{i+1/2j}\,\psi_{i+1j}\,\phi_{i+1j} + \nu^x_{i-1/2j}\,\psi_{i-1j}\,\phi_{i-1j} + \nu^y_{ij+1/2}\,\psi_{ij+1}\,\phi_{ij+1} + \nu^y_{ij-1/2}\,\psi_{ij-1}\,\phi_{ij-1} - \psi_{ij}(\nu^x_{i+1/2j} + \nu^x_{i-1/2j} + \nu^y_{ij+1/2} + \nu^y_{ij-1/2} + \omega_{ij})\,\phi_{ij} = S_{ij}\,. \tag{4}$$

For each row j five constants $(a_j, b_j, c_j, d_j, e_j)$ are to be chosen in conjunction with the as yet undetermined $\{\psi_{ij}\}$ to minimize the following error measure:

$$E = \sum_{j=1}^{M} E_j = \alpha \sum_{j=1}^{M}\sum_{i=1}^{N} [a_j - \nu^x_{i+1/2j}\,\psi_{i+1j}]^2 + \gamma \sum_{j=1}^{M}\sum_{i=1}^{N} [c_j - \nu^x_{i-1/2j}\,\psi_{i-1j}]^2 + \delta \sum_{j=1}^{M}\sum_{i=1}^{N} [d_{j+1} - \nu^y_{ij+1/2}\,\psi_{ij+1}]^2 + \epsilon \sum_{j=1}^{M}\sum_{i=1}^{N} [e_{j-1} - \nu^y_{ij-1/2}\,\psi_{ij-1}]^2 + \sum_{j=1}^{M}\sum_{i=1}^{N} [b_j - (\text{---})_{ij}\,\psi_{ij}]^2 \tag{5}$$

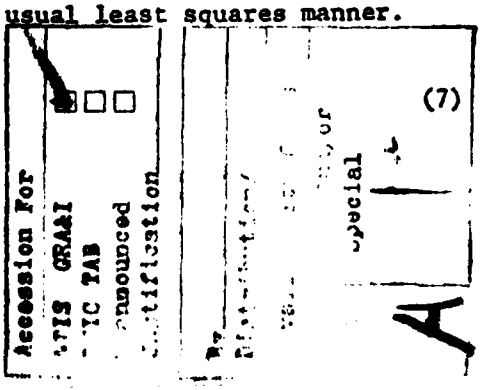
where I have designated

$$(\text{---})_{ij} \equiv (\nu^x_{i+1/2j} + \nu^x_{i-1/2j} + \nu^y_{ij+1/2} + \nu^y_{ij-1/2} + \omega_{ij}). \tag{6}$$

The undetermined row constants $\{a_j\}$, $\{c_j\}$, $\{d_j\}$, and $\{e_j\}$, along with the undetermined potential multipliers $\{\psi_{ij}\}$ satisfy a set of equations derived by setting the derivatives of E to zero in the usual least squares manner.

From

$$\frac{\partial E}{\partial \psi_{ij}} = 0 \tag{7}$$



one finds

$$\psi_{ij} = F_{ij}\,[\alpha\,\nu^{x}_{i-1/2j}\,a_j + \gamma\nu^{x}_{i+1/2j}\,c_j + \delta\,\nu^{y}_{ij-1/2}\,d_j + \epsilon\,\nu^{y}_{ij+1/2}\,e_j + (\text{---})_{ij}], \quad (8)$$

where the freedom to choose the multiplicative scale of ψ_{ij} in Eq. (3) permits us to set $\{b_j\} = 1$ without loss of generality. Here I have defined

$$F_{ij} \equiv 1/[\alpha(\nu^{x}_{i-1/2j})^2 + \gamma(\nu^{x}_{i+1/2j})^2 + \delta(\nu^{y}_{ij-1/2})^2 + \epsilon(\nu^{y}_{ij+1/2})^2 + (\text{---})^2_{ij}]. \quad (9)$$

The row constants are also found by minimizing E . Setting $\partial E/\partial a_j = 0$, $\partial E/\partial b_j = 0$, etc. gives the additional four equations for $\{a_j\}$, $\{c_j\}$, $\{d_j\}$ and $\{e_j\}$.

$$Na_j = \sum_{i=1}^{N} \nu^{x}_{i+1/2j}\,F_{i+1j}\,[\alpha\nu^{x}_{i+1/2j}\,a_j + \gamma\nu^{x}_{i+3/2j}\,c_j + \delta\nu^{y}_{i+1j-1/2}\,d_j + \epsilon\nu^{y}_{i+1j+1/2}\,e_j + (\text{---})_{i+1j}], \quad (10a)$$

$$Nc_j = \sum_{i=1}^{N} \nu^{x}_{i-1/2j}\,F_{i-1j}\,[\alpha\nu^{x}_{i-3/2}\,a_j + \gamma\nu^{x}_{i-1/2}\,c_j + \delta\nu^{y}_{i-1j-1/2}\,d_j + \epsilon\nu^{y}_{i-1j+1/2}\,e_j + (\text{---})_{i-1j}], \quad (10c)$$

$$Nd_j = \sum_{i=1}^{N} \nu^{y}_{ij-1/2}\,F_{ij}\,[\alpha\nu^{x}_{i-1/2j}\,a_j + \gamma\nu^{x}_{i+1/2j}\,c_j + \delta\nu^{y}_{i-1/2}\,d_j + \epsilon\nu^{y}_{ij+1/2}\,e_j + (\text{---})_{ij}], \text{ and} \quad (10d)$$

$$Ne_j = \sum_{i=1}^{N} \nu^{y}_{ij+1/2}\,F_{ij}\,[\alpha\nu^{x}_{i-1/2j}\,a_j + \gamma\nu^{x}_{i+1/2j}\,c_j + \delta\nu^{y}_{ij-1/2}\,d_j + \epsilon\nu^{y}_{ij+1/2}\,e_j + (\text{---})_{ij}]. \quad (10e)$$

These four linear inhomogeneous equations for a_j, c_j, d_j and e_j can be solved without reference to the results derived from any other row. A block tridiagonal system does not result.

Given these row coefficients, which can be seen not to depend on the source term $\{S_{ij}\}$, Eq. (8)yields the multiplicative coefficients $\{\psi_{ij}\}$ for each of the unknown potential values $\{\phi_{ij}\}$. The finite-difference equation which results to describe $\{\phi_{ij}\}$ has nearly constant coefficients, row by row, the complicating spatial variation being largely cancelled out by the derived variations of $\{\psi_{ij}\}$.

III. The Implementation

This algorithm has not yet been implemented numerically. Using the row constants derived from Eqs. (10) and the multiplication coefficients $\{\psi_{ij}\}$, the left hand side of Eq. (1) can be expanded as follows:

$$\underline{\nabla}\cdot\tilde{\underline{v}}(x,y)\ \underline{\nabla}\,\tilde{P}(x,y) - \tilde{\omega}(x,y)\,\tilde{P}(x,y) = D^2\phi(x,y) + [\nabla\cdot\tilde{\underline{v}}\ \nabla\tilde{\psi}\,\phi(x,y) - \tilde{\omega}\,\tilde{\psi}\,\phi(x,y) - D^2\phi(x,y)]. \tag{11}$$

The operator D^2, by construction, is Fourier-transformable in the x direction and results in decoupled tridiagonal equations in the y direction. The second term on the right in (11), which I shall call R^2, has been minimized in a simple least squares sense. Rewriting Eq. (1) gives

$$D^2(x,y)\,\phi^{m+1}(x,y) = \tilde{S}(x,y) - R^2(x,y)\,\phi^m(x,y) \tag{12}$$

as an iterative form for solving (1) which should converge rapidly. The superscript m indicates which level of iteration has been passed. The right hand side of (12) is initialized using $\phi^o(x,y)$, a guess at the expected potential.

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