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QUASILINEARIZATION SOLUTION OF THE PROPORTIONAL NAVIGATION PROBLEM

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An approximate solution is presented in this report for the trajectory of a constant-speed missile following a planar intercept trajectory based on proportional navigation guidance. The technique of quasilinearization is used to generate a sequence of solutions. An explicit solution is developed using the first recursive solution. The accuracy of this solution is discussed in the context of specific sets of parameters. It is shown that this explicit solution provides excellent accuracy over a wide range of intercept trajectories.

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INTRODUCTION

Many current interceptor systems use a guidance system based on proportional navigation. In a proportional navigation scheme, the direction of the interceptor is controlled in such a way as to make the rate of rotation of the interceptor velocity vector proportional to the rate of rotation of the line-of-sight (LOS) from the interceptor to the target [1], [2]. The derivation of the proportional navigation equations is carried out below.

From time to time in the analysis and design of automatic control systems, it is desirable or even necessary to have a closed form analytic solution for nominal system response. Determination of optimal control laws and stability studies often require that such nominal solutions be available. In the present paper, we obtain a closed form approximate solution to the classical proportional navigation problem. This solution is obtained through the application of quasilinearization to the governing nonlinear differential equations. As it will be demonstrated, this analytic solution exhibits very good accuracy when compared to numerical solutions of the nonlinear equations of motion.

DERIVATION OF EQUATIONS

In the following paragraphs, we derive the differential equations governing the trajectory of an interceptor following a proportional navigation guidance law. The geometry of the target and interceptor are shown in Figure 1. The interception is assumed to take place in a plane. The position of the interceptor relative to a fixed reference frame is given by the position vec-

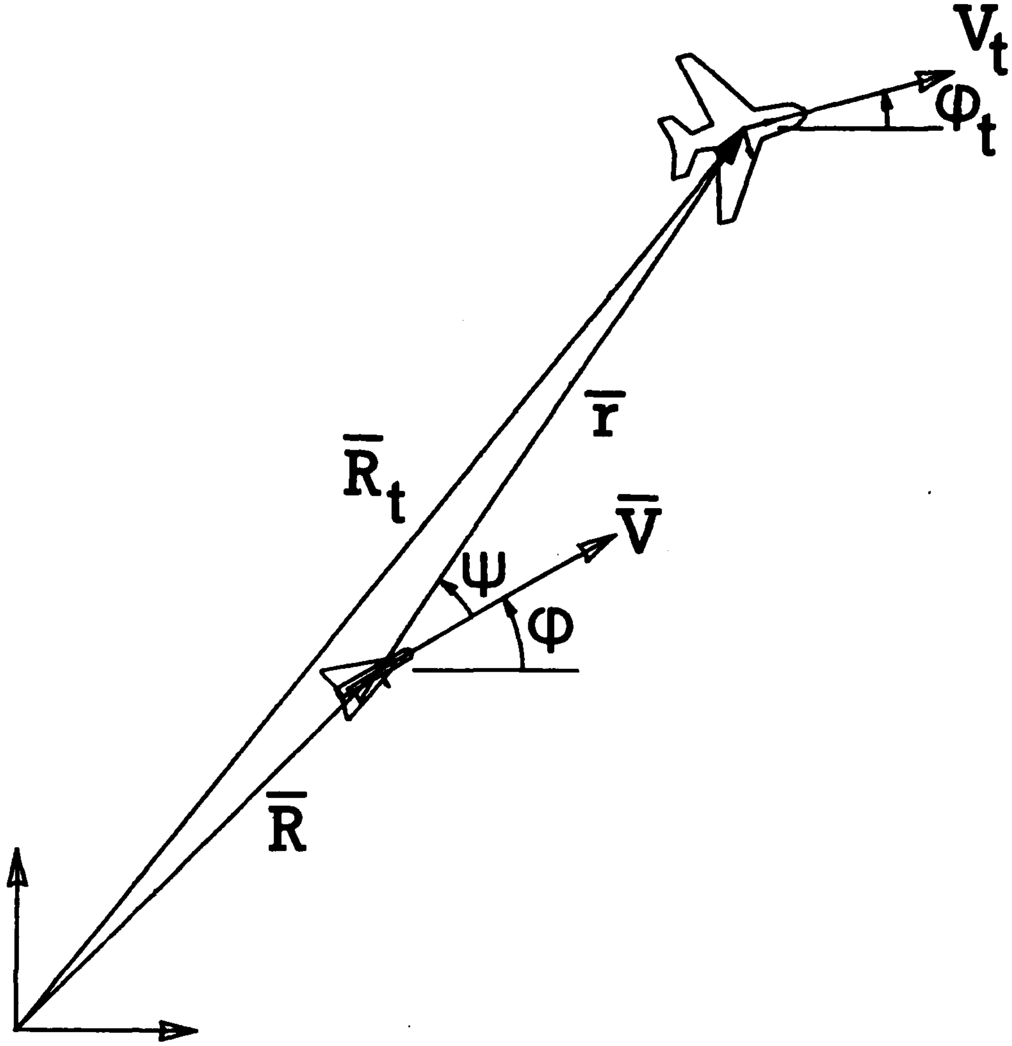

Figure 1. Geometry of interceptor and target.

tor $\overline{R}$. The target is located at $\overline{R}_T$. The position of the target relative to the interceptor is $\overline{r}$. The velocities of the interceptor and target are $\overline{V}$ and $\overline{V}_T$ respectively. As shown in Figure 1., the heading angle of the interceptor is ϕ and that of the target is ϕ_T. The angle between the velocity $\overline{V}$ and the LOS, $\overline{r}$ is ψ . Since

$$\overline{r} = \overline{R}_T - \overline{R} \quad , \tag{1}$$

$$\frac{d\overline{r}}{dt} = \overline{v} = \overline{V}_T - \overline{V} \quad . \tag{2}$$

Writing the relative velocity vector $\overline{v}$ in polar coordinates, we find:

$$\overline{v} = \dot{r}\overline{e}_r + r(\dot{\phi} + \dot{\psi})\overline{e}_\theta \tag{3}$$

where, $\overline{e}_r$ and $\overline{e}_\theta$ are radial and transverse unit vectors. By inspection of Figure 1,

$$\overline{v} = [V_T \mathrm{Cos}(\phi + \psi - \phi_T) - V\mathrm{Cos}\psi]\overline{e}_r - [V_T \mathrm{Sin}(\phi + \psi - \phi_T) - V\mathrm{Sin}\psi]\overline{e}_\theta \quad . \tag{4}$$

Comparison of Equations (3) and (4) shows that

$$\dot{r} = V_T \text{Cos}(\phi + \psi - \phi_T) - V\text{Cos}\psi \tag{5}$$

and

$$\dot{\phi} + \dot{\psi} = -V_T \text{Sin}(\phi + \psi - \phi_T) + V\text{Sin}\psi \quad . \tag{6}$$

Proportional navigation is defined by the choice of the control variable $\dot{\phi}$ as:

$$\dot{\phi} = \lambda\dot{\psi} \quad . \tag{7}$$

In this equation λ is called the navigation constant. Clearly if λ is a constant, Equation (7) admits a first integral of the form

$$\phi = \phi_o + \lambda(\psi - \psi_o) \tag{8}$$

where the subscript o indicates that the variables are evaluated at some initial instant. From Equation (8) we can write

$$\phi + \psi = \alpha\psi + \beta \tag{9}$$

where $\alpha \equiv 1 + \lambda$ and $\beta \equiv \phi_O - \lambda\psi_O$. Upon substitution of Equations (7), (8) and (9) in Equations (5) and (6) we obtain:

$$\dot{r} = V_T \mathrm{Cos}(\alpha\psi + \beta - \phi_T) - V\mathrm{Cos}\psi \tag{10}$$

and

$$r(1 + \lambda)\dot{\psi} = -V_T \mathrm{Sin}(\alpha\psi + \beta - \phi_T) + V\mathrm{Sin}\psi \tag{11}$$

For the purposes of this paper, we transform the two above equations to make r the independent variable. This is easily accomplished by dividing Equation (11) by Equation (10) and by taking the reciprocal of Equation (10). Thus

$$\frac{dt}{dr} = \frac{1}{[V_T \mathrm{Cos}(\alpha\psi + \beta - \phi_T) - V\mathrm{Cos}\psi]} \tag{12}$$

and

$$\frac{d\psi}{dr} = \frac{1}{r(1 + \lambda)} \frac{[-V_T \mathrm{Sin}(\alpha\psi + \beta - \phi_T) + V\mathrm{Sin}\psi]}{[V_T \mathrm{Cos}(\alpha\psi + \beta - \phi_T) - V\mathrm{Cos}\psi]} \tag{13}$$

DERIVATION OF ALGORITHM

In the next few paragraphs, an algorithm for obtaining an approximate analytic solution to the equations derived in the previous section will be developed. This algorithm is based on a technique called quasilinearization. Quasilinearization is an iterative technique which can be applied to the solution of nonlinear differential equations. The technique was apparently introduced and developed by Bellman and Kalaba [3]. Using this technique, the solution of a non-linear Two Point Boundary Value Problem (TPBVP) is reduced to the recursive solution of a set of linear TPBVP°s. The convergence of this technique can be shown to be quadratic. This is in contrast to the linear convergence rate of the well known Picard iteration.

In that which follows, the quasilinearization algorithm will be briefly derived. For further explanation of the ideas utilized, the reader is directed to the work of Bellman and Kalaba previously cited. While the field of quasilinearization is quite broad and rich in both applications and theory, the central idea lies in the approximation of certain non-linear functions by means of Taylor series expansions in function space. It should be noted that this type of expansion is not unique to the work of Bellman, et al. However, it seems clear that the technique utilized in this paper should be attributed to Bellman.

We shall restrict the present solution to cases where V and V_T are specified constants and $\phi_T = 0$. These restrictions are consistent with approximations usually made in the analysis of control systems. The second assumption simply reflects a rotation of the reference coordinate system. In cases where the heading angle of the target is constant, there is no loss of generality in utilizing this assumption. Introducing these assumptions into Equations (13) and (12), we write

$$r(1 + \lambda)\frac{d\psi}{dr} = g(\psi) \tag{14}$$

and

$$\frac{dt}{dr} = d(\psi) \tag{15}$$

where

$$g(\psi) \equiv \frac{-V_T \mathrm{Sin}(\alpha\psi + \beta) + V\mathrm{Sin}\psi}{V_T \mathrm{Cos}(\alpha\psi + \beta) - V\mathrm{Cos}\psi} \tag{16}$$

and

$$d(\psi) \equiv \frac{1}{V_T \mathrm{Cos}(\alpha\psi + \beta) - V\mathrm{Cos}\psi} \tag{17}$$

Note that if the solution for $\psi=\psi(r)$ can be found, the solution for $t=t(r)$ can be found by quadrature of Equation (15).

We begin the derivation of the solution by expanding the right side of Equation (14) in a Taylor series in function space, i.e.,

$$r(1 + \lambda)\frac{d\psi_n}{dr} = g(\psi_{n-1}) + \frac{\partial g}{\partial \psi}\bigg|_{\psi_{n-1}}(\psi_n - \psi_{n-1})\ldots \tag{18}$$

where we have truncated after the linear term. In this equation, the subscripts are used to indicate the iteration number. For simplicity let

$$h \equiv \frac{\partial g}{\partial \psi} \quad .$$

By differentiation of Equation (16) we easily find:

$$h = \frac{-\alpha V_T^2 - V^2 + (1+\alpha)VV_T \text{Cos}(\lambda\psi + \beta)}{[V_T \text{Cos}(\alpha\psi + \beta) - V\text{Cos}\psi]^2} \tag{19}$$

Utilizing the definition of h, we can write

Equation (18) as

$$r(1+\lambda)\frac{d\psi_n}{dr} - h_{n-1}\psi_n = g_{n-1} - h_{n-1}\psi_{n-1} \quad . \tag{20}$$

With this equation in mind, the recursive algorithm becomes apparent. That is, a solution ψ_o is assumed which satisfies the terminal condition $\psi(R_0) = \Psi_0$.[2] This solution is used to compute the functions $g_0(r) \equiv g(\psi = \psi_0)$ and $h_o(r) \equiv h(\psi = \psi_o)$. Those explicit functions of r are then substituted into Equation (20) and the differential equation solved subject to the initial condition $\psi_1(R_0) = \Psi_0$. This process is repeated recursively until satisfactory convergence is obtained.

As previously noted, if an analytic solution for ψ_1 is known, the solution for $t(r)$ can be obtained by quadrature of Equation (15). Note, however, that the functional form of the right hand side makes it difficult, if not impossible, to obtain the integral. Again, recourse to the technique of quasilinearization provides a recursive algorithm for obtaining the desired solution. We begin by expanding Equation (15) in a Taylor series in function space,

$$\frac{dt_n}{dr} = d(\psi_{n-1}) + \frac{\partial d}{\partial \psi}\bigg|_{\psi_{n-1}}(\psi_n - \psi_{n-1}) + \cdots \quad . \tag{21}$$

2. Note, Ψ_0 is the LOS angle at the initial position $r = R$.

As before, the subscripts indicate the iteration number. Again, for simplicity let

$$e(\psi) \equiv \frac{\partial d}{\partial \psi} \quad .$$

By differentiation we obtain:

$$e(\psi) = \frac{\alpha V_T \mathrm{Sin}(\alpha\psi + \beta) - V\mathrm{Sin}\psi}{[V_T \mathrm{Cos}(\alpha\psi + \beta) - V\mathrm{Cos}\psi]^2} \quad . \qquad (22)$$

Thus, we are able to write

$$\frac{dt_n}{dr} = d_{n-1} + e_{n-1}(\psi_n - \psi_{n-1}) + \ldots \quad . \qquad (23)$$

The above equation defines a simplified quadrature algorithm for the determination of the distance-to-go and time-to-go. This equation is integrated subject to the terminal condition $t(0) = 0$.

SOLUTION

Having derived the algorithm in the previous section, we now turn to the application of this algorithm. In this section we shall assume an initial solution (subscript 0) and obtain the next approximation (subscript 1). The accuracy of the solution will be discussed in the following section.

The quasilinearization algorithm requires that an initial approximation to the solution be assumed. Obviously, it is advantageous that the initial approximation be as accurate as possible. Our choice for the initial approximation is based on the well known constant bearing solution [1].

It is easily shown that for a constant speed target moving on a straight line path , it is possible to choose a straight line interceptor trajectory which will intercept the target [2]. This type of interception is known as a constant bearing interception since in such cases, the angle between the interceptor velocity vector and the LOS remains constant. Note that if the initial engagement conditions in a proportional navigation interception are such that the angular rate of the LOS is zero, a constant bearing trajectory will result.

For our purposes, we choose the initial approximation $\psi_0(r)$ to be given by

$$\psi_0(r) = \Psi_0 \tag{24}$$

2. Providing that the interceptor speed is greater than that of the target.

where, of course, Ψ_0 is the initial bearing angle to the target. When Equation (24) is substituted into Equations (16), (17), (19), and (22), the following constants are defined:

$$g_0(\Psi_0) \equiv G_0 = \frac{-V_T\mathrm{Sin}(\alpha\Psi_0+\beta)+V\mathrm{Sin}\Psi_0}{V_T\mathrm{Cos}(\alpha\Psi_0+\beta)-V\mathrm{Cos}\Psi_0} \tag{25}$$

$$h_0(\Psi_0) \equiv H_0 = \frac{-\alpha V_T^2-V^2+(1+\alpha)VV_T\mathrm{Cos}(\lambda\Psi_0+\beta)}{[V_T\mathrm{Cos}(\alpha\Psi_0+\beta)-V\mathrm{Cos}\Psi_0]^2} \tag{26}$$

$$d_0(\Psi_0) \equiv D_0 = \frac{1}{V_T\mathrm{Cos}(\alpha\Psi_0+\beta)-V\mathrm{Cos}\Psi_0} \tag{27}$$

$$e_0(\Psi_0) \equiv E_0 = \frac{\alpha V_T\mathrm{Sin}(\alpha\Psi_0+\beta)-V\mathrm{Sin}\Psi_0}{[V_T\mathrm{Cos}(\alpha\Psi_0+\beta)-V\mathrm{Cos}\Psi_0]^2}. \tag{28}$$

When the above four constants are substituted into Equations (20) and (23) we obtain the following constant coefficient, linear differential equations:

$$r(1+\lambda)\frac{d\psi_1}{dr} - H_0\psi_1 = G_0 - H_0\Psi_0 \tag{29}$$

and

$$\frac{dt_1}{dr} = D_0 + E_0(\psi_1 - \Psi_0) \quad . \tag{30}$$

When these two equations are solved subject to the initial conditions, $\Psi(R_0) = \Psi_0$ and $t(R_0)=0$, we obtain

$$\psi_1 = \Psi_0 + \left(\frac{G_0}{H_0}\right)\left[\left(\frac{r}{R_0}\right)^{\frac{H_0}{1+\lambda}} - 1\right] \tag{31}$$

and

$$t_1 = R_0 \left[D_0 - \frac{E_0 G_0}{H_0}\right]\left(\frac{r}{R_0} - 1\right)$$

$$+ \frac{E_0 G_0}{H_0} \frac{\alpha}{(H_0+\alpha)} \left[\left(\frac{r}{R_0}\right)^{\frac{H_0+\alpha}{\alpha}} - 1\right] \quad . \tag{32}$$

Equation (32) can be used to obtain the time-of-flight by substituting $r = 0$. Thus

$$T_f = R_0 \left[\frac{E_0 G_0}{H_0+\alpha} - D_0\right] \tag{33}$$

RESULTS

In order to gain insight into the accuracy of the first iteration solution, a series of comparisons of predictions made by the quasilinearization solution and numerical integration of the exact equations has been carried out. Some of these comparisons are reproduced in this paper. It should be noted that the number of parameters in the problem makes it difficult to draw general conclusions. Instead, we present specific results and attempt to infer more general results.

Table 1 shows a comparison of a trajectory computation using the exact solution and the quasilinearization solution. To obtain this solution, the following parameters were used:

$$V = 900 \text{ m/sec}$$

$$V_T = 300 \text{ m/sec}$$

$$\lambda = 4$$

$$\phi = 15^{\circ}$$

$$\phi_T = 0^{\circ}$$

Note that the exact equation for ψ has a singularity at r=0. Thus, the numerical solution is not reliable in the vicinity of r=0. For this reason, the last entry column 3 of Table 1 is indicated by asterisks.

TABLE 1. Comparison of Exact and Quasilinearization Solutions of an Interception Trajectory

	PSI		t	
r	QUASI	EXACT	QUASI	EXACT
1.000	-.0146	-.0144	.0000	.0000
.900	-.1523	-.1523	1.6401	1.6401
.800	-.3195	-.3194	3.2831	3.2830
.700	-.5052	-.5049	4.9293	4.9288
.600	-.7145	-.7137	6.5791	6.5777
.500	-.9554	-.9535	8.2330	8.2298
.400	-1.2407	-1.2365	9.8917	9.8854
.300	-1.5933	-1.5845	11.5561	11.5448
.200	-2.0630	-2.0439	13.2279	13.2081
.100	[illegible].7971	-2.7503	14.9102	14.8755
0.000	-8.9066	****	16.6155	16.5449

Examination of Table 1 shows that, for this case at least, the quasilinearization solution provides an accurate solution. In general, if the initial LOS angle was small, quasilinearization provided an accurate solution.

As the initial LOS angle became much greater than the LOS angle required for a constant bearing solution, the accuracy of quasilinearization solutions degraded somewhat.

Clearly, the error in the solution is greatest at the end of the trajection ($r=0$). It would thus appear that a reasonable measure of accuracy would be the percent error of the time of interception. A number of solutions were carried out to obtain the set of error curves shown in Figure 2. In this figure, the independent variable is the initial LOS angle (Ψ_o). The initial azmiuth angle (measured clockwise from a direction parallel to the target velocity vector) was varied parametrically to generate the family of plots. Examination of this figure shows that the maximum error in the use of the quasilinearization solution remains remarkably small over a substantial range of trajectory conditions. It appears that for practical cases where engagements would be made with LOS angles within ten or fifteen degrees of a constant bearing solution, the error induced by using the approximate solution will be less than two or three percent.

CONCLUSIONS

Having derived an approximate solution for a proportional navigation trajectory in an earlier section and having compared the approximate solution to numerical integrations of the exact equations in the last section, we are in a position to comment on the efficacy of the solution technique. As was previously noted, in cases where the initial LOS angle Ψ_o is relatively near the lead angle for a constant bearing solution, the quasilinearization solution is

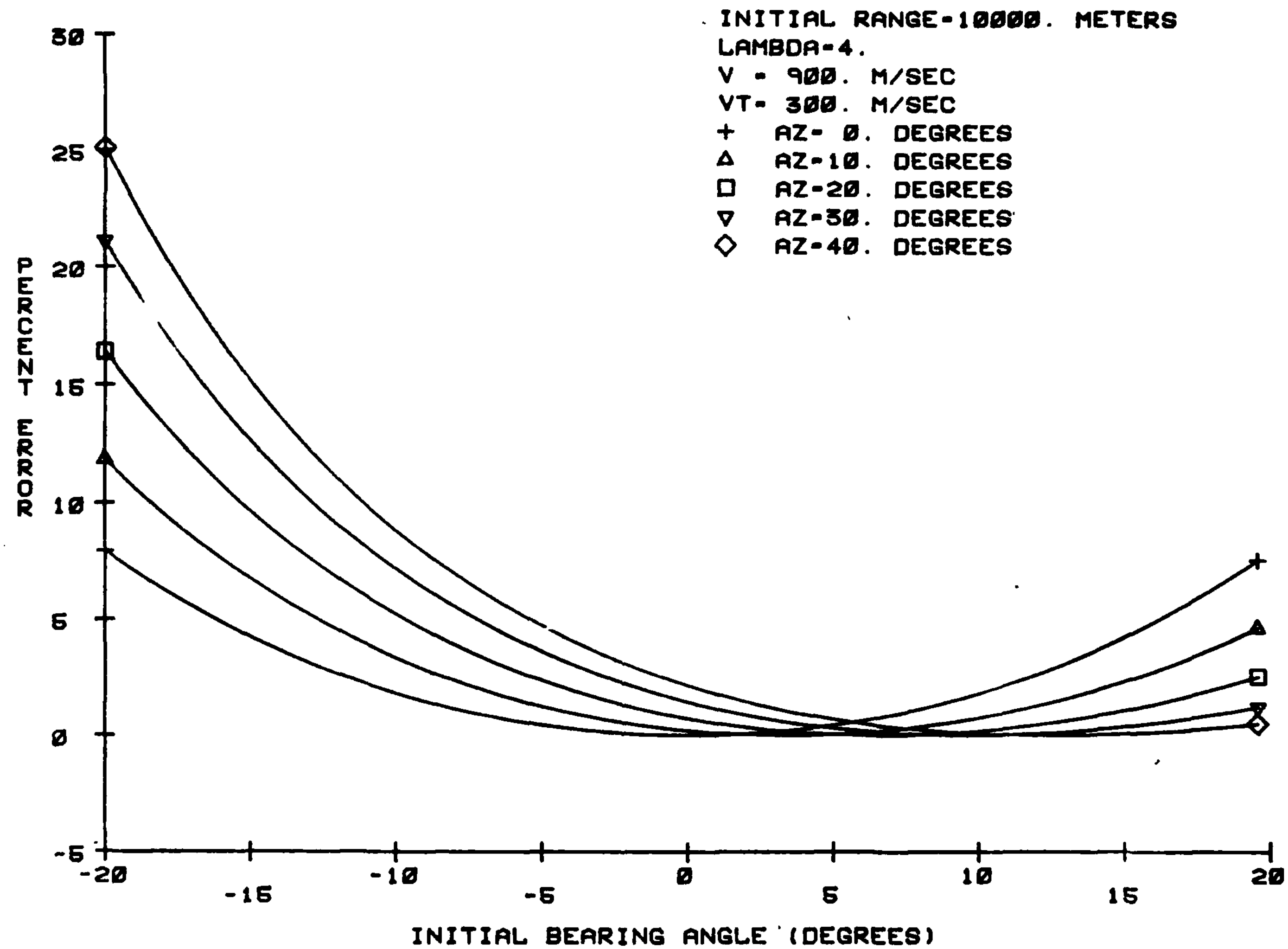

Figure 2. Error in the estimation of time-to-intercept using the quasilinearization.

very accurate. As this angle becomes larger, the accuracy degrades. Since the quasilinearization procedure produces a sequence of solutions which converges quadratically, an additional iteration will provide even more accuracy.

Since in certain aspects of control system design, it is advantageous to have an explicit approximate solution, the use of quasilinearization can be of real benefit. In such control applications, it is usually desired to use time as the independent variable. The present solution utilizes range-to-go as the independent variable. Since, in general, the first iteration results in a nonlinear function of range, analytic inversion of the solution to functions of time is not practical. However, the form of the solutions is simple enough that numerical inversion may be practical even in on-line applications.

Overall, it is clear that the use of the first iteration of the quasilinearization solution for proportional navigation provides a straightforward and accurate approximate solution. In a subsequent paper, we shall explore the relative accuracy of the second iteration.

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