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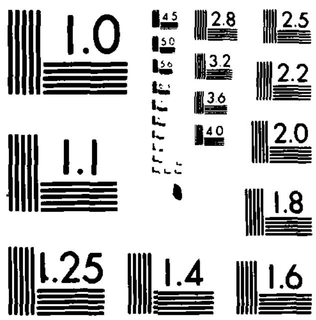

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GLOBAL NONEXISTENCE OF CLASSICAL SOLUTIONS IN ONE-DIMENSIONAL NONLINEAR VISCOELASTICITY†

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Results are presented which show that smooth global solutions of the one-dimensional nonlinear viscoelasticity initial-boundary value problem

$$u_{tt}(x,t) = \sigma(u_x(x,t))_x + \int_0^t a'(t-\tau)\sigma(u_x(x,\tau))_x d\tau + g(x,t), \quad 0 < x < 1, \quad t > 0$$

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ITEM #20, CONTINUED:

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad 0 \le x \le 1 \\ u(0,t) = 0 \qquad u(1,t) = 0, \quad t > 0 \end{cases}$$

cannot exist in the smoothness class

$$C^2([0,1] \times [0,\infty)) \cap A,$$

$$A = \{w(x,t) \mid w \in L^\infty([0,\infty);L^2(0,1)) \cap L^1([0,\infty);L^2(0,1))$$

$$\text{and} \quad w_t \in L^\infty([0,\infty):L^2(0,1))\}$$

if the data chosen are so that $\int_0^1 u_0^2(x)dx$ is sufficiently small while $\int_0^1 \Sigma(u_0'(x))dx$ is negative and sufficiently large in magnitude. The physical implication is that in the deformation of a nonlinear visco-elastic bar a shock will develop if the initial displacement is sufficiently small but has a sufficiently large gradient.

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1. Introduction

Many authors have, in recent years, considered the inital-boundary value problem

$$(1.1)\quad \begin{cases} u_{tt}(x,t) = \sigma(u_x(x,t))_x + \int_0^t a'(t-\tau)\sigma(u_x(x,\tau))_x d\tau \\ \qquad + g(x,t), \quad 0 < x < 1, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad 0 < x < 1 \\ u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \end{cases}$$

which arises naturally in a theory of one-dimensional nonlinear viscoelastic deformations of a solid continua in which the stress $\hat{\sigma}$ at the point x, $0 \le x \le 1$, and the time $t > 0$ is given by the constitutive relation

$$(1.2)\quad \hat{\sigma}(x,t) = \sigma(u_x(x,t)) + \int_0^t a'(t-\tau)\sigma(u_x(x,\tau))d\tau.$$

In (1.1), $u(x,t)$ is the deformation, with associated velocity $u_t(x,t)$ and deformation gradient $u_x(x,t)$, $\sigma(\cdot)$ is a nonlinear constitutive function, and $a(t)$ gauges the memory of the viscoelastic body; $g(x,t)$ represents an external forcing function, i.e., the intrinsic body force. The initial-boundary value problem (1.1) has been studied by MacCamy [1], Dafermos and Nohel [2] and Hattori [3]. In both [1] and [2] it is proven that under appropriate assumptions on $\sigma(\cdot)$, $a(\cdot)$, and g global smooth solutions will exist if the initial data functions $u_0(\cdot)$ and $v_0(\cdot)$, and their gradients, are small in an appropriate sense. The results in [1] are obtained by combining certain energy estimates with a Riemann invariants argument which is due, essentially, to Nishida [4] while the results in [2] are obtained by using Matsumura's refinement [5] of the Courant, Fredrichs, Lewy [6] energy method; this latter paper [2] also establishes the decay, as $t \to \infty$, of smooth solutions of (1.1) and is applicable to higher dimensional problems.

In [1] MacCamy conjectured that smooth (classical) solutions of (1.1) breakdown in finite-time if the initial data is sufficiently large; such a result is well-known for the damped nonlinear wave equation

$$(1.3) \qquad u_{tt}(x,t) + \alpha u_t(x,t) = \sigma(u_x(x,t))_x, \quad \alpha > 0$$

having been established by Slemrod [7] during the course of his study of *rectilinear shearing* flows in a nonlinear visco-elastic fluid. The damping mechanism in (1.1), however, is much weaker than the strong damping present in (1.3), where breakdown of class C^2 solutions occurs when the gradients of the initial-data are, pointwise, sufficiently large. Thus, the breakdown conjecture of MacCamy appeared to remain open for (1.1) until the recent Ph.D. thesis of Hattori [3] in which the author reformulates the evolution equation in terms of the resolvent kernel $k(t)$ associated with $a'(t)$, a device due to Dafermos and Nohel [2], reduces this equation to a damped first order system (with forcing and convolution terms) for u_t and u_x, and then rewrites the resulting system in Riemann invariant form; the author [3] then applies the method of Rozhdestvenskii [8] to the applicable first order system of equations for the Riemann invariants, the method consisting of showing that characteristic curves of the same family, which start off sufficiently close together on the initial line, must cross in finite-time and that at the instant where these curves cross the associated Riemann invariant must assume contradictory values (as one approaches the point of intersection along the respective characteristic curves). Hattori's result is obtained by assuming special forms for the inital values of the Riemann invariants; if the result in [3] is correct then, when combined with the global existence results in [1] and [2], it would seem to imply that smooth solutions of (1.1) must breakdown in finite-time when the gradients $u_0'(x)$, $v_0'(x)$ of the initial-data are sufficiently large in an appropriate sense, although it is not clear (at least to this author) how large the gradients of the data must be in [3] in order to produce breakdown of smooth solutions. In what follows we propose to present an elementary proof of breakdown (global nonexistence) of classical solutions to (1.1) which retains the basic assumptions of [1], [2], and [3], concerning the memory function $a(t)$ and the nonlinearity $\sigma(\zeta)$, and employs the dissipative nature of the resolvent kernel $k(t)$ associated with $a'(t)$, a modified concavity (differential inequality) argument, and a convergent integral argument to obtain an upper bound for the time of existence of a smooth solution.

2. Basic Assumptions and an Energy Lemma

The usual assumptions on the pertinent functions appearing in the equation (1.1) are as follows [1], [2]:

(A) $a : [0,\infty) \to R^1$

$a(\cdot) \in C^3[0,\infty)$, a, a', a'', a''' bounded on $[0,\infty)$

$a(t) = a_\infty + A(t)$, $a_\infty > 0$, $a(0) = 1$

$(-1)^m A^{(m)}(t) \geq 0$, $0 \leq t < \infty$, $m = 0,1,2$

$t^j A^{(m)}(t) \in L^1(0,\infty)$, $j,m = 0,1,2,3.$

(B) $\sigma : R^1 \to R^1$

$\sigma \in C^3(R^1)$, $\sigma(0) = 0$, $\sigma'(\zeta) \geq \varepsilon > 0$, $\forall \zeta \in R^1.$

(C) $g : [0,\infty) \times R^1 \to R^1$

$g, g_t \in L^1([0,\infty); L^2(R^1))$

$g_x, g_{tt}, g_{tx} \in L^2([0,\infty); L^2(R^1)).$

The assumption (C) will not be relevant in §3 where, for the sake of simplicity, we take the external forcing $g \equiv 0$; in (B) we need, in fact, only require that $\sigma \in C^1(R^1)$. If we introduce the resolvent kernel $k(t)$ associated with $a'(t)$ via

(2.1) $$k(t) + (a' * k)(t) = -a'(t), \quad 0 \leq t < \infty$$

where

$$(a' * k)(t) = \int_0^t a'(t-\tau)k(\tau)d\tau$$

then it can be shown that $k(\cdot) \in C^2(0,\infty)$ if $a(\cdot) \in C^3[0,\infty)$. Furthermore $k(t)$ satisfies k, k', k'' bounded on $[0,\infty)$, $k^{(m)}(t) \in L^1[0,\infty)$ $m = 0,1,2$ and the important dissipation property

$$(2.2) \qquad \begin{cases} \int_0^t v(\tau) \frac{\partial}{\partial\tau} \int_0^\tau k(\tau-\lambda)v(\lambda)d\lambda \, d\tau \geq 0, \\ \forall t > 0, \ \forall v(\cdot) \in L^2(0,t). \end{cases}$$

In what follows we will also make the assumption that

$$(D) \quad \int_0^\infty k'^2(\lambda)d\lambda < \infty$$

and we will append to (B) the mild assumption that $\forall\zeta \in R^1$ and some $\alpha > 0$

$$(2.3) \qquad \alpha\Sigma(\zeta) \geq \zeta\Sigma'(\zeta)$$

where $\Sigma(\zeta) = \int_0^\zeta \sigma(\rho)d\rho$. Growth restrictions of this kind on the nonlinearity σ have appeared before in studies on non-linear elasticity [9], [10]. Using the definition (2.1) of $k(t)$ it is not difficult (see [2], §2) to reduce the evolution equation in (1.1) to

$$(2.4) \quad u_{tt}(x,t) + \frac{\partial}{\partial t}\int_0^t k(t-\tau)u_t(x,\tau)d\tau = \sigma(u_x(x,t))_x + \Phi(x,t), \ 0 < x < 1, \ t > 0$$

where

$$(2.5) \quad \Phi(x,t) = g(x,t) + \int_0^t k(t-\tau)g(x,\tau)d\tau + k(t)v_0(x).$$

In §3 we will assume both that $g(x,t) \equiv 0$, $0 \leq x \leq 1$, $t > 0$ and $u_t(x,0)$, $0 \leq x \leq 1$ so that $\Phi(x,t) = 0$, $0 \leq x \leq 1$, $t > 0$. We now introduce the energy functional

$$E(t) = \frac{1}{2}\int_0^1 u_t^2(x,t)dx + \int_0^1 \Sigma(u_x(x,t))dx \tag{2.6}$$

and prove the following

<u>Lemma</u>. For as long as smooth (classical) solutions of (1.1) exist

$$E(T) \leq E(0) + \int_0^1\int_0^T u_t(x,t)k(t)u_t(x,0)dt\,dx. \tag{2.7}$$

<u>Proof</u>: We multiply (2.4) by $(u_t(x,t)$ and integrate over $[0,1] \times [0,T)$, $T > 0$, so as to obtain

$$\int_0^1\int_0^T u_t(x,t)u_{tt}(x,t)dt\,dx$$

$$+ \int_0^1\int_0^T u_t(x,t)\left(\frac{\partial}{\partial t}\int_0^t k(t-\tau)u_t(x,\tau)d\tau\right)dt\,dx$$

$$= \int_0^1\int_0^T u_t(x,t)\sigma(u_x(x,t))_x dt\,dx$$

$$+ \int_0^1\int_0^T u_t(x,t)k(t)u_t(x,0)dt\,dx.$$

By (2.2) if $u_t(x,\cdot) \in L^2(0,T)$, $0 \leq x \leq 1$ then

$$\int_0^T u_t(x,t)\left(\frac{\partial}{\partial t}\int_0^t k(t-\tau)u_t(x,\tau)d\tau\right)dt \geq 0$$

for all x, $0 \leq x \leq 1$. It therefore follows that

$$\frac{1}{2}\int_0^1\int_0^T \frac{\partial}{\partial t}\, u_t^2(x,t)dt\; dx$$
$$\le \int_0^T\int_0^1 u_t(x,t)\sigma(u_x(x,t))_x dx\; dt$$
$$+ \int_0^1\int_0^T u_t(x,t)k(t)u_t(x,0)dx\; dt.$$

Integrating the first term on the right hand side of this estimate by parts and using the boundary conditions in (1.1) we obtain

$$\frac{1}{2}\int_0^1 u_t^2(x,t)\Big|_0^T dx \le -\int_0^T\int_0^1 u_{xt}(x,t)\sigma(u_x(x,t))dx\; dt$$
$$+ \int_0^1\int_0^T u_t(x,t)k(t)u_t(x,0)dt\; dx$$
$$= -\int_0^T \frac{\partial}{\partial t}\left(\int_0^1 \Sigma(u_x(x,t))dx\right)dt$$
$$+ \int_0^1\int_0^T u_t(x,t)k(t)u_t(x,0)dx\; dt$$

by virtue of the definition of Σ. Thus

$$\frac{1}{2}\int_0^1 u_t^2(x,t)\Big|_0^T dx + \int_0^1 \Sigma(u_x(x,t)dx\Big|_0^T$$
$$\le \int_0^1\int_0^T u_t(x,t)k(t)u_t(x,0)dt\; dx$$

and the Lemma follows directly from the definition of $E(t)$; in particular, if $u_t(x,0) \equiv 0$, $0 \le x \le 1$, then for as long as classical solutions of (1.1) exist, $E(t) \le E(0)$.

Q.E.D.

3. A Differential Inequality for Solutions of (1.1)

In this section we derive a basic differential inequality which is satisfied by a simple functional defined on solutions of the initial-boundary value problem (1.1). By a smooth global solution of (1.1) in this section we shall understand a function

$$u(x,t) \in C^2([0,1] \times [0,\infty)) \cap \mathbf{A}$$

where

$$\text{(3.1)} \quad \mathbf{A} = \{w(x,t) \mid w \in L^\infty([0,\infty);\ L^2(0,1)) \cap L^1([0,\infty);\ L^2(0,1)) \text{ and } w_t \in L^\infty([0,\infty);\ L^2(0,1))\}.$$

Thus, if $u \in \mathbf{A}$, $\exists$ $C_\infty < \infty$ such that

$$\text{(3.2)} \quad \sup_{[0,\infty)} \left[\int_0^1 (w^2(x,t) + w_t^2(x,t))dx\right] + \int_0^\infty\!\!\int_0^1 w_t^2(x,t)dx\,dt \le C_\infty .$$

Our aim in §4 will be to show that under appropriate assumptions on the initial-data there cannot exist solutions $u(x,t) \in C^2([0,1] \times [0,\infty)) \cap \mathbf{A}$ of (1.1). To this end, let $\beta \ge 0$, $t_0 \ge 0$ be arbitrary nonnegative constants and consider the functional

$$\text{(3.3)} \quad \mathcal{U}(t) \equiv \int_0^1 u^2(x,t)dx + \beta(t + t_0)^2, \quad t > 0$$

If $u(x,t)$ is a solution of (1.1) which belongs to the indicated smoothness class then we claim the following:

Lemma. If $u(x,t) \in C^2([0,1]\times[0,\infty)) \cap A$ is a solution of (1.1) then for all $t > 0$, $U(t)$ as given by (3.3) satisfies

$$(3.4)\qquad U\ddot{U} - \left(\frac{\alpha+2}{2}\right)\dot{U}^2 \geq -2\alpha U\left(E(0) + \frac{\beta}{2} + C_\infty\right).$$

Proof: We are assuming in this section that $u_t(x,0) = 0$, $0 \leq x \leq 1$, and that $g(x,t) = 0$, $(x,t) \in [0,1]\times[0,\infty)$ so that $\Phi(x,t) \equiv 0$, $(x,t) \in [0,1]\times[0,\infty)$. Directly from (3.3) and the form (2.4) of the evolution equation, with $\Phi \equiv 0$, we compute

$$(3.5)\qquad \dot{U}(t) = 2\int_0^1 uu_t dx + 2\beta(t + t_0)$$

and

$$\begin{aligned}(3.6)\qquad \ddot{U}(t) &= 2\int_0^1 uu_{tt}dx + 2\int_0^1 u_t^2 dx + 2\beta \\ &= 2\int_0^1 u(x,t)\sigma(u_x(x,t))_x dx \\ &\quad - 2\int_0^1 u(x,t)\frac{\partial}{\partial t}\int_0^t k(t-\tau)u_\tau(x,\tau)d\tau\, dx \\ &\quad + 2\int_0^1 u_t^2(x,t)dx + \beta \\ &= -2\int_0^1 u_x(x,t)\sigma(u_x(x,t))dx \\ &\quad - 2\int_0^1 u(x,t)\,\frac{\partial}{\partial t}\int_0^t k(t-\tau)u_t(x,\tau)d\tau\, dx \\ &\quad + 2\int_0^1 u_t^2(x,t)dx + 2\beta.\end{aligned}$$

by adding and subtracting the quantity $2\alpha \int_0^1 \mathcal{G}(u_x(x,t))dx$ to (3.6_3) and employing the hypothesis (2.3) we obtain the estimate

$$(3.7)\quad \ddot{U}(t) \ge 2\int_0^1 u_t^2(x,t)dx - 2\alpha \int_0^1 \mathcal{G}(u_x(x,t))dx$$

$$- 2\int_0^1 u(x,t)\, \frac{\partial}{\partial t} \int_0^t k(t-\tau)u_t(x,\tau)d\tau\, dx + 2\beta.$$

We now replace $\int_0^1 \mathcal{G}(u_x(x,t))dx$ in (3.7) by $E(t) - \frac{1}{2}\int_0^1 u_t^2(x,t)dx$ and obtain from (3.7)

$$(3.8)\quad \ddot{U}(t) \ge (2+\alpha)\int_0^1 u_t^2(x,t)dx - 2\alpha E(t)$$

$$- 2\int_0^1 u(x,t)\, \frac{\partial}{\partial t} \int_0^t k(t-\tau)u_t(x,\tau)d\tau\, dx + 2\beta.$$

However, by the Lemma of §2 and our assumption that $u_t(x,0) = 0$, $0 \le x \le 1$, $E(t) \le E(0)$, $t > 0$ and, thus

$$(3.9)\quad \ddot{U}(t) \ge (2+\alpha)\int_0^1 u_t^2(s,t)dx - 2\alpha E(0) + 2\beta$$

$$- 2\int_0^1 u(x,t)\, \frac{\partial}{\partial t} \int_0^t k(t-\tau)u_t(x,\tau)d\tau\, dx.$$

As $\dot{U}^2(t) \le 2(\int_0^1 uu_t d_x)^2 + 2\beta^2(t+t_0)^2$, we then compute that

$$\ddot{U}(t)U(t) - (\frac{\alpha+2}{2})\dot{U}^2(t) \geq$$

$$(2+\alpha)[\int_0^1 u_t^2\, dx \int_0^1 u^2\, dx - (\int_0^1 uu_t dx)^2]$$

$$+ (2+\alpha)\beta(t+t_0)^2 \int_0^1 u_t^2\, dx$$

$$+ [-2\, E(0) + 2\beta + \delta(t)][\int_0^1 u^2\, dx + \beta(t+t_0)^2]$$

$$- (2+\alpha)\beta^2(t+t_0)^2$$

$$\geq \quad - (2+\alpha)\beta^2(t+t_0)^2$$

$$+ (-2\alpha E(0) + 2\beta + \delta(t)U(t)$$

where

$$\delta(t) \quad \equiv \quad -2 \int_0^1 u(x,t) \frac{\partial}{\partial t} \int_0^t k(t-\tau)u_t(x,\tau)d\tau\; dx.$$

We rewrite this last estimate as

$$(3.10) \quad \ddot{U}(t)U(t) - (\frac{\alpha+2}{2})\dot{U}^2(t) \geq$$

$$(-2\alpha E(0) + \delta(t))U(t)$$

$$+ 2\beta U(t) - (2+\alpha)\beta U(t)$$

$$+ (2+\alpha)\beta U(t) - (2+\alpha)\beta^2(t+t_0)^2 \geq$$

$$(-2\alpha E(0) + \delta(t))U(t) - \alpha\beta U(t)$$

where

$$(2+\alpha)\beta U(t) - (2+\alpha)\beta^2(t+t_0)^2 \quad = \quad (2+\alpha)\beta \int_0^1 u^2\; dx > 0.$$

Thus

$$(3.11)\qquad \ddot{U}(t)U(t) - \left(\frac{\alpha+2}{2}\right)\dot{U}^2(t) \geq -2\alpha U(t)\left(E(0) + \frac{\beta}{2} - \frac{\delta(t)}{2\alpha}\right).$$

By the definition of $\delta(t)$

$$-\frac{\delta(t)}{2\alpha} = \frac{1}{\alpha}\int_0^1 u(x,t)\,\frac{\partial}{\partial t}\int_0^t k(t-\tau)u_t(x,\tau)d\tau\,dx.$$

Therefore,

$$\begin{aligned}
(3.12)\qquad -\frac{\delta(t)}{2\alpha} &\leq \frac{1}{\alpha}\left|\int_0^1 u(x,t)\,\frac{\partial}{\partial t}\int_0^t k(t-\tau)u_t(x,\tau)d\tau\,dx\right| \\
&= \frac{1}{\alpha}\left|k(0)\int_0^1 uu_t dx + \int_0^1 u(x,t)\int_0^t k_t(t-\tau)u_t(x,\tau)d\tau dx\right| \\
&\leq \frac{1}{\alpha}\,k(0)|\left(\int_0^1 u^2dx\right)^{\frac{1}{2}}\left(\int_0^1 u_t^2dx\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{\alpha}\left(\int_0^1 u^2dx\right)^{\frac{1}{2}}\left(\int_0^1\left(\int_0^t k_t(t-\tau)u_t(x,\tau)d\tau\right)^2dx\right)^{\frac{1}{2}} \\
&= \frac{1}{\alpha}\left(\int_0^1 u^2dx\right)^{\frac{1}{2}}\Big[|k(0)|\left(\int_0^1 u_t^2dx\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^1\left(\int_0^t k_t(t-\tau)u_t(x,\tau)d\tau\right)^2dx\right)^{\frac{1}{2}}\Big].
\end{aligned}$$

Now,

$$\left(\int_0^1\left(\int_0^t k_t(t-\tau)u_t(x,\tau)d\tau\right)^2dx\right)^{\frac{1}{2}} \leq$$

$$\left(\int_0^1\left[\int_0^t k_t^2(t-\tau)d\tau\int_0^t u_t^2(x,\tau)d\tau\right]dx\right)^{\frac{1}{2}}.$$

But, for all $t < \infty$

$$\int_0^t k_t^2(t-\tau)d\tau = \int_0^t k'^2(\lambda)d\lambda \leq \int_0 k'^2(\lambda)d\lambda < \infty$$

by assumption (D).

Therefore, for all t, $0 \le t < \infty$

$$\left(\int_0^1 \left(\int_0^t k_t(t-\tau)u_t(x,\tau)d\tau\right)^2 dx\right)^{\frac{1}{2}} \le$$

$$\left(\int_0^\infty k'^2(\lambda)d\lambda\right)^{\frac{1}{2}}\left(\int_0^1\int_0^t u_t^2(x,\tau)d\tau dx\right)^{\frac{1}{2}}.$$

Combining our estimates with (3.12) and using our hypothesis that $u \in \mathbf{A}$ we find that

$$(3.13)\quad -\frac{\delta(t)}{2\alpha} \le \frac{1}{\alpha} \sup_{[0,\infty)} \left(\int_0^1 u^2(x,t)dx\right)^{\frac{1}{2}}\Big[|k(0)| \sup_{[0,\infty)} \left(\int_0^1 u_t^2(x,t)dx\right)^{\frac{1}{2}}$$

$$+ \left(\int_0^\infty k'^2(\lambda)d\lambda\right)^{\frac{1}{2}}\left(\int_0^\infty\int_0^1 u_t^2(x,\tau)d\tau dx\right)^{\frac{1}{2}}\Big]$$

$$\equiv C_\infty < \infty.$$

Thus, by (3.11) we have, for all $t < \infty$

$$(3.14)\quad \ddot{U}(t)U(t) - \left(\frac{\alpha+2}{2}\right)\dot{U}^2(t) \ge$$

$$- 2\alpha U(t)\left(E(0) + \frac{\beta}{2} + C_\infty\right) \qquad \text{Q.E.D.}$$

4. Nonexistence of a Class of Smooth Solutions

In this section we will show that solutions $u \in C^2([0,1] \times [0, \ )) \cap A$ of (1.1) cannot exist if $\int_0^1 \Sigma(u_0'(x))dx$ is negative and sufficiently large in magnitude. Our approach is to assume that $u \in C^2([0,1] \times [0,\infty))$ is a solution of (1.1) and then derive a contradiction from the differential inequality (3.14). It is rather simple at this stage to derive a restricted but still somewhat interesting first result for if $\beta > 0$ and

$$(4.1) \qquad \begin{cases} E(0) \equiv \int_0^1 \Sigma(u_0'(x))dx < 0 \quad \text{with} \\ \left|\int_0^1 \Sigma(u_0'(x))dx\right| > \beta/2 + C_\infty \end{cases}$$

then (3.14) implies that for all $t > 0$

$$(4.2) \qquad \ddot{U}(t)U(t) - (\gamma+1)\dot{U}^2(t) \geq 0; \quad \gamma = \alpha/2 > 0.$$

But (4.2) is equivalent to $\frac{d^2}{dt^2}(U^{-\gamma}(t)) \leq 0, \ 0 \leq t < \infty$ which implies that

$$(4.3) \qquad U(t) \geq U(0)[1 - \gamma(\frac{U'(0)}{U(0)})t]^{-1/\gamma}, \quad 0 \leq t < \infty.$$

However, $U(0) = \int_0^1 u_0^2(x)dx + \beta t_0^2$ and $U'(0) = 2\beta t_0$ as $u_t(x,0) = 0, \ 0 \leq x \leq 1$. Therefore, for $0 \leq t < \infty$ any solution $u \in C^2([0,1] \times [0,\infty)) \cap A$ satisfying (4.1) is such that the functional $U(t)$ is bounded from below by a function which blows up as

$$(4.4) \qquad t \to t_\infty \doteq \int_{-0}^{1} \frac{u_0^2(x)dx + \beta t_0^2}{2\beta\gamma t_0} .$$

As $U(t) = \int_0^1 u^2(x,t)dx + \beta(t+t_0)^2$, the $L^2(0,1)$ norm of $u(\cdot,t)$ cannot be finite for all $t > 0$ contradicting the assumption that $u(x,t)$ is a smooth solution of (1.1), i.e. a solution in the class $C^2([0,1] \times [0,\infty)) \cap A$. We state this first result as

<u>Theorem 1</u>. If $g(x,t) \equiv 0$, $(x,t) \in (0,1) \times [0,\infty)$, $u_t(x,0) = 0$, $0 \le x \le 1$, then under the hypotheses of §2 on $a(t)$, $k(t)$, and σ no solution of (1.1) can exist in $C^2([0,1] \times [0,\infty)) \cap A$ which satisfies (4.1).[1]

A much better result than Theorem 1 can be obtained if we rewrite (3.14) in the form

$$(4.1) \qquad \ddot{U}(t)U(t) - (\gamma+1)\dot{U}^2(t) \ge -2\nu^2(2\gamma+1)U(t)$$

where $\gamma = \alpha/2$ and $\nu^2 = (\frac{\alpha}{\alpha+1})(E(0) + \beta/2 + C_\infty)$.

In the analysis that follows we will want the initial data $u(x,0) \equiv u_0(x)$ to be such that $\dot{U}^2(0) > 4\nu^2 U(0)$; this latter condition is easily seen to be equivalent to

$$(4.2) \qquad \frac{\beta^2 t_0^2}{\int_0^1 u_0^2(x)dx + \beta t_0^2} > (\frac{\alpha}{\alpha+1})(\int_0^1 \Sigma(u_0'(x))dx + \beta/2 + C_\infty) .$$

If, as before, we choose $u_0(x)$ so that $\int_0^1 \Sigma(u_0'(x))dx < 0$ then (4.2) is equivalent to

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(1) Thus if $u \in C^2([0,1] \times [0,\infty)) \cap A$ is a solution of (1.1) then we must have the lower bound $C_\infty + \beta/2 \ge |\int_0^1 \Sigma(u_0'(x))dx|$ for all $\beta > 0$.

$$(4.3)\qquad \left|\int_0^1 \Sigma(u_0'(x))dx\right| \;>\; C_\infty + (\beta/2) - \left(\frac{\alpha+1}{\alpha}\right)\left(\frac{\beta^2 t_0^2}{\int_0^1 u_0^2(x)dx+\beta t_0^2}\right).$$

If we choose $\int_0^1 u_0^2(x)dx$ to be very small then

$$C_\infty + \beta/2 - \left(\frac{\alpha+1}{\alpha}\right)\left(\frac{\beta^2 t_0^2}{\int_0^1 u_0^2(x)dx+\beta t_0^2}\right)$$

$$= C_\infty + \beta/2 - \left(\frac{\alpha+1}{\alpha}\right)\left(\frac{\beta}{\frac{1}{\beta t_0^2}\left(\int_0^1 u_0^2(x)dx\right)+1}\right)$$

$$\approx C_\infty + \beta\left(\frac{1}{2} - \frac{\alpha+1}{\alpha}\right) = C_\infty - \beta\left(\frac{1}{2}+\frac{1}{\alpha}\right) > 0$$

for β chosen sufficiently small. Thus for $\int_0^1 u_0^2(x)dx$ and $\beta > 0$ chosen sufficiently small the right-hand side of (4.3) is positive so that the condition (4.3), which requires that $\left|\int_0^1 \Sigma(u_0'(x))dx\right|$ be sufficiently large, is not vacuous. We now choose β to be small enough so that $C_\infty - \beta(\frac{1}{2}+\frac{1}{\alpha}) > 0$ but large enough so that $\left|\int_0^1 \Sigma(u_0'(x))dx\right| > C_\infty - \beta(\frac{1}{2}+\frac{1}{\alpha})$.[2]

As $\dot{U}(0) = 2\beta t_0 > 0$, $\dot{\overline{U^{-\gamma}}}(0) = -\gamma U^{-(\gamma+1)}(0)\dot{U}(0) < 0$. By continuity $\dot{\overline{U^{-\gamma}}}(t) < 0$ for t sufficiently small. If $\dot{\overline{U^{-\gamma}}}(t) \nless 0$ for all $t > 0$ then let $t = t^*$ be the first time such that $\dot{\overline{U^{-\gamma}}}(t^*) = 0$; we want to show that no such t^* can exist. Since $U(t) > 0$, $t \in [0,t^*]$, we may rewrite (4.1) as

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(2) A precise specification of what is meant by taking β to be sufficiently large is given in the Remarks following Theorem 2.

(4.4) $$\overset{\cdot\cdot}{\overline{u^{-\gamma}}}(t) \le 2\gamma\nu^2(2\gamma+1)u^{-(\gamma+1)}(t), \quad t \in [0,t^*].$$

On $[0,t^*)$, $\dot{\overline{u^{-\gamma}}}(t) < 0$. Thus if we multiply (4.4) through by $2\overset{\cdot\cdot}{\overline{u^{-\gamma}}}(t)$ we obtain

$$2\,\overset{\cdot\cdot}{\overline{u^{-\gamma}}}(t)\,\dot{\overline{u^{-\gamma}}}(t) \ge 4\gamma\nu^2(2\gamma+1)u^{-(\gamma+1)}(t)\,\dot{\overline{u^{-\gamma}}}(t)$$

which is easily seen to be equivalent to

(4.5) $$\frac{d}{dt}[\dot{\overline{u^{-\gamma}}}(t)]^2 \ge 4\gamma^2\nu^2\frac{d}{dt}u^{-(2\gamma+1)}(t).$$

Integrating this inequality over $[0,t]$, $t \in [0,t^*]$ we obtain

(4.6) $$[\dot{\overline{u^{-\gamma}}}(t)]^2 - 4\gamma^2\nu^2\,u^{-(2\gamma+1)}(t) \ge H_0 > 0$$

where

(4.7) $$H_0 \equiv \gamma^2\,u^{-(2\gamma+1)}(0)[u^{-1}(0)\dot{u}^2(0)-4\nu^2] > 0$$

by virtue of the hypothesis (4.3). If we now write (4.6) in the form

(4.8) $$\{\dot{\overline{u^{-\gamma}}}(t) - 2\gamma\nu\,u^{-(\gamma+1/2)}(t)\} \times$$
$$\{\dot{\overline{u^{-\gamma}}}(t) + 2\gamma\nu u^{-(\gamma+1/2)}\} \ge H_0 > 0.$$

Since $\dot{\overline{u^{-\gamma}}}(t) < 0$, $t \in [0,t^*)$, the first factor on the left-hand side of (4.8) is negative and thus

(4.9) $$\dot{\overline{u^{-\gamma}}}(t) < -2\gamma\nu\,u^{-(\gamma+1/2)}(t), \quad t < t^*.$$

By continuity this last relation must hold at $t = t^*$, thus contradicting the assumption that $\dot{\overline{U^{-\gamma}}}(t^*) = 0$. Hence, $\dot{\overline{U^{-\gamma}}}(t) < 0$ for all $t > 0$ if $u \in C^2([0,1] \times [0,\infty)) \cap A$ is a solution of (1.1). The estimates (4.6) and (4.9) are, therefore, also valid for all $t > 0$ if $u \in C^2([0,1] \times [0,\infty)) \cap A$. Clearly we may rewrite the estimate (4.6) in the form

$$(4.10) \quad [-\gamma U^{-(\gamma+1)}(t)\dot{U}(t)]^2 \geq H_0 + 4\gamma^2\nu^2 U^{-(2\gamma+1)}(t)$$

from which it follows that

$$|-\gamma U^{-(\gamma+1)}(t)\dot{U}(t)| \geq (H_0+4\gamma^2\nu^2 U^{-(2\gamma+1)}(t))^{1/2}.$$

However, $-\gamma U^{-(\gamma+1)}(t)\dot{U}(t) < 0$, $t > 0$, so this last estimate is equivalent to

$$(4.11) \qquad \dot{U}(t) \geq (4\nu^2 U(t) + H_0\gamma^{-2} U^{2(\gamma+1)})^{1/2}.$$

If we let $T > 0$ and integrate (4.11) over $[0,T)$ we obtain the estimate

$$(4.12) \qquad T \leq \int_{U(0)}^{U(T)} \frac{dV}{(4\nu^2 V+H_0\gamma^{-2}V^{2(\gamma+1)})^{1/2}} < \infty.$$

Since the integral on the right-hand side of (4.12) is convergent, for all nonnegative values of $U(T)$, the estimate (4.12) clearly implies that $T \leq t_{max} < \infty$, thus contradicting our assumption that $u \in C^2([0,1] \times [0,\infty)) \cap A$ is a solution of (1.1); this last estimate also provides an upper bound on the maximal time interval $[0,t_{max}]$ of existence of a smooth solution of (1.1). We state our result as

Theorem 2. If $\int_0^1 u_0^2(x)dx$ is chosen sufficiently small while $\int_0^1 \sum(u_0'(x))dx < 0$ with $|\int_0^1 \sum(u_0'(x))dx|$ sufficiently large then no solution of (1.1) can exist in the class $C^2([0,1] \times [0,\infty)) \cap A$ and the maximal time of existence of a smooth solution must satisfy the estimate

$$t_{max} \leq \int_{U(0)}^{\infty} \frac{dU}{(4\nu^2 U + H_0 \gamma^{-2} U^{2(\gamma+1)})^{1/2}}.$$

Remarks. We elaborate briefly here on the hypothesis of Theorem 2. If $\int_0^1 u_0^2(x)dx$ is chosen sufficiently small then (4.3) will be satisfied provided $|\int_0^1 \sum(u_0'(x))dx| > C_\infty - \beta(\frac{1}{2} + \frac{1}{\alpha})$. We view this latter condition as follows: Suppose that $u \in C^2([0,1] \times [0,\infty)) \cap A$ is a solution of (1.1) corresponding to a specification of $u(x,0) = u_0(x)$. Then we may certainly choose δ, $0 < \delta < 1$, so small that $|\int_0^1 \sum(u_0'(x))dx| > \delta C_\infty$. If we now choose β so large that

$$\delta C_\infty > C_\infty - \beta(\frac{1}{2} + \frac{1}{\alpha})$$

i.e., if we take $\beta > \dfrac{C_\infty(1-\delta)}{(\frac{1}{2} + \frac{1}{\alpha})}$ then for this choice of β we would have $|\int_0^1 \sum(u_0'(x))dx| > C - \beta(\frac{1}{2} + \frac{1}{\alpha})$, which in turn leads to a contradiction of the assumption that $u \in C^2([0,1] \times [0,\infty)) \cap A$.

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