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A BAYES RULE FOR SELECTING THE LARGEST COMPONENT. (U)  
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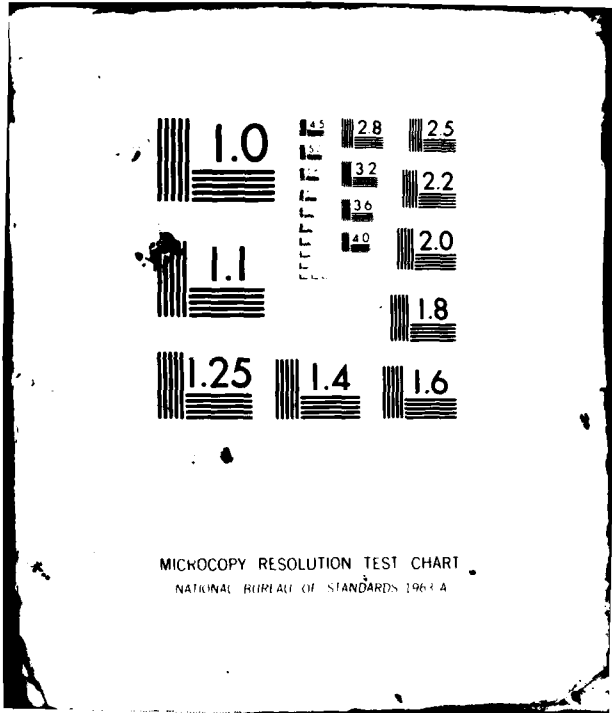
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A BAYES RULE FOR SELECTING  
THE LARGEST COMPONENT

By

Khursheed Alam\*

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A BAYES RULE FOR SELECTING THE LARGEST COMPONENT

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ABSTRACT

Let  $\underline{X} = (X_1, \dots, X_k)$  be a random vector whose distribution depends on an unknown vector parameter  $\underline{\theta} = (\theta_1, \dots, \theta_k)$ . The marginal distribution of  $X_i$  depends on  $\theta_i$  only,  $i = 1, \dots, k$ . This paper deals with the problem of selecting the largest component of  $\underline{\theta}$  and the analogous problem of selecting a subset of the components of  $\underline{\theta}$  which includes the largest component. We consider the selection problem in a general decision theoretic framework and derive Bayes rules for selecting the largest component. The Bayes rules are shown to have certain optimal properties. The ordinary selection rules are shown to be Bayes rules, with respect to a special loss function.

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1. Introduction. Let  $\underline{X} = (X_1, \dots, X_k)$  be a random vector whose distribution depends on an unknown vector parameter  $\underline{\theta} = (\theta_1, \dots, \theta_k)$ . The marginal distribution of  $X_i$  depends on  $\theta_i$  only, for each  $i = 1, \dots, k$ . We consider the problem of selecting the largest component of  $\underline{\theta}$  given  $\underline{x}$ , an observed value of  $\underline{X}$ .

There are two formulations of the selection problem which have been generally considered in the literature. In one the goal is to select the largest component with a "high" probability. In the other the goal is to select a subset of the  $k$  components which includes the largest component with a high probability and includes any of the remaining components with a "low" probability. In the second case, the selection would be correct if the largest component is included in the selected subset.

In the standard formulation of the selection problem a minimum probability is pre-assigned, equal to  $P^*$ , say, such that the probability of a correct selection (PCS) should be at least as large as  $P^*$ . This is called the  $P^*$ -condition. To meet the  $P^*$ -condition it needs to find a "least favorable" configuration (lfc) of the parameter space for which the PCS is minimized. The lfc is found easily in some special cases which have been considered in the literature for the underlying distribution of  $\underline{X}$ . In other cases the minimization of the PCS is not so straightforward. Consider, for example, the case where  $\underline{X}$  is distributed according to a multivariate normal distribution with mean  $\underline{\theta}$  and covariance matrix  $\Sigma$ ; where  $\Sigma$  is known. A simple rule can be given for selecting the largest component of  $\underline{\theta}$  in the special case when the components of  $\underline{X}$  have a common variance and are equi-correlated (see Gibbons, Olkin and Sobel (1977), § 15.2.1). It is not simple to find an optimal selection rule when  $\Sigma$  is defined more generally. The difficulty arises even in the case where the components of  $\underline{X}$  are uncorrelated but they have unequal variances.

For this case various rules have been proposed in the literature for selecting a subset which includes the largest component of  $\theta$ . Berger and Gupta (1980) have examined these rules and compared them, applying certain criterion of optimality.

In this paper we consider the selection problem in a Bayesian framework. The Bayes formulation involves the specification of a loss function and the assumption of a prior distribution for the parameter  $\theta$ . Given the loss function and the prior distribution of  $\theta$ , it is fairly easy to find an optimal selection rule. The optimal rule is called a Bayes rule. The Bayes solution does not involve the minimization problem of finding the least favorable configuration. Therefore, at least from the point of view of mathematical simplicity, a Bayes solution of the selection problem should be more attractive than the standard method, discussed above.

In the following section we give a decision theoretic formulation of the selection problem and derive the Bayes solution for a general loss function. We illustrate our result with an example from the multivariate normal distribution.

Berger and Gupta (1980) have considered a monotonicity property for an optimal selection rule. A rule is said to be just if it has that property. In Section 3 we show that our Bayes rules are just if certain conditions with regard to the distribution of  $X$  and the loss function are met.

2. Bayes selection rules. We formulate the selection problem in a decision theoretic framework, as follows: A rule for selecting the largest component of  $\underline{\theta}$  (selecting a subset of the  $k$  components, which includes the largest component) is given by a vector  $\phi(\underline{x}) = (\phi_1(\underline{x}), \dots, \phi_k(\underline{x}))$ , where  $\phi_i(\underline{x})$  denotes the probability that the  $i$ th component is selected (included in the selected subset) when  $\underline{x}$  is the observed value of  $\underline{X}$ . For the problem of selecting the largest component we have

$$(2.1) \quad \sum_{i=1}^k \phi_i(\underline{x}) = 1, \quad \forall \underline{x}.$$

First, consider the subset selection problem. We call it Problem I. Let  $L_i(\underline{\theta})$  denote the loss incurred due to including the  $i$ th component in the selected subset, and let  $L_i^*(\underline{\theta})$  denote the loss due to excluding the  $i$ th component from the selected subset. The total loss due to selecting a subset  $(\delta)$  is given by

$$(2.2) \quad \begin{aligned} L(\delta, \underline{\theta}) &= \sum_{i=1}^k \delta_i L_i(\underline{\theta}) + \sum_{i=1}^k (1-\delta_i) L_i^*(\underline{\theta}) \\ &= \sum_{i=1}^k \delta_i (L_i(\underline{\theta}) - L_i^*(\underline{\theta})) + \sum_{i=1}^k L_i^*(\underline{\theta}) \end{aligned}$$

where  $\delta = (\delta_1, \dots, \delta_k)$  and  $\delta_i = 1(0)$  if the  $i$ th component is included in (excluded from) the selected subset. We assume that

$$(2.3) \quad \sum_{i=1}^k (L_i(\underline{\theta}) - L_i^*(\underline{\theta})) \leq 0, \quad \forall \underline{\theta}.$$

The above inequality implies that the loss due to including all the components in the selected subset is  $\leq$  the loss due to excluding all the components from



the selected subset. Therefore, we include at least one component in the selected subset.

Next, consider the problem of selecting the largest component. We call it Problem II. Using the same generic notation for the loss function as in Problem I, we let  $L_i(\underline{\theta})$  denote the loss due to selecting the  $i$ th component as the largest component, and let  $L_i^*(\underline{\theta})$  be the loss due to not selecting the  $i$ th component. The total loss due to selecting a component ( $\delta$ ) is given by (2.2), where now  $\delta_i = 1(0)$  if the  $i$ th component is selected (not selected) for the largest component.

Consider a special case of the loss function, given above. Let

$$(2.4) \quad L_i(\underline{\theta}) = \begin{cases} 0 & \text{if } \theta_i = \theta_{[k]} \\ 1 & \text{if } \theta_i \neq \theta_{[k]} \end{cases}$$
$$L_i^*(\underline{\theta}) = c(1 - L_i(\underline{\theta}))$$

where  $c$  is a positive number and  $\theta_{[k]} = \max(\theta_1, \dots, \theta_k)$ . We let  $c \geq k-1$  for Problem I and  $c = 1$  for Problem II. In Problem I the value of  $c$  measures the loss due to excluding the largest component from the selected subset, relative to the loss due to including a wrong component in the selected subset. The inequality (2.3) holds since  $c \geq k-1$ :

The risk, that is, the expected loss due to a selection rule  $\phi = \phi(\underline{x})$  is given by

$$(2.5) \quad R_\phi(\underline{\theta}) = \sum_{i=1}^k (L_i(\underline{\theta}) - L_i^*(\underline{\theta})) E \phi_i(\underline{x}) + \sum_{i=1}^k L_i^*(\underline{\theta})$$

In Problem I the risk for the loss function given by (2.4), is equal to the sum of  $c(1-PCS)$  and the expected number of wrong components included in the

selected subset. In Problem II the risk for the same loss function is equal to  $(1+c)(1-PCS)$ .

Let  $P_{\underline{\theta}}$  denote the conditional distribution of  $\underline{X}$  given  $\underline{\theta}$ . We suppose that the distribution has a density  $p_{\underline{\theta}}(\underline{x})$  with respect to a  $\sigma$ -finite measure  $\mu$  on  $R^k$ . For the Bayes formulation of the selection problem we assume that  $\underline{\theta}$  is distributed a priori according to a probability distribution  $G$ , say. The optimal selection rule is a functional  $\phi$  which minimizes the average of the risk function with respect to the given prior distribution of  $\underline{\theta}$ , given by

$$R_{\phi} = \int R_{\phi}(\underline{\theta}) dG(\underline{\theta}).$$

Let

$$(2.6) \quad M_i(\underline{x}) = \int (L_i(\underline{\theta}) - L_i^*(\underline{\theta})) p_{\underline{\theta}}(\underline{x}) dG(\underline{\theta})$$

$$(2.7) \quad M(\underline{x}) = \min (M_1(\underline{x}), \dots, M_k(\underline{x})) .$$

By virtue of (2.5) a Bayes rule for Problem I is given by

$$(2.8) \quad \phi_i(\underline{x}) = \begin{cases} 1 & \text{if } M_i(\underline{x}) \leq 0 \\ 0 & \text{otherwise} \end{cases} .$$

The Bayes rule for Problem II is given by

$$(2.9) \quad \phi_i(\underline{x}) = \begin{cases} 1 & \text{if } M_i(\underline{x}) = M(\underline{x}) \\ 0 & \text{otherwise} \end{cases} .$$

If  $M_i(\underline{x}) = M(\underline{x})$  for several values of  $i$ , we select the smallest among the tied values of  $i$  for the largest component. We note that the Bayes rules (2.8) and (2.9) are both non-randomized selection rules.

We illustrate our results by the following example.

Example. Let  $\underline{X}$  be distributed according to a multivariate normal distribution  $N(\underline{\theta}, \Sigma)$ , where the covariance  $\Sigma$  is a diagonal matrix, the  $i$ th element on the diagonal being denoted by  $\sigma_i^2$ . Let the loss function be given by (2.4) and let  $\underline{\theta}$  be distributed a priori according to  $N(\underline{0}, \tau^2 \mathbf{I})$ , where  $\mathbf{I}$  denotes an identity matrix. Let  $\phi(x)$  and  $\Phi(x)$  denote the standard normal density and cdf, respectively, and let

$$\lambda_i^2 = \tau^2 / (\tau^2 + \sigma_i^2) .$$

By direct computation we get

$$(2.10) \quad M_j(\underline{x}) = p(\underline{x}) [1 - (1+c) \int_{-\infty}^{\infty} \prod_{j=1(j \neq i)}^k \phi \left( \frac{\lambda_i \sigma_j}{\lambda_j \sigma_j} u + \frac{\lambda_i^2 x_i - \lambda_j^2 x_j}{\lambda_j \sigma_j} \right) \phi(u) du]$$

where

$$\begin{aligned} p(\underline{x}) &= \int p_{\underline{\theta}}(\underline{x}) dG(\underline{\theta}) \\ &= \left( \prod_{i=1}^k (2\pi(\sigma_i^2 + \tau^2))^{-\frac{1}{2}} \right) \exp \left( -\frac{1}{2} \sum_{i=1}^k x_i^2 / (\sigma_i^2 + \tau^2) \right) \end{aligned}$$

denotes the marginal density of  $\underline{X}$ .

If we let  $\tau \rightarrow \infty$ , so that the prior distribution of  $\underline{\theta}$  tends to be non-informative, then

$$(2.11) \quad M_j(\underline{x}) = p(\underline{x}) [1 - (1+c) \int_{-\infty}^{\infty} \prod_{j=1(j \neq i)}^k \phi \left( \frac{\sigma_j}{\sigma_j} u + \frac{x_i - x_j}{\sigma_j} \right) \phi(u) du] .$$

Hence, the  $i$ th component is included in the selected subset if

$$(2.12) \quad \int_{-\infty}^{\infty} \prod_{j=1(j \neq i)}^k \phi \left( \frac{\sigma_j}{\sigma_j} u + \frac{x_i - x_j}{\sigma_j} \right) \phi(u) du \geq \frac{1}{1+c} .$$

The above inequality holds for sufficiently large values of  $(x_i - x_j)/\sigma_j$ ,  $j \neq i$ . In Problem II we select the  $r$ th component if the quantity on the left side of (2.12) is maximized for  $i=r$ . For  $k=2$  this quantity is equal to

$$\phi \left( (x_i - x_j) / (\sigma_i^2 + \sigma_j^2)^{1/2} \right).$$

Therefore, we select the component associated with the larger of the two values  $x_1$  and  $x_2$ .

Let  $\sigma_1 = \dots = \sigma_k = \sigma$ , say, giving  $\lambda_1 = \dots = \lambda_k = \lambda$ , say. We have

$$(2.13) \quad M_i(\underline{x}) = p^0(\underline{x}) [1 - (1+c) \int_{-\infty}^{\infty} \prod_{j=1, j \neq i}^k \phi \left( u + \frac{\lambda}{\sigma} (x_i - x_j) \right) \phi(u) du]$$

where  $p^0(\underline{x})$  is obtained from  $p(\underline{x})$  by substituting  $\sigma$  for  $\sigma_i$ ,  $i = 1, \dots, k$ .

We find that  $M_i(\underline{x}) \leq 0$  for  $x_i - x_j \geq 0$ ,

$j = 1, \dots, i-1, i+1, \dots, k$ . Therefore, we include the  $i$ th component in the selected subset if none of these differences is negatively large. Also,  $M_i(\underline{x})$  is minimized for the value of  $i$  associated with the largest component of  $\underline{x}$ . Therefore, in Problem II we select the component associated with the largest value among  $x_1, \dots, x_k$ . These are ordinary selection rules. We see them as Bayes rules.

3. Just rules. First we define a stochastically increasing property (SIP) of a class of multivariate distributions. A set  $A \subset R^k$  is said to be monotone if, if  $\underline{x} \in A$  and  $y_i \geq x_i$ ,  $i=1, \dots, k$  then  $\underline{y} \in A$ . Let  $P_{\underline{\theta}}$  be a family of probability distributions on  $R^k$  indexed by a vector parameter  $\underline{\theta} = (\theta_1, \dots, \theta_k)$ . Let  $\Omega \subset R^k$  denote the parameter space. The family of distributions  $P_{\underline{\theta}}$  is said to have SIP with respect to  $\underline{\theta}$  if, if  $\underline{\theta} \in \Omega$ ,  $\underline{\theta}' \in \Omega$  and  $\theta_i \leq \theta'_i$ ,  $i = 1, \dots, k$  then  $P_{\underline{\theta}}(A) \leq P_{\underline{\theta}'}(A)$  for all monotone sets  $A$ . A character-

ization of the SIP, due to Lehmann (1955), is given as follows: A function  $\psi(\underline{x})$  is said to be nondecreasing in  $\underline{x}$  if, if  $x_i \leq x'_i$ ,  $i = 1, \dots, k$  then  $\psi(\underline{x}) \leq \psi(\underline{x}')$ . A family of distributions  $P_{\underline{\theta}}$  is said to have SIP with respect to  $\underline{\theta}$  if and only if, if  $\psi(\underline{x})$  is nondecreasing in  $\underline{x}$  then  $\int \psi(\underline{x}) dP_{\underline{\theta}}(\underline{x})$  is nondecreasing in  $\underline{\theta}$ .

Now we define a just rule. A selection rule  $\phi$  is said to be just if  $\phi_i(\underline{x})$  is nondecreasing in  $x_i$  and nonincreasing in  $x_j$  ( $j \neq i$ ) for  $i, j = 1, \dots, k$ . Theorem 1 below, shows that the Bayes rules given by (2.8) and (2.9) are just if the following assumptions are valid.

Assumption 1 - The posterior distribution of  $\underline{\theta}$  given  $\underline{x}$ , has SIP with respect to  $\underline{x}$ .

Assumption 2 - The function  $L_i(\underline{\theta}) - L_i^*(\underline{\theta})$  is nonincreasing in  $\theta_i$  and nondecreasing in  $\theta_j$  ( $j \neq i$ ).

Assumption 3 -  $M_i(\underline{x})$  is a continuous function of  $\underline{x}$ .

Assumption 3 is valid if, for example, the loss functions  $L_i(\underline{\theta})$  and  $L_i^*(\underline{\theta})$  are bounded and  $p_{\underline{\theta}}(\underline{x})$  is continuous in  $\underline{x}$  uniformly for  $\underline{\theta} \in \Omega$ .

Theorem 1. If assumptions 1 and 2 hold then the Bayes rule (2.8) is just. If moreover Assumption 3 holds then the Bayes rule (2.9) is just.

Proof: From the characterization of the SIP given above, and Assumptions 1 and 2, it follows that  $M_i(\underline{x})$  as given by (2.6), is nonincreasing in  $x_i$  and nondecreasing in  $x_j$  ( $j \neq i$ ). Therefore, the function  $\phi_i(\underline{x})$  as given by (2.8), is nondecreasing in  $x_i$  and nonincreasing in  $x_j$  ( $j \neq i$ ). Hence, the Bayes rule (2.8) is just if Assumptions 1 and 2 hold. If moreover, Assumption 3 holds then it follows from the continuity and monotonicity property of  $M_i(\underline{x})$  that  $\phi_i(\underline{x})$ , given by (2.9), is nondecreasing in  $x_i$  and nonincreasing in  $x_j$  ( $j \neq i$ ). Hence, the Bayes rule (2.9) is just.  $\square$

In the application of Theorem 1 it would be interesting to find for a given family of distributions  $P_{\underline{\theta}}$  which is stochastically increasing in  $\underline{\theta}$ , the family of prior distributions for which Assumption 1 holds. We have not investigated this problem to any length. We discuss below some cases in which an appropriate prior distribution can be found for which Assumption 1 holds.

Clearly,  $P_{\underline{\theta}}(\underline{x})$  has SIP with respect to  $\underline{\theta}$  if  $\underline{\theta}$  is a location parameter of the conditional distribution. The posterior distribution of  $\underline{\theta}$  with respect to a non-informative prior distribution  $G$  which is uniform on  $R^k$ , has SIP with respect to  $\underline{x}$ . Similarly,  $P_{\underline{\theta}}(\underline{x})$  has SIP with respect to  $\underline{\theta}$  if  $\underline{\theta}$  is a scale parameter (component-wise) of the conditional distribution. The posterior distribution of  $\underline{\theta}$  with respect to the non-informative prior on  $R^k$  with density function  $g(\underline{\theta}) \propto (\theta_1 \dots \theta_k)$  has SIP with respect to  $\underline{x}$ .

The prior distributions for the location and scale parameters considered above, which lead to the posterior distribution with SIP are both improper distributions. We give now an example of a proper prior distribution for a location parameter. Let  $P_{\underline{\theta}}$  denote the multivariate normal distribution  $N(\underline{\theta}, \Sigma)$  and let  $\underline{\theta}$  be distributed a priori according to  $N(\underline{0}, \Omega)$ . A posteriori,  $\underline{\theta}$  is distributed according to the normal distribution  $N(A \underline{x}, (\Sigma^{-1} + \Omega^{-1})^{-1})$ , where

$$A = (\Sigma^{-1} + \Omega^{-1})^{-1} \Omega^{-1}.$$

If the elements of  $A$  are non-negative then the posterior distribution of  $\underline{\theta}$  has SIP with respect to  $\underline{x}$ . The elements of  $A$  are positive if the covariance matrices  $\Sigma = (\sigma_{ij})$  and  $\Omega = (\omega_{ij})$  are given by  $\sigma_{ii} = 1$ ,  $\sigma_{ij} = \rho$  ( $i \neq j$ ),  $\rho < 0$ ,  $\omega_{ii} = \frac{1}{\lambda}$  and  $\omega_{ij} = 0$  ( $i \neq j$ ). The matrix  $A$  is given by

$$A = (1 + \lambda(1 - \rho))^{-1} \left( I - \frac{\lambda \rho}{1 + \lambda + \lambda \rho(k-1)} J \right)$$

where  $I$  denotes the identity matrix and  $J$  denotes a matrix whose elements are each equal to 1.

## References

- [1] Berger, R. L. and Gupta, S. S. (1980). Minimax subset selection with applications to unequal variances (unequal sample size) problems. Scand. J. Statist. 7: 21-26.
- [2] Gibbons, J. D., Olkin, I. and Sobel, M. (1977). Selecting and Ordering Populations - A New Statistical Methodology. John Wiley and Sons, New York.
- [3] Lehmann, E. L. (1955). Ordered families of distributions. Ann. Math. Statist. (26) 399-419.

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