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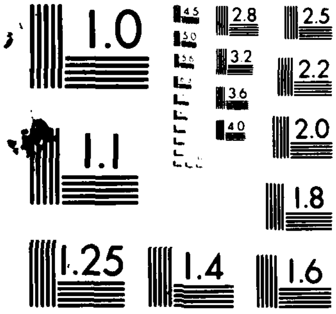

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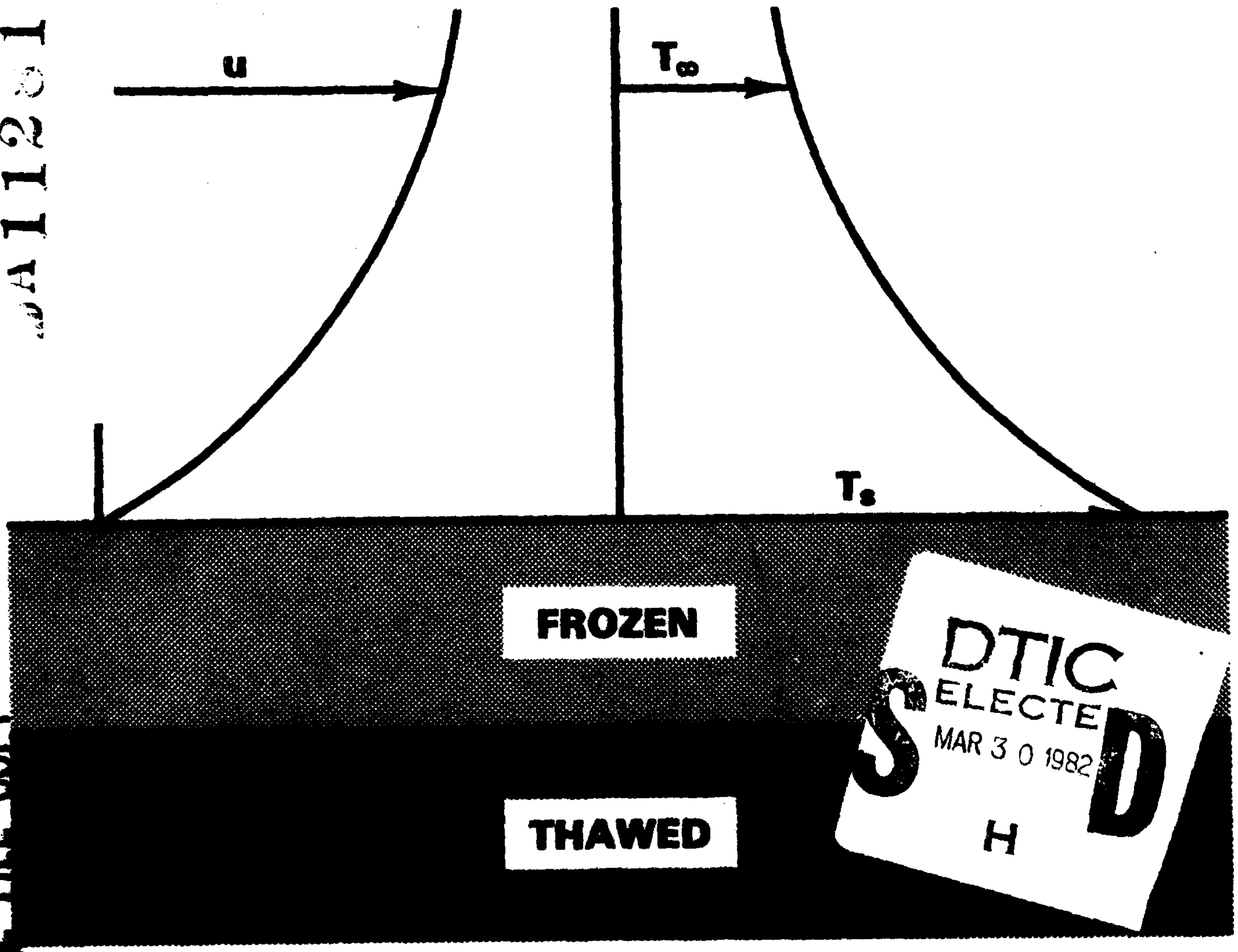

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CRREL Report 81-25

Application of the heat balance integral to conduction phase change problems

Virgil J. Lunardini

December 1981

UNITED STATES ARMY CORPS OF ENGINEERS
COLD REGIONS RESEARCH AND ENGINEERING LABORATORY
HANOVER, NEW HAMPSHIRE, U.S.A.

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The problem of heat conduction with phase change (often called the Stefan problem) includes some of the most intractable mathematical areas of heat transfer. Exact solutions are extremely limited and approximate methods are widely used. This report discusses the collocation method for the heat balance integral approximation. The method is applied to some standard problems of phase change (Neumann's problem) and a new solution is presented for the case of surface convection for a semi-infinite body. Numerical results are given for soil systems and also for materials of interest in latent heat thermal storage.

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PREFACE

This report was prepared by Dr. Virgil J. Lunardini, Mechanical Engineer, Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory.

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The numerical calculations and the computer program were prepared by T. Carpenter.

CONTENTS

ILLUSTRATIONS

TABLE

NOMENCLATURE

a_1, a_2	coefficients in eq 13
b	$\dfrac{2k_{21}\,\theta_m + \alpha_{21}}{\theta_\infty - \theta_m}$
B	$(\theta_\infty - \theta_m)b$
B_1	parameter defined by eq 18
c	mass specific heat
c_{21}	c_2/c_1
F	defined by eq 27
G	specified surface heat flux
h	surface coefficient of convection
k	thermal conductivity
k_{21}	k_2/k_1
ℓ	mass latent heat of fusion
q	surface heat transfer rate per unit area
q^*	$\dfrac{q}{h(T_\infty - T_f)}$
S	$\dfrac{GX}{\rho_1 \ell \alpha_1}$, dimensionless phase change depth
S_T	$\dfrac{c_1}{\ell}\,(T_s - T_f)$, Stefan number
S_{Tm}	$\dfrac{c_1}{\ell}\,(T_\infty - T_f)$, modified Stefan number
t	time
t_0	time at which phase change starts
T	temperature
x	Cartesian coordinate
x_ℓ	volumetric water content
X	phase change depth
α	thermal diffusivity
α_{21}	α_2/α_1
β	$\dfrac{T_f - T_0}{T_s - T_f}$
γ	$1 + 2S_{Tm}$
δ	temperature penetration depth
Δ	$\dfrac{G\delta}{\rho_1 \ell \alpha_1}$, dimensionless penetration depth
θ_m	$\dfrac{c_1}{\ell}\,(T_f - T_0)$
θ_w	$\dfrac{c_1}{\ell}\,[T_1\,(0,t) - T_0]$
θ_∞	$\dfrac{c_1}{\ell}\,(T_\infty - T_0)$
ξ	integrated temperature
ρ	density
σ	$\dfrac{h}{k_1}X$, dimensionless phase change depth
τ	$\dfrac{h^2\,(T_\infty - T_f)\,(t - t_0)}{\rho_1\,k_1 \ell}$, dimensionless time
τ^*	$\dfrac{G}{\rho_1^2 \ell^2 \alpha_1} \displaystyle\int_0^t \sqrt{G(t')dt'}$, dimensionless time
ϕ	$\dfrac{h}{k_1}\,\delta$, dimensionless temperature penetration depth

Subscripts

0	initial value
1,2	thawed and frozen regions, for thaw case
f	fusion value
s	surface value
∞	ambient value

CONVERSION FACTORS: U.S. CUSTOMARY TO METRIC (SI) UNITS OF MEASUREMENT

These conversion factors include all the significant digits given in the conversion tables in the ASTM Metric Practice Guide (E 380), which has been approved for use by the Department of Defense. Converted values should be rounded to have the same precision as the original (see E 380).

Multiply	*By*	*To obtain*
lbm/ft^3	16.01846	kg/m^3
Btu/lbm	2326.000*	J/kg
Btu/lbm °F	4186.800*	J/kg K
Btu/hr °F ft	1.730735	W/m K

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*Exact

APPLICATION OF THE HEAT BALANCE INTEGRAL TO CONDUCTION PHASE CHANGE PROBLEMS

Virgil J. Lunardini

INTRODUCTION

Problems of freezing and thawing arise frequently in such diverse applications as thermal design in permafrost regions, thermal storage of latent heat for solar systems, and the heat treatment of metals. One is often interested in the penetration rate of the phase change interface, the temperature field, and the boundary heat transfer rates. These problems fall into the category of conductive heat transfer with solidification phase change. From an engineering design viewpoint, exact solutions are sought for geometries and boundary conditions that are simple and yet representative of significant systems. Unfortunately the mathematical difficulties are such that exact solutions to this class of problems are limited to a few very special geometries and boundary conditions (Lunardini 1981). However, a number of approximate methods have been developed that can yield solutions acceptable for engineering design. This report describes one of these approximations: the heat balance integral method.

This method, which has been used with good results for phase change problems, involves the concept of the temperature penetration depth. Consider the semi-infinite solid shown in Figure 1. At a time t, after the surface temperature has jumped to T_s, the temperature in the solid will be disturbed to a depth $X(t) + \delta(t)$. Beyond this depth, the temperature of the solid remains at the initial temperature T_0 and no energy is transferred beyond this point. The penetration distance $X + \delta$ is analogous to the boundary layer thickness in fluid mechanics. The heat balance integral method is similar to the momentum integral method in that the basic equations are satisfied on average over the volume of thickness $X(t) + \delta(t)$. This avoids solving the partial differential equation at each point within the domain of interest.

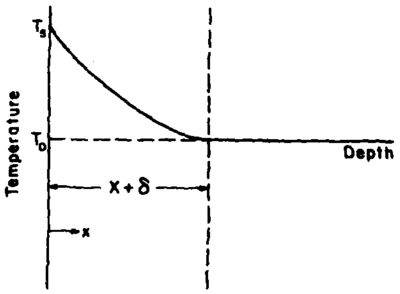

Figure 1. Temperature penetration depth.

The conduction equation, with constant thermal properties, is

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{1}$$

where α is thermal diffusivity. Now this equation is spatially integrated over the distance $X(t) + \delta(t)$. Thus

$$\int_0^{X+\delta} \alpha \frac{\partial^2 T}{\partial x^2} dx = \int_0^{X+\delta} \frac{\partial T}{\partial t} dx .$$

The left-hand side of this equation is

$$\alpha \int_0^{X+\delta} \frac{\partial^2 T}{\partial x^2} dx = \alpha \left[\frac{\partial T(X+\delta,t)}{\partial x} - \frac{\partial T(0,t)}{\partial x} \right] . \tag{2}$$

Leibniz's rule for a general function is

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f(b,t)\,\frac{db}{dt} - f(a,t)\,\frac{da}{dt} + \int_a^b \frac{\partial f}{\partial t}\,(x,t)\,dx\,.$$

Then

$$\int_0^{X+\delta} \frac{\partial T}{\partial t}\,dx = \frac{d}{dt}\int_0^{X+\delta} T(x,t)\,dx + T(X+\delta)\,\frac{d(X+\delta)}{dt}\,.$$

Let

$$\xi = \int_0^{X+\delta} T(x,t)\,dx\,. \qquad (3)$$

Then the heat balance integral equation is

$$\frac{d\xi}{dt} + \alpha\,\frac{\partial T(0,t)}{\partial x} - T_0\,\frac{d(X+\delta)}{dt} = 0\,. \qquad (4)$$

This equation is valid if there is no phase change. Consider the case of phase change with the properties of the frozen region different from those of the thawed region. Using the procedure outlined above, there will then be two integral equations as follows:

$$\frac{d\xi_1}{dt} - T_f\,\frac{dX}{dt} - \alpha_1\left[\frac{\partial T_1(X,t)}{\partial x} - \frac{\partial T_1(0,t)}{\partial x}\right] = 0 \qquad (5)$$

$$\frac{d\xi_2}{dt} - T_0\,\frac{d(X+\delta)}{dt} + T_f\,\frac{dX}{dt} + \alpha_2\,\frac{\partial T_2(X,t)}{\partial x} = 0 \qquad (6)$$

where

$$\xi_1 = \int_0^{X} T_1\,(x,t)\,dx$$

$$\xi_2 = \int_X^{X+\delta} T_2\,(x,t)\,dx$$

and $T_1(X,t) = T_2(X,t) = T_f$, $T_2(X+\delta,t) = T_0$ have been used. The energy balance at the phase change interface is

$$k_1\,\frac{\partial T_1(X,t)}{\partial x} - k_2\,\frac{\partial T_2(X,t)}{\partial x} = -\rho_1 \ell\,\frac{dX}{dt}\,. \qquad (7)$$

The solution of a general problem with superheating or subcooling (the initial temperature is above or below the fusion value) will involve two coupled, nonlinear differential equations for the parameters X and δ. The solution will normally be tedious and often requires a starting solution to handle the singularity at the origin. However, assume that the initial temperature is T_f. Then the problem reduces to only one differential equation since the penetration distance $X+\delta$ is now identical to the phase change depth X:

$$\frac{d\xi_1}{dt} - T_f\,\frac{dX}{dt} - \alpha_1\left[\frac{\partial T_1(X,t)}{\partial x} - \frac{\partial T_1(0,t)}{\partial x}\right] = 0 \qquad (8)$$

$$\xi_1 = \int_0^{X} T_1(x,t)\,dx\,. \qquad (9)$$

The heat balance integral method has been used extensively for single phase problems (Goodman 1958, 1964, Goodman and Shea 1960, Poots 1962, Lardner and Pohle 1961, Bell 1978) and also for the much more complicated two-phase problems (Lunardini 1980, Lunardini and Varotta 1981). The single phase problems are also referred to as nonsubcooling problems since the initial temperature is identical to the fusion temperature.

COLLOCATION METHOD

The usual heat balance integral equations for two-phase problems are coupled and the solution can be difficult. A slight variation of the heat balance integral method can be used to find an explicit functional relation between δ and X that will uncouple the equations and simplify the solution.

If eq 5-7 are added together the result will be the overall energy balance for the volume of interest:

$$\frac{d}{dt}\left[\rho_1 c_1 \xi_1 + \rho_2 c_2 \xi_2 + \rho_1 \ell X + (\rho_2 c_2 - \rho_1 c_1)\,T_f X - \rho_2 c_2 T_0 (X+\delta)\right] = -\,k_1\,\frac{\partial T_1(0,t)}{dx}\,. \qquad (10)$$

The term $(\rho_2 c_2 - \rho_1 c_1)\,T_f\,dX/dt$, in eq 10, is the net sensible flux of enthalpy at the phase change interface due to the sudden jump in the specific heats of the frozen and thawed volumes. This term was omitted in a recent study by Yuen (1980), although Yuen's derivations implicitly assumed that $\rho_2 c_2 \approx \rho_2 c_1$

at the phase change interface. The retention of the sensible enthalpy term gives better numerical comparisons to exact solutions.

Equation 7 can be rewritten as two collocation equations (see Lunardini 1981):

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} + k_2 \frac{\partial T_2(X,t)}{\partial x} = -\rho_1 \ell \alpha_1 \times \frac{\partial^2 T_1(X,t)}{\partial x^2} \Bigg/ \frac{\partial T_1(X,t)}{\partial x} \tag{11}$$

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} + k_2 \frac{\partial T_2(X,t)}{\partial x} = -\rho_2 \ell \alpha_2 \times \frac{\partial^2 T_2(X,t)}{\partial x^2} \Bigg/ \frac{\partial T_2(X,t)}{\partial x} . \tag{12}$$

For semi-infinite solids the following temperature approximations can be used:

$$T_1 = T_f + a_1(x-X) + a_2(x-X)^2 \tag{13}$$

$$T_2 = T_f - 2\,\frac{(T_f - T_0)}{\delta}\,(x-X) + \frac{(T_f - T_0)}{\delta^2}\,(x-X)^2 . \tag{14}$$

Equation 13, representing the temperature in the region which has changed phase, contains two unknown coefficients. One of these can be found from the specified boundary condition at $x = 0$. Combining eq 11-14 yields

$$\frac{a_2}{a_1} = -\frac{a_{21}}{2\delta} . \tag{15}$$

NEUMANN PROBLEM

The surface temperature of the volume changes to a constant T_s at the start of phase change (see Fig. 2). This problem has been solved exactly by Neumann (c. 1860) and approximately by Lunardini and Varotta (1981) using the heat balance integral. The solution to eq 10–15 is

$$X = 2\psi\sqrt{a_1 t} \tag{16}$$

$$\psi^2 = \frac{B_1 + \alpha_{21}}{(\alpha_{21} + 2B_1)\left[\frac{1}{2} + c_{21}\beta + \frac{1}{3}c_{21}B_1\beta + \frac{1}{S_T}\right] - \frac{1}{6}\alpha_{21}} . \tag{17}$$

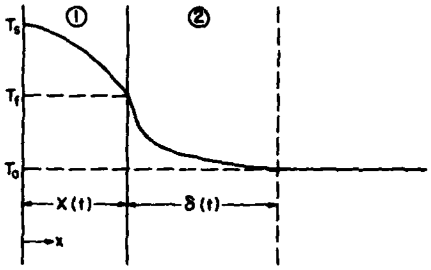

Figure 2. Geometry of the Neumann problem.

$$B_1 = k_{21}\beta + \frac{\alpha_{21}}{2S_T} + \sqrt{\left(k_{21}\beta + \frac{\alpha_{21}}{2S_T}\right)^2 + k_{21}\,\alpha_{21}\beta + \frac{\alpha_{21}^2}{2S_T}} \tag{18}$$

Equations 17 and 18 reduce to those of Lunardini and Varotta (1981) for the single phase case when $\beta = 0$. When $\beta \neq 0$, eq 17 and 18 agree well with the exact solution but are less accurate than the solution of Lunardini and Varotta (1981); the maximum errors of about 15% occur at low Stefan numbers with high β values.

Although this solution is for a step change in surface temperature, it has been shown that the solution is valid for a sinusoidal surface temperature if the step change temperature is the average value of the sinusoidal temperature over one half cycle.

SPECIFIED SURFACE HEAT FLUX

The problem of a specified surface heat flux can also be solved in a closed form. The surface temperature will increase from T_0 to the fusion value T_f when melting begins (see Fig. 3) and the phase change solution can then be obtained.

The surface boundary condition is

$$-k_1 \frac{\partial T_1(0,t)}{\partial x} = G(t) . \tag{19}$$

Equations 13 and 15 lead to

$$a_1 = \frac{G}{k_1}\left|\frac{a_{21}X}{\delta + a_{21}X} - 1\right|$$

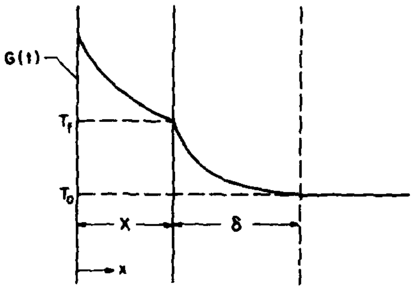

Figure 3. Specified surface heat flux for a semi-infinite medium.

$$a_2 = \frac{G\alpha_{21}}{2k_1\,(\delta + \alpha_{21}X)}\,.$$

The collocation method allows a simple relation to be derived between δ and X. By using equation 12 this is

$$\Delta = \frac{B}{2} + \sqrt{\frac{B^2}{4} + \alpha_{21}\,B\,S} \tag{20}$$

where

$$B = 2k_{21}\,\theta_m + \alpha_{21}.$$

Equation 10 can now be solved for the phase change depth S. The result is

$$\begin{aligned}\frac{\alpha_{21}}{6}\,S^3 &+ S^2\left[\frac{\Delta}{2} + \alpha_{21}\,(1 + c_{21}\,\theta_m)\right] \\ &+ S\left[(1 + c_{21}\,\theta_m)\Delta + \frac{1}{3}\,k_{21}\,\theta_m\,(\Delta - B)\right] \\ &+ \frac{1}{3}\,c_{21}\,\theta_m\,\Delta(\Delta - B) = \tau^*\,(\Delta + \alpha_{21}\,S)\,.\end{aligned} \tag{21}$$

There is no exact solution of this problem for comparison, but approximate solutions can be found for the single phase case when $\theta_m = 0$. Equation 21 then reduces to

$$\tau^* = \frac{S}{6}\,(5 + S + \sqrt{1+4S})\,. \tag{22}$$

This is exactly the equation obtained by Goodman (1958) with the usual heat balance integral. This solution has been shown to be in good agreement with an analog solution be Kreith and Romie (1955).

Lozano and Reemsten (1981) derived an exact solution for the single phase case. The solution for $S_{Tm} = 0.2$ was essentially identical to eq 22. Unfortunately the exact solution converges so slowly for large time values that it is inefficient for numerical computations. The surface temperature (for $t > t_0$) is given by

$$\theta_w - \theta_m = \frac{\alpha_{21}\,S^2 + 2S\Delta}{2(\Delta + \alpha_{21}\,S)}\,. \tag{23}$$

As has been pointed out by Goodman (1964), the solutions here are valid only if $G(t)$ is monotonically increasing with time or is a constant. Pulse type heat fluxes will not yield correct solutions.

CONVECTIVE SURFACE HEAT FLUX

A problem of importance is that of heat flow from the environment, by convection, to or from a volume which is undergoing phase change. The situation is shown in Figure 4 for thawing. This problem is physically more significant than the Neumann problem because the ambient temperature and convective heat transfer are specified rather than the surface temperature.

The surface boundary condition is

$$-k_1\,\frac{\partial T_1\,(0,t)}{\partial x} = h[T_\infty - T_1\,(0,t)]\,. \tag{24}$$

Equations 13 and 15 now yield

$$a_1 = \frac{-\ell}{c_1}\;\frac{(\theta_\infty - \theta_m)}{X\left(1 + \dfrac{\alpha_{21}}{2\delta}\,X\right) + \dfrac{k_1}{h}\left(1 + \dfrac{\alpha_{21}}{\delta}\,X\right)}$$

Again using eq 12

$$\phi = \frac{b\,(\sigma + 1)}{2} + \sqrt{\frac{b^2(\sigma + 1)^2}{4} + \alpha_{21}\,\sigma\left(\frac{\sigma}{2} + 1\right)b} \tag{25}$$

where

$$b = \frac{2k_{21}\,\theta_m + \alpha_{21}}{(\theta_\infty - \theta_m)}\,.$$

where

$$\phi = \frac{h}{k_1}\,\delta$$

$$\sigma = \frac{h}{k_1}\,X\,.$$

The energy balance equation, eq 10, can now be written as

$$\frac{dF}{d\tau} = \frac{2\,(\phi + \alpha_{21}\,\sigma)}{2\phi(\sigma + 1) + \alpha_{21}\,(\sigma + 2)} \tag{26}$$

and

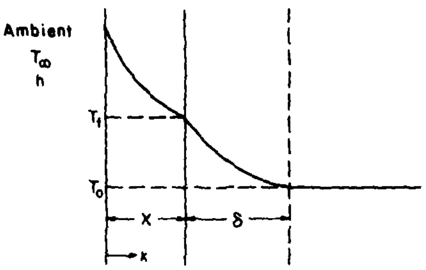

Figure 4. Surface convection for a semi-infinite body.

$$F = \frac{(\theta_\infty - \theta_m)\,\sigma^2\,(\phi + \frac{1}{3}\alpha_{21}\,\sigma)}{2\phi\,(\sigma+1) + \alpha_{21}\,\sigma(\sigma+2)}$$

$$+ \sigma(1 + c_{21}\theta_m) + \frac{1}{3}\,c_{21}\theta_m\phi. \qquad (27)$$

Equation 26 can be written as

$$2\tau = \int_0^\sigma Q\,d\sigma' \qquad (28)$$

where

$$\begin{aligned} PQ &= (2\phi + \alpha_{21}\,\sigma)\,\sigma S_{\mathrm{Tm}} + (1 + C_{21}\theta_m)g \\ &\quad +2\,[\sigma(1 + C_{21}\theta_m) + \tfrac{1}{3}\,C_{21}\theta_m\phi] \\ &\quad \times\,(P + \alpha_{21}) + S_{\mathrm{Tm}}\,\sigma^2 + \tfrac{1}{3}\,C_{21}\theta_m g \\ &\quad +2\,[\sigma(1 + C_{21}\theta_m) + \tfrac{1}{3}\,C_{21}\theta_m\phi]\,(\sigma+1) \\ &\quad -2\,(\sigma+1)F\,\frac{b(P+\alpha_{21})}{2\phi - b(\sigma+1)} - 2\,(P + \alpha_{21})\,F \end{aligned}$$

where $P = \phi + \alpha_{21}\sigma$ and $g = 2\,[P(1-\sigma) - \sigma\phi]$.

There is no exact solution of eq 26 for comparison but it can be shown that when $\theta_m = 0$ and $S_{\mathrm{Tm}} = (\theta_\infty - \theta_m) = 0$, eq 26 can be solved as

$$\tau = \frac{\sigma^2}{2} + \sigma \qquad (29)$$

or

$$\sigma = -1 + \sqrt{1 + 2\tau}. \qquad (30)$$

Physically this is a single phase problem with the latent heat predominating. Equation 30 is the quasi-steady solution (Lunardini 1981).

The numerical solution to eq 28, when $\theta_m = 0$, is identical to the heat balance integral solution of Goodman (1958):

$$\begin{aligned} 12\gamma\,\tau &= [(1+2\gamma) + (2+\gamma)\sigma]\,[1 + \gamma\sigma(2+\sigma)]^{1/2} \\ &\quad - \frac{2(\gamma-1)}{\sqrt{\gamma}}\,\ln\frac{[1+\gamma\sigma(2+\sigma)]^{1/2} + [(1+\sigma)\gamma]^{1/2}}{1+\sqrt{\gamma}} \\ &\quad - 4\gamma(\gamma-1)\,\ln\frac{-1+\gamma(2+\sigma) + [1+\gamma\sigma(2+\sigma)]^{1/2}}{2\gamma} \\ &\quad + (\gamma^2 + 5\gamma)\,\frac{\sigma^2}{2} + 2\,(\gamma^2 + 4\gamma - 2)\,\sigma - 1 - 2\gamma \qquad (31) \end{aligned}$$

where $\gamma = 1 + 2\,S_{\mathrm{Tm}}$. Equation 31 reduces to eq 29 when $S_{Tm} = 0$.

Cho and Sunderland (1981) presented an approximate method of solving this problem for the single phase case ($\theta_m \equiv 0$). Their results agree very well with eq 31, but they note that the zero-subcooling solution is a good approximation to the subcooling problem when $\theta_m \neq 0$. This is not true, as can be seen from the graphs presented here. The subcooling has a very significant effect upon the rate of phase change and may be ignored only at the risk of serious error.

The surface temperature is

$$\frac{T_1\,(0,t) - T_f}{T_\infty - T_f} = \frac{\sigma(2\phi + \alpha_{21}\,\sigma)}{\sigma(2\phi + \alpha_{21}\sigma) + 2(\phi + \alpha_{21}\sigma)}. \qquad (32)$$

The nondimensional surface heat transfer rate is

$$q^* = \frac{(\phi + \alpha_{21}\sigma)}{\sigma(\phi + \frac{1}{2}\,\alpha_{21}\sigma) + (\phi + \alpha_{21}\sigma)}. \qquad (33)$$

Equation 28 can be solved by simple, numerical, quadrature. Figures 5-14 are plots of the solution for some values of Stefan number and θ_m, with property ratios given as functions of the volumetric water content for soil systems. As has been noted, the heat balance integral method yields solutions that compare quite well with the few exact solutions. Thus the graphs presented here should be accurate for normal engineering design, especially since the soil thermal properties will normally be known only to within 10-20%.

Storage of thermal energy, as latent heat, is becoming more significant as solar energy becomes more important. In general, the storage of thermal energy will play an increasingly important role in energy

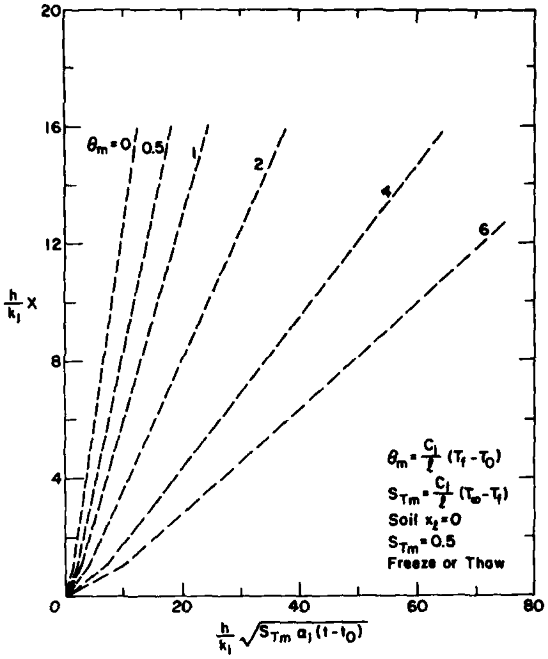

Figure 5. Surface convection for soil, x_ℓ (volumetric water content) = 0, S_{Tm} = 0.5

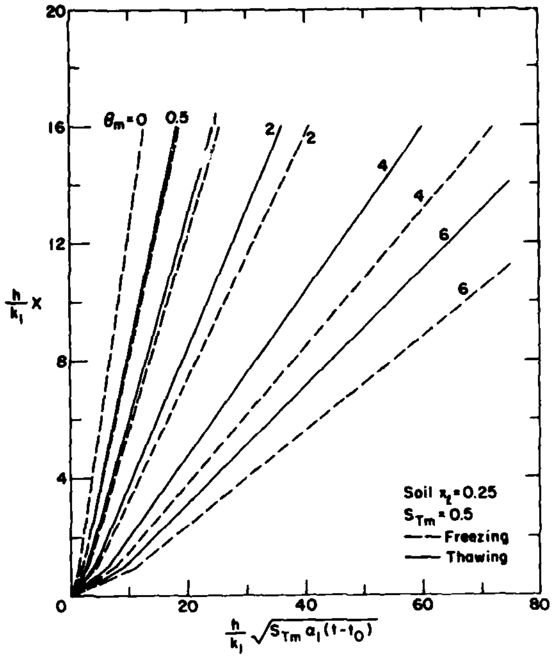

Figure 6. Surface convection for soil, x_ℓ = 0.25, S_{Tm} = 0.5.

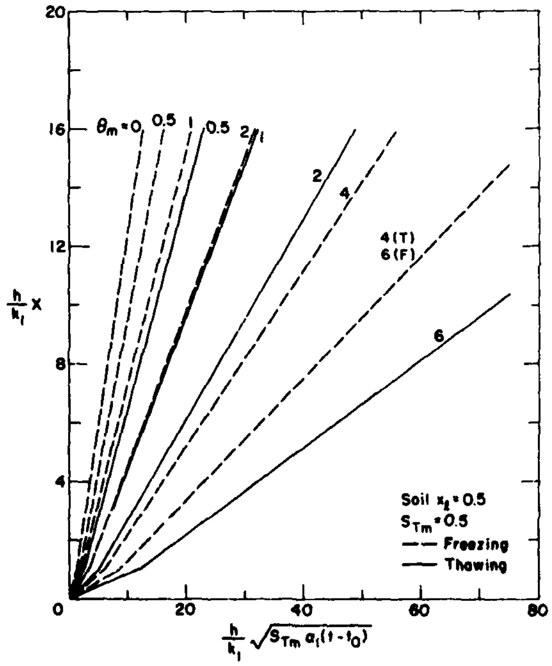

Figure 7. Surface convection for soil, x_ℓ = 0.50, S_{Tm} = 0.5

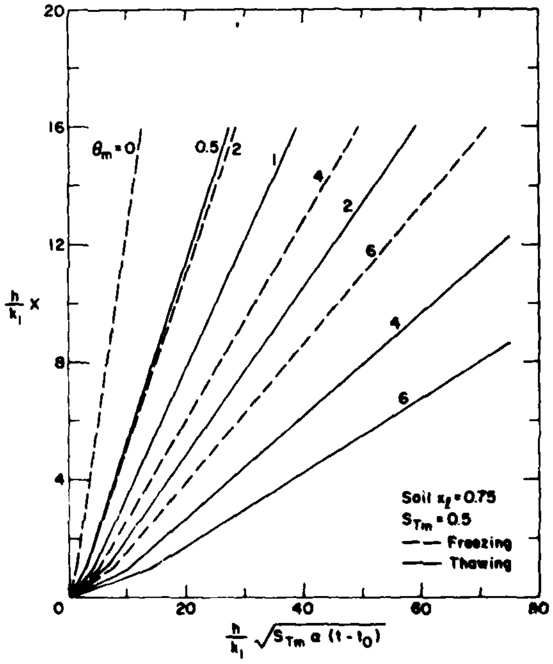

Figure 8. Surface convection for soil, x_ℓ = 0.75, S_{Tm} = 0.5.

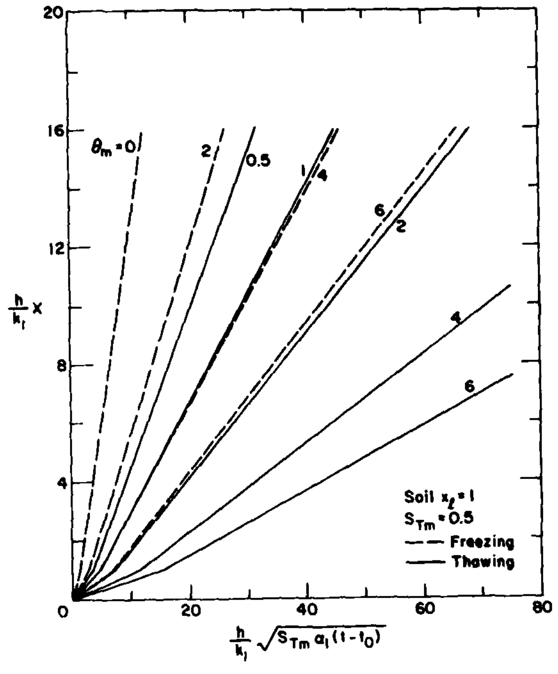

Figure 9. Surface convection for soil, $x_\ell = 1.0$, $S_{Tm} = 0.5$

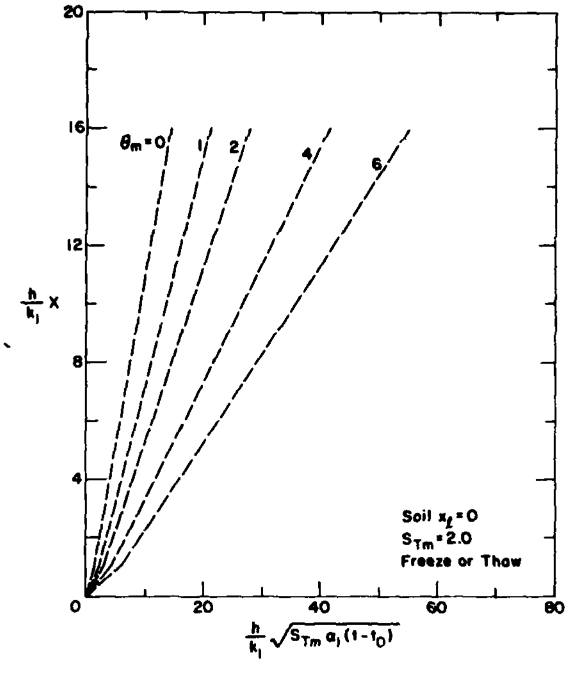

Figure 10. Surface convection for soil, $x_\ell = 0$, $S_{Tm} = 2$.

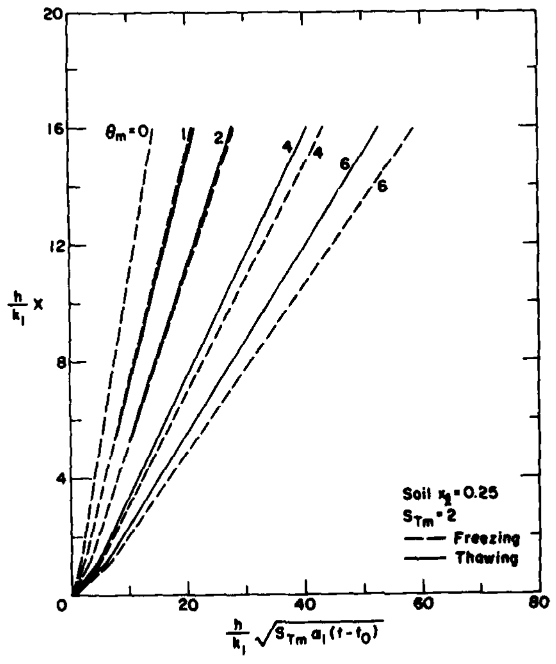

Figure 11. Surface convection for soil, $x_\ell = 0.25$, $S_{Tm} = 2$.

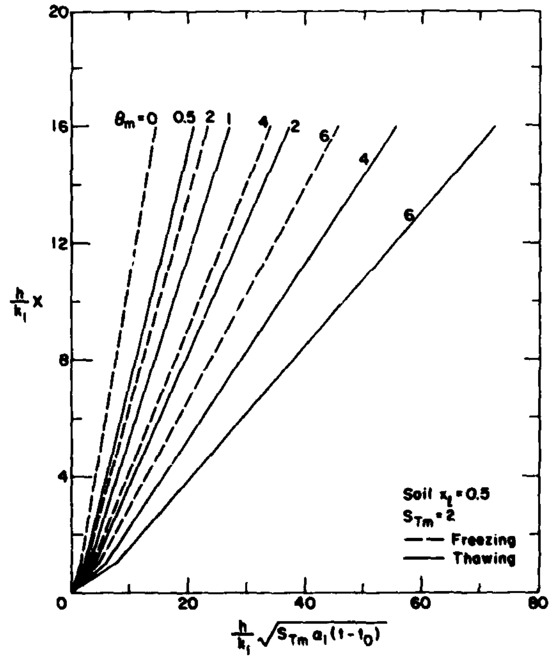

Figure 12. Surface convection for soil, $x_\ell = 0.50$, $S_{Tm} = 2$.

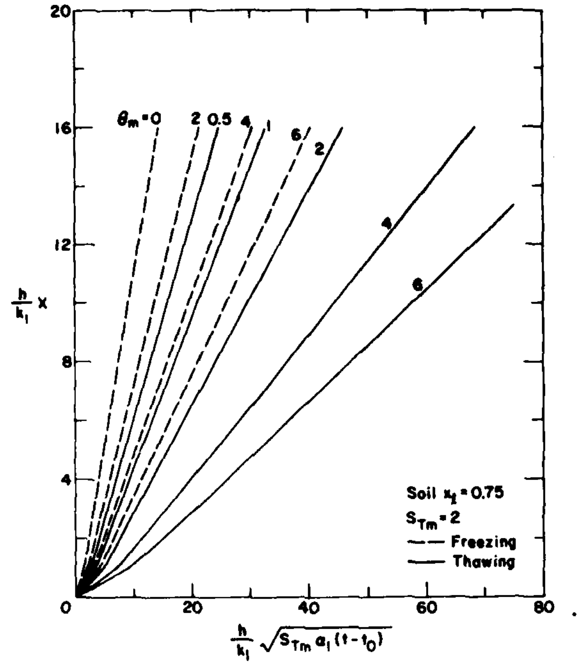

Figure 13. Surface convection for soil, $x_\ell = 0.75$, $S_{Tm} = 2$.

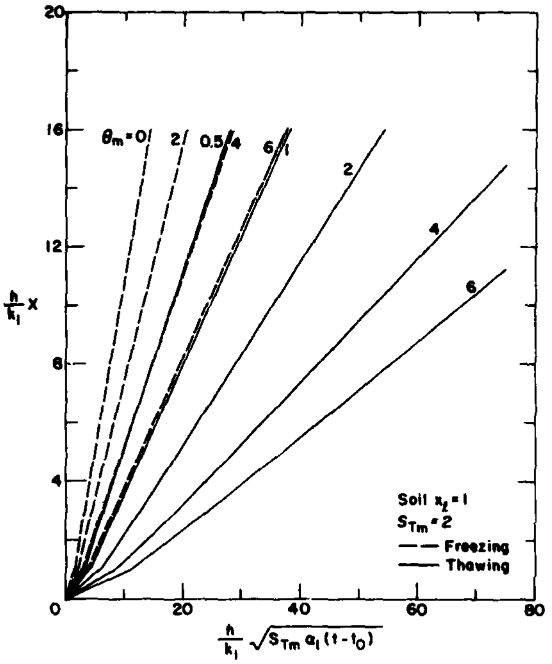

Figure 14. Surface convection for soil, $x_\ell = 1.0$, $S_{Tm} = 2$.

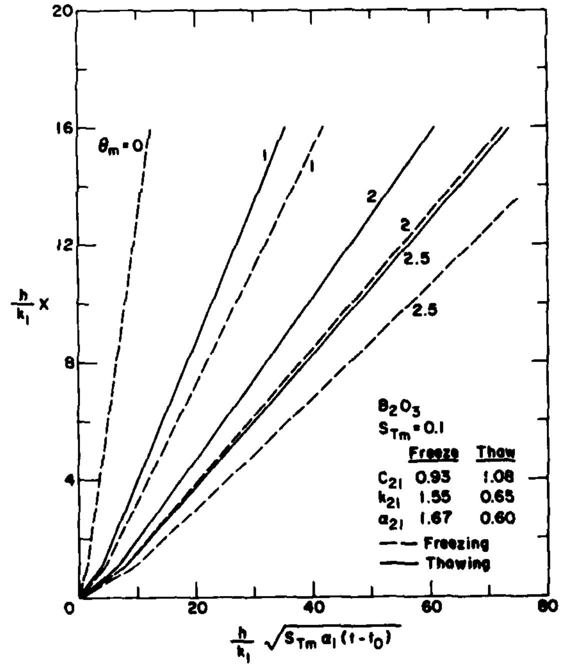

Figure 15. Surface convection, B_2O_3 $S_{Tm} = 0.1$.

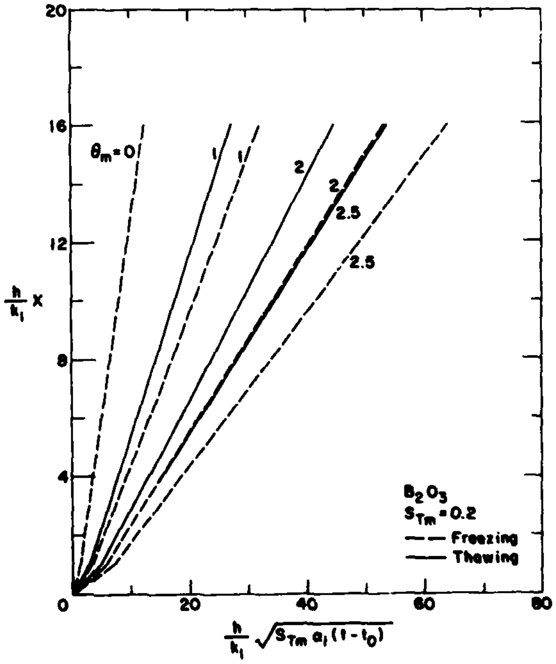

Figure 16. Surface convection, B_2O_3 $S_{Tm} = 0.2$.

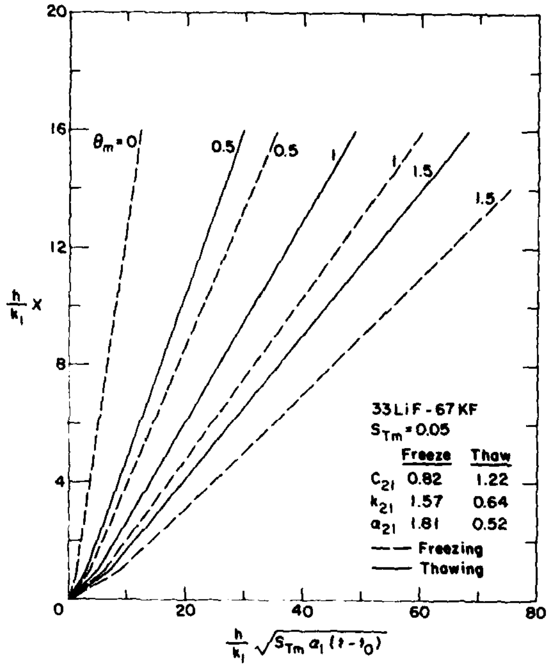

Figure 17. Surface convection, 33 LiF-67 KF, S_{Tm} = 0.05.

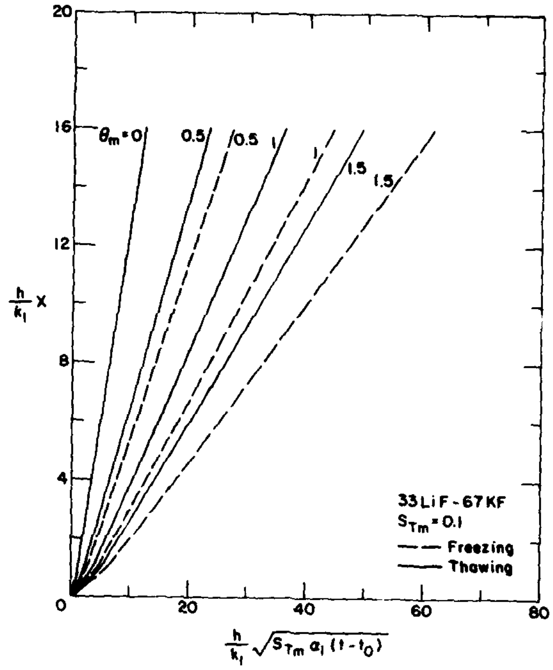

Figure 18. Surface convection, 33 LiF-67 KF, S_{Tm} = 0.1.

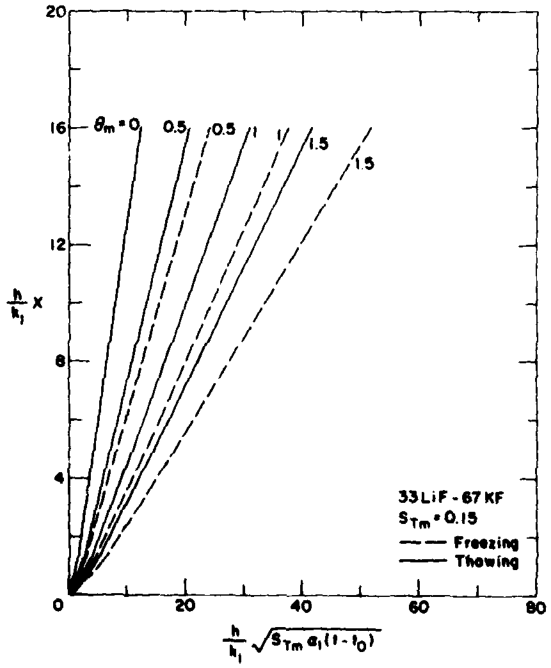

Figure 19. Surface convection, 33 LiF-67 KF, S_{Tm} = 0.15

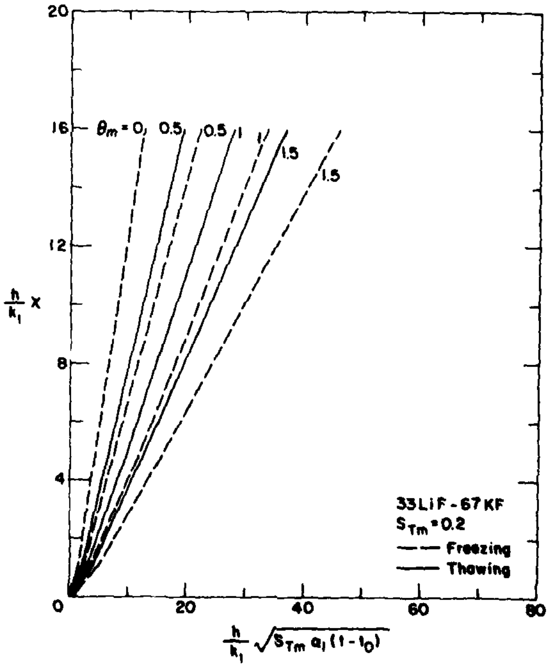

Figure 20. Surface convection, 33 LiF-67 KF, S_{Tm} = 0.20.

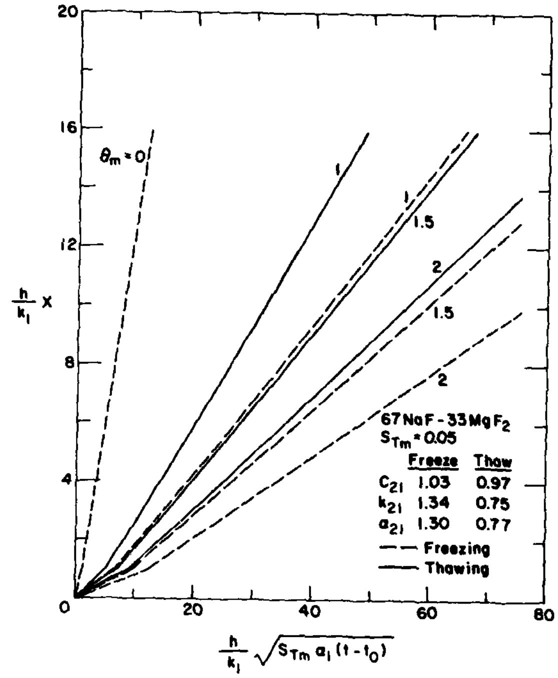

Figure 21. Surface convection, 67 NaF-33 MgF$_2$. S_{Tm} = 0.05.

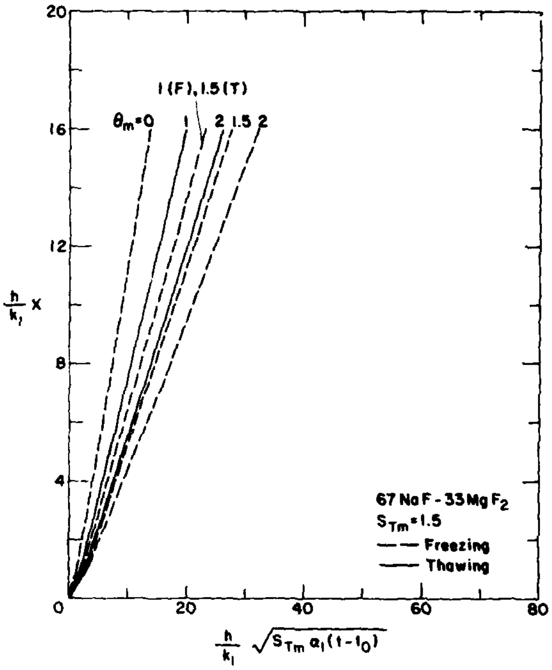

Figure 22. Surface convection, 67 NaF-33 MgF$_2$, S_{Tm} = 1.5.

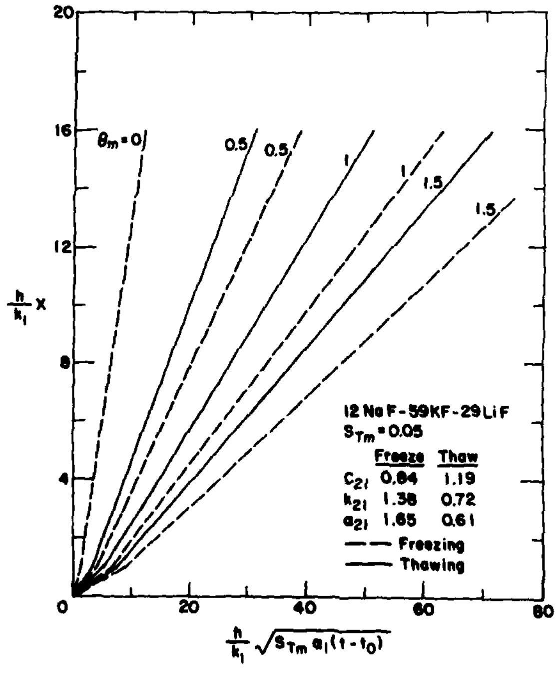

Figure 23. Surface convection, 12 NaF-59 KF-29 LiF, S_{Tm} = 0.05.

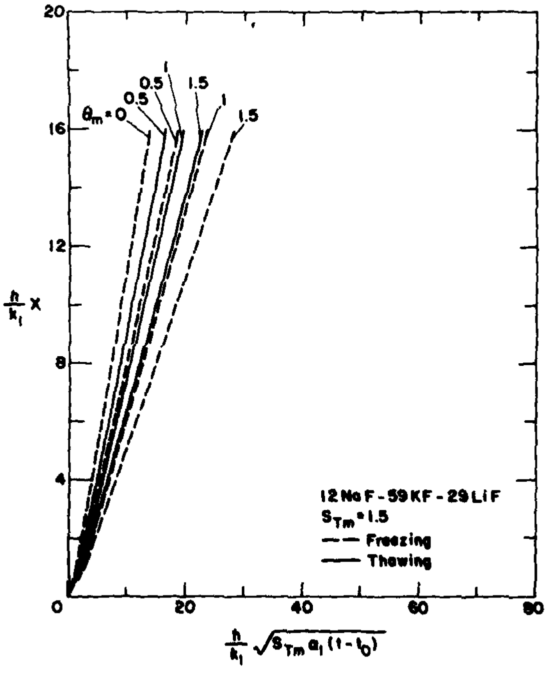

Figure 24. Surface convection, 12 NaF-59 KF-29 LiF, S_{Tm} = 1.5.

Table 1. Thermal properties of some phase change materials.*

Phase change material	Fusion temperature (°F)	Latent heat of fusion (Btu/lbm)	Specific heat at T_f (Btu/lbm °F)		Thermal conductivity at T_f (Btu/hr °F ft)		Density at 25°C (lbm/ft^3)
			Solid	Liquid	Solid	Liquid	
B_2O_3	842	142	0.41	0.44	0.9	0.58	115.5
33 LiF-67 KF	918	266	0.32	0.39	2.4-4.8	2.30	157.9
67 NaF-33 MgF_2	1530	265	0.34	0.33	2.4-4.8	2.69	133.6
12 NaF-59 KF-29 LiF	849	257	0.32	0.38	2.4-4.8	2.60	157.9

*ERDA (1976)

conservation for technically advanced countries. Figures 15-24 give the phase change depth vs time for some possible phase change materials with the properties listed in Table 1.

With these graphs the phase change depth, temperature, and heat flux can be predicted as a function of time. The computer listing is given for the numerical quadrature and can be used if materials with different properties are considered.

INSULATED SEMI-INFINITE BODY

Figures 5-24 can also be used for the case of a slab insulated with a layer of material when the insulation temperature is T_∞, as shown in Figure 25. The conductive resistance of the insulation must equal the convective resistance of the air layer. Then

$$\frac{d}{k_i} = \frac{1}{h}\,. \tag{34}$$

The dimensionless phase change depth is then given by

$$\sigma_c = \frac{k_i}{dk_1}\,X_c\,. \tag{35}$$

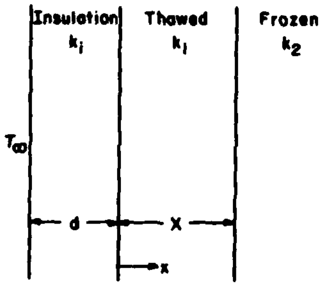

Figure 25. Semi-infinite body with insulation layer.

The graphs can then be used by assuming that the insulation layer has no latent heat and phase change starts at $t = t_0$ when the temperature of the insulation-slab interface reaches T_f.

The single-phase solution, with $S_{Tm} = 0$, eq 30, can be rewritten as

$$X_c = \sqrt{k_{1i}^2 d^2 + \frac{2k_1(T_\infty - T_f)(t - t_0)}{\rho_1 \ell}} - k_{1i}d\,. \tag{36}$$

Equation 36 is identical to the quasi-steady solution derived by Lunardini (1981).

CONCLUSION

The heat balance integral method can be applied to conductive heat transfer problems with phase change to obtain good, approximate, solutions. The method is particularly useful for soil systems since their nature often precludes obtaining accurate data on the soil thermal properties. Thus the use of approximate solutions will not increase the uncertainty of the design process.

The main value of the collocation method is that it provides an explicit functional relationship between the phase change depth and the temperature disturbance depth. This relationship will usually uncouple the system of differential equations for two-phase problems and can lead to closed form solutions or to reduced numerical effort. The collocation solution of the Neumann problem has been shown to be quite accurate with a worst case accuracy of less than 15%. For most soil systems the accuracy is within 5%. The collocation method is not quite as accurate as the usual heat balance integral method but it is easier to apply to two-phase problems.

Quantitative values have been obtained for the previously unsolved case of convection at the surface of an infinite medium. These results generalize the widely used Neumann solution and are applicable to the same physical situations as the Neumann problem.

The procedure can be used for any material if the appropriate thermal properties are supplied. The results of this report apply only to conductive heat transfer and should be considered as first approximations if convection occurs within the melted phase of the material.

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APPENDIX A. PROGRAM LISTING FOR NUMERICAL QUADRATURE OF EQUATION 28

This appendix includes the FORTRAN program for the numerical quadrature of the conduction phase change problem for a semi-infinite medium with a convective heat flux at the free surface.

```
C       FREEZING CASE


C       FREEZING CASE
C       CONDUCTION PHASE CHANGE
C       CONVECTIVE SURFACE FLUX
C       SEMI-INFINITE MEDIUM

        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        CALL CONTRL(2,'COMOUT',5)
        WRITE(1,782)
782     FORMAT(1,'OUTPUT WILL APPEAR IN  FILE COMOUT')
        WRITE(5,950)
950     FORMAT(1,5X,13HFREEZING CASE)
        Z=100.
        CALL TNOU('WHAT K21 VALUE WOULD YOU LIKE',29)
        READ(1,*)W
        CALL TNOU('WHAT A21 VALUE WOULD YOU LIKE',29)
        READ(1,*)A
        CALL TNOU('WHAT C21 VALUE WOULD YOU LIKE',29)
        READ(1,*)C
        CALL TNOU('WHAT STEFAN VALUE WOULD YOU LIKE',32)
        READ(1,*)ST
        CALL TNOU('HOW MANY THETA-M VALUES DO YOU HAVE',35)
        READ(1,*)ICOUNT
            DO 50 J=1,ICOUNT
                WRITE(1,780)J
780             FORMAT(1,'INPUT THETAM-',I3)
                READ(1,*)THETM
                WRITE(5,75)THETM,ST,W,C,A
75              FORMAT(1,/1,5X,10HTHETA-M = ,F3.1/
     1                   /1,5X,9HSTEFAN = ,F6.4
     1                   /1,5X,6HK21 = ,F6.2
     1                   /1,5X,6HC21 = ,F6.2
     1                   /1,5X,6HA21 = ,F6.2)
                WRITE(5,100)
100             FORMAT(//1,2X,5HSIGMA,15X,3HTAU,15X,3HPHI13X,8HSQRT TAU)
                TAU=0.
                BOW=0.
                DO 125 K=0,20
                    UPP=FLOAT(K)
                    CALL SIMP(CONST,THETM,SIGMA,BOW,UPP,ST,Z,PHI,TOTAL,W,C,A)
                    BOW=UPP
                    TAU=TAU+TOTAL
                    SQTAU=TAU**(1./2.)
                    WRITE(5,700)SIGMA,TAU,PHI,SQTAU
700                 FORMAT(1,2X,F4.1,10X,F12.5,6X,F12.5,6X,F12.5)
125             CONTINUE
50      CONTINUE
        CALL CONTRL(4,'COMOUT',5)
        CALL EXIT
        END

        SUBROUTINE SIMP(CONST,THETM,SIGMA,BOW,UPP,ST,Z,PHI,TOTAL,W,C,A)
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        D=0.
        TOTAL=0.
        H=(UPP-BOW)/Z
        SIGMA=BOW+H*D
        CALL FCT(CONST,THETM,SIGMA,TOT,ST,PHI,W,C,A)
        TOTAL=TOT
        D=D+1.
```

```
      SIGMA=BOW+H*D
      CALL FCT(CONST,THETM,SIGMA,TOT,ST,PHI,W,C,A)
      TOT=TOT*4.
      TOTAL=TOTAL+TOT
      D=D+1.
      SIGMA=BOW+H*D
      IF(D .EQ. Z)GO TO 200
      CALL FCT(CONST,THETM,SIGMA,TOT,ST,PHI,W,C,A)
      TOT=TOT*2.
      TOTAL=TOTAL+TOT
      GO TO 23
200   CALL FCT(CONST,THETM,SIGMA,TOT,ST,PHI,W,C,A)
      TOTAL=(TOTAL+TOT)*H/3.
      RETURN
      END
      SUBROUTINE FCT(CONST,THETM,SIGMA,TOT,ST,PHI,W,C,A)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

      BET=(2.*W*THETM+A)/ST

      PHI=BET*(SIGMA+1.)/2.+(BET**2*(SIGMA+1.)**2
     1/4.+A*SIGMA*(SIGMA/2.+1.)*BET)**(1./2.)

      HELP=2.*PHI*(SIGMA+1.)+A*SIGMA*(SIGMA+2.)

      FUNC=(ST*SIGMA**2*(PHI+1./3.*A*SIGMA))/(2.*PHI*(SIGMA
     1+1.)+A*SIGMA*(SIGMA+2.))+SIGMA*(1.+C*THETM)+1./3.
     1*C*THETM*PHI

      P=(PHI+A*SIGMA)*2.

      R=ST*SIGMA*(2.*PHI+A*SIGMA)+(1.+A*THETM)*HELP

      T=2.*(SIGMA*(1.+C*THETM)+(1./3.)*C*THETM*PHI)*(PHI+A*
     1(SIGMA+1.))

      Y=(ST*SIGMA**2+(1./3.)*C*THETM*HELP*SIGMA*(1.+C*THETM)+
     1(1./3.)*C*THETM*PHI)*2.*(SIGMA+1.)-2.*FUNC*(SIGMA+1.)

      U=((A*(SIGMA+1.)*BET+BET*PHI)/(2.*PHI-BET*(SIGMA+1.)))
      D=2.*FUNC*(PHI+A*(SIGMA+1.))

      Q=R+T+Y*U-D

      TOT=Q/P

      RETURN
      END
```

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