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SET THEORETIC SIGNED DOMINATION FOR COHERENT SYSTEMS.(U)
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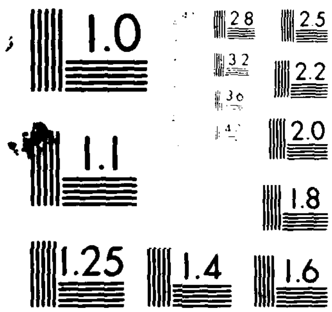

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SET THEORETIC SIGNED DOMINATION FOR COHERENT SYSTEMS

by
RICHARD E. BARLOW

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SET THEORETIC SIGNED DOMINATION FOR COHERENT SYSTEMS

Operations Research Center Research Report No. 82-1

Richard E. Barlow

February 1982

U. S. Army Research Office - Research Triangle Park

DAAG29-81-K-0160

Operations Research Center
University of California, Berkeley

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ORC 82-1	2. GOVT ACCESSION NO. AD-A112 699	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SET THEORETIC SIGNED DOMINATION FOR COHERENT SYSTEMS		5. TYPE OF REPORT & PERIOD COVERED Research Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Richard E. Barlow		8. CONTRACT OR GRANT NUMBER(s) DAAG29-81-K-0160
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS P-18195-M
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE February 1982
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

System Reliability
Graph Signed Domination
Coherent Structures
Computational Complexity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ACKNOWLEDGEMENT

I would like to acknowledge many fruitful discussions with Dr. A. Satyanarayana, Kevin Wood and Rubin Johnson on network reliability.

ABSTRACT

A recent combinatorial result relevant to the computational complexity of undirected networks is extended to include all coherent structures. This set-theoretic result provides computational insight for the problem of computing k-out-of-n system reliability, for example. All results are illustrated via simple networks.

-A

SET THEORETIC SIGNED DOMINATION FOR COHERENT SYSTEMS

by

Richard E. Barlow

1. INTRODUCTION

There have recently been several outstanding advances in our understanding of the computational complexity of algorithms for computing system reliability. Satyanarayana and Prabhakar (1978), in considering the problem of computing the reliability of two terminal *directed* networks, introduced the concept of domination and showed its significance relative to their algorithm for computing system reliability. They started with the classical inclusion-exclusion formula for computing reliability based on minimal path sets. In this formula, there will be a term corresponding to the probability that all system components work. The coefficient of this term (an integer, possibly positive, negative or zero) is called the *signed domination* of the network or system. In Satyanarayana and Chang (1981), it was shown that for *undirected* networks, the absolute value of the signed domination is a measure of the computational complexity of an algorithm for computing reliability using pivotal decomposition and series-parallel reductions. This result provides an interesting connection between two different methods for computing system reliability - namely inclusion-exclusion based on minimal path sets and pivotal-decomposition followed by series-parallel reductions. (The comparison assumes statistically independent components.) However, the absolute value of the signed domination does not provide a measure of the computational complexity of using pivotal decomposition followed by series-parallel reductions for *general* coherent systems.

For example, for a directed *cyclic* network, the signed domination is zero! [See R. R. Willie (1980).]

Networks and logic trees (or fault trees when the analysis is failure oriented) are two important system representations with respect to a system event of interest. At the present time, there are no computational complexity results with respect to probabilistic methods for analyzing logic trees as there are for undirected networks. The set theory generalization of the signed domination theorem provided in this paper may provide some insight into the more general reliability computational problem. The original signed domination theorem [see Property 5 in Satyanarayana and Chang (1981)] was proved only for undirected networks.

2. SIGNED DOMINATION

Let C be a set of components corresponding to a system of interest. Let $\mathcal{P} = [P_1, P_2, \ldots, P_p]$ be a family of success sets (e.g., minimal path sets); i.e., $P_i \subseteq C$; $\bigcup_{i=1}^{p} P_i = C$ and $P_i \subseteq P_j$ implies $i = j$. We call $(C,\mathcal{P})$ a *coherent system* [see Chapter 1, Barlow and Proschan (1975).] A *formation*, F , of $(C,\mathcal{P})$ is a family of sets such that $F \subseteq \mathcal{P}$ and $\bigcup_{P_i \in F} P_i = C$.

Example 2.1

The following simple example of an undirected network will be used to illustrate ideas.

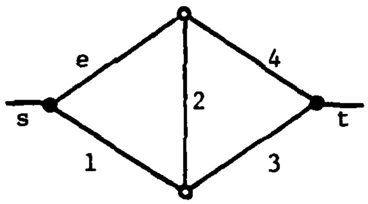

FIGURE 2.1

UNDIRECTED TWO TERMINAL NETWORK

For this example, $C = \{1,2,3,4,e\}$ while $\mathcal{P} = [\{1,3\},\{1,2,4\},\{e,4\},\{e,2,3\}]$. The formations of $(C,\mathcal{P})$ are:

$$\mathcal{F}_1 \begin{cases} F_0 = \mathcal{P} = [\{1,3\},\{1,2,4\},\{e,4\},\{e,2,3\}] \\ F_1 = [\{1,3\},\{1,2,4\},\{e,2,3\}] \\ F_2 = [\{1,3\},\{1,2,4\},\{e,4\}] \end{cases}$$

$$\mathcal{F}_2 \left\{ F_3 = [\{1,3\},\{e,4\},\{e,2,3\}] \right.$$

$$\mathcal{F}_3 \begin{cases} F_4 = [\{1,2,4\},\{e,2,3\}] \\ F_5 = [\{1,2,4\},\{e,4\},\{e,2,3\}] \end{cases}$$

Let $\mathcal{F}$ be a complete family of formations for $(C,\mathcal{P})$. In our example, $\mathcal{F} = [F_0,F_1,F_2,F_3,F_4,F_5]$.

The *signed domination*, $d(C,\mathcal{P})$, is the number of odd formations of $(C,\mathcal{P})$ minus the number of even formations. In our example, the number of odd formations is 4 and the number of even formations is 2 so that in this case $d(C,\mathcal{P}) = 2$. As mentioned earlier, the signed domination is the coefficient of the term in the inclusion-exclusion formula corresponding to the probability that all components in C are working. In Satyanarayana and Chang (1981), $|d(C,\mathcal{P})|$ is called the *domination.*

Pivoting

By pivoting on a component $e \in C$, we create two subsystems, corresponding to the system with e failed and to the system with e perfect, respectively. Let $\mathcal{P}(e) = [P_i \mid e \in P_i$ and $P_i \in \mathcal{P}]$ and $\mathcal{P}(e') = [P_i \mid e \notin P_i$ and $P_i \in \mathcal{P}]$. Then

$$\mathcal{P} = \mathcal{P}(e) \cup \mathcal{P}(e') .$$

In our example, $\mathcal{P}(e) = [\{e,4\},\{e,2,3\}]$ and $\mathcal{P}(e') = [\{1,3\},\{1,2,4\}]$. In all cases, $\bigcup_{P_i \in \mathcal{P}(e')} P_i \subseteq C-e$. In our example, $\bigcup_{P_i \in \mathcal{P}(e')} P_i = C-e$ so that $(C-e,\mathcal{P}(e'))$ is coherent and corresponds to our system with e failed. If $\bigcup_{P_i \in \mathcal{P}(e')} P_i \subset C-e$, then $(C-e,\mathcal{P}(e'))$ has *no* formations so that in this case $d(C-e,\mathcal{P}(e')) = 0$.

To describe a system with e perfect, let

$$\mathcal{P}-e = [P_1-e,P_2-e, \ldots, P_p-e] .$$

If $e \notin P_i$, then P_i is included as it is. Let $M[\mathcal{P} - e]$ be the set minimization of $\mathcal{P} - e$. In our example,

$$\mathcal{P}-e = [\{1,3\},\{1,2,4\},\{4\},\{2,3\}]$$

and

$$M[\mathcal{P} - e] = [\{1,3\},\{4\},\{2,3\}]$$

since $\{4\} \subset \{1,2,4\}$. In this case,

$$\bigcup_{A_i \in M[\mathcal{P}-e]} A_i = C - e \ ,$$

so that $(C - e, M[\mathcal{P} - e])$ corresponds to our example system with e perfect. In general, we only know that

$$\bigcup_{A_i \in M[\mathcal{P}-e]} A_i \subseteq C - e$$

so that $(C - e, M[\mathcal{P} - e])$ might be noncoherent.

The main theorem proved in Section 3 is

<u>Theorem</u> *0:*

For any $e \in C$,

$$d(C,\mathcal{P}) = d(C - e, M[\mathcal{P} - e]) - d(C - e, \mathcal{P}(e')) \ .$$

In our example, $d(C - e, M[\mathcal{P} - e]) = 1$ and $d(C - e, \mathcal{P}(e')) = -1$ so that $d(C,\mathcal{P}) = 2$ which agrees with Theorem 0.

In Satyanarayana and Chang (1981), it is shown that for undirected networks without replicated arcs

$$|d(C,P)| = |d(C-e, M[P-e])| + |d(C-e, P(e'))| \ . \tag{2.1}$$

This is their domination theorem and is the basis for asserting that the optimal binary computational structure using pivotal decomposition and series-parallel reductions has $|d(C,P)|$ leaves.

Example 2.2

(2.1) is *not* true in general. For example, the directed network in Figure 2.2 has domination zero since it contains a cycle. However, if we pivot on e, the subsystems each have domination 1 so that (2.1) is not true in this case. Theorem 0, on the other hand, is still valid.

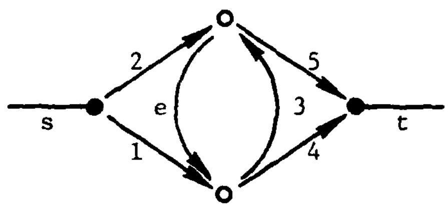

FIGURE 2.2

DIRECTED TWO TERMINAL NETWORK

k-out-of-n Systems

A k-out-of-n system functions if any k or more components function. Such systems *cannot* be represented as two terminal networks without replicated arcs. Using Theorem 0 and induction, Kevin Wood observed that the signed domination in this case is

$$d(C,P) = (-1)^{k+1}\binom{n-1}{k-1} \ .$$

Hence, the domination is $\binom{n-1}{k-1} \sim n^{k-1}$ for fixed k and large n. It is easy to show that for these systems (2.1) holds so that the computational complexity of a pivot and reduce algorithm is polynomial in n. Also, a pivot and reduce algorithm can be used even if components are statistically dependent. Of course, this would require knowledge of conditional probabilities and would require additional storage and computing time.

Assuming components are statistically independent in a k-out-of-n system, a better (i.e., n^2 computing time) algorithm would be based on the generating function

$$g(z) = \prod_{i=1}^{n} (q_i + p_i z)$$

where p_i is the probability component i works and $q_i = 1 - p_i$. We need only expand and sum the coefficients of $z^k, z^{k+1}, \ldots, z^n$. This is a well-known technique. However, it does require component statistical independence so that the domination result may be of interest relative to the statistically dependent component case.

3. PROOF OF THE SIGNED DOMINATION THEOREM

We wish to prove Theorem 0; i.e.,

Theorem 0:

For any $e \in C$,

$$d(C,\mathcal{P}) = d(C-e,\mathcal{M}[\mathcal{P}-e]) - d(C-e,\mathcal{P}(e')) \ .$$

To do this, we partition the class of all formations of $\mathcal{F}$ into 3 classes.

Class 1 Formations: $\mathcal{F}_1$

$F \in \mathcal{F}$ is a class 1 formation iff $\bigcup_{P_i \in F(e')} P_i = C - e$. Recall that $F(e') = \{P_i \mid e \notin P_i \text{ and } P_i \in F\}$.

In our example Figure 2.1, the following are class 1 formations:

$$F_0 = [\{1,3\},\{1,2,4\},\{e,4\},\{e,2,3\}]$$

$$F_1 = [\{1,3\},\{1,2,4\},\{e,2,3\}]$$

$$F_2 = [\{1,3\},\{e,4\},\{1,2,4\}] \ .$$

Theorem 1:

$$N_{odd}(\mathcal{F}_1) - N_{even}(\mathcal{F}_1) = -d(C-e,\mathcal{P}(e')) \ .$$

Proof:

[See also proof of Theorem 1 in Satyanarayana and Chang (1981).]

If $\mathcal{F}_1$ is empty, then $(C-e,\mathcal{P}(e'))$ is noncoherent since $\bigcup_{P_i \in \mathcal{P}(e')} P_i \subset C-e$. Hence, $(C-e,\mathcal{P}(e'))$ has no formations based on $\mathcal{P}(e')$ and $d(C-e,\mathcal{P}(e')) = 0$.

Assume $F = F(e) \cup F(e') \in \mathcal{F}_1$. Since $\bigcup_{P_i \in F(e')} P_i = C - e$, we need only add one $P_i \in \mathcal{P}(e)$ to $F(e')$ to obtain a formation for $(C,\mathcal{P})$. Let n_{odd} and n_{even} be the number of odd and even, respectively, formations of $(C-e,\mathcal{P}(e'))$. Let $|\mathcal{P}(e)| = x$. The number of odd formations in class 1 is

$$N_{odd}[\mathcal{F}] = 2^{x-1} n_{even} + (2^{x-1} - 1) n_{odd}$$

which is the number of ways to include an odd number of sets of $\mathcal{P}(e)$ with an even formation of $(C-e,\mathcal{P}(e'))$ plus the number of ways to include an even number of sets of $\mathcal{P}(e)$ with an odd formation of $(C-e,\mathcal{P}(e'))$. Similarly, the number of even formations in class 1 is

$$N_{even}[\mathcal{F}] = 2^{x-1} n_o + (2^{x-1} - 1) n_e .$$

Therefore, $N_{odd} - N_{even} = n_{even} - n_{odd} = -d(C-e,\mathcal{P}(e'))$. ||

In our example 2.1, $(C-e,\mathcal{P}(e'))$ has only the formation $[\{1,3\}, \{1,2,4\}]$ so that $n_{odd} - n_{even} = -1 = d(C-e,\mathcal{P}(e'))$. For the class 1 formations,

$$N_{odd}[\mathcal{F}_1] - N_{even}[\mathcal{F}_1] = 2 - 1 = 1$$

so that

$$N_{odd}[\mathcal{F}_1] - N_{even}[\mathcal{F}_1] = -d(C-e),\mathcal{P}(e'))$$

as claimed.

<u>Class 2 Formations</u>: $\mathcal{F}_2$

$F \in \mathcal{F}$ is a class 2 formation iff $\bigcup_{P_i \in F(e')} P_i \subset C - e$ but is not equal to $E - e$ and

$$F(e') \subset \mathcal{M}[\mathcal{P} - e] \ .$$

In our example,

$$\mathcal{P} - e = [\{1,3\},\{1,2,4\},\{4\},\{2,3\}]$$

and

$$\mathcal{M}[\mathcal{P} - e] = [\{1,3\},\{4\},\{2,3\}]$$

so that

$$F_3 = [\{1,3\},\{e,4\},\{e,2,3\}]$$

is the only class 2 formation in our example 2.1.

<u>Theorem 2</u>:

$$N_{odd}(F_2) - N_{even}(F_2) = d(C - e, \mathcal{M}[\mathcal{P} - e]) \ .$$

<u>Proof</u>:

<u>Case 1</u>:

$(C - e, \mathcal{M}[\mathcal{P} - e])$ is noncoherent; i.e., $\bigcup_{A_i \in \mathcal{M}[\mathcal{P}-e]} A_i \subset E - e$. This can only happen if some member of $\mathcal{P}(e')$ is a superset of some member of $\mathcal{P}(e) - e$. Hence,

$$\mathcal{M}[\mathcal{P} - e] = \left[P_{j_1}, P_{j_2}, \ \ldots, \ P_{j_s}, \ P_{r+1} - e, \ \ldots, \ P_p - e\right]$$

where $\left[P_{j_1}, \ \ldots, \ P_{j_s}\right] \subset \mathcal{P}(e')$ and $[P_{r+1}, \ \ldots, \ P_p] = \mathcal{P}(e)$ and

$$P_{j_1} \cup P_{j_2} \cup \ldots \cup P_{j_s} \cup P_{r+1} - e \cup \ldots \cup P_p - e \cup E - e \ .$$

Hence, $P_{j_1} \cup P_{j_2} \cup \ldots \cup P_{j_s} \cup P_{r+1} \cup \ldots \cup P_p \subset C$.

Let $F = \Big[\underbrace{F_{i_1}, \ldots, F_{i_k}}_{F(e')}, \underbrace{F_{i_{k+1}}, \ldots, F_{i_m}}_{F(e)}\Big]$ and suppose

$F(e') \subset M[P-e]$. Then in fact $F(e') \subset \Big[P_{j_1}, \ldots, P_{j_s}\Big]$ and $F(e) \subset [P_{r+1}, \ldots, P_p]$ so that

$$F(e') \cup F(e) \subset \Big[P_{j_1}, P_{j_2}, \ldots, P_{j_s}, P_{r+1}, \ldots, P_p\Big]$$

and

$$\bigcup_{A_i \in F(e') \cup F(e)} A_i \subset C$$

so that F is *not* a formation for (C,P) . It follows that in this case F_2 is empty and the theorem follows.

Case 2:

$(C-e, M[P-e])$ is coherent.

Let

$$M[P-e] = \Big[P_{j_1}, \ldots, P_{j_s}, P_{r+1}-e, \ldots, P_p-e\Big]$$

as before. Let $H = [H_1, \ldots, H_\ell, H_{\ell+1}, \ldots, H_t]$ be a formation for $(C-e, M[P-e])$ where

$$[H_1, \ldots, H_\ell] \subseteq \Big[P_{j_1}, \ldots, P_{j_s}\Big]$$

and

$$[H_{\ell+1}, \ldots, H_t] \subseteq [P_{r+1}-e, \ldots, P_p-e] .$$

Define U_1 so that $H \varepsilon U_1$ implies $[H_1, \ldots, H_\ell]$ is itself a formation for $(C-e, M[P-e])$. Claim *the number of odd formations in* U_1 *is equal to the number of even formations.* If U_1 is empty, the assertion is obvious. Hence, suppose U_1 is not empty. Choose any $P_i \varepsilon P$ such that $e \varepsilon P_i$. If $P_i - e \varepsilon [H_{\ell+1}, \ldots, H_t]$, then $H - (P_i - e)$ is also a formation for $(C-e, M[P-e])$ and $H^1 =$ $H - (P_i - e) \varepsilon U_i$. If $P_i - e \notin H$, then $H^1 = H + (P_i - e)$ is in U_1. Clearly, if H is odd, H^1 is even and vice versa. Therefore, we can pair the odd and even formations of U_1 with respect to P_i and the claim is obvious.

Let U_2 be the remaining formations of $(C-e, M[P-e])$. Hence, $H \varepsilon U_2$ implies

$$[H_1, \ldots, H_\ell] \text{ is not a formation for } (C-e, M[P-e]) .$$

Therefore, $[H_1, \ldots, H_\ell, H_{\ell+1}+e, \ldots, H_t+e]$ is a class 2 formation for (C,P).

Let $F = \Big[\underbrace{F_{i_1}, \ldots, F_{i_k}}_{F(e')}, \underbrace{F_{i_{k+1}}, \ldots, F_{i_m}}_{F(e)}\Big]$ be a class 2 formation for (C,P) so that $F(e')$ is *not* a formation for $(C-e, M[P-e])$. Then $F - e \varepsilon U_2$. It follows that there is a 1-1 correspondence between members of F_2 and U_2. Hence,

$$N_{odd}(F_2) - N_{even}(F_2) = d(C-e, M[P-e])$$

as was to be shown. ||

<u>Class 3 Formations</u>: F_3

$F \varepsilon F$ is a class 3 formation for (C,P) iff $\bigcup_{P_i \varepsilon F(e')} P_i \subset C-e$ and $F(e') \not\subset M[P-e]$. For our example 2.1,

$$F_4 = [\{1,2,4\},\{e,2,3\}]$$

and

$$F_5 = [\{1,2,4\},\{e,4\},\{e,2,3\}]$$

are class 3 formations and

$$N_{odd}[F_3] - N_{even}[F_3] = 0 .$$

<u>Theorem 3</u>:

$$N_{odd}[\mathcal{F}_3] - N_{even}[\mathcal{F}_3] = 0 .$$

<u>Proof</u>:

$$\text{Let } F = \Big[\underbrace{F_{i_1}, F_{i_2}, \ldots, F_{i_k}}_{F(e')}, \underbrace{F_{i_{k+1}}, \ldots, F_{i_m}}_{F(e)}\Big], \ F \in \mathcal{F}_3 .$$

Since $F(e') \not\subset M[\mathcal{P} - e]$, $F(e')$ must contain a superset of some member of $\mathcal{P}(e) - e$, say $P_i - e$ where $e \in P_i$. If $P_i \notin F$, then form $F^1 = F + P_i$ and F^1 is also in $\mathcal{F}_3$ but of opposite parity to F . If $P_i \in F$, then there must be at least one other member of $F(e)$, since otherwise $\bigcup_{j=1}^{m} F_{i_j} \subset C$. Since there is at least one other member of $F(e)$ containing e , we may form $F^1 = F - P_i$ and F^1 is a member of $\mathcal{F}_3$ but of opposite parity to F .

Hence, to every even formation in $\mathcal{F}_3$, there is an odd formation in $\mathcal{F}_3$ and vice versa. The theorem follows. ||

Theorem 0 is now proved since every formation of $(C,\mathcal{P})$ must be either a class 1, class 2 or class 3 formation. ||

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