

AD-A112 256

AERONAUTICAL RESEARCH LABS MELBOURNE (AUSTRALIA)  
STRAPDOWN INERTIAL NAVIGATION SYSTEMS: AN ALGORITHM FOR ATTITUDE--ETC(U)

F/B 17/7

OCT 80 R B MILLER

UNCLASSIFIED

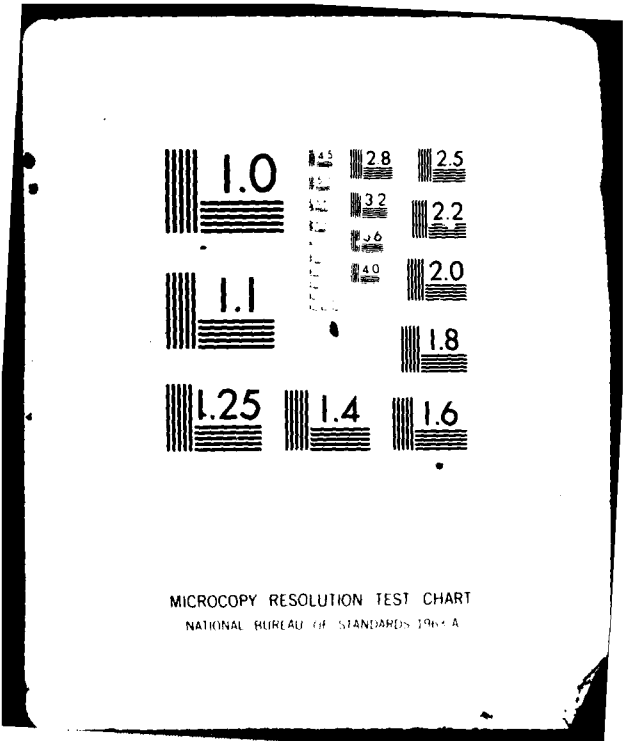
ARL/SYS-23

ML

101  
101

101

END  
DATE  
FILMED  
1-82  
DTIC



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS 1963-A

(2)



ADA112256

**DEPARTMENT OF DEFENCE  
DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION  
AERONAUTICAL RESEARCH LABORATORIES**

MELBOURNE, VICTORIA

SYSTEMS REPORT 23

**STRAPDOWN INERTIAL NAVIGATION SYSTEMS:  
AN ALGORITHM FOR ATTITUDE AND  
NAVIGATION COMPUTATIONS**

by

R. B. MILLER

Approved for Public Release.

DTIC  
S ELECTED  
MAR 22 1982  
A

© COMMONWEALTH OF AUSTRALIA 1981

COPY No

OCTOBER 1980

012

DTIC FILE COPY

DEPARTMENT OF DEFENCE  
DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION  
AERONAUTICAL RESEARCH LABORATORIES

SYSTEMS REPORT 23

**STRAPDOWN INERTIAL NAVIGATION SYSTEMS:  
AN ALGORITHM FOR ATTITUDE AND  
NAVIGATION COMPUTATIONS**

by

R. B. MILLER

*SUMMARY*

*An algorithm for strapdown inertial navigation and the associated theoretical analysis are presented. Vehicle attitude is maintained through quaternions, which are updated by a modified third order method; a split frame technique is used for solution of the navigation equation.*



---

POSTAL ADDRESS: Chief Superintendent, Aeronautical Research Laboratories,  
Box 4331, P.O., Melbourne, Victoria, 3001, Australia.

**DOCUMENT CONTROL DATA SHEET**

Security classification of this page: Unclassified

- |  |  |
|--|--|
| <p>1. Document Numbers</p> <p>(a) AR Number:<br/>AR-002-265</p> <p>(b) Document Series and Number:<br/>Systems Report 23</p> <p>(c) Report Number:<br/>ARL-Sys-Report-23</p> | <p>2. Security Classification</p> <p>(a) Complete document:<br/>Unclassified</p> <p>(b) Title in isolation:<br/>Unclassified</p> <p>(c) Summary in isolation:<br/>Unclassified</p> |
|--|--|

3. Title: STRAPDOWN INERTIAL NAVIGATION SYSTEMS: AN ALGORITHM FOR ATTITUDE AND NAVIGATION COMPUTATIONS

- |  |  |
|--|--|
| <p>4. Personal Author:<br/>Miller, R. B.</p> | <p>5. Document Date:<br/>October, 1980</p> |
|--|--|
6. Type of Report and Period Covered:

- |  |  |
|--|--|
| <p>7. Corporate Author:<br/>Aeronautical Research Laboratories</p> | <p>8. Reference Numbers</p> <p>(a) Task:<br/>AIR 78/084</p> <p>(b) Sponsoring Agency:<br/>DEFAIR</p> |
|--|--|
9. Cost Code:  
72 3370

- |   |   |
|---|---|
| <p>10. Imprint:<br/>Aeronautical Research Laboratories,<br/>Melbourne</p> | <p>11. Computer Program(s)<br/>(Title(s) and language(s)):<br/>NONE</p> |
|---|---|

12. Release Limitations (of the document):  
Approved for Public Release

12-0. Overseas:	N.O.		P.R.	1	A		B		C		D		E	
-----------------	------	--	------	---	---	--	---	--	---	--	---	--	---	--

13. Announcement Limitations (of the information on this page):  
No Limitations

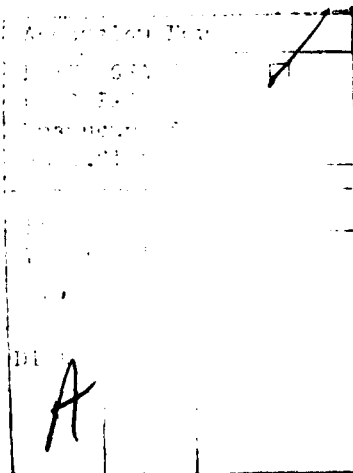
- |   |  |                                  |
|---|--|----------------------------------|
| <p>14. Descriptors:</p> <p>Inertial navigation</p> <p>Strapdown inertial guidance</p> <p>Navigation</p> <p>Avionics</p> | <p>Algorithms</p> <p>Attitude indicators</p> <p>Guidance computers</p> | <p>15. Cosati Code:<br/>1707</p> |
|---|--|----------------------------------|

16. **ABSTRACT**

*An algorithm for strapdown inertial navigation and the associated theoretical analysis are presented. Vehicle attitude is maintained through quaternions, which are updated by a modified third order method; a split frame technique is used for solution of the navigation equation.*

## CONTENTS

	Page No
<b>1. INTRODUCTION</b>	1
<b>1.1 The Concept of Strapdown I.N.S.</b>	1
<b>1.2 Computation in Strapdown I.N.S.</b>	1
<b>1.3 Content of this Paper</b>	2
<b>2. UPDATING THE ATTITUDE REFERENCE</b>	2
<b>2.1 Rotation Vector Updating Concept</b>	3
<b>2.1.1 Calculation of Rotation Vector from Gyroscope Outputs</b>	3
<b>2.1.2 Quaternion Update Using the Rotation Vector</b>	3
<b>2.2 Modification of the Attitude Algorithm</b>	4
<b>2.3 Impact on Sensor Compensation</b>	4
<b>3. SOLUTION OF THE NAVIGATION EQUATION</b>	4
<b>3.1 The Split-Frame Mechanisation</b>	5
<b>3.1.1 Body Axes Calculations (Fast Rate)</b>	5
<b>3.1.2 Navigation Axes Calculations (Intermediate Rate)</b>	6
<b>3.1.3 Navigation Frame Rotation</b>	6
<b>3.1.4 Velocity Transformation</b>	6
<b>3.1.5 Calculation of Velocity Change due to Gravitation Effects</b>	6
<b>3.1.6 Vertical Channel Damping</b>	7
<b>3.1.7 Position and Velocity Integration</b>	7
<b>3.2 Slow Rate Calculations</b>	7
<b>4. SUMMARY OF ALGORITHM</b>	7
<b>5. DISCUSSION OF ITERATION RATES</b>	8
<b>6. ALGORITHM PERFORMANCE</b>	8
<b>6.1 Attitude Updating</b>	8
<b>6.1.1 Computer Loadings</b>	9
<b>6.1.2 Sampling Rates</b>	9
<b>6.2 Navigation Performance</b>	9
<b>REFERENCES</b>	
<b>APPENDICES</b>	
<b>DISTRIBUTION</b>	



## 1. INTRODUCTION

The various aspects of Strapdown Inertial Navigation Systems have been subjects of investigation in the U.S.A. for the past 15 years or so. More recently, interest has been shown from other countries, having been stimulated by the demonstration of suitable sensors and computing hardware. Problems related to computation, which is fundamental to a strapdown system, have received the most attention. Reference 1 is a literature survey of strapdown technology, with some 300 references. Reference 2 contains a bibliography of approximately 230 references. Reference 3 is a review of the fundamentals of strapdown I.N. systems.

Papers dealing with strapdown inertial navigation systems are often rather fragmentary in that they deal with certain aspects in isolation, and may be obscure to those not familiar with the complete subject. In this and following papers an attempt is being made to document a complete and viable system in a form which may be followed by those less familiar with the subject. To this end, a survey of all the alternative methods has not been included, except for a few examples for comparison purposes.

### 1.1 The Concept of Strapdown I.N.S.

An inertial measurement unit consisting of a minimum of three gyroscopes and three accelerometers is mounted directly or perhaps with vibration isolators to the body of the vehicle. Associated with this is a computer which processes the sensors' outputs and performs the navigation calculations. The gyroscopes are usually arranged with their sensitive axes nominally mutually perpendicular. The accelerometers are similarly arranged.

At a particular time, the en-route vehicle has a certain position, velocity, and attitude relative to a specified reference. After a short time, the vehicle has moved to a new position, and the velocity and attitude have changed. During this period, the sensor outputs are observed. The gyroscope outputs are used to calculate the change in attitude over the period. The updated attitude of the vehicle is then calculated. With allowances for the effects of gravity, the accelerometer outputs are used to calculate increments of velocity over the period. Because the accelerometers are fixed to the vehicle, the direction of these increments changes as the vehicle attitude changes. However, because the attitude of the vehicle is known, these velocity increments can be expressed relative to the reference, and so the change in position can be calculated.

This process has been going on continuously since the commencement of the flight, so assuming position, velocity, and attitude were known at the start, navigation has been achieved. Before commencement of navigation, a "levelling and alignment" sequence is performed. This is the process of acquiring the initial conditions, and is often referred to simply as "alignment".

### 1.2 Computation in Strapdown I.N.S.

The computations performed by a strapdown navigator may be regarded as comprising two major parts: propagation of the attitude reference, and solution of the navigation equation. The former uses the gyro outputs to calculate the attitude of the body coordinate frame with respect to a reference coordinate frame. The latter uses this relationship to transform the coordinates of vectors (which may include accelerometer outputs, velocities, gravity effects, Earth rotation and curvature effects, etc.) between frames and hence to calculate velocity and position of the body.

Other calculations performed in the navigator include sensor compensation (e.g. correction of gyro outputs for known drifts, etc.) and, before the commencement of navigation, levelling and alignment of the system's internal references. Alignment may be carried out with the I.N.S. nominally at rest, such as when an aircraft is parked on the ground, or when the system is in motion, if another source of navigation information is available. The latter is known as transfer alignment.

### 1.3 Content of this Paper

In this paper an algorithm for solution of the attitude and navigation equations is presented. The algorithm and the theory behind it are discussed. Where practical, the mathematics have been confined to appendices, for reference as required.

This analysis is thought to be unique in its presentation in that the process has been split into sequentially performed modules each of which may be analysed in isolation. This approach allows clearer insight into the workings of the process and considerably facilitates modification of the algorithm to provide greater or lesser accuracy (in return for greater or lesser computer loading) as required.

In particular, the attitude updating part of the algorithm has been split into two parts—the solution for a “rotation vector”, and the update of the quaternions. The navigation part of the algorithm uses a split frame technique whereby body related quantities are evaluated in body axis coordinates, and navigation frame related quantities are evaluated in navigation axis coordinates. Additionally, the algorithm is partitioned into three sections which are performed at different rates according to the application.

## 2. UPDATING THE ATTITUDE REFERENCE

All strapdown system mechanisations must maintain a relationship between the body frame and some reference frame. The accuracy of this relationship is critical to the successful operation of the system.

This relationship may be the actual coordinate transformation matrix between the two frames: that is the direction cosine matrix. Alternatively it may be some parameter set from which a transformation may be obtained later. The latter approach, using quaternions (see Appendix 2), is the preferred technique, although direct updating of the direction cosine matrix is sometimes reported.

Updating is conventionally achieved by the solution of the differential equation governing the parameter:

for a direction cosine matrix the equation is

$$\dot{C} = C\Omega$$

and for quaternions the equation is

$$\dot{Q} = \frac{1}{2}Q * \omega$$

(see Appendix 1 for notation).

These equations are derived in Appendix 3. The method of solution is usually by either 3rd order Taylor Series or 4th order Runge-Kutta (see Appendix 4). Higher order methods seem to be neither necessary nor practical in these real-time applications.

The analysis presented here uses what will be referred to as the Rotation Vector method.

This concept has not received much attention in the literature. It was used by Bortz (Ref. 4) in a proposal for a hybrid system. In practice, as here, its application gives an end result very similar to the Taylor Series of equivalent order. The concept is further developed here because it is considered that it provides a most useful insight into the workings of the mathematical process. This facilitates any modification of the resultant algorithm to provide greater or lesser accuracy as required.

Finite rotations (such as occur in a gyro sampling period) are non-commutative. This means that the actual net rotation of the vehicle (which is required) is not equal to the integral of angular rate (the gyro output), unless the rotation of the vehicle is about a fixed axis during the sample period. In a real system the axis is not usually fixed, so a fast rate of sampling and updating must be used. For a given level of system accuracy, the use of the higher order algorithms mentioned above is usually cost-effective in use of computer time—the cost of increased complexity is more than saved by the gain from the lower iteration rate. In both the 3rd and 4th order methods mentioned above (and in the rotation vector method), the assumption is made that, during the iteration period, the gyro outputs follow a square law and may therefore be approximated by a second order polynomial. This requires two gyro samples per iteration, and affords a substantial correction for non-commutativity effects.



## 2.1 Rotation Vector Updating Concept

In this method, the gyro outputs are corrected for non-commutativity effects by calculating an equivalent fixed axis rotation vector, which is then used to update the quaternion.

### 2.1.1 Calculation of Rotation Vector from Gyroscope Outputs

For a finite rotation about a fixed axis, a "rotation vector" may be defined, its direction being along the axis of rotation, and its length the rotation magnitude. In Appendix 2 a quaternion is shown as  $\bar{Q} = C, \theta S$ : here,  $\theta$  is the rotation vector. The relationship between rotation vector and angular velocity is derived in Appendix 5:

$$\dot{\theta} = \omega + \frac{1}{2}(\theta \times \omega) + A\theta \times (\theta \times \omega) \quad \text{where} \quad A = \frac{1}{\theta_0^2} \left\{ 1 - \frac{\theta_0}{2 \tan(\frac{1}{2}\theta_0)} \right\}$$

$$\text{and} \quad \theta_0^2 = \theta \cdot \theta$$

Euler's Theorem states that, for any finite rotation, there is an equivalent fixed axis rotation. The rotation vector approach to the non-commutativity problem is to use the gyro outputs to solve the rotation vector equation, obtaining the equivalent fixed axis rotation over the period, and to use this rotation for updating the attitude quaternion. For this case, where two gyro samples are taken per interval, and the triple vector product (which is very small) is neglected, the numerical solution for the rotation vector  $\theta$  is given by

$$\theta = \delta_1 + \delta_2 + \frac{1}{2}(\delta_1 \times \delta_2)$$

where  $\delta_1$  and  $\delta_2$  are the incremental gyro samples, taken at the mid-point and end-point of the interval. This equation is derived in Appendix 6.

### 2.1.2 Quaternion Update Using the Rotation Vector

The "equivalent rotation" is about a fixed axis; if the rotation vector  $\theta$  is expressed as a rotation quaternion  $\bar{\theta}$ , then the updated quaternion is given by the standard rules of quaternion multiplication:

$$\bar{Q}(t + \delta t) = \bar{Q}(t) * \bar{\theta} \quad \text{where} \quad \bar{\theta} = C, \theta S \text{ as in Appendix 2}$$

$$\text{i.e.} \quad C = \cos(\frac{1}{2}\theta_0), \quad S = (1/\theta_0) \sin(\frac{1}{2}\theta_0)$$

$$\text{and} \quad \theta_0^2 = \theta \cdot \theta$$

For a given value of  $\theta$ , the above result is exact, subject to calculation of  $C$  and  $S$ . These can be obtained from the series expansions for sine and cosine. The third order expansion has  $C = 1 - \frac{1}{8}\theta_0^2$  and  $S = \frac{1}{4} - \frac{1}{48}\theta_0^2$ .

For fixed axis rotation, Wilcox (Ref. 5) showed how modified second order expansions of  $C$  and  $S$  ( $C = 1 - \frac{1}{12}\theta_0^2$  and  $S = 0.5$ ) could give drift performance slightly superior to that of a third order expansion. This simplification is achieved at the expense of increased scale errors. There is little to choose between this third order and the modified second order methods in accuracy for fixed axis rotation. In computer loading, the third order requires an extra 4 multiplications and 1 addition per iteration, with occasional normalisation; the modified second order requires more frequent normalisation, which involves 8 multiplications and 4 additions. These figures do not include multiplication or division by 2. On balance, the modified second order method has been found preferable.

In this mechanisation, the equivalent rotation vector is computed from the gyro outputs, and the rotation update quaternion is obtained from this using the modified second order expansion for  $C$  and  $S$ . This may be considered as a modified third order method: in Appendix 4, it is shown how the rotation vector calculation, together with the third order expansion of  $C$  and  $S$ , is numerically almost equivalent to the third order Taylor Series solution of the quaternion differential equation.

## 2.2 Modification of the Attitude Algorithm

The Rotation Vector Concept gives a useful insight into the workings of the mathematical process, by splitting the non-commutativity correction procedures from the quaternion update procedures.

Consider the equation

$$\theta = \delta_1 + \delta_2 + \frac{2}{3}(\delta_1 \times \delta_2)$$

the quantity  $(\delta_1 + \delta_2)$  is of course the sum of the gyro outputs over the whole period;  $\frac{2}{3}(\delta_1 \times \delta_2)$  is the non-commutativity correction. In an application requiring less accuracy, one sample  $\delta$  per iteration may be taken: then  $\theta = \delta$  and fixed axis rotation is assumed (i.e.  $\delta_1 \times \delta_2 = 0$ ).

Alternatively, more than two gyro samples per iteration may be taken. For example, if three samples  $\delta_1, \delta_2, \delta_3$  are taken, the gyro outputs are assumed cubic, and the solution of the rotation vector equation has the form:

$$\theta = \delta_1 + \delta_2 + \delta_3 + \text{cross product terms.}$$

Similarly, the quaternion update accuracy may be varied by taking fewer or more terms in the expansions for  $C$  and  $S$ .

## 2.3 Impact on Sensor Compensation

A considerable amount of computer time may be taken by sensor compensation—allowing for known biases and acceleration sensitivities, etc. It may appear that taking several gyro samples per iteration would multiply this time. However, the rotation vector solution is of the form

$$\theta = \text{sum term} + \text{cross product term.}$$

The cross-product term is much smaller than the sum term, so for most applications it should be possible to compensate only the sum of the gyro outputs at the end of the period, and calculate the cross-product term from uncorrected intermediate outputs.

## 3. SOLUTION OF THE NAVIGATION EQUATION

The function of the navigation algorithm is to accept the accelerometer outputs and the attitude parameters, and to calculate position and velocity of the vehicle.

Accelerometers respond to specific force, which is the difference between inertial and gravitational ("mass attraction") acceleration. An allowance must therefore be made for gravitation. In practice, it is usual to calculate the value of "gravity", which is defined as the resultant of gravitational and centrifugal acceleration due to Earth's rotation. The Coriolis effect, which arises from the measurement of velocities relative to rotating axes, must also be allowed for.

In order to measure vector quantities, a coordinate frame must be specified. Many coordinate frames are used in inertial navigation analysis (Ref. 6), but for present purposes, only two are necessary: see Figure 1.

The aircraft Body frame is defined as the orthogonal set having the Roll axis pointing forward, the Pitch axis pointing out the starboard side, and the Yaw axis pointing "down" relative to the aircraft. The origin of the body frame is at the aircraft centre of mass, not coincident with the I.N.S.

The Geographic frame has its origin at the system's location and its axes aligned with the local North, East, and Down directions. Down is defined as normal to the Earth's reference ellipsoid. In this discussion, the geographic frame is the navigation frame.

For any terrestrial inertial navigation system analysis, the concept of an inertial frame is also required. This frame has its origin at the mass centre of the Earth, and is non-rotating relative to the stars. For present purposes, it is not necessary to specify the directions of the axes of the inertial frame.

The set of sensor axes may be arranged in any attitude relative to the body frame, although conceptually it is useful to consider them as coincident with the body axes. (Whatever their orientation, a nominally constant transformation will give their outputs relative to the body axes.) For purposes of this discussion, the sensor and body axes will be assumed coincident except where mentioned otherwise.

Conventionally, the navigation algorithm takes the attitude reference and calculates the coordinate transformation matrix (if the reference is not already the direction cosine matrix). It uses this to transform the accelerometer outputs from body to geographic (or whatever set of axes are being used for navigation) axes coordinates. Present position and velocity are used to calculate the gravity and coriolis effects. These are added to the accelerometer outputs in the appropriate coordinates, and the resulting quantities are integrated to get velocity and position. In a digital system the accelerometers are usually arranged to act as acceleration-integrating sensors, so the outputs are obtained as velocity increments, integrated along body axes.

Vehicle attitude and transformation matrix are likely to be changing rapidly, therefore the coordinate transformations must be performed at a fast rate. This imposes a considerable burden on the computer.

The navigation equation for strapdown inertial navigation is derived in Appendix 7: for any reference frame  $K$ ;

$$\left[ \frac{d\mathbf{V}_E}{dt} \right]_K = \mathbf{F} + \mathbf{g} - (2\Omega_{IE} + \Omega_{EK}) \mathbf{V}_E$$

for example, if  $K$  is a "geographic" frame  $G$ , and the quantities are expressed in  $G$  frame coordinates, we get

$$[\Delta \mathbf{V}_E]^G = \int_T^{T+\Delta t} C_B^G [\mathbf{F}]^B dt + \int_T^{T+\Delta t} \{[\mathbf{g}]^G - (2\Omega_{IE}^G + \Omega_{EG}^G) [\mathbf{V}_E]^G\} dt.$$

In a "conventional" mechanisation this equation is solved at a fast rate. The  $\mathbf{g}$  and  $\Omega$  terms may not have to be evaluated at the fast rate, but  $\int C_B^G [\mathbf{F}]^B dt$  does.

In the equation,  $C_B^G$  is the body to geographic coordinate transformation matrix, which is varying rapidly as the aircraft attitude changes. The output of an integrating accelerometer is  $\int [\mathbf{F}]^B dt$ . The solution of  $\int C_B^G [\mathbf{F}]^B dt$  has not been widely discussed in the literature, and is usually solved by rectangular or, preferably, trapezoidal integration. Higher orders have been recommended: e.g. Levinson (Ref. 7) considers that a third order solution is required. Such a solution fits a second order polynomial to both  $\Omega$  and  $\mathbf{F}$ , and requires considerable computer time.

The aircraft attitude rate, of up to several hundred degrees per second, may be far greater than the navigation frame rotation rate, which is unlikely to exceed one degree per minute. Changes in gravity, and effects of Earth rotation and curvature are functions of the position of the navigation axes relative to Earth: although they change slowly, their effects on each of the system accelerometers change at the body rate.

The split frame mechanisation of the navigation equation takes advantage of the different frame rates, by performing body-axes-related calculations at a fast rate in body coordinates, and navigation-axes-related calculations at slower rates in navigation axes coordinates. This leads to considerable savings in computer time.

### 3.1 The Split-Frame Mechanisation

This concept is not new, but it has not had much attention in the literature. It was mentioned by Wilcox (Ref. 5), but it did not appear to attract further study until a mathematical analysis was reported by Bar-Itzhak (Ref. 8). Wray and Flynn (Ref. 9), in a comparison of various solutions of the navigation equation, concluded that it was the most efficient in use of computer time.

The basis of the split frame mechanisation is the fact that changes in position or velocity caused by specific force and changes in position or velocity caused by gravitation may be calculated separately. The actual change is the sum of these. The effects of specific force are evaluated at a fast rate in body axes coordinates. The effects of gravitation are evaluated at a slower rate in navigation axes coordinates. Coordinates of the incremental velocity are transformed from body to navigation axes at the slower rate, leading to a significant saving in computer loading.

#### 3.1.1 Body Axes Calculations (Fast Rate)

In this section, the accelerometer outputs and the rotation vector are used to calculate increments in velocity caused by specific force. This is performed at a fast rate in body axes.

The velocity of the body, with respect to an inertial frame, caused by specific force, is given by the equation

$$[\dot{V}_{IF}]^B = [F]^B - \Omega_{IB}^B [V_{IF}]^B.$$

This is derived in Appendix 8. This equation is solved numerically at a fast rate, using the accelerometer output and the rotation vector calculated for the attitude update. The solution of this equation is derived in Appendix 9: for present purposes the assumption is made of constant angular velocity and specific force during the calculation interval. This has been found to give an adequate trade off between computer loading and accuracy. The resulting algorithm is

$$[V_{IF}]^B \leftarrow [V_{IF}]^B + [A]^B - [\theta_{IB}]^B \times \{[V_{IF}]^B + \frac{1}{2}[A]^B\}$$

where  $[A]^B = \int [F]^B dt$  - the accelerometer output, and  $[\theta_{IB}]^B$  is the rotation vector.

In Appendix 10 the relationship between  $V_{IF}$  and  $V_{EF}$  is discussed and it is shown that subject to restraints on the frequency of the navigation frame calculations, it can be assumed that

$$[V_{IF}]^B = [V_{EF}]^B.$$

The  $[V_{IF}]^B$  are calculated and accumulated at the fast rate until the next navigation frame calculation is to take place. The assumption is then made that  $[V_{EF}]^B = [V_{IF}]^B$ , and  $V_{EF}$  is transformed into navigation coordinates. The initial condition for fast rate calculations is then applied ( $V_{IF} = R_{IF} = 0$ ) and a new series begins.

### 3.1.2 Navigation Axes Calculations (Intermediate Rate)

During the intermediate rate calculation cycle, the attitude quaternion is updated, to account for rotation of the navigation axes since the last intermediate cycle, normalised, and then used to transform the coordinates of  $V_{EF}$  from body to navigation axes. This is added to the previously calculated  $[V_{EG}]^N$ , giving the net change in velocity, relative to Earth, over the interval.

The total velocity relative to Earth, and the change in position since the last slow cycle, are then updated. Using these values, the navigation frame rotation and the  $[V_{EG}]^N$  are calculated for use in the next intermediate cycle.

### 3.1.3 Navigation Frame Rotation

The attitude quaternion required for velocity coordinate transformation relates the body axes to navigation axes. However, during the fast rate cycles, the quaternion is being updated with rotations of the body relative to an inertial frame. (The gyroscope outputs are relative to inertial space). It is therefore necessary to allow for the rotation of the navigation frame relative to the inertial frame, caused by Earth rotation and aircraft movement over the curved surface of the Earth. Appendix 11 shows how a quaternion multiplication is used to account for this rotation. The quaternion represents the rotation, over the intermediate cycle, of the navigation frame relative to the inertial frame, and is expressed in navigation frame coordinates.

### 3.1.4 Velocity Transformation

The  $V_{EF}$  coordinates are transformed from body to navigation axes using the quaternion. However, if body/navigation attitude information is required (e.g. for flight control, bomb aiming, etc.), it is usually more economical to calculate the direction cosine matrix, and use that for the transformation. These procedures are well known, but are listed for reference in Appendix 12.

### 3.1.5 Calculation of Velocity Change due to Gravitation Effects

The velocity of the body, relative to Earth, caused by the effects of gravitation, is given by the equation

$$[\dot{V}_{EG}]^N = [g]^N - (2\Omega_{IE}^N + \Omega_{EN}^N) [V_{EG}]^N$$

This is derived in Appendix 8. It can be seen that this equation is of a similar form to the  $\dot{V}_{IF}$  equation: it is solved numerically using the same procedure, the solution being

$$[V_{EG}]^N = [g]^N t_I - (2\Omega_{IE}^N + \Omega_{EN}^N) ([V_E]^N + \frac{1}{2}[g]^N t_I) t_I.$$

In this mechanisation, only the vertical component of  $g$  is considered.

### 3.1.6 Vertical Channel Damping

The vertical channel of a pure inertial system has an exponential instability with a time constant near the earth of about 9.5 minutes. This arises because the value of  $g$  is computed on the basis of the calculated altitude. See Appendix 14.

It is common practice to stabilise the vertical channel by an external altitude reference, usually the barometric altimeter. For optimal mixing a Kalman filter is used (e.g. Ref. 10), although in cases where the ultimate in accuracy is not required, or where computer capacity is limited, such as in the present algorithm, a fixed gains system is used. A typical third order mechanisation is employed, taken from Reference 11. The equations for this are given in Appendix 14, and the triple pole time constant of 100 seconds is retained.

### 3.1.7 Position and Velocity Integration

Having obtained  $[\Delta V_E]^N = [V_{EF}]^N + [V_{EG}]^N$ , the change in position is calculated:

$$[\Delta X]^N \leftarrow [\Delta X]^N + \{[V_E]^N + \frac{1}{2}[\Delta V_E]^N\} t_I,$$

and the updated velocity:

$$[V_E]^N \leftarrow [V_E]^N + [\Delta V_E]^N.$$

In this mechanisation, the  $[\Delta X]^N$  are themselves changes in position since the last slow rate cycle

## 3.2 Slow Rate Calculations

During the slow rate cycle, updates of vehicle position, in latitude and longitude, and also of the sine and cosine of latitude are made. In this mechanisation, the sine and cosine terms are updated as shown in Appendix 13. New values of Earth radii and gravity are calculated each slow cycle: the equations used are given in Appendix 13.

## 4. SUMMARY OF ALGORITHM

Gyroscopes are sampled twice per iteration period, accelerometers once. The attitude reference is maintained as a quaternion. Gyroscope outputs are used to calculate an equivalent (over the period) rotation vector, which is used to update (to third order) the attitude quaternion.

A split frame method is used for solution of the navigation equation: body related quantities are evaluated in body axes coordinates, and navigation frame related quantities are evaluated in navigation axes coordinates.

The algorithm is partitioned into 3 rates: fast, intermediate, and slow. Body axes calculations are performed at the fast rate, navigation axes calculations at the intermediate rate, and Earth-related quantities are evaluated at the slow rate.

Body axes are Roll, Pitch, Yaw; Navigation axes are the Geographic axes North, East, Down.

Incremental gyro samples are  $\delta_1$  and  $\delta_2$  at the mid-point and end of the fast calculation period. Incremental accelerometer samples are  $A$  at the end of the period.

The operations are as follows:

Fast rate: (body axes coordinates)

Rotation Vector	$\theta = \delta_1 + \delta_2 + \frac{1}{2}(\delta_1 \times \delta_2)$
Attitude Update	$\bar{Q} \leftarrow \bar{Q} * \{(1 - \frac{1}{2}\theta, 0), \frac{1}{2}\theta\}$
Body axes Velocity	$V_{IF} \leftarrow V_{IF} + A - \theta \times (V_{IF} + \frac{1}{2}A).$

Intermediate rate (navigation axes coordinates):

$$\begin{aligned}
 \text{Earth rotation and transport} \quad \bar{Q} &\leftarrow \{(1 - \frac{1}{12}\varphi \cdot \varphi), -\frac{1}{2}\varphi\} * \bar{Q} \\
 \text{Normalise quaternion} \quad QL &= 1.5 - 0.5 \sum (Q_i)^2 \quad i = 0, 3 \\
 &\bar{Q} \leftarrow QL \cdot \bar{Q} \\
 \text{Body to nav. axes} \quad [V_{EF}]^N &= \bar{Q} * [V_{EF}]^B * \bar{Q}^{-1} \quad V_{EF} \doteq V_{IF} \\
 \text{coord. transformation} & \\
 &\text{then } V_{IF} = 0 \\
 \text{Total vel. increment} \quad \Delta V_E &= V_{EF} + V_{EG} \\
 \text{Vertical channel} \quad H_c &= H - H_B \quad a \leftarrow a + H_c \\
 &\Delta V_E(D) \leftarrow \Delta V_E(D) - (k_2 H_c + k_3 a t_1) t_1 \\
 &H \leftarrow H - \{V_E(D) + \frac{1}{2} \Delta V_E(D)\} t_1 - k_1 H_c t_1 \\
 &V_E(D) \leftarrow \dot{V}_E(D) + \Delta V_E(D) \\
 \text{Horizontal channels} \quad \Delta X &\leftarrow \Delta X + (V_E + \frac{1}{2} \Delta V_E) t_1 \\
 &V_E \leftarrow V_E + \Delta V_E \\
 V_{EG} \text{ for next cycle} \quad V_{EG} &= g t_1 - (2\Omega_{IE} + \Omega_{EN}) (V_E + \frac{1}{2} g t_1) t_1 \\
 \varphi \text{ for next cycle} \quad \varphi &= \omega_{IN} t_1.
 \end{aligned}$$

Slow rate:

Update latitude, longitude, sine and cosine of latitude from the  $\Delta X$ ; calculate gravity and Earth radius. Then  $\Delta X = 0$ .

## 5. DISCUSSION OF ITERATION RATES

In the literature, the usual rate for "fast" calculation cycles is around 100 Hz, in the range 50 to 200 Hz. Some mechanisations have a "slow" position cycle in the range of about 0.5 to 2 Hz.

Similar rates are envisaged for this algorithm: the "intermediate" rate calculation cycle is limited in respect of its minimum rate as shown in Appendix 10, by an acceleration error of magnitude approximately  $36t_1$  parts per million (p.p.m.), where  $t_1$  is the period of the intermediate cycle. Thus, if say 10 p.p.m. were acceptable for this error, a higher limit of  $t_1 =$  about 0.25 sec., that is, a minimum frequency of about 4 Hz is imposed.

The other major factor influencing the choice of intermediate rate is vehicle speed—a high speed aircraft, particularly if operating at high latitudes, would require a higher iteration rate, whereas a helicopter, for example, would not. The attitude and body axis calculations have to contend with the full range of vehicle rates and vibration: their iteration rate should be as high as possible, unless the environment is unusually benign.

In practice, the capability of the computer is a limiting factor on the iteration rate of any strapdown system.

## 6. ALGORITHM PERFORMANCE

This will be discussed in two parts—attitude updating, and navigation.

### 6.1 Attitude Updating

Presentation of attitude performance test data will be limited to an example of coning motion. This is a standard test for evaluating attitude algorithms, as it is a highly non-commutative environment, yet with an analytical solution.

The results demonstrate the effects of iteration and sampling rates and of computer loading; see Figure 2. Performance of the third order algorithm taking two samples per iteration was found to be virtually identical to that of the fourth order method.

### 6.1.1 Computer Loadings

The third order methods are almost identical in this respect. The modified third order is slightly more economical, requiring approximately 29 multiplications and 27 additions per iteration. The fourth order Runge-Kutta method imposes almost twice the loading of the third order methods. A third order method could therefore be run at almost twice the rate of the fourth order, and, as can be seen from Figure 2, this gives considerably more than twice the accuracy in the coning tests.

### 6.1.2 Sampling Rates

In order to effect the non-commutativity correction, two gyro samples are required. In this algorithm, as in the Runge-Kutta method, two samples are taken per iteration. However, a third order Taylor series method often appears in the literature (e.g. Ref. 12), in which only one sample per iteration is taken. This single sample method has an accuracy equivalent to that of a two-sample-per-iteration method running at half the iteration rate. In other words, by taking two samples per iteration instead of one, the computing load may be almost halved.

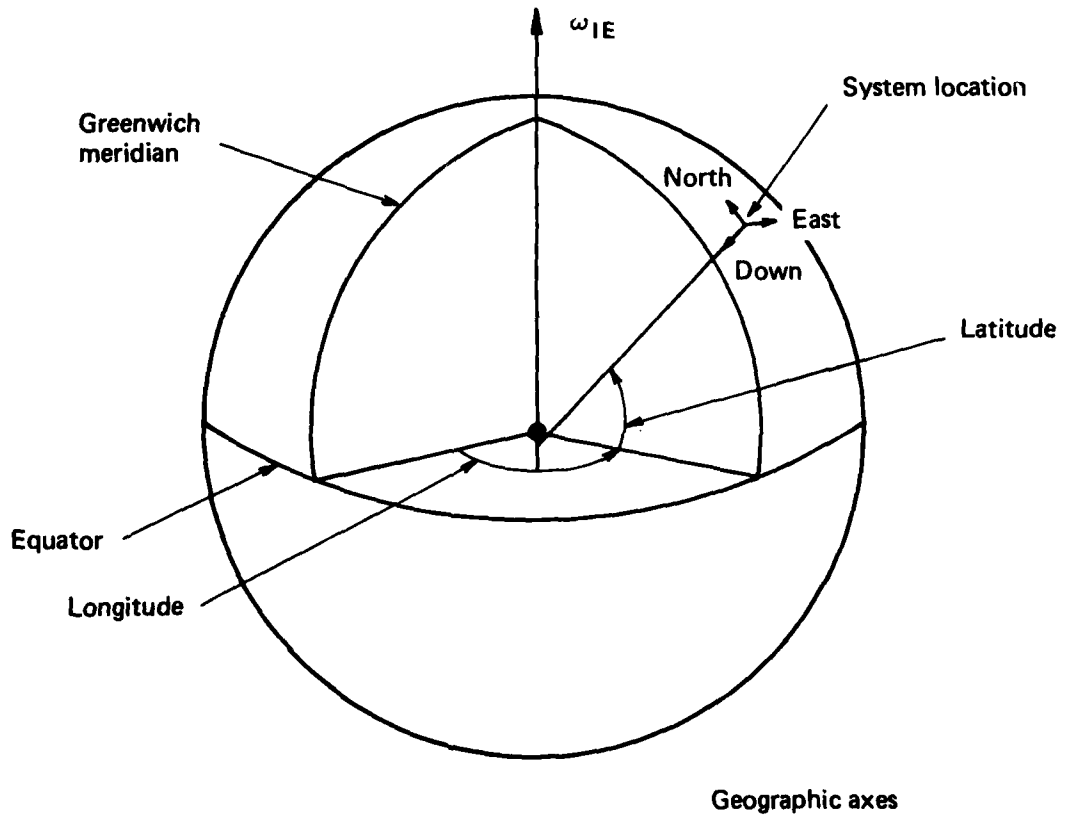
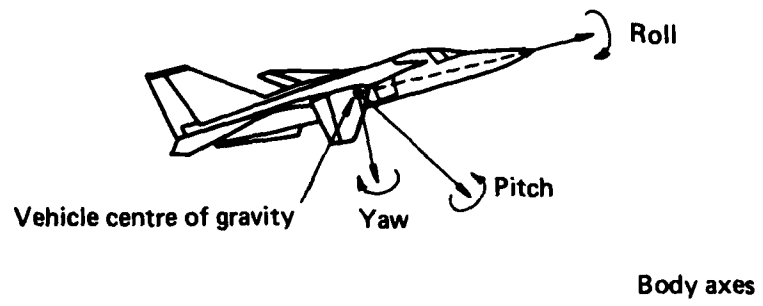
This is seen in the results, where the single-sample method must run at 200 Hz to give the same accuracy as the two-sample methods running at 100 Hz.

## 6.2 Navigation Performance

A program (Ref. 13) which simulates the somewhat idealised movement of an aircraft above the surface of an ellipsoidal, rotating Earth, was used to exercise the algorithm. This provided the outputs of a strapdown inertial measurement unit (here assumed error-free) in the aircraft, which performed manoeuvres including balanced horizontal and vertical turns, with prescribed angular and linear acceleration and turning rates. The program also outputs the position, velocity, and attitude of the aircraft during flight. The simulation is open loop; i.e. there is no feedback of the aircraft state to the flight control system, so vibration effects have been excluded. However, the effects of the dynamic environment of the I.M.U. are primarily of importance to the attitude segment of the algorithm: this exercise is considered to be a valid demonstration of the navigation segment.

Details of the flight are shown in Figure 3.

The strapdown algorithm was implemented on a 27 bit mantissa machine: results of two runs are shown to illustrate the magnitudes of errors which arise from differing navigation cycle calculation rates. Figure 4 shows the result of running all segments at the fast rate—100 Hz. Figure 5 was obtained with the fast, intermediate, and slow segments running at 100 Hz, 10 Hz, and 1 Hz respectively. For many applications, the intermediate and slow segments could run at even slower rates.



The Inertial Frame has its origin at the earth centre. Inertial Axes are not defined.

FIG. 1 CO-ORDINATE FRAMES



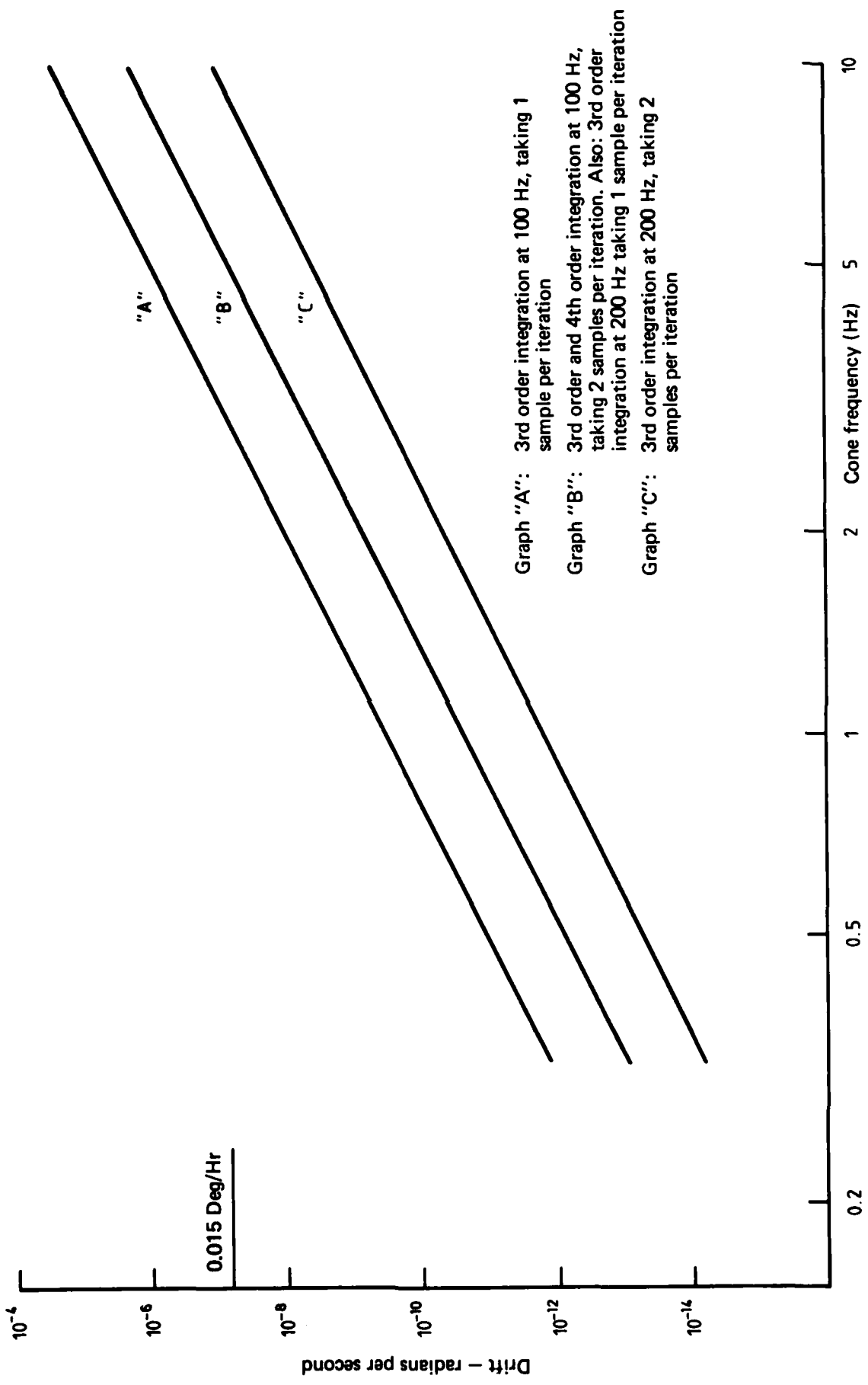


FIG. 2: ALGORITHM DRIFT FOR 1° HALF ANGLE CONING

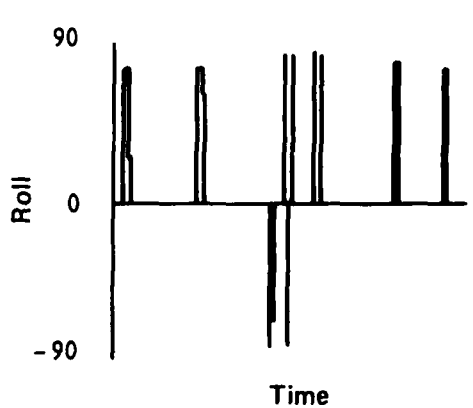
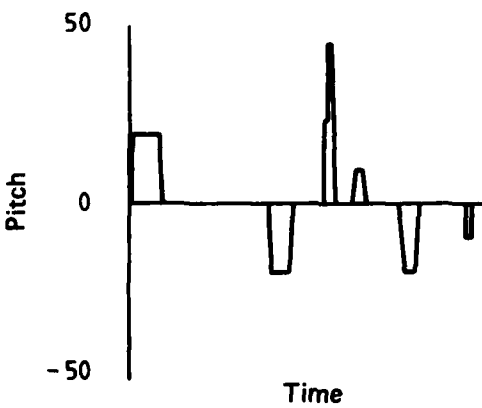
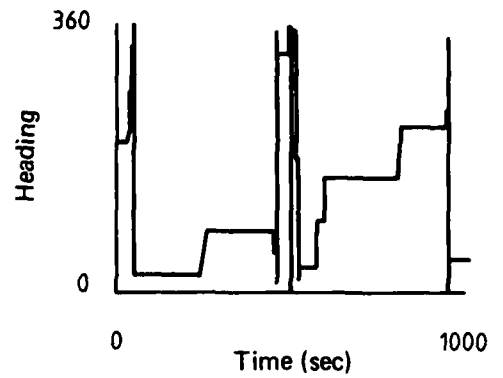
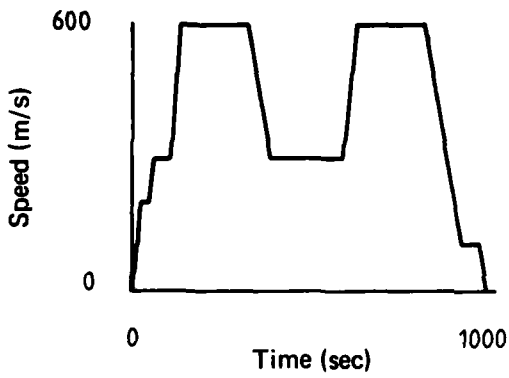
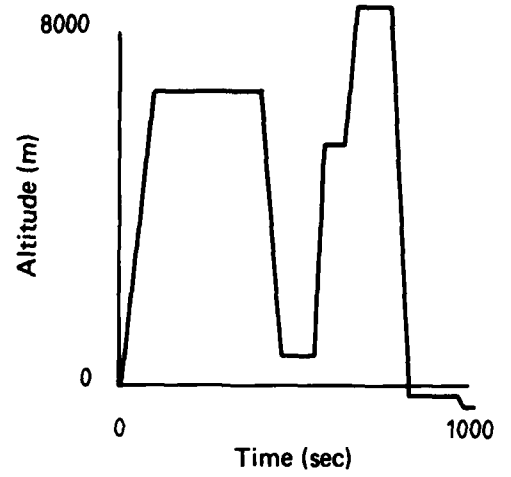
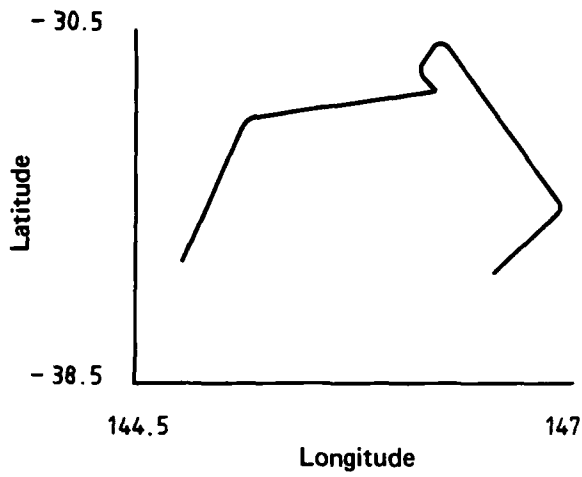


FIG. 3: SIMULATION FLIGHT DETAILS

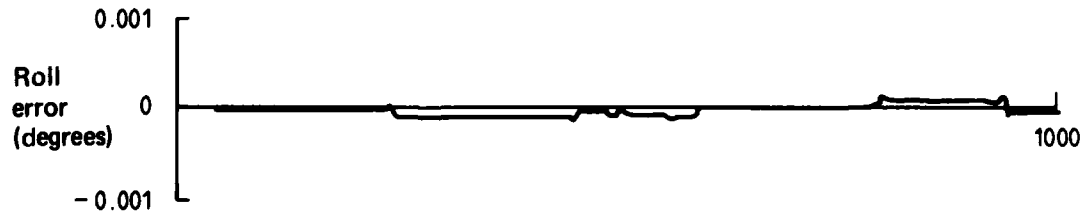
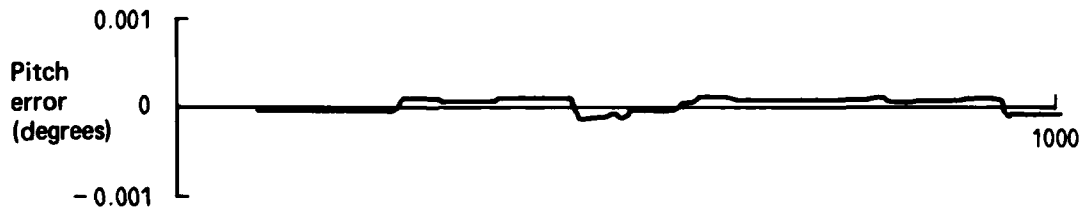
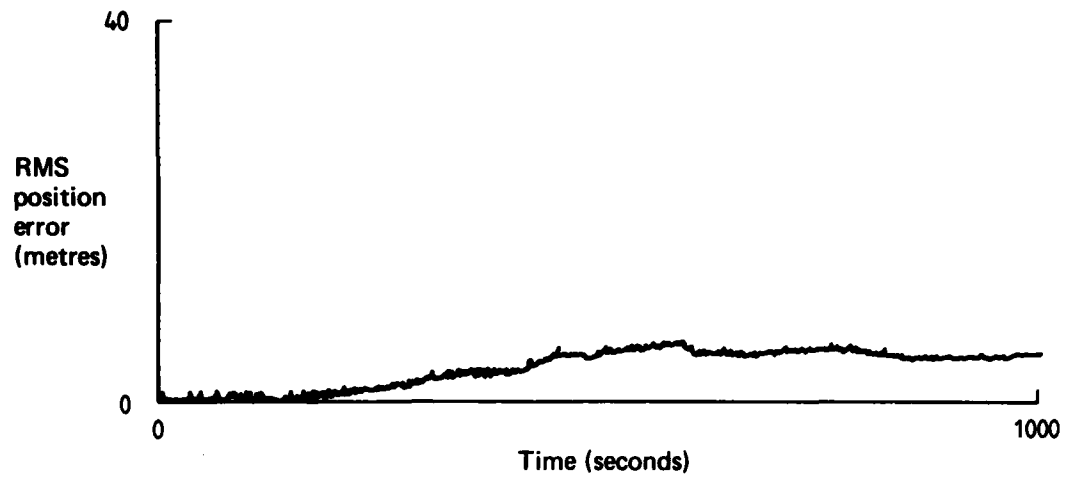


FIG. 4: ALGORITHM ERRORS – ALL SEGMENTS AT FAST (100 Hz) RATE

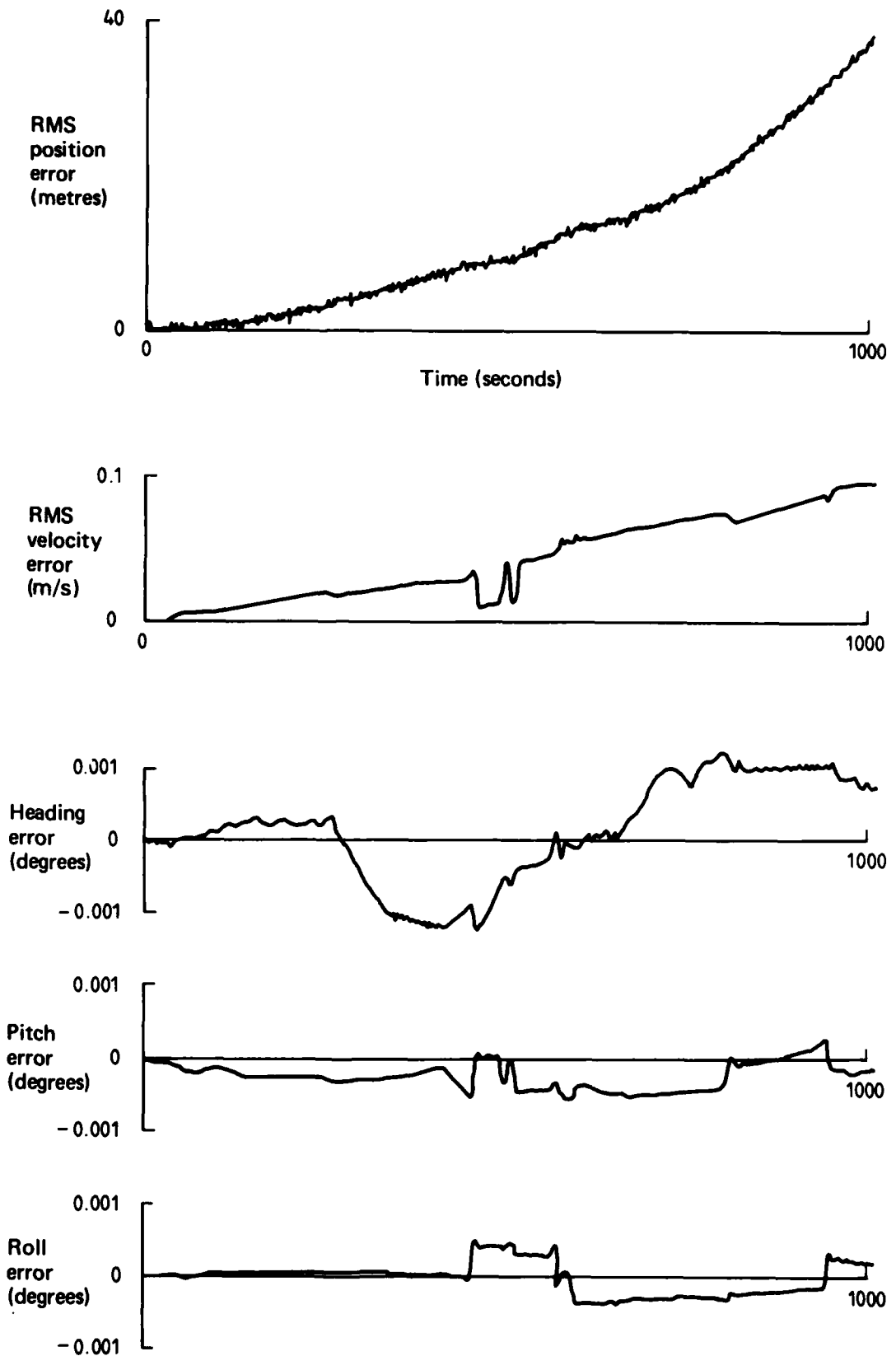


FIG. 5: ALGORITHM ERRORS: FAST CYCLE 100 Hz,  
 INTERMEDIATE CYCLE 10 Hz, SLOW CYCLE 1 Hz

## REFERENCES

1. Garg, S. C., Morrow, L. D., Mamen, R.: Strapdown Navigation Technology: A Literature Survey. A.I.A.A. Paper 77-1105, Aug. 1977.
2. AGARD LS-95: Strapdown Inertial Systems, June 1978.
3. Lloyd, I. V.: Fundamentals of Strapdown Inertial Navigation. A.R.L. Systems Group Report S.O.G. 11, Jan. 1975.
4. Bortz, J. E.: A New Mathematical Formulation for Strapdown Inertial Navigation. I.E.E.E. Trans. Aerospace and Electronic Systems, Vol. AES, No. 1, Jan. 1971.
5. Wilcox, J. C.: A New Algorithm for Strapped-Down Inertial Navigation. I.E.E.E. Trans. Aerospace and Electronic Systems, Vol. AES-3, No. 5 Sept. 1967.
6. Britting K. R.: Inertial Navigation Systems Analysis. Wiley Interscience, 1971.
7. Levinson, E.: Laser Gyro Strapdown Inertial System Applications. (In Reference 2.)
8. Bar-Itzhack, I. Y.: On the Navigation Computation in Terrestrial Strapdown Inertial Navigation Systems. I.E.E.E. Trans. Aerospace and Electronic Systems, Vol. AES-13, 1977.
9. Wray, G. L., Flynn, D. J.: An Assessment of Various Solutions of the Navigation Equation for a Strapdown Inertial System. R.A.E. TR-79017, Jan. 1979.
10. Farrell, J. L.: Integrated Aircraft Navigation. Academic Press, 1976.
11. Widnall, W. S., Grundy, P. A.: I.N.S. Error Models. Intermetrics Inc., Cambridge, Mass., TR-03-73, May 1973.
12. Nurse, R., Prohaska, J., Riegsecker, D.: A New Baseline for the Inertial Navigation Strapdown Simulator Program, Vol. II. Charles Stark Draper Laboratory Inc., Cambridge, Mass., July 1978.
13. Miller, R. B.: A Flight Profile Generator Program. (To be published.)

## APPENDIX 1

### Notation

$\mathbf{M}$	a vector
$[\mathbf{M}]^R$	vector $\mathbf{M}$ in $R$ frame coordinates
$\left[\frac{d\mathbf{M}}{dt}\right]_R$	the rate of change of $\mathbf{M}$ with respect to frame $R$
$\left[\frac{d\mathbf{M}}{dt}\right]_R^B$	is $\left[\frac{d\mathbf{M}}{dt}\right]_R$ in $B$ frame coordinates. N.B. $\left[\frac{d\mathbf{M}}{dt}\right]_R^R = [\dot{\mathbf{M}}]^R$
$C_R^B$	the transformation matrix (direction cosines) to transform from $R$ to $B$ coordinates. i.e. $[\mathbf{M}]^B = C_R^B[\mathbf{M}]^R$
$\omega_{IE}$	the (vector) angular velocity of the $E$ frame with respect to the $I$ frame
$\times$	vector cross product operator
$\Omega_{IE}^N$	the skew symmetric matrix $[\omega_{IE}]^N \times$
$*$	quaternion "multiplication" operator
$\bar{Q}_{RB}$	a quaternion representing a rotation from frame $R$ to frame $B$
$\mathbf{F}$	specific force
$\mathbf{R}$	position vector from Earth centre
$\mathbf{V}_I$	velocity relative to inertial space
$\mathbf{V}_{IF}$	velocity relative to inertial space caused by specific force
$\mathbf{V}_E$	velocity relative to Earth
$\mathbf{V}_{EF}$	velocity relative to Earth caused by specific force
$\mathbf{V}_{EG}$	velocity relative to Earth caused by gravitation
$\Delta\mathbf{V}$	increment in $\mathbf{V}$
$\mathbf{X}$	position vector
$H$	altitude
$H_B$	barometric altitude
$V_E(D)$	vertical velocity component
$\mathbf{g}_m$	mass attraction (gravitation)
$\mathbf{g}$	gravity (includes Earth rotation effects)
$\omega$	angular velocity
$\delta$	gyro output <span style="float: right;"><math>\delta = \int \omega dt</math></span>
$\theta$	"rotation vector"

$h$  a time interval  
 $t_F$  fast cycle time period  
 $t_I$  intermediate cycle time period  
 $t_S$  slow cycle time period  
 $\varphi$  rotation of navigation axes relative to inertial during  $t_I$

## APPENDIX 2

### Some Properties of Quaternions

The uses of four parameter techniques to represent rotations are well established in dynamics. Some properties of quaternions as applicable to this work are listed below:

A quaternion representing a rotation may be expressed as a scalar and a three element vector: (the "Euler Parameters")

$$\bar{Q} = \cos \frac{1}{2}\theta_0, e_1 \sin \left(\frac{1}{2}\theta_0\right), e_2 \sin \left(\frac{1}{2}\theta_0\right), e_3 \sin \left(\frac{1}{2}\theta_0\right).$$

This may be interpreted as a rotation through an angle  $\theta_0$  measured from reference to body axes, about a unit vector defined (in both body and reference axes) by its components  $e_1, e_2, e_3$ .

Alternatively,

$$\bar{Q} = \cos \left(\frac{1}{2}\theta_0\right), (\theta_1/\theta_0) \sin \left(\frac{1}{2}\theta_0\right), (\theta_2/\theta_0) \sin \left(\frac{1}{2}\theta_0\right), (\theta_3/\theta_0) \sin \left(\frac{1}{2}\theta_0\right),$$

where the  $\theta_i$  are the components of the rotation, and  $\theta_0 = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2}$ . This may be written as  $\bar{Q} = C, \theta S$  where  $C = \cos \left(\frac{1}{2}\theta_0\right)$  and  $S = (1/\theta_0) \sin \left(\frac{1}{2}\theta_0\right)$ .

#### *Quaternion "Multiplication"*

Quaternion "multiplication" may be defined as follows: given the quaternions  $\bar{A} = A_0, \mathbf{A}$  and  $\bar{B} = B_0, \mathbf{B}$ , then the product  $\bar{C} = C_0, \mathbf{C}$ , of these is

$$\bar{C} = \bar{A} * \bar{B} = (A_0 B_0 - \mathbf{A} \cdot \mathbf{B}), \{A_0 \mathbf{B} + \mathbf{A} B_0 + (\mathbf{A} \times \mathbf{B})\}.$$

For a physical interpretation of this, consider a body, with a quaternion  $\bar{A}$  representing the rotation from reference to body axes. Give the body a rotation such that the quaternion  $\bar{B}$  represents the rotation from old to new body axes. The quaternion  $\bar{C}$  as defined above now represents a rotation from the reference axes to the new body axes.

#### *Coordinate Transformation by Quaternions*

The "unit length" rotation quaternion  $\bar{Q} = C, \theta S$  has a conjugate  $\bar{Q}^* = C, -\theta S$  which is  $\bar{Q}^{-1}$ .

Any 3 dimensional vector  $\mathbf{M}$  may be regarded as a quaternion  $\bar{M}$  with its scalar part zero: then if  $\bar{Q}$  represents the rotation from reference to body axes,

$$[\mathbf{M}]^R = \bar{Q} * [\mathbf{M}]^B * \bar{Q}^{-1} \quad \{[\bar{M}]^B \equiv [\mathbf{M}]^B\}.$$

#### *Normalisation of Quaternions*

For a rotation quaternion, the sum of the squares of the elements is nominally unity. The square root of this is sometimes referred to as the "length" of the quaternion. Any departure from unity in this value causes a "scale error".

This effect can be removed by "normalisation", in which each element of the quaternion is divided by the "length".

A "scale error" occurs in a vector whose coordinates are transformed between two coordinate frames, where its length does not remain constant.

A "drift error" occurs in a vector whose coordinates are transformed between two coordinate frames, where its direction does not remain constant.



## APPENDIX 3

### Attitude Propagation

#### A3.1 Direction Cosines

For any vector  $M$ :

$$[M]^B = C_A^B [M]^A \quad (1)$$

also,

$$[dM/dt]_A^B = C_A^B [\dot{M}]^A \quad (2)$$

Time derivative of (1):

$$[\dot{M}]^B = \dot{C}_A^B [M]^A + C_A^B [\dot{M}]^A. \quad (3)$$

If  $\omega_{AB}$  is the angular velocity of frame  $B$  relative to frame  $A$ , then Coriolis' equation, in  $B$  frame coordinates, gives:

$$[dM/dt]_A^B = [\dot{M}]^B + \Omega_{AB}^B [M]^B. \quad (4)$$

Use (1)–(3) in (4):

$$C_A^B [\dot{M}]^A = C_A^B [\dot{M}]^A + \dot{C}_A^B [M]^A + \Omega_{AB}^B C_A^B [M]^A$$

i.e.

$$\{\dot{C}_A^B + \Omega_{AB}^B C_A^B\} [M]^A = 0$$

$M$  is any vector,

$$\therefore \dot{C}_A^B = -\Omega_{AB}^B C_A^B.$$

Now

$$[-\Omega_{AB}^B]^{-1} \equiv \Omega_{AB}^B, \text{ and } \Omega^{-1} C^{-1} \equiv [C\Omega]^{-1},$$

$$\therefore \dot{C}_A^B = C_A^B \Omega_{AB}^B. \quad (5)$$

This is the differential equation for propagation of the matrix  $C_A^B$ .

#### A3.2 Quaternions

For any vector  $M$ :

$$[M]^B = \bar{Q}_{AB}^{-1} * [M]^A * \bar{Q}_{AB}. \quad (6)$$

Differentiate w.r.t. time, and put  $[M]^A = \bar{Q}_{AB} * [M]^B * \bar{Q}_{AB}^{-1}$ :

$$[\dot{M}]^B = \dot{\bar{Q}}_{AB}^{-1} * \bar{Q}_{AB} * [M]^B + [M]^B * \dot{\bar{Q}}_{AB}^{-1} * \bar{Q}_{AB} + \bar{Q}_{AB}^{-1} * [\dot{M}]^A * \bar{Q}_{AB} \quad (7)$$

where

$$\dot{\bar{Q}}_{AB}^{-1} \text{ means } d/dt (\bar{Q}_{AB}^{-1}).$$

Now

$$\bar{Q}_{AB}^{-1} * [\dot{M}]^A * \bar{Q}_{AB} \equiv C_A^B [\dot{M}]^A.$$

Using (2) and (4), we get

$$\bar{Q}_{AB}^{-1} * [\dot{M}]^A * \bar{Q}_{AB} = [\dot{M}]^B + \Omega_{AB}^B [M]^B. \quad (8)$$

Consider the identity  $q^{-1} * q = 1$ , where  $q$  is any rotation quaternion:

$$q = q_0, \mathbf{q} \text{ and } q^{-1} = q_0, -\mathbf{q}.$$

Differentiate w.r.t. time:

$$\dot{q}^{-1} * q + q^{-1} * \dot{q} = 0. \quad (9)$$

Now, by definition

$$\dot{q}^{-1} * q = (\dot{q}_0 q_0 + \dot{\mathbf{q}} \cdot \mathbf{q}), \{\dot{q}_0 \mathbf{q} - \dot{\mathbf{q}} q_0 - (\dot{\mathbf{q}} \times \mathbf{q})\} = p_0, \mathbf{p} \text{ (say)}$$

and,

$$q^{-1} * \dot{q} = (q_0 \dot{q}_0 + \mathbf{q} \cdot \dot{\mathbf{q}}), \{q_0 \dot{\mathbf{q}} - \mathbf{q} \dot{q}_0 - (\mathbf{q} \times \dot{\mathbf{q}})\} = r_0, \mathbf{r} \text{ (say)}.$$

It can be seen that  $p_0 = r_0$  and  $\mathbf{p} = -\mathbf{r}$

however, from (9):  $p_0, \mathbf{p} + r_0, \mathbf{r} = 0$  therefore  $p_0 = r_0 = 0$ .

In equation (7), let  $\dot{Q}_{AB}^{-1} * Q_{AB} = \mathbf{p}$  and  $\dot{Q}_{AB}^{-1} * \dot{Q}_{AB} = -\mathbf{p}$ .

Using (8), (7) may then be written

$$[\mathbf{M}]^B = \mathbf{p} * [\mathbf{M}]^B - [\mathbf{M}]^B * \mathbf{p} + [\mathbf{M}]^B + \Omega_{AB}^B [\mathbf{M}]^B.$$

Evaluating the quaternion products, and writing  $[\omega_{AB}]^B \times$  for  $\Omega_{AB}^B$ ,

$$\therefore 0 = -\mathbf{p}[\mathbf{M}]^B + \mathbf{p} \times [\mathbf{M}]^B + \mathbf{p} \cdot [\mathbf{M}]^B - [\mathbf{M}]^B \times \mathbf{p} + [\omega_{AB}]^B \times [\mathbf{M}]^B$$

$$\therefore \{2\mathbf{p} + [\omega_{AB}]^B\} \times [\mathbf{M}]^B = 0.$$

$\mathbf{M}$  is any vector,

$$\therefore \frac{1}{2}[\omega_{AB}]^B = -\mathbf{p} = \dot{Q}_{AB}^{-1} * \dot{Q}_{AB}$$

$$\therefore \dot{Q}_{AB} = \frac{1}{2}\dot{Q}_{AB} * [\omega_{AB}]^B. \quad (10)$$

Alternatively, if  $[\omega_{AB}]^A$  is available, (10) becomes  $\dot{Q}_{AB} = \frac{1}{2}[\omega_{AB}]^A * \dot{Q}_{AB}$

(10) is the differential equation for the propagation of quaternion  $Q_{AB}$ .

## APPENDIX 4

### Quaternion Update Methods

#### A4.1 Taylor Series Expansion (to Third Order)

We wish to use the equation

$$\bar{Q}(t) = \frac{1}{2} \bar{Q}(t) * \bar{\omega}(t) \quad (1)$$

to obtain

$$\bar{Q}(t+h).$$

Using a Taylor Series expansion, we get:

$$\bar{Q}(t+h) = \bar{Q}(t) + h\dot{\bar{Q}}(t) + (h^2/2!)\ddot{\bar{Q}}(t) + (h^3/3!)\dddot{\bar{Q}}(t) + \dots \quad (2)$$

[for clarity, the  $(t)$  will be omitted].

Differentiate (1) w.r.t. time:

$$\dot{\bar{Q}} = \frac{1}{2}(\dot{\bar{Q}} * \bar{\omega}) * \bar{\omega} + \frac{1}{2}\bar{Q} * \dot{\bar{\omega}}.$$

Now,

$$\begin{aligned} (\bar{Q} * \bar{\omega}) * \bar{\omega} &\equiv \bar{Q} * (\bar{\omega} * \bar{\omega}) = \bar{Q}(-\omega \cdot \omega) \\ \therefore \dot{\bar{Q}} &= \bar{Q} * \left\{ \frac{1}{2}\dot{\bar{\omega}} - \frac{1}{4}(\omega \cdot \omega) \right\}. \end{aligned} \quad (3)$$

Differentiate again:

$$\ddot{\bar{Q}} = \frac{1}{2}(\dot{\bar{Q}} * \bar{\omega}) * \dot{\bar{\omega}} + \frac{1}{2}\bar{Q} * \ddot{\bar{\omega}} - \frac{1}{2}(\omega \cdot \dot{\omega})\bar{Q} - \frac{1}{4}(\omega \cdot \omega)(\frac{1}{2}\dot{\bar{Q}} * \bar{\omega}).$$

Now,

$$\begin{aligned} (\bar{Q} * \bar{\omega}) * \dot{\bar{\omega}} &\equiv \bar{Q} * (\bar{\omega} * \dot{\bar{\omega}}) \\ \therefore \ddot{\bar{Q}} &= \bar{Q} * \left\{ \frac{1}{4}(\bar{\omega} * \dot{\bar{\omega}}) - \frac{1}{2}(\omega \cdot \dot{\omega}) - \frac{1}{8}(\omega \cdot \omega)\bar{\omega} + \frac{1}{2}\ddot{\bar{\omega}} \right\}. \end{aligned} \quad (4)$$

Using (1), (3), and (4) in (2), we get

$$\bar{Q}(t+h) = \bar{Q}(t) * \bar{U}(t) \quad (5)$$

where:

$$\bar{U} = 1 + \frac{1}{2}h\bar{\omega} + \frac{1}{2}h^2 \left\{ \frac{1}{2}\ddot{\bar{\omega}} - \frac{1}{4}(\omega \cdot \omega) \right\} + \frac{1}{8}h^3 \left\{ \frac{1}{4}(\bar{\omega} * \dot{\bar{\omega}}) - \frac{1}{2}(\omega \cdot \dot{\omega}) - \frac{1}{8}(\omega \cdot \omega) + \frac{1}{2}\ddot{\bar{\omega}} \right\} \quad (6)$$

Expand  $\bar{U}$  into its scalar and vector components: [N.B.  $(\bar{\omega} * \dot{\bar{\omega}}) = -\omega \cdot \dot{\omega} + (\omega \times \dot{\omega})$ .]

$$U_0, \mathbf{U} = \left\{ 1 - \frac{1}{8}h^2(\omega \cdot \omega) - \frac{1}{8}h^3(\omega \cdot \dot{\omega}), \left\{ \frac{1}{2}h\bar{\omega} + \frac{1}{4}h^2\dot{\bar{\omega}} + \frac{1}{24}h^3(\omega \times \dot{\omega}) - \frac{1}{8}h^3(\omega \cdot \omega)\bar{\omega} + \frac{1}{12}h^3\ddot{\bar{\omega}} \right\} \right\}. \quad (7)$$

The values of  $\omega(t)$ ,  $\dot{\omega}(t)$  and  $\ddot{\omega}(t)$  must now be found from the gyro outputs. It is assumed that a second order polynomial may be fitted to the gyro outputs: i.e.,  $\delta(t) = \mathbf{a}t + \mathbf{b}t^2$ . Now,  $\delta(t) = \int \omega(t)dt$  over the sample period: this implies that  $\dot{\omega}(t) = 0$ , and  $\omega(t)$  is constant over the period.

If the gyros are sampled at the mid-point and end of each iteration period, giving  $\delta_1$  from  $t$  to  $(t + \frac{1}{2}h)$ , and  $\delta_2$  from  $(t + \frac{1}{2}h)$  to  $(t + h)$ , we get  $\omega(t) = (3\delta_1 - \delta_2)/h$  and  $\ddot{\omega}(t) = 4(\delta_2 - \delta_1)/h^2$ .

Substituting in (7), rearranging, and putting  $\delta = (\delta_1 + \delta_2)$ , we get

$$U_0 = 1 - \frac{1}{2}(\delta \cdot \delta) + \frac{1}{2}(\delta_1 - \delta_2) \cdot (\delta_1 - \delta_2)$$

$$U = \frac{1}{2}\delta + \frac{1}{2}(\delta_1 \times \delta_2) - \frac{1}{48}\{(\delta \cdot \delta) - 8\delta_1 \cdot (\delta_2 - \delta_1)\} \{\delta - 2(\delta_2 - \delta_1)\}.$$

The last term in each part of  $\bar{U}$  may be assumed small: if in these terms only, we put  $\delta_1 = \delta_2$ , then we get

$$U_0 = 1 - \frac{1}{2}(\delta \cdot \delta)$$

$$U = \frac{1}{2}\delta + \frac{1}{2}(\delta_1 \times \delta_2) - \frac{1}{48}(\delta \cdot \delta)\delta.$$

It is interesting to compare this result with the solution obtained from the rotation vector method using a third order expansion of  $C$  and  $S$ : in that case;

$$C, S\theta = (1 - \frac{1}{2}\theta_0^2), (\frac{1}{2}\theta - \frac{1}{48}\theta_0^2\theta)$$

where

$$\theta = \delta_1 + \delta_2 + \frac{2}{3}(\delta_1 \times \delta_2) \text{ and } \theta_0^2 = \theta \cdot \theta.$$

If  $(\delta_1 \times \delta_2)^2$  and  $\theta_0^2(\delta_1 \times \delta_2)$ , which are small, are neglected, we get

$$C = 1 - \frac{1}{2}(\delta \cdot \delta)$$

$$S\theta = \frac{1}{2}\delta + \frac{1}{2}(\delta_1 \times \delta_2) - \frac{1}{48}(\delta \cdot \delta)\delta$$

i.e.

$$U_0, U \doteq C, S\theta.$$

A method of estimating  $\omega(t)$  and  $\dot{\omega}(t)$  which often appears in the literature (e.g. Ref. 12) uses the incremental gyro output  $\delta$  over the whole calculation period, that is, between time  $t$  and time  $(t + h)$ , and the previous output  $\delta'$ , between times  $(t - h)$  and  $t$ .

Using these values, we get  $\omega(t) = (\delta + \delta')/2h$  and  $\dot{\omega}(t) = (\delta - \delta')/h^2$ .

When these are substituted into (7), we get, after rearrangement,

$$U_0 = 1 - \frac{1}{2}(\delta \cdot \delta) + \frac{1}{2}(\delta - \delta') \cdot (\delta - \delta')$$

$$U = \frac{1}{2}\delta + \frac{1}{4}(\delta' \times \delta) - \frac{1}{96}\{(\delta \cdot \delta) - \frac{1}{4}(3\delta + \delta') \cdot (\delta - \delta')\}(\delta + \delta').$$

The last term in each part of  $\bar{U}$  may be assumed small: in these terms, if one assumes that  $\delta = \delta'$ , the result is obtained:

$$U_0 = 1 - \frac{1}{2}(\delta \cdot \delta)$$

$$U = \frac{1}{2}\delta + \frac{1}{4}(\delta' \times \delta) - \frac{1}{48}(\delta \cdot \delta)\delta.$$

#### A4.2 Quaternion Update by Fourth Order Runge-Kutta

Runge-Kutta methods are standard tools in numerical analysis. The application of a fourth order method to the attitude quaternion update is as follows:

Given the equation  $\dot{Q}(t) = \frac{1}{2}Q(t) * \bar{\omega}(t)$ : to find  $Q(t + h)$ .

Incremental gyro outputs  $\delta_1$  and  $\delta_2$  are taken at the mid-point (time  $t + \frac{1}{2}h$ ) and end (time  $t + h$ ) of the period respectively. Fitting a second order polynomial to these allows calculation of the angular velocity at the start, mid-point, and end of the period:

$$\omega(t) = (3\delta_1 - \delta_2)/h, \quad \omega(t + \frac{1}{2}h) = (\delta_1 + \delta_2)/h, \quad \omega(t + h) = (3\delta_2 - \delta_1)/h$$

Let

$$\begin{aligned} H\omega_0 &= h \cdot \omega(t) & \text{and} & & H\omega_1 &= h \cdot \omega(t + \frac{1}{2}h) & \text{and} & & H\omega_2 &= h \omega(t + h) \\ &= (3\delta_1 - \delta_2) & & & &= (\delta_1 + \delta_2) & & & &= (3\delta_2 - \delta_1). \end{aligned}$$

The method is:

$$1) \quad \bar{k}_0 = h\dot{Q}(t) = \frac{1}{2}Q(t) * H\omega_0$$

$$2) \quad Q'(t + \frac{1}{2}h) = Q(t) + \frac{1}{2}\bar{k}_0$$

$$\bar{k}_1 = h\dot{Q}'(t + \frac{1}{2}h) = \frac{1}{2}Q'(t + \frac{1}{2}h) * H\omega_1$$

$$3) \quad Q''(t + \frac{1}{2}h) = Q'(t) + \frac{1}{2}\bar{k}_1$$

$$\bar{k}_2 = h\dot{Q}''(t + \frac{1}{2}h) = \frac{1}{2}Q''(t + \frac{1}{2}h) * H\omega_1$$

$$4) \quad Q'(t + h) = Q'(t) + \bar{k}_2$$

$$\bar{k}_3 = h\dot{Q}'(t + h) = \frac{1}{2}Q'(t + h) * H\omega_2$$

$$5) \text{ update: } \quad Q(t + h) = Q(t) + \frac{1}{4}(\bar{k}_0 + 2(\bar{k}_1 + \bar{k}_2) + \bar{k}_3).$$

## APPENDIX 5

### Rotation Vector Equation

Consider a quaternion  $\bar{Q} = C, S\theta$ :  $C = \cos(\frac{1}{2}\theta_0)$ ,  $S = (1/\theta_0) \sin(\frac{1}{2}\theta_0)$ ,  $\theta_0 = (\theta \cdot \theta)^{1/2}$ ,  $\theta$  is the rotation vector.

Differentiate:  $\dot{\theta}_0 = \frac{\frac{1}{2}(\dot{\theta} \cdot \theta + \theta \cdot \dot{\theta})}{(\theta \cdot \theta)^{1/2}} = \frac{\dot{\theta} \cdot \theta}{\theta_0}$

also  $\dot{C} = -\sin(\frac{1}{2}\theta_0) \cdot \frac{1}{2}\dot{\theta}_0 = -\frac{1}{2}S(\dot{\theta} \cdot \theta)$

and  $\dot{S} = -\frac{\dot{\theta}_0}{\theta_0^2} \sin(\frac{1}{2}\theta_0) + \frac{1}{\theta_0} \cos(\frac{1}{2}\theta_0) \cdot \frac{1}{2}\dot{\theta}_0 = \frac{(\dot{\theta} \cdot \theta)}{2\theta_0^2} (C - 2S)$

and  $\dot{\bar{Q}} = \dot{C}, (\dot{S}\theta + S\dot{\theta})$

therefore  $\dot{\bar{Q}} = \frac{1}{2}\{-S(\dot{\theta} \cdot \theta), \frac{(\dot{\theta} \cdot \theta)}{\theta_0^2} (C - 2S)\theta + 2S\dot{\theta}\}$ .

Put  $(\dot{\theta} \cdot \theta)\theta = \theta \times (\theta \times \dot{\theta}) + \theta_0^2 \dot{\theta}$

$$\therefore \dot{\bar{Q}} = \frac{1}{2}\{-S(\dot{\theta} \cdot \theta), C\dot{\theta} + \frac{(C - 2S)}{\theta_0^2} \theta \times (\theta \times \dot{\theta})\} \quad (1)$$

also:  $\dot{\bar{Q}} = \frac{1}{2}\bar{Q} * \bar{\omega} = \frac{1}{2}\{-(Q \cdot \omega), Q_0\omega + (Q \times \omega)\}$ .

Now,  $(C, S\theta) \equiv (Q_0, Q)$ ,

$$\therefore \dot{\bar{Q}} = \frac{1}{2}\{-S(\theta \cdot \omega), C\omega + S(\theta \times \omega)\}. \quad (2)$$

Equating scalar parts of  $\dot{\bar{Q}}$  in (1) and (2), we get

$$\theta \cdot \omega = \theta \cdot \dot{\theta} \quad (3)$$

i.e. the component of  $\dot{\theta}$  parallel to  $\theta$  is equal to the component of  $\omega$  parallel to  $\theta$ .

Equating vector parts of  $\dot{\bar{Q}}$  in (1) and (2), we get

$$C\dot{\theta} + \frac{(C - 2S)}{\theta_0^2} \theta \times (\theta \times \dot{\theta}) = C\omega + S(\theta \times \omega). \quad (4)$$

Consider the relations

$$\theta \times (\theta \times \dot{\theta}) \equiv (\dot{\theta} \cdot \theta)\theta - (\theta \cdot \theta)\dot{\theta}$$

and

$$\theta \times (\theta \times \omega) \equiv (\theta \cdot \omega)\theta - (\theta \cdot \theta)\omega.$$

Subtracting these, and using (3), we get

$$\theta \times (\theta \times \dot{\theta}) = \theta \times (\theta \times \omega) - \theta_0^2(\dot{\theta} - \omega).$$

Substitute in (4):

$$C\dot{\theta} + \frac{(C - 2S)}{\theta_0^2} \theta \times (\theta \times \omega) - (C - 2S)(\dot{\theta} \cdot \omega)\theta = C\omega + S(\theta \times \omega).$$

Simplifying, we get

$$\dot{\theta} = \omega + \frac{1}{2}(\theta \times \omega) + (1/\theta_0^2)[1 - (C/2S)]\theta \times (\theta \times \omega). \quad (5)$$

Making the approximations  $C = 1 - \frac{1}{4}\theta_0^2$  and  $S = \frac{1}{2}(1 - \frac{1}{2}\theta_0^2)$ , we get

$$\dot{\theta} = \omega + \frac{1}{2}(\theta \times \omega) + \frac{1}{12}\theta \times (\theta \times \omega).$$

## APPENDIX 6

### Solution for the Rotation Vector

For present purposes, the very small triple vector product term will be neglected, thus:

$$\dot{\theta}(t) = \omega(t) + \frac{1}{2}\{\theta(t) \times \omega(t)\}. \quad (1)$$

Assume second order gyro output variation:

$$\delta(t) = \mathbf{a}t + \mathbf{b}t^2 \quad (2)$$

where  $T < t < T + h$ :  $T$  is time at the start of the period of length  $h$ . Therefore  $\omega(t) = \mathbf{a} + 2\mathbf{b}t$ ,  $\dot{\omega}(t) = 2\mathbf{b}$ , and  $\ddot{\omega}(t) = 0$ .

Now,  $\theta(T) = 0$ ,  $\omega(T) = \mathbf{a}$ ,  $\dot{\omega}(T) = 2\mathbf{b}$ .

equation (1) gives

$$\dot{\theta}(T) = \omega(T) = \mathbf{a}$$

differentiate and substitute:

$$\ddot{\theta}(T) = \dot{\omega}(T) = 2\mathbf{b}$$

differentiate and substitute:

$$\ddot{\theta}(T) = \frac{1}{2}\omega(T) \times \dot{\omega}(T) = \mathbf{a} \times \mathbf{b}$$

and  $\ddot{\theta}(T)$  is a triple vector product, which is neglected.

Apply a Taylor Series solution to equation (1):

$$\theta(T + h) = \theta(T) + h\dot{\theta}(T) + (1/2!)h^2\ddot{\theta}(T) + (1/3!)h^3\ddot{\theta}(T) + \dots$$

therefore

$$\theta(T + h) = \mathbf{a}h + \mathbf{b}h^2 + \frac{1}{2}h^2(\mathbf{a} \times \mathbf{b}). \quad (3)$$

If  $\delta_1$  is the incremental gyro output at the mid point of the period, and  $\delta_2$  is that at the end, then (eq. 2)

$$\delta(T + \frac{1}{2}h) = \delta_1 = \frac{1}{2}\mathbf{a}h + \frac{1}{4}\mathbf{b}h^2$$

and

$$\delta(T + h) = \delta_2 = \mathbf{a}h + \mathbf{b}h^2.$$

Solving for  $\mathbf{a}$  and  $\mathbf{b}$ , we get

$$\mathbf{a}h = 3\delta_1 - \delta_2 \text{ and } \mathbf{b}h^2 = 2(\delta_2 - \delta_1).$$

Therefore

$$\frac{1}{2}h^2(\mathbf{a} \times \mathbf{b}) = \frac{1}{3}(\delta_1 \times \delta_2)$$

and equation (3) may be written

$$\theta(T + h) = \delta_1 + \delta_2 + \frac{1}{3}(\delta_1 \times \delta_2).$$

The same procedure may be used for higher order solutions if necessary.

## APPENDIX 7

### Derivation of the Navigation Equation

Apply Coriolis' equation to  $\mathbf{R}$ :

$$\left(\frac{d\mathbf{R}}{dt}\right)_I = \left(\frac{d\mathbf{R}}{dt}\right)_E + \Omega_{IE}\mathbf{R} = \mathbf{V}_E + \Omega_{IE}\mathbf{R}.$$

Differentiate

$$\begin{aligned} \left(\frac{d^2\mathbf{R}}{dt^2}\right)_I &= \left(\frac{d\mathbf{V}_E}{dt}\right)_I + \Omega_{IE}\left(\frac{d\mathbf{R}}{dt}\right)_I \\ \therefore \left(\frac{d^2\mathbf{R}}{dt^2}\right)_I &= \left(\frac{d\mathbf{V}_E}{dt}\right)_I + \Omega_{IE}\mathbf{V}_E + \Omega_{IE}\Omega_{IE}\mathbf{R}. \end{aligned}$$

Specific force is given by  $\mathbf{F} = (d^2\mathbf{R}/dt^2)_I - \mathbf{g}_m$  where  $\mathbf{g}_m$  is mass attraction

$$\therefore \left(\frac{d\mathbf{V}_E}{dt}\right)_I = \mathbf{F} + (\mathbf{g}_m - \Omega_{IE}\Omega_{IE}\mathbf{R}) - \Omega_{IE}\mathbf{V}_E.$$

Gravity is defined as the resultant of mass attraction and centrifugal force: i.e.

$$\begin{aligned} \mathbf{g} &= \mathbf{g}_m - \Omega_{IE}\Omega_{IE}\mathbf{R} \\ \therefore \left(\frac{d\mathbf{V}_E}{dt}\right)_I &= \mathbf{F} + \mathbf{g} - \Omega_{IE}\mathbf{V}_E. \end{aligned}$$

For any frame  $K$ , Coriolis' equation applied to  $\mathbf{V}_E$  gives

$$\left(\frac{d\mathbf{V}_E}{dt}\right)_I = \left(\frac{d\mathbf{V}_E}{dt}\right)_K + \Omega_{IK}\mathbf{V}_E.$$

Put  $\Omega_{IK} = \Omega_{IE} + \Omega_{EK}$ , and substitute for  $(d\mathbf{V}_E/dt)_I$ :

$$\left(\frac{d\mathbf{V}_E}{dt}\right)_K = \mathbf{F} + \mathbf{g} - (2\Omega_{IE} + \Omega_{EK})\mathbf{V}_E.$$



## APPENDIX 8

### Split-Frame Mechanisation

The concept applied here is that changes in position caused by specific force ( $\mathbf{R}_F$ ) and those caused by gravitation ( $\mathbf{R}_G$ ) may be calculated separately. The actual change in position ( $\mathbf{R}$ ) is given by  $\mathbf{R} = \mathbf{R}_F + \mathbf{R}_G$ .

Specific force is given by  $\mathbf{F} = (d^2\mathbf{R}/dt^2)_I - \mathbf{g}_m$ ;  $\mathbf{g}_m$  is mass attraction vector then

$$\left(\frac{d^2\mathbf{R}_F}{dt^2}\right)_I = \mathbf{F}$$

and

$$\left(\frac{d^2\mathbf{R}_G}{dt^2}\right)_I = \mathbf{g}_m.$$

The initial conditions are  $\mathbf{R}_F = 0$ ;  $(d\mathbf{R}_F/dt)_I = \mathbf{V}_{IF} = 0$ ;  $(d\mathbf{R}_F/dt)_E = \mathbf{V}_{EF} = 0$ , and  $\mathbf{R}_G = \mathbf{R}$ ;  $(d\mathbf{R}_G/dt)_I = \mathbf{V}_{IG} = \mathbf{V}_I$ ;  $(d\mathbf{R}_G/dt)_E = \mathbf{V}_{EG} = \mathbf{V}_E$ .

#### A8.1 The Specific Force Component

Coriolis' equation applied to  $\mathbf{V}_{IF}$ :

$$\left(\frac{d\mathbf{V}_{IF}}{dt}\right)_I = \left(\frac{d\mathbf{V}_{IF}}{dt}\right)_B + \Omega_{IB} \mathbf{V}_{IF}$$

but

$$\left(\frac{d\mathbf{V}_{IF}}{dt}\right)_I = \left(\frac{d^2\mathbf{R}_F}{dt^2}\right)_I = \mathbf{F}$$

$$\therefore \left(\frac{d\mathbf{V}_{IF}}{dt}\right)_B = \mathbf{F} - \Omega_{IB} \mathbf{V}_{IF}.$$

When this is expressed in body frame coordinates, it may be written:

$$[\dot{\mathbf{V}}_{IF}]^B = [\mathbf{F}]^B - \Omega_{IB}^B [\mathbf{V}_{IF}]^B.$$

This is the equation for the velocity component caused by specific force.

#### A8.2 The Gravitation Component

Coriolis' equation applied to  $\mathbf{R}_G$ :

$$\left(\frac{d\mathbf{R}_G}{dt}\right)_I = \left(\frac{d\mathbf{R}_G}{dt}\right)_E + \Omega_{IE} \mathbf{R}_G$$

i.e.

$$\mathbf{V}_{IG} = \mathbf{V}_{EG} + \Omega_{IE} \mathbf{R}_G.$$

Differentiate:

$$\left(\frac{d\mathbf{V}_{IG}}{dt}\right)_I = \left(\frac{d\mathbf{V}_{EG}}{dt}\right)_I + \Omega_{IE}\left(\frac{d\mathbf{R}_G}{dt}\right)_I$$

Therefore

$$\left(\frac{d\mathbf{V}_{IG}}{dt}\right)_I = \mathbf{g}_m = \left(\frac{d\mathbf{V}_{EG}}{dt}\right)_I + \Omega_{IE}\mathbf{V}_{EG} + \Omega_{IE}\Omega_{IE}\mathbf{R}_G$$

Coriolis' equation applied to  $\mathbf{V}_{EG}$  for any frame  $K$ :

$$\left(\frac{d\mathbf{V}_{EG}}{dt}\right)_I = \left(\frac{d\mathbf{V}_{EG}}{dt}\right)_K + \Omega_{IK}\mathbf{V}_{EG}$$

Substitute for  $(d\mathbf{V}_{EG}/dt)_I$ :

$$\mathbf{g}_m = \left(\frac{d\mathbf{V}_{EG}}{dt}\right)_K + (\Omega_{IK} + \Omega_{IE})\mathbf{V}_{EG} + \Omega_{IE}\Omega_{IE}\mathbf{R}_G$$

$\mathbf{R}_F$  is small compared with the distance to Earth centre  $\mathbf{R} = \mathbf{R}_G + \mathbf{R}_F$ , so we may put

$$\mathbf{g} = \mathbf{g}_m - \Omega_{IE}\Omega_{IE}\mathbf{R}_G$$

also

$$\Omega_{IK} = \Omega_{IE} + \Omega_{EK}$$

therefore

$$\left(\frac{d\mathbf{V}_{EG}}{dt}\right)_K = \mathbf{g} - (2\Omega_{IE} + \Omega_{EK})\mathbf{V}_{EG}$$

This is the equation for the velocity component caused by the effect of gravitation.

## APPENDIX 9

### Solution of Body Axis Velocity Equation

We wish to use the equation (Appendix 8)

$$\dot{\mathbf{V}}(t) = \mathbf{F}(t) - \Omega(t) \mathbf{V}(t) \quad (1)$$

to obtain  $\mathbf{V}(t + h)$ .

Using a Taylor Series expansion, we get [omitting the  $(t)$ ]:

$$\mathbf{V}(t + h) = \mathbf{V} + h\dot{\mathbf{V}} + (h^2/2!) \ddot{\mathbf{V}} + \dots \quad (2)$$

Differentiate (1):

$$\ddot{\mathbf{V}} = \dot{\mathbf{F}} - \dot{\Omega}\mathbf{V} - \Omega(\dot{\mathbf{F}} - \Omega\mathbf{V}). \quad (3)$$

Using (1) and (3) in (2):

$$\mathbf{V}(t + h) = \mathbf{V} + h\mathbf{F} - h\Omega\mathbf{V} + (h^2/2)(\dot{\mathbf{F}} - \dot{\Omega}\mathbf{V} - \Omega\mathbf{F} + \Omega\Omega\mathbf{V}).$$

Assuming that specific force and angular velocity are constant during the interval, and that the term  $(h^2/2)\Omega\Omega\mathbf{V}$  may be neglected, we get

$$\mathbf{V}(t + h) = \mathbf{V} + h\mathbf{F} - h\Omega(\mathbf{V} + \frac{1}{2}h\mathbf{F}).$$

Now,  $h\mathbf{F}$  is the accelerometer output  $\mathbf{A}$ , and  $h\Omega$  is the skew symmetric of the rotation vector, i.e.  $h\Omega = [\theta \times]$ .

$$\therefore \mathbf{V}(t + h) = \mathbf{V}(t) + \mathbf{A} - \theta \times [\mathbf{V}(t) + \frac{1}{2}\mathbf{A}].$$

## APPENDIX 10

### Relationship between $V_{IF}$ and $V_{EF}$

Coriolis' equation applied to  $R_F$  gives:

$$\left(\frac{dR_F}{dt}\right)_I = \left(\frac{dR_F}{dt}\right)_E + \Omega_{IE} R_F$$

or, in terms of velocity, over an interval  $h$ ;

$$V_{IF} = V_{EF} + \Omega_{IE} \int_0^h V_{IF} dt$$

(at the start of the interval,  $V_{IF} = V_{EF} = 0$ ).

If constant specific force  $F$  is assumed through the interval

$$\int_0^h V_{IF} dt = \frac{1}{2} V_{IF} \cdot h$$

therefore

$$V_{EF} = (1 - \frac{1}{2} \Omega_{IE} h) V_{IF}$$

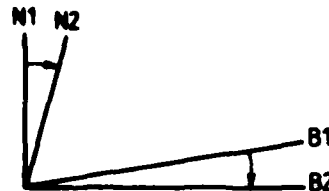
The  $V_{EF}$  error per interval caused by assuming that  $V_{EF} = V_{IF}$  is  $\frac{1}{2} \Omega_{IE} V_{IF} h$  this is equivalent to a specific force error  $\frac{1}{2} \Omega_{IE} Fh$ , i.e. approx.  $36h$  parts per million.

## APPENDIX 11

### Rotation of Navigation Axes (relative to Inertial)

Body axis rotation obtained via the gyros is relative to inertial space. The attitude quaternion stored in the computer relates body axes to navigation axes. Therefore, between navigation updates, the navigation axes rotation must be allowed for.

In the diagram, N1 and B1, and N2 and B2 respectively represent the navigation and body axes at times T1 and T2. Rotations are relative to inertial space.



For time T1, the quaternion of rotation from N1 to B1 is in the computer ( $\overline{Q1}$ ).

Between T1 and T2, the quaternion of rotation from B1 to B2 is calculated from the gyro outputs ( $\overline{QG}$ ), and the attitude quaternion relative to the N1 axes is updated:

$$\overline{Q1a} = \overline{Q1} * \overline{QG}$$

( $\overline{Q1a}$  is in N1 and B2 coordinates, and represents the rotation from N1 to B2).

The rotation from N1 to N2 during this period is calculated, based on the conditions at T1. This is represented by  $\overline{QN}$  (in N1 and N2 coordinates).

For time T2, the quaternion  $\overline{Q2}$  representing the rotation N2 to B2 is calculated as follows:

the rotation B2 to N1 is represented by  $\overline{Q1a}^{-1}$

the rotation N1 to N2 is represented by  $\overline{QN}$

therefore the rotation B2 to N2 is represented by  $\overline{Q1a}^{-1} * \overline{QN}$  (in B2 and N2 coordinates)

therefore

$$\overline{Q2} = [\overline{Q1a}^{-1} * \overline{QN}]^{-1} = \overline{QN}^{-1} * \overline{Q1a}$$

This is the required attitude quaternion at T2.

## APPENDIX 12

### Transformation by Quaternion

For any vector  $V$ :

$$[V]^G = \bar{Q} * [V]^B * \bar{Q}^{-1} \quad (1)$$

where  $\bar{Q}$  is the quaternion representing coordinate rotation from the  $G$  to the  $B$  axes.

Write  $\bar{Q} = (Q_0, \mathbf{Q})$ ,  $\bar{Q}^{-1} = (Q_0, -\mathbf{Q})$ , and evaluate (1) using the rules for quaternion multiplication:  
this gives

$$[V]^G = Q_0^2 [V]^B + (\mathbf{Q} \cdot [V]^B) \mathbf{Q} + 2Q_0 (\mathbf{Q} \times [V]^B) + \mathbf{Q} \times (\mathbf{Q} \times [V]^B).$$

Now,

$$(\mathbf{Q} \cdot [V]^B) \mathbf{Q} = \mathbf{Q} \times (\mathbf{Q} \times [V]^B) + (\mathbf{Q} \cdot \mathbf{Q}) [V]^B, \text{ also } (Q_0^2 + \mathbf{Q} \cdot \mathbf{Q}) = 1$$

therefore

$$[V]^G = [V]^B + 2Q_0 (\mathbf{Q} \times [V]^B) + 2\mathbf{Q} \times (\mathbf{Q} \times [V]^B) \quad (2)$$

Writing the elements of  $[V]^B$  as  $VB_1, VB_2, VB_3$ , and those of  $\mathbf{Q}$  as  $Q_1, Q_2, Q_3$ , (2) may be written in full:

$$[V]^G = \begin{bmatrix} VB_1 + 2[Q_0(Q_2 \cdot VB_3 - Q_3 \cdot VB_2) + Q_2(Q_1 \cdot VB_2 - Q_2 \cdot VB_1) - Q_3(Q_3 \cdot VB_1 - Q_1 \cdot VB_3)] \\ VB_2 + 2[Q_0(Q_3 \cdot VB_1 - Q_1 \cdot VB_3) + Q_3(Q_2 \cdot VB_3 - Q_3 \cdot VB_2) - Q_1(Q_1 \cdot VB_2 - Q_2 \cdot VB_1)] \\ VB_3 + 2[Q_0(Q_1 \cdot VB_2 - Q_2 \cdot VB_1) + Q_1(Q_3 \cdot VB_1 - Q_1 \cdot VB_3) - Q_2(Q_2 \cdot VB_3 - Q_3 \cdot VB_2)] \end{bmatrix}$$

this is the quaternion transformation.

If we collect the  $VB$ 's together and take them outside, we get the Direction Cosine Matrix in terms of the quaternion:

$$[V]^G = \begin{bmatrix} 1 - 2(Q_2^2 + Q_3^2) & 2(Q_1 \cdot Q_2 - Q_0 \cdot Q_3) & 2(Q_0 \cdot Q_2 + Q_1 \cdot Q_3) \\ 2(Q_0 \cdot Q_3 + Q_1 \cdot Q_2) & 1 - 2(Q_1^2 + Q_3^2) & 2(Q_2 \cdot Q_3 - Q_0 \cdot Q_1) \\ 2(Q_1 \cdot Q_3 - Q_0 \cdot Q_2) & 2(Q_0 \cdot Q_1 + Q_2 \cdot Q_3) & 1 - 2(Q_1^2 + Q_2^2) \end{bmatrix} \cdot \begin{bmatrix} VB_1 \\ VB_2 \\ VB_3 \end{bmatrix}$$

i.e.  $[V]^G = C_B^G [V]^B$ .

## APPENDIX 13

### Slow Cycle Operations

#### A13.1 Update of Position

Position is maintained as latitude and longitude angles. At the start of each slow cycle, new values  $XN$  and  $XE$  of incremental (since the previous slow cycle) distance travelled in north and east directions are available.

The change in latitude is given by  $XN/RN$ ; in longitude by  $XE/RE \cos(\lambda)$ .

#### A13.2 Update of Sine and Cosine of Latitude ( $L$ )

The angular change ( $\lambda$ ) in latitude is  $\lambda = XN/RN$ .

Now,

$$\sin(L + \lambda) = \sin(L) \cos(\lambda) + \cos(L) \sin(\lambda), \text{ or, if } (\lambda) \text{ is small,}$$

then

$$\sin(L + \lambda) = \sin(L) + \lambda \cos(L)$$

similarly

$$\cos(L + \lambda) = \cos(L) - \lambda \sin(L).$$

For a vehicle travelling at approximately 640 m/s, and a slow cycle period of one second, this angle is approximately 0.0001 radian, and the errors in  $\sin(L + \lambda)$  and  $\cos(L + \lambda)$  would be less than 0.01 part per million per second. If this were unacceptable, then a faster slow cycle rate would be used.

#### A13.3 Update of Earth Constants

The shape of the Earth may be approximated to an ellipsoid of revolution. This approximation is widely used to obtain formulae for calculation of gravity and local radius of Earth.

Formulae for the vertical component of gravity  $g$ , and the north  $RN$ , and east  $RE$ , values of the Earth radius of curvature are given in Reference 11:

$$g = g_0 (1 + 0.0052884 S - 3.157 \times 10^{-6} H + \text{smaller terms})$$

$$RN = R (1 - 2e + 3e S)$$

$$RE = R (1 + e S)$$

where  $S$  is  $\sin(\text{latitude})$ ,  $H$  is altitude (metres),  $g_0 = 9.78049 \text{ m/s}^2$ ,  $R = 6378160 \text{ m}$ , and  $e = 1/298.25$ ,

these may be written

$$g = 9.78049 + 0.051723 S - 3.088 \times 10^{-6} H$$

$$RN = 6335389 + 64155.84 S$$

$$RE = 6378160 + 21385.28 S.$$

## APPENDIX 14

### Vertical Channel Damping

#### A14.1 Pure Inertial (Undamped) System

The vertical component of gravity is calculated from  $g = g_0(1 + A \sin^2 L - 2H/R)$  where  $H$  is altitude,  $L$  is latitude,  $R = 6378160$ ,  $A = 0.05288$ , and  $g_0 = 9.78049$ .

For an error  $\tilde{H}$  in estimated altitude, there is an error in  $g$  given by

$$\tilde{g} = -\tilde{H} \cdot 2g_0/R.$$

The vertical channel equations are  $\dot{H} = v$  and  $\dot{v} = f - g + \text{Coriolis terms}$ .

The vertical channel error equations are  $\dot{\tilde{H}} = \tilde{v}$  and  $\dot{\tilde{v}} = -\tilde{g}$

therefore

$$\ddot{\tilde{H}} = \ddot{\tilde{H}} = \tilde{H} \cdot 2g_0/R.$$

The solution of this has the form  $\tilde{H} = A \exp[\sqrt{2g_0/R} t] + B \exp[-\sqrt{2g_0/R} t]$ .

The time constant of the instability is therefore  $\sqrt{R/2g_0} = \text{approx. } 570 \text{ sec}$ .

#### A14.2 Baro-Inertial Third Order System

Given a measurement of the barometric altitude  $H_B$ , the vertical channel equations are modified to:

$$\dot{H} = v - K1(H - H_B) \tag{1}$$

$$\dot{v} = f - g - K2(H - H_B) - a + \text{Coriolis terms} \tag{2}$$

where

$$a = K3(H - H_B). \tag{3}$$

Differentiate (2), substitute for  $a$ , then write as error equations:

$$\dot{\tilde{H}} = \tilde{v} - K1(\tilde{H} - \tilde{H}_B) \tag{4}$$

$$\dot{\tilde{v}} = \tilde{H} \cdot 2g_0/R - K2(\dot{\tilde{H}} - \dot{\tilde{H}}_B) - K3(\tilde{H} - \tilde{H}_B). \tag{5}$$

Differentiate (4) twice, and substitute for  $\dot{\tilde{v}}$ :

$$\ddot{\tilde{H}} + K1 \ddot{\tilde{H}} + (K2 - 2g_0/R) \dot{\tilde{H}} + K3 \dot{\tilde{H}} = K1 \dot{\tilde{H}}_B + K2 \dot{\tilde{H}}_B + K3 \dot{\tilde{H}}_B. \tag{6}$$

Following Reference 11, the characteristic equation is of the form

$$(s + 1/\tau)^3 \tilde{H} = 0$$

this is satisfied if  $K1 = 3/\tau$ ,  $K2 = 3/\tau^2 + 2g_0/R$ ,  $K3 = 1/\tau^3$ .

So, for  $\tau = 100$  seconds,  $K1 = 0.03 \text{ s}^{-1}$ ,  $K2 = 3.03 \times 10^{-4} \text{ s}^{-2}$ ,  $K3 = 10^{-6} \text{ s}^{-3}$ .



## DISTRIBUTION

Copy No.

### AUSTRALIA

#### Department of Defence

##### Central Office

Chief Defence Scientist	1
Deputy Chief Defence Scientist	2
Superintendent, Science and Technology Programmes	3
Australian Defence Scientific and Technical Rep. (U.K.)	—
Counsellor, Defence Science (U.S.A.)	—
Defence Central Library	4
Document Exchange Centre, D.I.S.B.	5-21
Director General — Army Development (NCO)	22
Joint Intelligence Organisation	23

##### Aeronautical Research Laboratories

Chief Superintendent	24
Library	25
Superintendent Systems	26
Divisional File — Systems	27
Author: R. B. Miller	28

##### Materials Research Laboratories

Library	29
---------	----

##### Defence Research Centre

Library	30
---------	----

##### Central Office

Defence Industry and Material Policy, FAS	31
---	----

##### Central Studies Establishment

Information Centre	32
--------------------	----

##### RAN Research Laboratory

Library	33
---------	----

##### Victorian Regional Office

Library	34
---------	----

##### Navy Office

Naval Scientific Adviser	35
Directorate of Naval Aircraft Engineering	36

##### Army Office

Army Scientific Adviser	37
Royal Military College Library	38

##### Air Force Office

Air Force Scientific Adviser	39
Technical Division Library	40
Director General Aircraft Engineering	41

Director General Operational Requirements	42	
HQ Operational Command (SENGSO)	43	
HQ Support Command (SENGSO)	44	
RAAF Academy, Point Cook	45	
<b>Department of Industry and Commerce</b>		
<b>Government Aircraft Factories</b>		
Manager	46	
Library	47	
<b>Statutory and State Authorities and Industry</b>		
CSIRO Applied Physics Division, Library	48	
Commonwealth Aircraft Corporation, Library	49	
<b>Universities and Colleges</b>		
Adelaide	Barr Smith Library	50
Melbourne	Engineering Library	51
Monash	Hargrave Library	52
N.S.W.	Physical Sciences Library	53
Queensland	Library	54
Tasmania	Engineering Library	55
Western Australia	Library	56
<b>CANADA</b>		
NRC Aeronautical and Mechanical Engineering Library	57	
<b>Universities and Colleges</b>		
Toronto Institute for Aerospace Studies	58	
<b>FRANCE</b>		
AGARD, Library	59	
ONERA, Library	60	
<b>GERMANY</b>		
ZLDI	61	
<b>INDIA</b>		
Defence Ministry, Aero Development Establishment, Library	62	
<b>ISRAEL</b>		
Technion-Israel Institute of Technology, Professor J. Singer	63	
<b>ITALY</b>		
Associazione Italiana di Aeronautica e Astronautica, Professor A. Evla	64	
<b>JAPAN</b>		
National Aerospace Laboratory, Library	65	
<b>NETHERLANDS</b>		
National Aerospace Laboratory (NLR), Library	66	
<b>NEW ZEALAND</b>		
Defence Scientific Establishment, Library	67	
<b>SWEDEN</b>		
Aeronautical Research Institute, Library	68	
<b>SWITZERLAND</b>		
F+W (Swiss Federal Aircraft Factory)	69	

**UNITED KINGDOM**

Royal Aircraft Establishment Farnborough, Library	70
British Library, Science Reference Library	71
British Library, Lending Division	72
Ferranti Inertial Systems Department	73
Sperry Gyroscope	74
Marconi-Elliott Avionic Systems	75
British Aerospace, Precision Products Group	76

**UNITED STATES OF AMERICA**

N.A.S.A. Scientific and Technical Information Facility	77
--	----

**Universities and Colleges**

Harvard	Professor G. F. Carrier, Applied Mathematics	78
Stanford	Library, Department of Aeronautics	79
M.I.T.	Library	80
Polytechnic Institute (New York)	Library, Aerospace Lab.	81
California Inst. of Technology	Library, Graduate Aeronautical Labs.	82

Spares	83-92
--------	-------

