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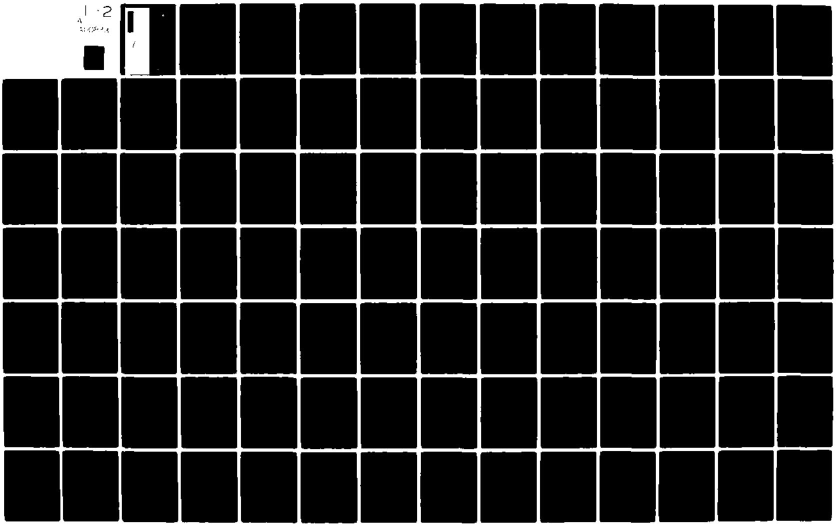

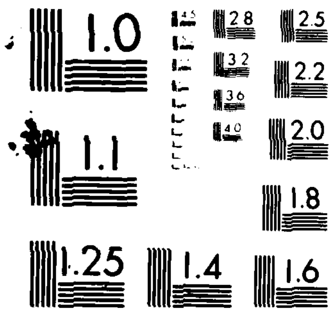
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THE LINEARIZED UNSTEADY LIFTING SURFACE THEORY APPLIED TO THE PUMP-JET PROPULSIVE SYSTEM

by

W.R. Jacobs, S. Tsakonas, and Ping Liao

Co-sponsored by the

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THE LINEARIZED UNSTEADY LIFTING SURFACE THEORY APPLIED TO THE PUMP-JET PROPULSIVE SYSTEM

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Approved:

John P. Breslin, Director

ABSTRACT

A theoretical analysis is developed treating the "Pump-Jet" propulsive unit comprised of stator, rotor and enshrouding nozzle by using the unsteady lifting surface theory. The analysis takes into account accurate geometry, realistic flow conditions and hydrodynamic interactions between all lifting surfaces of finite thickness distribution. The system is assumed to be immersed in a non-uniform flow of an incompressible fluid.

Expressions have been developed for loadings on all interacting surfaces and corresponding resulting forces evaluated at proper frequencies dictated mainly by those of the rotor.

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KEYWORDS

Hydrodynamics

Propulsion

TABLE OF CONTENTS

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NOMENCLATURE

$\bar{A}^{(\ell N,\nu,\bar{n})}$	coefficients of chordwise loading distribution on duct
a_R	$\Omega r_o/U = \pi/J$ (for rotor)
a_S	$1/\rho_S \tan\theta_{P_S}(\rho_S)$ at $\rho_S = 0.7r_{RO}$ (for stator)
C_D	semichord of cylindrical duct
c	expanded chord of rotor or stator
D	subscript index of duct
d_o	semithickness of duct at trailing edge
$F_{x,y,z}$	rotor or stator hydrodynamic forces
$F_{D_{x,y,z}}$	duct hydrodynamic forces
$I^{(\bar{m})}(x)$	defined in Appendix A
$I_m(x)$	modified Bessel function of order m of first kind
i	index of control point
$J_m(x)$	Bessel function of order m
J	$U/2nr_o$, advance ratio
j	index of loading point
$K_m(x)$	modified Bessel function of order m of second kind
$K_m'(x)$	$= \partial K_m(x)/\partial x$
K_{ji}	kernel of integral equation
$\bar{K}_{ji}$	kernel after θ_α- and φ_α-integrations
k	variable of integration
L_j	loading, lb/ft
$L_D^{(\ell N_R)}(x_D)$	chordwise loading distribution on duct at rotor blade frequency
$L_R^{(q_R)}(r_R)$	spanwise loading distribution on rotor blade at frequency q_R

$L_S^{(\ell N_R)}(r_S)$	spanwise loading distribution on stator blade at rotor blade frequency
$L_R^{(q_R,\bar{n})}(\rho_R)$	coefficients of chordwise loading distribution on rotor blade
$L_S^{(\ell N_R,\bar{n})}(\rho_S)$	coefficients of chordwise loading distribution on stator blade
ℓ	integer multiple
$M_{Dy,z}$	duct hydrodynamic moments
$\bar{m}$	order of lift operator mode
m_k	index of summation
N	number of blades
$\bar{n}$	order of chordwise mode
n	blade index
$Q_{x,y,z}$	rotor or stator hydrodynamic moments
q	order of harmonic of inflow field
R	Descartes distance
R	subscript index of rotor
R_D	radius of cylindrical duct
r	radial coordinate of control point
r_{RO}	rotor radius
S	subscript index of stator
S_j	lifting surface
t	time, sec
t_o	maximum thickness of blade section or duct section
U	free stream velocity, ft/sec
u	variable of integration
$V^{(q)}(r)$	Fourier coefficients of onset velocity normal to blade of rotor or stator
W_R	downwash velocity distribution normal to rotor at control point

W_S	downwash velocity distribution normal to stator at control point
W_D	downwash velocity distribution normal to duct at control point
x,r,φ	cylindrical coordinate system of control points
α	conicity angle of duct
ϵ_D	axial distance between rotor plane and duct midchord (positive)
ϵ_S	axial distance between rotor plane and stator plane (negative)
$\Theta(\bar{n})$	chordwise modes
$\theta_{RO,SO}$	angular position of loading point with respect to midchord line in projected plane
θ_α	angular chordwise location of loading point
θ_b	projected semichord length of rotor or stator, radians
$\bar{\theta}_n$	$(2\pi/N)(n-1)$, $n=1,2,\ldots,N$
θ_p	geometric pitch angle
$\Lambda^{(\bar{n})}(x)$	defined in Appendix A
λ_k	positive integer
μ	index of summation of Fourier series
ν	order of peripheral mode
ξ,ρ,θ	cylindrical coordinate system of loading points
ρ	radial coordinate of loading point
ρ_f	fluid density, slugs/ft^3
σ	angular measure of skewness, radians
τ	variable of integration
Φ	velocity potential
$\Phi(\bar{m})$	orthogonal functions used in generalized lift operator
$\varphi_{RO,SO}$	angular position of control point with respect to midchord line in projected plane
φ_α	angular chordwise location of control point
Ω	magnitude of rotor angular velocity

INTRODUCTION

Previous investigations at Davidson Laboratory have been concerned with the adaptation of linearized unsteady lifting-surface theory to the cases of a marine propeller operating in a nonuniform inflow field,[1,2*] of counterrotating propeller systems,[3,4] and of ducted propellers,[5,6] where the exact geometry of the systems, the realistic inflow conditions and the mutual interaction of all lifting surfaces are taken into account.

In the case of the single propeller with enshrouding nozzle, both accelerating and decelerating ducts were discussed, the accelerating (Kort) nozzle offering the advantage over conventional propellers of increasing the flow rate through the propeller, reducing the loading and thereby increasing the efficiency, and the decelerating type of reducing the flow rate, thus delaying cavitation inception and lowering noise level.

The present study treats the pump-jet configuration, which is a type comprised of stator, rotor and enshrouding nozzle. The stator vanes, in addition to their structural support of the nozzle, are presumed to homogenize the inflow to the rotor blades, reducing further the vibratory loading and resulting forces and the radiated noise. To assess the advantages or disadvantages of the system, a theoretical analysis and corresponding computer program are developed which will reveal the steady state and vibratory characteristics of this propulsive device as a function of various geometric parameters of the system.

This study was co-sponsored by the Naval Sea Systems Command Exploratory Development Program and General Hydromechanics Research Program under Contract N00014-77-C-0298, administered by the David W. Taylor Naval Ship Research and Development Center.

*Superior numbers in text matter refer to similarly numbered references listed at the end of this technical report.

STATEMENT OF THE INTERACTION PROBLEM

A pump-jet configuration comprised of stator, rotor and enshrouding nozzle is immersed in a nonuniform flow of an ideal incompressible fluid. Figure 1 shows the relative location of each member and the corresponding coordinate system. Figure 2 exhibits the definitions of the angular measures of the rotor.

The kinematic boundary conditions on all interacting lifting surfaces expressing the impermeability of the boundaries can be written in the general form as

$$W_R = \iint_{S_R} L_R \overset{(1)}{K_{RR}} dS_R + \iint_{S_D} L_D \overset{(2)}{K_{DR}} dS_D + \iint_{S_S} L_S \overset{(3)}{K_{SR}} dS_S \tag{1}$$

$$W_S = \iint_{S_R} L_R \overset{(4)}{K_{RS}} dS_R + \iint_{S_D} L_D \overset{(5)}{K_{DS}} dS_D + \iint_{S_S} L_S \overset{(6)}{K_{SS}} dS_S \tag{2}$$

$$W_D = \iint_{S_R} L_R \overset{(7)}{K_{RD}} dS_R + \iint_{S_D} L_D \overset{(8)}{K_{DD}} dS_D + \iint_{S_S} L_S \overset{(9)}{K_{SD}} dS_S \tag{3}$$

where subscripts R, S, D, refer to rotor, stator, duct lifting surfaces, respectively.

The kernel function K_{ij} represents the induced velocity on element j due to an oscillating load L_i of unit amplitude on element i. The kernel function K_{jj} is the self-induced velocity at a point of the particular lifting surface due to unit load at each and every point on the same surface. The kernels with two different subscripts represent the interaction effects from neighboring surfaces. The integrations on surfaces S_R, S_S, and S_D, are over the rotor blades, the stator vanes and the enshrouding nozzle, respectively.

The terms W_j on the left-hand (L-H) side of the equations are the known velocity distributions normal to the lifting surfaces, nondimensionalized by the free stream velocity U. The velocities normal to the respective lifting surfaces are the perturbations from the basic flow due to nonuniformity of the flow field (wake), camber, incident flow, and thicknesses of the respective lifting surfaces. In the linear theory, their effects can simply be added.

We consider two basic flows: a) one generated from the hull wake and measured in the plane of the stator in the absence of all interacting surfaces, and b) the other generated by the presence of the hull and stator together, measured at the plane of the rotor in the absence of duct and rotor. Thus, any harmonic content of the viscous and potential wake generated by the presence of the hull and the stator will be included as an input to the interaction problem. (See Note at end of this section.)

The flow disturbances considered in the present study are:

1) The basic flows (hull wake and combinations of the hull and stator wakes) both of which will affect the steady and unsteady loadings of all interacting lifting surfaces. In fact, the former will be utilized to calculate the steady and unsteady loadings on the stator and the latter will be used to determine the loadings on the rotor and enshrouding nozzle, as will be demonstrated later on in the development.

2) The thickness distributions of all lifting surfaces affect, in principle, both steady and unsteady loadings of the interacting surfaces as will be seen in the analysis. These effects sometimes are omitted because of the presence of the axisymmetric duct configuration and sometimes because the effect is very small in magnitude, e.g., being at the blade-blade crossing frequency.

3) The camber and flow angle (i.e., incident angle) of the respective surfaces will affect their steady-state loadings only.

Thus, W_R, the velocity normal to the rotor, is due to basic flow disturbances in the presence of hull and stator wakes, which affect both steady and unsteady loadings; the rotor blade camber and incidence angle affects only the steady state rotor loading whereas the effects of duct and stator thickness distributions may be present in both steady and unsteady state rotor loadings.

The flow disturbances W_S are made up of the normal velocities on the stator due to the hull wake, stator blade camber and incidence angle, and duct and rotor blade thickness distributions. Details of these contributions will be seen later on in the development.

In the linearized version of the interaction problem, the duct is assumed to be a cylinder with zero conicity angle (i.e., $\alpha = 0$). The flow disturbances W_D are those due to non-zero α (conic form) and to duct camber, both of which affect steady-state duct loading only, and those due to rotor and stator blade thicknesses.

The surface integrals of Equations 1, 2, and 3, are reduced to line integrals by approximating the chordwise loadings on stator, rotor and duct by appropriate mode shapes, as in References 1, 5, and 6. The blades of stator and rotor are divided into small spanwise strips and the spanwise loading coefficients of the chordwise modes are assumed constant over each small strip so that only the kernels need be integrated over the span. The collocation method is used together with the generalized lift operator technique,[7] as in the references cited, to determine the spanwise loading coefficients. In the case of loading on the duct of circular section, the peripheral loading is expressed in terms of a Fourier series so that the peripheral integration is easily performed, and the chordwise loading coefficients are obtained by the collocation and generalized lift operator methods.

The kernel functions are derived by means of the acceleration potential, K_{RR} as in References 1 and 2 for the propeller alone, and K_{DR}, K_{RD}, and K_{DD}, as in Reference 5 for the propeller-duct interaction. The kernels K_{SR} and K_{RS} representing the interaction of stator and rotor will be developed following the approach of References 3 and 4 for the counter-rotating propeller system. The remaining kernels K_{SD}, K_{DS}, and K_{SS}, will be derived following References 5, 6 and 1.

The three integral equations are solved by an iteration procedure. It will be assumed at first that duct and rotor have no effect on stator loading which will be obtained from Equation (2) by ignoring the first and third integrals. On substituting that value of L_S in Equations (1) and (3), those equations will be solved by the iteration procedure outlined in References 5 and 6, thus obtaining values of L_R and L_D. The values obtained for L_R and L_D are then substituted in Equation (2), which is solved for a new L_S. The new L_S is next used in Equations (1) and (3) which are put through the iteration process again. The procedure is repeated until stabilized values are secured.

The first set of iterations will yield first approximations of the loadings by solving

$$W_S = \iint_{S_S} L_{SO}K_{SS}dS_S$$

$$W_R - \iint_{S_S} L_{SO}K_{SR}dS_S = \iint_{S_R} L_{RO}K_{RR}dS_R$$

$$W_D - \iint_{S_S} L_{SO}K_{SD}dS_S = \iint_{S_R} L_{RO}K_{RD}dS_R + \iint_{S_D} L_{DO}K_{DD}dS_D$$

Second approximations of the loadings will be obtained from

$$W_S - \iint_{S_R} L_{RO}K_{RS}dS_R - \iint_{S_D} L_{DO}K_{DS}dS_D = \iint_{S_S} L_{S1}K_{ss}dS_S$$

$$W_R - \iint_{S_S} L_{S1}K_{SR}dS_S = \iint_{S_R} L_{R1}K_{RR}dS_R + \iint_{S_D} L_{D1}K_{DR}dS_D$$

$$W_D - \iint_{S_S} L_{S1}K_{SD}dS_S = \iint_{S_R} L_{R1}K_{RD}dS_R + \iint_{S_D} L_{D1}K_{DD}dS_D \quad ,$$

and so forth.

NOTE: If measurements are not available of the flow generated by the presence of both hull and stator at the plane of the rotor, in the absence of duct and rotor, corrections to the velocity on the L-H side of Eq.(1) must be introduced to take into account the effects on the rotor, which operates in the race of the stator, due to both viscous and potential wake of the stator.

THE VELOCITY DISTRIBUTIONS

1. W_R, Normal to the Rotor

At q=0, the steady-state velocity distribution normal to the R-H* rotor on the L-H side of Equation (1), after the lift operator of order $\bar{m}$ has been applied to both sides of the equation, is made up of

$$\bar{W}_R^{(0,\bar{m})}(r_R) = \bar{W}_W^{(0,\bar{m})}(r_R) + \bar{W}_{R_{c+f}}^{(0,\bar{m})}(r_R) + \bar{W}_{D_tR}^{(0,\bar{m})}(r_R) \tag{4}$$

The wake component $\bar{W}_W$ (nondimensionalized by U) is derived from [1,5]

$$\bar{W}_W^{(q_R,\bar{m})}(r_R) = \frac{1}{\pi}\int_0^{\pi} \Phi(\bar{m}) \frac{V_W^{(q_R)}}{U}(r_R)e^{-iq_R\varphi_{RO}}d\varphi_\alpha \tag{5}$$

where

$V_W^{(q_R)}$ = q_R-harmonic of wake velocity normal to the rotor blade in the presence of the hull and stator

φ_{RO} = $\sigma_R-\theta_{bR}\cos\varphi_\alpha$, angular position of control point with respect to midchord-line, radians

σ_R = angular position of midchord-line of the projected blade from the reference line through the hub

θ_{bR} = projected semichord-length of the blade in radians

$\Phi(\bar{m})$ = lift operator function

With $I^{(\bar{m})}(x) = \frac{1}{\pi}\int_0^{\pi}\Phi(\bar{m})e^{ix\cos\varphi_\alpha}d\varphi_\alpha$ (see Appendix A), the wake harmonic component is defined as

$$\bar{W}_W^{(q_R,\bar{m})}(r_R) = \frac{V_W^{(q_R)}}{U}(r_R)e^{-iq_R\sigma_R}\, I^{(\bar{m})}(q_R\theta_{bR}) \tag{6}$$

and

$$\bar{W}_W^{(0,\bar{m})}(r_R) = \frac{V_W^{(0)}}{U}(r_R)\, I^{(\bar{m})}(0) \tag{7}$$

*Right-handed

The nondimensional normal velocity component $\bar{W}_{R_{c+f}}$, due to effects of camber and incident flow angle, which is present only in the steady state (q_R=0) since the blades are considered rigid, is given as the sum $\bar{W}_{R_c} + \bar{W}_{R_f}$, where

$$\bar{W}_{R_f}^{(0,\bar{m})}(r_R) = -\sqrt{1 + a_R^2\, r_R^2}\left[\theta_{PR}(r_R) - \beta(r_R)\right] I^{(\bar{m})}(0) \tag{8}$$

where

$a_R = \Omega\, r_{RO}/U$

Ω = magnitude of angular velocity of rotor

r_{RO} = radius of rotor

θ_{PR} = geometric pitch angle of rotor blade

$\beta = \tan^{-1}(1/a_R r_R)$ = hydrodynamic pitch angle of assumed helicoidal surface

$$\bar{W}_{R_c}^{(0,\bar{m})}(r_R) = \frac{\sqrt{1 + a_R^2\, r_R^2}}{\pi C_R(r_R)} \int_0^{\pi} \Phi(\bar{m}) \frac{\partial f(r_R, S_R)}{\partial S_R}\, d\varphi_\alpha \tag{9}$$

where

$f(r_R, S_R)$ = camberline ordinates from the face pitch-line

$S_R = (1-\cos\varphi_\alpha)/2$, chordwise location as fraction of chord length C_R

C_R = chord length

(This component is derived in Reference 8 for arbitrary camber shape.)

The nondimensional normal velocity component due to the effect of duct thickness on the rotor is derived in Reference 6 for a modified lenticular chordwise section (see Figure 3 represented by

$$f(\theta_\alpha) \approx \tfrac{1}{2}\{[t_o - d_o]\sin^2\theta_\alpha + d_o(1-\cos\theta_\alpha)\}\ ,\ 0 \le \theta_\alpha \le \pi)$$

as

$$\bar{W}_{D_tR}^{(0,\bar{m})}(r_R) = -\frac{2R_D r_R a_R}{\pi\sqrt{1+a_R^2 r_R^2}} \int_0^\infty \Big[k\, I_o(kr_R)K_o(kR_D) \Big]$$

$$\cdot R.P.\left\{\Big[(2t_o-2d_o)F(k)+i\,d_o\,G(k)\Big] e^{-ik\left(\frac{\sigma_R}{a_R}-\epsilon_D\right)} I^{(\bar{m})}\left(\frac{k\theta_{bR}}{a_R}\right)\right\}dk \tag{10}$$

where

R_D = radius of cylindrical duct

t_o = maximum duct thickness

d_o = semi-thickness of duct at trailing edge

ϵ_D = axial distance between rotor plane and duct midchord (ϵ_D is positive)

and $F(k)=[\sin(kC_D)-kC_D\cos(kC_D)]/(kC_D)^2$

$G(k) = \sin(kC_D)/kC_D$

C_D = semichord of cylindrical duct

$I_o(\ )$ and $K_o(\ )$ are modified Bessel functions (see Eq.(19) of Ref.6)

When $q_R \neq 0$ (unsteady cases), the velocity distribution normal to the rotor on the L-H of Eq.(1), after the lift operator has been applied to both sides of the equation, is made up of

$$\bar{W}_R^{(q_R,\bar{m})}(r_R) = \bar{W}_W^{(q_R,\bar{m})}(r_R) \tag{11}$$

where the $\bar{W}_W$ is given by Eq.(6).

As shown in Reference 6, for an axisymmetric duct, with d_o = constant over the circumference, (as in the pump-jet system), there is no effect of duct thickness on the rotor when $q_R \neq 0$.

As noted in the preceding section, if the wake of the stator has not been measured, additional normal velocity components must be included due to the potential and viscous effects on the rotor of the race of the stator. These are derived in Appendices L and M as suggested by Dr. John Breslin.

2. W_S, Normal to the Stator

In the steady state (q_S=0), the nondimensional velocity distribution normal to the stator* on the L-H of Eq.(2), after the lift operator of order $\bar{m}$ has been applied to both sides of the equation, is

$$\bar{W}_S^{(0,\bar{m})}(r_S) = \bar{W}_W^{(0,\bar{m})}(r_S) + W_{S_{c+f}}^{(0,\bar{m})}(r_S) + \bar{W}_{D_tS}^{(0,\bar{m})}(r_S) + \bar{W}_{R_tS}^{(0,\bar{m})}(r_s) \tag{12}$$

Here

$$\bar{W}_W^{(0,\bar{m})}(r_S) = \frac{V_W^{(0)}}{U}(r_S)\,I^{(\bar{m})}(0) \quad \text{(wake of hull alone in plane of stator)} \tag{12a}$$

$$\bar{W}_{S_f}^{(0,\bar{m})}(r_S) = -\sqrt{1+a_S^2 r_S^2}\left[\theta_{PS}(r_S) - \beta(r_S)\right] I^{(\bar{m})}(0) \tag{12b}$$

where

$$a_S = \frac{1}{r_S \tan\theta_{PS}(r_S)} \quad \text{at} \quad r_S = 0.7 r_{RO}$$

$$\beta = \theta_{PS}(0.7 r_{RO})$$

$$\bar{W}_{S_c}^{(0,\bar{m})}(r_S) = \frac{\sqrt{1+a_S^2 r_S^2}}{\pi\, C_S(r_S)} \int_0^{\pi} \Phi(\bar{m})\, \frac{\partial f(r_S, s_S)}{\partial s_S}\, d\varphi_\alpha \tag{12c}$$

(cf. Eq.(9) for details.)

The velocity due to the effect of duct thickness on the stator can be shown (see Ref. 6) to be equal to

$$\bar{W}_{D_tS}^{(0,\bar{m})} = -\frac{2R_D r_S a_S}{\pi\sqrt{1+a_S^2 r_S^2}} \int_0^{\infty} \{k I_o(kr_S) K_o(kR_D)\}$$

$$\cdot \mathrm{R.P.}\left\{\left[(2t_o - 2d_o)F(k) + i d_o G(k)\right] e^{-ik\left(\frac{\sigma_S}{a_S} - \epsilon_D + \epsilon_S\right)} I^{(\bar{m})}\left(\frac{k\theta_{bS}}{a_S}\right)\right\} dk \tag{13}$$

which is Eq.(10) with stator geometry substituted for rotor geometry. Note the factor $\exp(-ik\epsilon_S)$ which is the result of the substitution $x_S' = \varphi_{S0}/a_S + \epsilon_S$, where ϵ_S is the axial distance of the stator from the rotor and

*The stator has the geometry of a left-handed propeller.

$\varphi_{S0} = \sigma_S - \theta_{bS}\cos\varphi_\alpha$ (see Eq.(5)). In this case the stator being forward of the rotor ϵ_S is negative.

The component $\bar{W}_{R_tS}$ due to the effect of rotor blade thickness on the stator can be shown (see Appendix B) to be given in the steady-state by

$$\bar{W}_{R_tS}^{(0,\bar{m})}(r_S) = -\frac{4a_R^2 a_S N_R r_S}{\pi^2\sqrt{1+a_S^2 r_S^2}} \int_{\rho_R} \frac{\rho_R}{\theta_{bR}} \frac{t_o}{c}(\rho_R)\sqrt{1+a_R^2\rho_R^2} \int_0^\infty u\, F(u,\rho_R)(IK)_o$$

$$\cdot\ R.P.\left\{e^{-i\left(\frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R} + \epsilon_S\right)u}\, I^{(\bar{m})}(u\theta_{bS}/a_S)\right\}du\,d\rho_R \qquad (14)$$

where

$$(IK)_o = \begin{cases} I_o(u\rho_R)K_o(ur_S) & \text{for } \rho_R < r_S \\ I_o(ur_S)K_o(u\rho_R) & \text{for } r_S < \rho_R \end{cases}$$

$$F(u,\rho_R) = \left\{\sin(u\theta_{bR}/a_R) - (u\theta_{bR}/a_R)\cos(u\theta_{bR}/a_R)\right\}/u^2$$

N_R = number of blades of rotor

In the axisymmetric duct case, which is presently under consideration, there is no effect of the duct thickness on the stator or rotor for the unsteady flow case, i.e., $q_S \neq 0$ (see Eq.(2) of Ref. 6), so that the velocity distribution normal to the stator is

$$\bar{W}_S^{(q_S,\bar{m})}(r_S) = \bar{W}_W^{(q_S,\bar{m})}(r_S) + \bar{W}_{R_tS}^{(\ell N_R,\bar{m})}(r_S) \quad , \quad \ell = 1,2,3, \ldots \qquad (15)$$

where

$\bar{W}_W^{(q_S,\bar{m})}$ (cf.Eq.(6)) is due to the wake of the hull only measured at the plane of the stator (15a)

and the effect of the rotor thickness (see Appendix B) is

$$\bar{W}_{R_t S}^{(\ell N_R,\bar{m})}(r_S) = -\frac{4a_R^2 N_R r_S}{\pi^2\sqrt{1+a_S^2 r_S^2}}\, e^{i\ell N_R\left[\sigma_S\left(1+\frac{a_R}{a_S}\right)+a_R\epsilon_S\right]}$$

$$\cdot \int_{\rho_R} \frac{\rho_R}{\theta_{bR}} \frac{t_o}{c}(\rho_R)\sqrt{1+a_R^2\rho_R^2}\int_0^\infty F(u,\rho_R)[G_2(u)-G_2(-u)]\,du\,d\rho_R$$

$$(\ell = +1,+2,\ \dots) \qquad (15b)$$

$$G_2(u)=I_{\ell N_R}\left(\left|u+a_R\ell N_R\right|\rho_R\right)K_{\ell N_R}\left(\left|u+a_R\ell N_R\right|r_S\right)\left[a_S u+\ell N_R\left(a_S a_R-\frac{1}{r_S^2}\right)\right]$$

$$\cdot\, e^{iu(\sigma_S/a_S-\sigma_R/a_R+\epsilon_S)}\, I^{(\bar{m})}\left[\left(-\ell N_R\left(1+\frac{a_R}{a_S}\right)-\frac{u}{a_S}\right)\theta_{bS}\right]$$

3. W_D, Normal to the Duct

In the steady state ($q=\ell=0$), the nondimensionalized velocity distribution normal to the duct on the L-H of Eq.(3), after the lift operator of order $\bar{m}$ has been applied to both sides of the equation, is

$$\bar{W}_D^{(0,\bar{m})} = \alpha I^{(\bar{m})}(0) + \bar{W}_{D_c}^{(0,\bar{m})} + \bar{W}_{R_t D}^{(0,\bar{m})} + \bar{W}_{S_t D}^{(0,\bar{m})} \qquad (16)$$

where the first component is due to conicity angle α, the second to duct camber, the third to rotor blade thickness, and the fourth to stator blade thickness.

It should be noted that in the linearized version of the interaction problem, with the duct assumed to be a cylinder with no conicity angle, it is assumed that there is no contribution to the normal velocity on the duct surface (i.e., in the radial direction) due to the wake generated in the presence of both hull and stator together.

Reference 6 shows that for axisymmetric ducts, and assuming a modified lenticular camber distribution, namely,

$$c(\varphi_\alpha) \approx \left(m_x + \frac{d_o}{2}\right)\sin^2\varphi_\alpha - \frac{1}{2}d_o(1-\cos\varphi_\alpha) \quad , \quad 0 \le \varphi_\alpha \le \pi$$

where m_x is maximum camber and d_o the semi-thickness of the trailing edge of the duct, the component due to duct camber (see Eq.54 of Ref.6) is:

$$\bar{W}_{D_c}^{(0,\bar{m})} = \frac{1}{C_D}\left\{(2m_x + d_o) I_1^{(\bar{m})}(0) - \frac{d_o}{2} I^{(\bar{m})}(0)\right\} \tag{17}$$

($I_1^{(\bar{m})}(0)$ is defined in Appendix A.)

The same reference derives the velocity component due to the effect of rotor (propeller) thickness on the duct (see Eq.43 of Ref.6) as

$$\bar{W}_{R_tD}^{(0,\bar{m})} = -\frac{4a_R^2 N_R}{\pi^2} \int_{\rho_R} \sqrt{1+a_R^2\rho_R^2}\ \frac{\rho_R}{\theta_{bR}}\ \frac{t_o}{c}(\rho_R)$$

$$\cdot \int_0^\infty u I_o(u\rho_R) K_1(uR_D) F(u,0,\rho_R) \cdot \mathrm{Im\ Part}\left\{e^{-iu(\epsilon_D-\sigma_R)a_R} I^{(\bar{m})}(uC_D)\right\} du\, d\rho_R \tag{18}$$

where

$$F(u,0,\rho_R) = \frac{\sin(u\theta_{bR}/a_R) - (u\theta_{bR}/a_R)\cos(u\theta_{bR}/a_R)}{u^2}$$

By analogy with the above, the component due to the effect of stator thickness on the duct would be

$$\bar{W}_{S_tD}^{(0,\bar{m})} = -\frac{4a_S^2 N_S}{\pi^2} \int_{\rho_S} \sqrt{1+a_S^2\rho_S^2}\ \frac{\rho_S}{\theta_{bS}}\ \frac{t_o}{c}(\rho_S)$$

$$\cdot \int_0^\infty u I_o(u\rho_S) K_1(uR_D) F(u,0,\rho_S) \cdot \mathrm{Im\ Part}\left\{e^{-iu(\epsilon_D-\epsilon_S-\sigma_S/a_S)} I^{(\bar{m})}(uC_D)\right\} du\, d\rho_S \tag{19}$$

where

$$F(u,0,\rho_S) = \frac{\sin(u\theta_{bS}/a_S) - (u\theta_{bS}/a_S)\cos(u\theta_{bS}/a_S)}{u^2}$$

When $\ell \neq 0$, unsteady flow, Reference 6 gives the nondimensionalized velocity component as $W_{R_tD}^{(\ell N_R,\bar{m})} e^{i\ell N_R\varphi_D} e^{i\ell N_R\Omega t}$, where

$$W_{R_tD}^{(\ell N_R,\bar{m})} = \frac{i4a_R^2N_R}{\pi^2} \int_{\rho_R} \sqrt{1+a_R^2\rho_R^2}\ \frac{\rho_R}{\theta_{bR}} \frac{t_o}{c}(\rho_R)$$

$$\cdot\int_0^\infty u I_{\ell N_R}(u\rho_R)K'_{\ell N_R}(uR_D)e^{-i\ell N_R\sigma_R}\Big[e^{iu(\epsilon_D-\sigma_R/a_R)}I^{(\bar{m})}(-uC_D)F(u,-\ell N_R,\rho_R)$$

$$-e^{-iu(\epsilon_D-\sigma_R/a_R)}I^{(\bar{m})}(uC_D)F(u,\ell N_R,\rho_R)\Big]du\,d\rho_R \qquad (20)$$

and

$$F(u,\ell N_R,\rho_R)=\frac{\sin\Big((u-a_R\ell N_R)\theta_{bR}/a_R\Big)-\Big(u-a_R\ell N_R\Big)\Big(\theta_{bR}/a_R\Big)\cos\Big((u-a_R\ell N_R)\theta_{bR}/a_R\Big)}{\Big(u-a_R\ell N_R\Big)^2}$$

A similar formula can be derived for $W_{S_tD}^{(\ell N_S,\bar{m})}$, which however can only be effective when $\ell = N_R$ (blade crossing frequency), since N_S is usually not an integer multiple of N_R, and thus is negligibly small.

COMPONENTS OF THE SYSTEM OF INTEGRAL EQUATIONS

1) Kernel Function K_{RR}

From References 1 and 2, the first integral of Eq.(1) can be shown to be equivalent for each frequency q_R to

$$I_1 = e^{iq_R\Omega t} \int_{\rho_R} \sum_{\bar{m}=1}^{\bar{n}} \sum_{\bar{n}=1}^{\infty} L_R^{(q_R,\bar{n})}(\rho_R)\bar{K}_{RR}\left(r_R,\varphi_{R_o},\rho_R,\theta_{R_o};q_R\right)d\rho_R$$

where

$$\bar{K}_{RR}^{(\bar{m},\bar{n})} = \left(\frac{-N_R}{4\pi\rho_f U^2 r_{RO}}\right)\frac{r_R e^{-iq_R\Delta\sigma}}{a_R\sqrt{1+a_R^2 r_R^2}} \sum_{\substack{m_1=-\infty \\ m_1=q_R+\ell_1 N_R}}^{\infty}\left\{g_1(0)-\frac{i}{\pi}\int_0^{\infty}\frac{g_1(u)-g_1(-u)}{u}du\right\} \tag{21}$$

where

$$g_1(u) = (IK)_{m_1} B(u) e^{i\frac{u}{a_R}\Delta\sigma}$$

$$(IK)_{m_1} = \begin{cases} I_{m_1}(|u+a_R\ell_1 N_R|\rho_R)K_{m_1}(|u+a_R\ell_1 N_R|r_R) & \text{for } \rho_R < r_R \\ I_{m_1}(|u+a_R\ell_1 N_R|r_R)K_{m_1}(|u+a_R\ell_1 N_R|\rho_R) & \text{for } r_R < \rho_R \end{cases}$$

$$B(u) = (a_R u + a_R^2\ell_1 N_R + \frac{m}{r_R^2})\,(a_R u + a_R^2\ell_1 N_R + \frac{m}{\rho_R^2}) \cdot I^{(\bar{m})}((q_R - \frac{u}{a_R})\theta_b{}^r)\Lambda^{(\bar{n})}((q_R - \frac{u}{a_R})\theta_b{}^\rho)$$

ρ_f = fluid mass density, slugs/ft^3

r_{RO} = rotor radius, ft

$\Delta\sigma = \sigma^r - \sigma^\rho$ = difference between skewness of the blade at control point r and skewness at a loading point ρ, radians

a_R = $\Omega\ r_{RO}/U$ and ρ and r are also nondimensionalized by r_o

Ω = angular velocity of rotor, radians/sec, U= freestream velocity, ft/sec

$\theta_b^r, \theta_b^\rho$ = subtended angle of projected semichord of blade at r, at ρ, radians

$I_m(\ ), K_m(\ )$ modified Bessel functions of the first and second kind

and $\ell_1 = 0, \pm 1, \pm 2, \ldots$

The $L_R^{(q_R,\bar{n})}(\rho_R)$ are the unknown spanwise loading coefficients after approximation of the unknown loading function $L^{(q_R)}(\rho_R, \theta_{RO})$ in the chordwise direction by Birnbaum mode shape

$$L^{(q_R)}(\rho_R, \theta_{RO}) = \sum_{\bar{n}=1}^{\infty} L^{(q_R,\bar{n})}(\rho_R)\Theta(\bar{n}) =$$

$$= \frac{1}{\pi}\left\{L_R^{(q_R,1)}(\rho_R)\cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\infty} L^{(q_R,\bar{n})}(\rho_R)\sin(\bar{n}-1)\theta_\alpha\right\}$$

where $\theta_{RO} = \sigma^\rho - \theta_b^\rho\cos\theta_\alpha$ and after the subsequent chordwise integration over θ_{RO}

$$\Lambda^{\bar{n}}(y) = \frac{1}{\pi}\int_0^\pi \Theta(\bar{n})e^{-iy\cos\theta_\alpha}\sin\theta_\alpha d\theta_\alpha$$

(See Appendix A.)

The $\varphi_{RO}(=\sigma^r - \theta_b^r\cos\varphi_\alpha)$ dependence is eliminated by operating on both sides of the integral equation by the "generalized" lift operators $\Phi(\bar{m})$. The factor $I^{(\bar{m})}(x)$ in the kernel function is the result of this:

$$I^{(\bar{m})}(x) = \frac{1}{\pi}\int_0^\pi \Phi(\bar{m})e^{ix\cos\varphi_\alpha}\, d\varphi_\alpha$$

(See Appendix A.) Equation (21) has an integrable singularity at u=0, the value of which is determined by L'Hospital's rule as shown in Appendix C.

2) Kernel Function K_{DR}

When the control point is on the rotor and the loading point is on the cylindrical duct, the induced velocity, nondimensionalized by U, normal to the rotor blades, is

$$I_2 = \sum_{\lambda_2=0}^{\infty} \iint_{S_D} L_D'^{(\lambda_2)}(\xi_D, \rho_D, \theta_D)e^{i\lambda_2\Omega t}K_{DR}(x_R, r_R, \varphi_{RO}; \xi_D, \rho_D, \theta_D; \lambda_2)dS_D$$

or

$$I_2 = \sum_{\lambda_2=0}^{\infty} \int_0^{2\pi} \int_{2C_D} L_D^{(\lambda_2)} e^{i\lambda_2\Omega t} K_{DR}\rho_D d\theta_D d\xi_D \tag{22}$$

where $L_D^{(\lambda_2)} = L_D^{\prime(\lambda_2)} \cdot \rho_D$ = duct loading, lb/ft (see Reference 5).

$$K_{DR} = -\frac{1}{4\pi\rho_f U^2} \lim_{\substack{x_R \to \varphi_{R0}/a_R \\ \rho_D \to R_D}} \frac{\partial}{\partial n_R^{\prime}} \frac{\partial}{\partial n_D} \int_{-\infty}^{x_R-\xi_D} \frac{e^{i\lambda_2 a_R(\tau - x_R + \xi_D)}}{R_{DR}} d\tau$$

$$\frac{\partial}{\partial n_R^{\prime}} = \frac{r_R}{\sqrt{1+a_R^2 r_R^2}} \left(a_R \frac{\partial}{\partial x_R} - \frac{1}{r_R^2} \frac{\partial}{\partial \varphi_{R0}}\right)$$

$$\frac{\partial}{\partial n_D} = \frac{\partial}{\partial \rho_D}$$

$$R_{DR} = \left\{\tau^2 + r_R^2 + \rho_D^2 - 2r_R\rho_D \cos(+\theta_D - \varphi_{R0} + \Omega t)\right\}^{\frac{1}{2}}$$

$$\varphi_{R0} = \sigma_R - \theta_{bR}\cos\varphi_\alpha \quad , \quad 0 \le \varphi_\alpha \le \pi$$

The loading will be expressed in a Fourier series as

$$L_D^{(\lambda_2)}(\xi_D, \rho_D, \theta_D) = \sum_{\mu=-\infty}^{\infty} L_D^{(\lambda_2,\mu)}(\xi_D) e^{-i\mu\theta_D} \tag{23}$$

at $\rho_D = R_D$. The reciprocal Descartes distance $1/R$ can be expanded in the form

$$\frac{1}{R_{DR}} = \frac{1}{\pi} \sum_{m_2=-\infty}^{\infty} e^{im_2(+\theta_D - \varphi_{R0} + \Omega t)} \int_{-\infty}^{\infty} I_{m_2}(|k| r_R) K_{m_2}(|k|\rho_D) e^{i\tau k} dk \tag{24}$$

($r_R < \rho_D$ in the limit as $\rho_D \to R_D$.)

From the θ_D-integration, it is determined that $m_2 = \mu$, since

$$\int_0^{2\pi} e^{i(m_2-\mu)\theta_D} d\theta_D = \begin{cases} 2\pi \text{ for } m_2 - \mu = 0 \\ 0 \text{ otherwise} \end{cases} \tag{25}$$

Also, since the L-H of Eq.(1) is an $\exp(iq_R\Omega t)$ function of time and I_2 is an $\exp[i(\lambda_2+m_2)\Omega t]$ function,

$$\lambda_2 + m_2 = q_R \tag{26}$$

where $q_R \ge 0$, $\lambda_2 \ge 0$. The double series can thus be reduced to a single infinite series. Equation (22) becomes

$$I_2 = \sum_{\substack{\lambda_2=0 \\ m_2=q_R-\lambda_2}}^{\infty} - \frac{2e^{iq_R\Omega t}}{4\pi\rho_f U^2} \lim_{\substack{x_R\to\varphi_{R0}/a_R \\ \rho_D\to R_D}} \int_{2C_D} L_D^{(\lambda_2,m_2)}(\xi_D) \frac{\partial}{\partial n_R'}\frac{\partial}{\partial n_D}$$

$$\cdot e^{-im_2\varphi_{R0}} e^{i(m_2-q_R)a_R(x_R-\xi_D)} \int_{-\infty}^{x_R-\xi_D}$$

$$\cdot e^{-i(m_2-q_R)a_R\tau} \int_{-\infty}^{\infty} I_{m_2}(|k|r_R)K_{m_2}(|k|\rho_D)\, e^{i\tau k}\,dk\,d\tau\,d\xi_D \tag{27}$$

After the τ-integration is performed and the derivatives and limits are taken,[5] the generalized lift operators[7] are applied. Equation (2) becomes

$$I_2 = \sum_{\bar m=1} \sum_{\substack{\lambda_2=0 \\ m_2=q_R-\lambda_2}} \int_{2C_D} L_D^{(\lambda_2,m_2)}(\xi_D)e^{iq_R\Omega t}\, \bar K_{DR}^{(m_2,\bar m)}\, d\xi_D$$

where $\bar K_{DR}^{(m_2,\bar m)}$ is the modified kernel after the φ_α-integration:

$$\bar K_{DR}^{(m_2,\bar m)} = \frac{1}{4\pi\rho_f U^2 r_{R0}} \frac{r_R}{\sqrt{1+a_R^2r_R^2}}\, e^{-im_2\sigma_R}$$

$$\cdot\Big\{i\pi a_R|m_2-q_R|\Big[a_R^2(m_2-q_R)+\frac{m_2}{r_R^2}\Big]I_{m_2}(a_R|m_2-q_R|r_R)\Big[K_{m_2-1}(a_R|m_2-q_R|R_D)$$

$$+K_{m_2+1}(a_R|m_2-q_R|R_D)\Big]\, e^{-ia_R(m_2-q_R)(\xi_D-\sigma_R/a_R)}\, I^{(\bar m)}(q_R\theta_{bR})$$

$$+a_R\int_{-\infty}^{\infty} \frac{k|k|I_{m_2}(|k|r_R)\Big[K_{m_2-1}(|k|R_D)+K_{m_2+1}(|k|R_D)\Big]I^{(\bar m)}\Big((m_2-\frac{k}{a_R})\theta_{bR}\Big)e^{-ik(\xi_D-\frac{\sigma_R}{a_R})}\,dk}{k-a_R(m_2-q_R)}$$

(Cont'd)

$$+\frac{m_2}{r_R^2}\int_{-\infty}^{\infty}\frac{|k|\, I_{m_2}(|k|r_R)\left[K_{m_2-1}(|k|R_D)+K_{m_2+1}(|k|R_D)\right] I^{(\bar{m})}\left((m_2-\frac{k}{a_R})\theta_{bR}\right) e^{-ik(\xi_D-\frac{\sigma_R}{a_R})}\, dk}{k-a_R(m_2-q_R)}\Bigg\} \tag{28}$$

evaluated at $m_2=q_R-\lambda_2$. The rotor radius r_{RO} is introduced in the denominator of the first factor because now r_R, R_D, and ξ_D are fractions of r_{RO} and $a_R= r_{RO}\Omega/U$. The k-integrals have integrable Cauchy-type singularities. There is no other singularity, as can be seen from the original kernel of Eq.(22) since r_R is always less than $\rho_D=R_D$.

If the chordwise loading on the duct is approximated by the Birnbaum mode shapes

$$L_D^{(\lambda_2,m_2)}(\xi_D) = \frac{1}{\pi}\left\{A^{(\lambda_2,m_2,1)}\cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\infty} A^{(\lambda_2,m_2,\bar{n})}\sin(\bar{n}-1)\theta_\alpha\right\} \tag{29}$$

where θ_α is defined by $\xi_D = \epsilon_D - C_D\cos\theta_\alpha$, $0 \le \theta_\alpha \le \pi$ (see Figure 1), then the integration over ξ_D is easily accomplished.

$$\int_{2C_D} L_D^{(\lambda_2,m_2)}(\xi_D)\bar{K}_{DR}^{(m_2,\bar{m})}d\xi_D = \frac{1}{\pi}\sum_{\bar{n}=1}^{\infty}\int_0^{\pi} A^{(\lambda_2,m_2,\bar{n})}\Theta(\bar{n})\bar{K}_{DR}^{(m_2,\bar{m})}C_D\sin\theta_\alpha d\theta_\alpha$$
$$= \sum_{\bar{n}=1}^{\infty}\bar{A}^{(\lambda_2,m_2,\bar{n})}\bar{K}_{DR}^{(m_2,\bar{m},\bar{n})} \tag{30}$$

where $\Theta(\bar{n})$ are the chordwise mode shapes given in Eq.(29),

and $\bar{A}^{(\lambda_2,m_2,\bar{n})} = C_D A^{(\lambda_2,m_2,\bar{n})}$

and
$$\bar{K}_{DR}^{(m_2,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U^2 r_{RO}}\frac{r_R}{\sqrt{1+a_R^2r_R^2}}e^{-im_2\sigma_R}\Bigg\{i\pi a_R|m_2-q_R|\left[a_R^2(m_2-q_R)+\frac{m_2}{r_R^2}\right]$$
$$\cdot I_{m_2}(a_R|m_2-q_R|r_R)\left[K_{m_2-1}(a_R|m_2-q_R|R_D)+K_{m_2+1}(a_R|m_2-q_R|R_D)\right]$$
$$\cdot e^{-ia_R(m_2-q_R)(\epsilon_D-\sigma_R/a_R)}\, I^{(\bar{m})}(q_R\theta_{bR})\Lambda^{(\bar{n})}(-a_R(m_2-q_R)C_D)$$

$$+\int_{-\infty}^{\infty}\left(a_Rk+\frac{m_2}{r_R^2}\right)\frac{|k|\, I_{m_2}(|k|r_R)\left[K_{m_2-1}(|k|R_D)+K_{m_2+1}(|k|R_D)\right]I^{(\bar{m})}((m_2-\frac{k}{a_R})\theta_{bR})\Lambda^{(\bar{n})}(-kC_D)e^{-ik(\epsilon_D-\frac{\sigma_R}{a_R})}dk}{k-a_R(m_2-q_R)}\Bigg\} \tag{31}$$

Then letting $u = k - a_R(m_2-q_R)$

$$\bar{K}_{DR}^{(m_2=q_R-\lambda_2,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U^2 r_{RO}}\frac{r_R}{\sqrt{1+a_R^2r_R^2}}e^{-im_2\sigma_R}e^{-ia_R(m_2-q_R)(\epsilon_D-\frac{\sigma_R}{a_R})}$$

(Cont'd)

$$\cdot \left\{ i\pi \; g_2(0) + \int_0^\infty [g_2(u) - g_2(-u)] \frac{du}{u} \right\} \qquad (31a)$$

where

$$g_2(u) = \left[a_R u + a_R^2(m_2 - q_R) + \frac{m_2}{r_R^2}\right] \left[\left|u + a_R(m_2 - q_R)\right|\right] e^{-iu(\epsilon_D - \frac{\sigma_R}{a_R})}$$

$$\cdot I_{m_2}\left(\left|u + a_R(m_2 - q_R)\right| r_R\right) \left[K_{m_2-1}\left(\left|u + a_R(m_2 - q_R)\right| R_D\right) + K_{m_2+1}\left(\left|u + a_R(m_2 - q_R)\right| R_D\right)\right]$$

$$\cdot I^{(\bar{m})}\left(\left(-\frac{u}{a_R} + q_R\right)\theta_{bR}\right) \Lambda^{(\bar{n})}\left(\left(-u - a_R(m_2 - q_R)\right) c_D\right)$$

See Appendix D for the evaluation of the singular part of Eq.(31a).

3) Kernel Function K_{SR}

When the control point is at (x_R, r_R, φ_R) on the rotor and the loading points are at $(\xi_S', \rho_S, \theta_{S0})$ on the stator, the nondimensional induced velocity *normal to the rotor* is,[2,3]

$$I_3 = \sum_{\lambda_3=0}^{\infty} \int_0^{\pi} \int_{\rho_S} L_S^{(\lambda_3)}(\rho_S, \theta_\alpha) e^{i\lambda_3 \Omega t} K_{SR}' \frac{\sqrt{1 + a_S^2 \rho_S^2}}{a_S \rho_S} \sin\theta_\alpha d\theta_\alpha d\rho_S \qquad (32)$$

where

$$\theta_{S0} = \sigma_S^\rho - \theta_{bS}^\rho \cos\theta_\alpha$$

$$K_{SR}' = -\frac{1}{4\pi\rho_f U^2} \sum_{n=1}^{N_S} \lim_{\delta_{SR} \to 0} \frac{\partial}{\partial n_R^\intercal} \int_{-\infty}^{x_R} e^{i\lambda_3 \left[a_R(\tau' - x_R) - \bar{\theta}_{Sn}\right]} \frac{\partial}{\partial n_S}\left(\frac{1}{R_{SR}}\right) d\tau'$$

$$R_{SR} = \left\{\left(\tau' - \xi_S'\right)^2 + r_R^2 + \rho_S^2 - 2 r_R \rho_S \cos\left(\theta_S - \varphi_R\right)\right\}^{\frac{1}{2}}$$

$$= \left\{\left(\tau' - \xi_S'\right)^2 + r_R^2 + \rho_S^2 - 2 r_R \rho_S \cos\left(-\theta_{S0} - \varphi_{R0} + \Omega t - \bar{\theta}_{Sn}\right)\right\}^{\frac{1}{2}}$$

and $\delta_{SR} \to 0$ means that $x_R \to \varphi_{R0}/a_R$ and $\xi_S' \to \theta_{S0}/a_S + \epsilon_S$

The inverse Descartes distance $1/R_{SR}$ is expanded as

$$\frac{1}{R_{SR}} = \frac{1}{\pi}\sum_{m_3=-\infty}^{\infty} e^{im_3\beta_{SR}} \int_{-\infty}^{\infty} e^{i(\tau'-\xi_S')k}\, I_{m_3}(|k|\rho_S)K_{m_3}(|k|r_R)\,dk \tag{33}$$

(for $\rho_S < r_R$) with $\beta_{SR} = -\theta_{S0} - \varphi_{R0} + \Omega t - \bar{\theta}_{Sn}$.

The derivative

$$\frac{\partial}{\partial n_S} = \frac{\rho_S}{\sqrt{1+a_S^2\rho_S^2}}\left(a_S \frac{\partial}{\partial \xi_S'} - \frac{1}{\rho_S^2}\frac{\partial}{\partial\theta_{S0}}\right)$$

is the directional derivative normal to the stator blade.

After the τ'-integration and derivatives are taken, the kernel function K_{SR} becomes

$$\begin{aligned} K_{SR} &= K'_{SR}\,\frac{\sqrt{1+a_S^2\rho_S^2}}{a_S\rho_S} \\ &= -\frac{1}{4\pi\rho_f U^2 a_S}\,\frac{r_R}{\sqrt{1+a_R^2 r_R^2}}\sum_{n=1}^{N_S}\sum_{m_3=-\infty}^{\infty} e^{-i(\lambda_3+m_3)\bar{\theta}_{Sn}}\, e^{im_3\Omega t}\, e^{-im_3(\theta_{S0}+\varphi_{R0})} \\ &\quad\cdot\left\{\left(a_S a_R\lambda_3 + \frac{m_3}{\rho_S^2}\right)\left(a_R^2\lambda_3 - \frac{m_3}{r_R^2}\right) e^{-ia_R\lambda_3\left(\frac{\varphi_{R0}}{a_R} - \frac{\theta_{S0}}{a_S} - \epsilon_S\right)} I_{m_3}(a_R\lambda_3\rho_S)K_{m_3}(a_R\lambda_3 r_R)\right. \\ &\quad\left. - \frac{i}{\pi}\int_{-\infty}^{\infty}\left(a_S k - \frac{m_3}{\rho_S^2}\right)\left(a_R k + \frac{m_3}{r_R^2}\right)\frac{I_{m_3}(|k|\rho_S)K_{m_3}(|k|r_R)}{k + a_R\lambda_3}\, e^{ik\left(\frac{\varphi_{R0}}{a_R} - \frac{\theta_{S0}}{a_S} - \epsilon_S\right)} dk\right\} \end{aligned} \tag{34}$$

From the time-dependent factors on both sides of Eq.(1)

$$\lambda_3 + m_3 = q_R$$

and from the summation over the stator blades

$$\sum_{n=1}^{N_S} e^{-i(\lambda_3+m_3)\bar{\theta}_{Sn}} = \begin{cases} N_S & \text{for } (\lambda_3+m_3) = \ell_3 N_S \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_3 + m_3 = q_R = \ell_3 N_S \quad , \quad \ell_3 \geq 0$$

Thus the frequencies q_R of the first equation are limited to zero and positive multiples of the number of stator blades.

Since $\lambda_3 = q_R - m_3 \geq 0$, $m_3 \leq q_R$ and the double series over λ_3 and m_3 can be reduced to a singl e infinite series.

The unknown loading function $L_S(\rho_S, \theta_{S0})$ is approximated as before in chordwise direction by Birnbaum mode shapes. After the chordwise integration over θ_α and application of the generalized lift operator, I_3 can be written

$$I_3 = e^{iq_R\Omega t} \sum_{\lambda_3=0}^{\infty} \int_{\rho_S} \sum_{\bar{m}=1}^{\infty} \sum_{\bar{n}=1}^{\infty} L_S^{(\lambda_3,\bar{n})}(\rho_S) \bar{K}_{SR}^{(\bar{m},\bar{n})}(m_3 = q_R - \lambda_3)\, d\rho_S \tag{35}$$

where the modified kernel is

$$\bar{K}_{SR}^{(\bar{m},\bar{n})}(m_3 = q_R - \lambda_3) = \Bigg\{ -\frac{N_S}{4\pi\rho_f U^2 r_{R0}} \; \frac{r_R}{a_S\sqrt{1+a_R^2 r_R^2}} \Bigg\}$$

$$\cdot \Bigg\{ \Big[a_S a_R (q_R - m_3) + \frac{m_3}{\rho_S^2} \Big] \Big[a_R^2 (q_R - m_3) - \frac{m_3}{r_R^2} \Big] I_{m_3}\big(a_R(q_R-m_3)\rho_S\big) K_{m_3}\big(a_R(q_R-m_3) r_R\big)$$

$$\cdot e^{-im_3\sigma_S} e^{-iq_R\sigma_R} e^{ia_R(q_R-m_3)(\epsilon_S + \sigma_S/a_S)} \Lambda^{(\bar{n})}\Big(\big(\frac{a_R}{a_S}(q_R - m_3) - m_3\big)\theta_{bS}\Big) \Gamma^{(\bar{m})}(q_R\theta_{bR})$$

$$-\frac{i}{\pi} \int_{-\infty}^{\infty} \Big[\Big(a_S k - \frac{m_3}{\rho_S^2}\Big)\Big(a_R k + \frac{m_3}{r_R^2}\Big) \frac{I_{m_3}(|k|\rho_S) K_{m_3}(|k| r_R)}{k + a_R(q_R - m_3)} e^{-ik\epsilon_S} e^{-i(m_3 - \frac{k}{a_R})\sigma_R} e^{-i(m_3 + \frac{k}{a_S})\sigma_S} \Big]$$

$$\cdot \; \Lambda^{(\bar{n})}\Big((-m_3 - \frac{k}{a_S})\theta_{bS}\Big) \Gamma^{(\bar{m})}\Big((m_3 - \frac{k}{a_R})\theta_{bR}\Big) dk \Bigg\} \tag{36}$$

where now, a,k,r, and ρ are nondimensionalized with respect to r_{R0}.

Let $u = k + a_R(q_R - m_3)$. The kernel may be written as

$$\bar{K}_{SR}^{(\bar{m},\bar{n})}(m_3 = q_R - \lambda_3) = \Bigg\{ -\frac{N_S}{4\pi\rho_f U^2 r_{R0}} \; \frac{r_R}{a_S\sqrt{1+a_R^2 r_R^2}} e^{-im_3\sigma_S} e^{-iq_R\sigma_R} e^{ia_R(q_R-m_3)(\epsilon_S + \frac{\sigma_S}{a_S})} \Bigg\}$$

$$\cdot \Big\{ g_3(0) - \frac{i}{\pi} \int_0^{\infty} \frac{g_3(u) - g_3(-u)}{u} du \Big\} \tag{37}$$

where

$$g_3(u) = \left[a_S u - a_S a_R(q_R - m_3) - \frac{m_3}{\rho_S^2}\right]\left[a_R u - a_R^2(q_R - m_3) + \frac{m_3}{r_R^2}\right]$$

$$\cdot I_{m_3}(|u - a_R(q_R - m_3)|\rho_S) K_{m_3}(|u - a_R(q_R - m_3)| r_R)\, e^{-iu\left(\epsilon_S - \frac{\sigma_R}{a_R} + \frac{\sigma_S}{a_S}\right)}$$

$$\cdot \Lambda^{(\bar{n})}\left(\left(-m_3\left(1 + \frac{a_R}{a_S}\right) + \frac{a_R}{a_S} q_R - \frac{u}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}\left(\left(q_R - \frac{u}{a_R}\right)\theta_{bR}\right)$$

for $\rho_S < r_R$. Note that for $r_R < \rho_S$, these are interchanged in the modified Bessel functions.

See Appendix E for the evaluation of the singularity of K_{SR} as $u \to 0$.

4) Kernel Function K_{RS}

If the control point is at (x_S', r_S, φ_S) on the stator and the loading points are at $(\xi_R, \rho_R, \theta_R)$ on the rotor, then following the development[2,3] for the other kernel functions it can be shown that the nondimensional induced velocity at the control point due to N_R-blades of the rotor will be given by

$$I_4 = \sum_{\lambda_4=0}^{\infty} \int_0^{\pi} \int_{\rho_R} L_R^{(\lambda_4)}(\rho_R, \theta_\alpha) e^{i\lambda_4 \Omega t} K_{RS}' \frac{\sqrt{1 + a_R^2 \rho_R^2}}{a_R \rho_R} \sin\theta_\alpha d\theta_\alpha d\rho_R \tag{38}$$

where

$$K_{RS}' = -\frac{1}{4\pi\rho_f U^2} \sum_{n=1}^{N_R} \lim_{\delta_{RS}\to 0} \frac{\partial}{\partial n_S'} \int_{-\infty}^{x_S'} e^{i\lambda_4[a_R(\tau' - x_S') - \bar{\theta}_{Rn}]} \frac{\partial}{\partial n_R}\left(\frac{1}{R_{RS}}\right) d\tau'$$

$$\frac{\partial}{\partial n_S'} = \frac{r_S}{\sqrt{1 + a_S^2 r_S^2}}\left(a_S \frac{\partial}{\partial x_S'} - \frac{1}{r_S^2}\frac{\partial}{\partial \varphi_{S0}}\right)$$

$$x_S' = \varphi_{S0}/a_S + \epsilon_S \qquad (\epsilon_S \text{ negative})$$

$$a_S = \frac{1}{r_S \tan\theta_{PS}(r_S)} \qquad \text{at } r_S = 0.7 \text{ radius}$$

$$R_{RS} = \left\{(\tau' - \xi_R)^2 + r_S^2 + \rho_R^2 - 2 r_S \rho_R \cos[\theta_{R0} + \varphi_{S0} - \Omega t + \bar{\theta}_{Rn} - a_R(\tau' - x_S)]\right\}^{\frac{1}{2}}$$

and by $\delta_{RS} \to 0$ is meant $x_S' \to \varphi_{S0}/a_S + \epsilon_S$ and $\xi_R \to \theta_{R0}/a_R$.

Expanding the Descartes distance,

$$\frac{1}{R_{RS}} = \frac{1}{\pi} \sum_{m_4=-\infty}^{\infty} e^{im_4\beta_{RS}} \int_{-\infty}^{\infty} e^{i(\tau'-\xi_R)k} I_{m_4}(|k|\rho_R)K_{m_4}(|k|r_S)dk \tag{39}$$

for $\rho_R < r_S$, otherwise ρ_R and r_S are interchanged. Here

$$\beta_{RS} = \theta_{RO} + \varphi_{SO} - \Omega t + \bar{\theta}_{Rn} - a_R(\tau'-x_S') \ .$$

After performing the τ'-integration and taking the derivatives in the proper order, and taking the limit

$$K_{RS} = K'_{RS} \frac{\sqrt{1+a_R^2\rho_R^2}}{a_R\rho_R} =$$

$$= - \frac{1}{4\pi\rho_f U^2 a_R} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} \sum_{n=1}^{N_R} \sum_{m_4=-\infty}^{\infty} e^{-im_4\Omega t} e^{i(m_4-\lambda_4)\bar{\theta}_{Rn}} e^{im_4(\theta_{RO}+\varphi_{SO})}$$

$$\cdot \left\{ \left[a_S a_R(m_4-\lambda_4) - \frac{m_4}{r_S^2}\right]\left[a_R^2(m_4-\lambda_4) + \frac{m_4}{\rho_R^2}\right] e^{ia_R(m_4-\lambda_4)\left(\frac{\varphi_{SO}}{a_S} + \epsilon_S - \frac{\varphi_{RO}}{a_R}\right)}\right.$$

$$\cdot I_{m_4}(a_R|m_4 - \lambda_4|\rho_R)K_{m_4}(a_R|m_4 - \lambda_4|r_S)$$

$$\left. - \frac{i}{\pi} \int_{-\infty}^{\infty}\left(a_S k - \frac{m_4}{r_S^2}\right)\left(a_R k + \frac{m_4}{\rho_R^2}\right) \frac{I_{m_4}(|k|\rho_R)K_{m_4}(|k|r_S)}{k-a_R(m_4-\lambda_4)} e^{ik\left(\frac{\varphi_{SO}}{a_S} + \epsilon_S - \frac{\theta_{RO}}{a_R}\right)} dk \right. \tag{40}$$

The time-dependent factor on the L-H of Eq.(2) is $\exp(iq_S\Omega_R t)$ and the time-dependent factor of I_4 on the R-H side is $\exp[i(\lambda_4-m_4)\Omega t]$, therefore $q_S = \lambda_4-m_4$.

Also $\displaystyle\sum_{n=1}^{N_R} e^{i(m_4-\lambda_4)\theta_{Rn}} = \begin{cases} N_R & \text{for } (m_4-\lambda_4) = \ell_4 N_R,\ \ell_4 = 0, \pm1, \ldots \\ 0 & \text{otherwise} \end{cases}$

so that

$$q_S = \ell_4 N_R,\ \ell_4 = 0, +1, +2, \ldots \qquad \text{and} \quad \lambda_4-m_4 \geq 0 \ .$$

After the chordwise integration over θ_α is performed, by representing the chordwise loading distribution by the appropriate mode shapes $\Theta(\bar{n})$ (see I_1) and the generalized lift operator $\Phi(\bar{m})$ is applied, the integral I_4 becomes for each q_S, $\bar{m}$ and $\bar{n}$

$$I_4 = e^{iq_S\Omega t} \int_{\rho_R} \sum_{\lambda_4=0}^{\infty} L_R^{(\lambda_4,\bar{n})}(\rho_R)\bar{K}_{RS}^{(\bar{m},\bar{n})} d\rho_R \tag{41}$$

where the modified kernel is

$$\bar{K}_{RS}^{(\bar{m},\bar{n})}(m_4 = \lambda_4 - q_S) = -\frac{N_R}{4\pi\rho_f U^2 a_R r_{R0}} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} e^{im_4(\sigma_R+\sigma_S)}$$

$$\cdot\left\{\left(a_S a_R q_S + \frac{m_4}{r_S^2}\right)\left(a_R^2 q_S - \frac{m_4}{\rho_R^2}\right) e^{-ia_R q_S\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)} I_{m_4}(a_R q_S \rho_R) K_{m_4}(a_R q_S r_S)\right.$$

$$\cdot\ \Lambda^{(n)}\Big((m_4 + q_S)\theta_{bR}\Big)\ I^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)$$

$$-\frac{i}{\pi}\int_{-\infty}^{\infty}\left(a_S k - \frac{m_4}{r_S^2}\right)\left(a_R k + \frac{m_4}{\rho_R^2}\right) e^{ik\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)} I_{m_4}(|k|\rho_R) K_{m_4}(|k| r_S)$$

$$\left.\cdot\ \Lambda^{(\bar{n})}\left(\left(m_4 - \frac{k}{a_R}\right)\theta_{bR}\right)\ I^{(\bar{m})}\left(\left(-m_4 - \frac{k}{a_S}\right)\theta_{bS}\right) \frac{dk}{k + a_R q_S}\right\} \tag{42}$$

Let $u = k + a_R q_S$, then

$$\bar{K}_{RS}^{(\bar{m},\bar{n})}(m_4 = \lambda_4 - q_S) = -\frac{N_R}{4\pi\rho_f U^2 a_R r_{R0}} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} e^{im_4(\sigma_R+\sigma_S)}$$

$$e^{-ia_R q_S\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)} \left\{g_4(0) - \frac{i}{\pi}\int_0^{\infty} \frac{g_4(u) - g_4(-u)}{u}\, du\right\}$$

where

$$g_4(u) = \left(a_S u - a_S a_R q_S - \frac{m_4}{r_S^2}\right)\left(a_R u - a_R^2 q_S + \frac{m_4}{\rho_R^2}\right) e^{iu(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R})}$$

$$\cdot\ I_{m_4}\left(|u - a_R q_S|\rho_R\right) K_{m_4}\left(|u - a_R q_S| r_S\right)$$

$$\cdot\ \Lambda^{(\bar{n})}\left((m_4 + q_S - \frac{u}{a_R})\theta_{bR}\right) I^{(\bar{m})}\left((-m_4 + \frac{a_R}{a_S} q_S - \frac{u}{a_S})\theta_{bS}\right) \tag{43}$$

(for $\rho_R < r_S$). See Appendix F for the singularity of K_{RS} at $u = 0$.

5) Kernel Function K_{DS}

When the control point is on the stator and the loading point is on the cylindrical duct, the nondimensional induced velocity normal to the stator blades, the second integral of Eq.(2), is

$$I_5 = \sum_{\lambda_5=0}^{\infty} \int_0^{2\pi} \int_{2C_D} L_D^{(\lambda_5)} e^{i\lambda_5 \Omega t} K_{DS} d\theta_D d\xi_D \tag{44}$$

where

$$K_{DS} = -\frac{1}{4\pi\rho_f U^2} \lim_{\substack{x_S \to \varphi_S/a_S + \epsilon_S \\ \rho_D \to R_D}} \frac{\partial}{\partial n_S'} \frac{\partial}{\partial \rho_D} \int_{-\infty}^{x_S - \xi_D} \frac{e^{i\lambda_5 a_R(\tau - x_S + \epsilon_D)}}{R_{DS}} d\tau$$

$$R_{DS} = \left\{\tau^2 + r_S^2 + \rho_D^2 - 2 r_S \rho_D \cos(\theta_D - \varphi_S)\right\}^{\frac{1}{2}}$$

$$= \left\{\tau^2 + r_S^2 + \rho_D^2 - 2 r_S \rho_D \cos(\theta_D + \varphi_{S0})\right\}^{\frac{1}{2}}$$

The loading will be expressed in a Fourier series as

$$L_D^{(\lambda_5)}(\xi_D, \rho_D, \theta_D) = \sum_{\mu=-\infty}^{\infty} L_D^{(\lambda_5,\mu)}(\xi_D) e^{-i\mu\theta_D} \tag{45}$$

at $\rho_D = R_D$. The reciprocal of the Descartes distance can be expanded in the form

$$\frac{1}{R_{DS}} = \frac{1}{\pi} \sum_{m_5=-\infty}^{\infty} e^{im_5(\theta_D + \varphi_{S0})} \int_{-\infty}^{\infty} I_{m_5}(|k| r_S) K_{m_5}(|k|\rho_D) e^{i\tau k} dk \tag{46}$$

since $r_S \le \rho_D$ in the limit as $\rho_D \to R_D$.

From the θ_D-integration it is determined that $m_5=\mu$, because

$$\int_0^{2\pi} e^{i(m_5-\mu)\theta_D}\, d\theta_D = \begin{cases} 2\pi & \text{for } m_5-\mu=0 \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

Since the L-H side of Eq.(2) is an $\exp(iq_S\Omega_R t)$ function of time and I_5 is an $\exp(i\lambda_5\Omega t)$ function of time

$$\lambda_5 = q_S = \ell N_R \tag{48}$$

With the substitution of (45,(46),(47) and (48), and after the τ-integration and the derivatives and limits are taken and the generalized lift operator is applied using the complete orthogonal set of functions designated as $\Phi(\bar{m})$, I_5 becomes for each $\bar{m}$, order of lift operator,

$$I_5 = \sum_{q_S=0}^{\infty} \sum_{m_5=-\infty}^{\infty} \int_{2C_D} L_D^{(q_S,m_5)}(\xi_D) e^{iq_S\Omega t} \bar{K}_{DS}^{(m_5,\bar{m})} d\xi_D \tag{49}$$

where the modified kernel (after the φ_α-integration) is

$$\bar{K}_{DS}^{(m_5,\bar{m})} = \frac{1}{4\pi\rho_f U^2 r_{RO}} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} e^{im_5\sigma_S}$$

$$\cdot \left\{ -i\pi a_R q_S \left(a_S a_R q_S + \frac{m_5}{r_S^2}\right) e^{iq_S a_R\left(\xi_D-\epsilon_S-\frac{\sigma_S}{a_S}\right)} I_{m_5}(a_R q_S r_S)\right.$$

$$\cdot \left[K_{m_5-1}(a_R q_S R_D) + K_{m_5+1}(a_R q_S R_D)\right] I^{(\bar{m})}\left(\left(-m_5 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)$$

$$+ \int_{-\infty}^{\infty} |k| \left(a_S k - \frac{m_5}{r_S^2}\right) e^{-ik\left(\xi_D-\epsilon_S-\frac{\sigma_S}{\sigma_S}\right)} I_{m_5}(|k| r_S) \left[K_{m_5-1}(|k|R_D) + K_{m_5+1}(|k|R_D)\right]$$

$$\left. \cdot \frac{I^{(\bar{m})}\left(\left(-m_5 - \frac{k}{a_S}\right)\theta_{bS}\right)}{k+a_R q_S} dk \right\} \tag{50}$$

The expansion scheme has introduced an integrable Cauchy-type singularity in the k-integrals. There is no other singularity.

If the chordwise loading on the duct is approximated by the Birnbaum mode shapes as in I_2

$$L_D^{(q_S,m_5)}(\xi_D) = \frac{1}{\pi}\left\{A^{(q_S,m_5,1)}\cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\infty} A^{(q_S,m_5,\bar{n})}\sin(\bar{n}-1)\theta_\alpha\right\}$$

(see Eq.(29)) then the integration over ξ_D is easily accomplished.

$$\int_{2C_D} L_D^{(q_S,m_5)}(\xi_D)\bar{K}_{DS}^{(m_5,\bar{m})}d\xi_D = \frac{1}{\pi}\sum_{\bar{n}=1}^{\infty}\int_0^{\pi} A^{(q_S,m_5,\bar{n})}\Theta(\bar{n})\bar{K}_{DS}^{(m_5,\bar{m})} C_D\sin\theta_\alpha d\theta_\alpha$$

$$= \sum_{\bar{n}=1}^{\infty}\bar{A}^{(q_S,m_5,\bar{n})}\bar{K}_{DS}^{(m_5,\bar{m},\bar{n})} \qquad (51)$$

where $\bar{A}^{(\cdots)} = C_D A^{(\cdots)}$ (see Eq.(30))

and

$$\bar{K}_{DS}^{(m_5,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U^2 r_{RO}}\,\frac{r_S}{\sqrt{1+a_S^2r_S^2}}\,e^{im_5\sigma_S}$$

$$\cdot\left\{-i\pi a_R q_S\left(a_S a_R q_S + \frac{m_5}{r_S^2}\right)e^{iq_Sa_R(\epsilon_D-\epsilon_S-\frac{\sigma_S}{a_S})}\, I_{m_5}(a_Rq_Sr_S)\right.$$

$$\cdot\left[K_{m_5-1}(a_Rq_SR_D) + K_{m_5+1}(a_Rq_SR_D)\right] I^{(\bar{m})}\left(\left(-m_5 + \frac{a_R}{a_S}q_S\right)\theta_{bS}\right)\Lambda^{(\bar{n})}(a_Rq_SC_D)$$

$$+\int_{-\infty}^{\infty}|k|\left(a_Sk - \frac{m_5}{r_S^2}\right)e^{-ik(\epsilon_D-\epsilon_S-\frac{\sigma_S}{\sigma_S})}\, I_{m_5}(|k|r_S)\left[K_{m_5-1}(|k|R_D)+K_{m_5+1}(|k|R_D)\right]$$

$$\left.\cdot\frac{\Lambda^{(\bar{n})}(-kC_D)I^{(\bar{m})}\left(\left(-m_5 - \frac{k}{a_S}\right)\theta_{bS}\right)}{k+a_Rq_S}dk\right\} \qquad (52)$$

Let $u = k + a_R q_S$

$$\bar{K}_{DS}^{(m_5,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U^2 r_{RO}} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} e^{im_5\sigma_S} e^{ia_R q_S(\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S})}$$

$$\cdot \left\{ i\pi g_5(0) + \int_0^\infty [g_5(u) - g_5(-u)] \frac{du}{u} \right\} \tag{52a}$$

where

$$g_5(u) = |u - a_R q_S| \left(a_S u - a_S a_R q_S - \frac{m_5}{r_S^2} \right) e^{-iu(\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S})}$$

$$\cdot I_{m_5}\left(|u - a_R q_S| r_S\right)\left[K_{m_5-1}\left(|u - a_R q_S| R_D\right) + K_{m_5+1}\left(|u - a_R q_S| R_D\right)\right]$$

$$\cdot I^{(\bar{m})}\left(\left(-\frac{u}{a_S} + \frac{a_R}{a_S} q_S - m_5\right)\theta_{bS}\right) \Lambda^{(\bar{n})}\left(\left(-u + a_R q_S\right) c_D\right)$$

Equation (52) has an integrable singularity at $k = -a_R q_S$. The value of the integrand at that point is determined by means of L'Hospital's rule as shown in Appendix G.

6) Kernel Function K_{SS}

The third integral of Eq.(2), the nondimensional self-induced velocity at a point $(x_S', r_S, \varphi_{S0})$ on the stator due to the loading at points $(\xi_S', \rho_S, \theta_{S0})$ of all N_S blades of the stator, is given as

$$I_6 = \sum_{\lambda_6=0}^{\infty} \int_{\rho_S} L_S^{(\lambda_6)}(\rho_S, \theta_\alpha) e^{i\lambda_6 \Omega t} K_{SS}' \frac{\sqrt{1+a_S^2 \rho_S^2}}{a_S \rho_S} \sin\theta_\alpha d\theta_\alpha d\rho_S \tag{53}$$

where

$$K'_{SS} = -\frac{1}{4\pi\rho_f U^2}\sum_{n=1}^{N_S}\lim_{\delta_{SS}\to 0}\frac{\partial}{\partial n'_S}\int_{-\infty}^{x'_S} e^{i\lambda_6[a_R(\tau'-x'_S)-\bar{\theta}_{Sn}]}\frac{\partial}{\partial n_S}\left(\frac{1}{R_{SS}}\right)d\tau'$$

$$R_{SS} = \left\{(\tau'-\xi'_S)^2 + r_S^2 + \rho_S^2 - 2r_S\rho_S\cos(-\theta_{S0} + \varphi_{S0} - \bar{\theta}_{Sn})\right\}^{\frac{1}{2}}$$

$\delta_{SS}\to 0$ means that $x'_S \to \varphi_{S0}/a_S + \epsilon_S$ and $\xi'_S \to \theta_{S0}/a_S + \epsilon_S$

The inverse Descartes distance is expanded as

$$\frac{1}{R_{SS}} = \frac{1}{\pi}\sum_{m_6=-\infty}^{\infty} e^{im_6\beta_{SS}}\int_{-\infty}^{\infty} I_{m_6}(|k|\rho_S)K_{m_6}(|k|r_S)e^{i(\tau'-\xi'_S)k}dk \qquad (54)$$

(for $\rho_S < r_S$, otherwise ρ_S and r_S are interchanged in the modified Bessel functions) with $\beta_{SS} = \theta_{S0} - \varphi_{S0} + \bar{\theta}_{Sn}$. The summation over the blades becomes

$$\sum_{n=1}^{N_S} e^{i(m_6-\lambda_6)\bar{\theta}_{Sn}} = \begin{cases} N_S & \text{for } m_6-\lambda_6 = \ell_6 N_S,\ \ell_6 = 0,\pm1,\pm2,\ \ldots \\ 0 & \text{otherwise}\end{cases} \qquad (55)$$

Also, since the L-H side of Eq.(2) is an $\exp(iq_S\Omega t)$ function of time and I_6 is an $\exp(i\lambda_6\Omega t)$ function of time,

$$\lambda_6 = q_S = \ell N_R \quad , \ \ell=0,+1,+2,\ \ldots \qquad (56)$$

With these substitutions, after the τ'-integration and the derivatives are taken in the proper order, the kernel function $K_{SS} = K'_{SS}\sqrt{1+a_S^2\rho_S^2}/a_S\rho_S$ becomes

$$K_{SS} = -\frac{N_S}{4\pi^2\rho_f U^2 a_S r_{R0}}\frac{r_S}{\sqrt{1+a_S^2r_S^2}}\lim_{\delta_{SS}\to 0}\sum_{\substack{m_6=-\infty \\ m_6=q_S+\ell_6 N_S}}^{\infty} e^{im_6(\theta_{S0}-\varphi_{S0})}$$

[Cont'd]

$$\cdot \left\{\pi\left(a_S a_R q_S - \frac{m_6}{r_S^2}\right)\left(a_S a_R q_S - \frac{m_6}{\rho_S^2}\right)e^{-iq_S a_R(x_S'-\xi_S')} I_{m_6}(a_R q_S \rho_S) K_{m_6}(a_R q_S r_S)\right.$$

$$\left. -i\int_{-\infty}^{\infty}\left(a_S k + \frac{m_6}{r_S^2}\right)\left(a_S k + \frac{m_6}{\rho_S^2}\right) e^{ik(x_S'-\xi_S')} \frac{I_{m_6}(|k|\rho_S)K_{m_6}(|k|r_S)}{k + a_R q_S} dk\right\} \tag{57}$$

where a, k, r and ρ are nondimensionalized by r_{RO} the rotor radius.

After taking the limit and substituting $\theta_{SO} = -\theta_{bS}^{\rho}\cos\theta_{\alpha}$ and $\varphi_{SO} = -\theta_{bS}^{r}\cos\varphi_{\alpha}$ (0^{o} skew), the chordwise integration over φ_{α} can be performed by representing the chordwise loading distribution by the appropriate mode shapes $\Theta(\bar{n})$ (see I_3) and the generalized lift operator $\Phi(\bar{m})$ can be applied. The integral becomes for each q_S, $\bar{m}$ and $\bar{n}$

$$I_6 = \int_{\rho_S} L_S^{(q_S,\bar{n})}(\rho_S) e^{iq_S\Omega t} \bar{K}_{SS}^{(\bar{m},\bar{n})} d\rho_S \tag{58}$$

where the modified kernel is

$$\bar{K}_{SS}^{(\bar{m},\bar{n})} = -\frac{N_S}{4\pi\rho_f U^2 a_S r_{RO}} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} \sum_{\substack{m_6=-\infty \\ m_6=q_S+\ell_6 N_S}}^{\infty} \left\{\left(a_S a_R q_S - \frac{m_6}{r_S^2}\right)\left(a_S a_R q_S - \frac{m_6}{\rho_S^2}\right)\right.$$

$$\cdot I_{m_6}(a_R q_S \rho_S) K_{m_6}(a_R q_S r_S)\, I^{(\bar{m})}\left(\left(m_6 + \frac{a_R}{a_S} q_S\right)\theta_{bS}^{r}\right)\Lambda^{(\bar{n})}\left(\left(m_6 + \frac{a_R}{a_S} q_S\right)\theta_{bS}^{\rho}\right)$$

$$\left. -\frac{i}{\pi}\int_{-\infty}^{\infty}\left(a_S k + \frac{m_6}{r_S^2}\right)\left(a_S k + \frac{m_6}{\rho_S^2}\right) \frac{I_{m_6}(|k|\rho_S)K_{m_6}(|k|r_S)}{k + a_R q_S} I^{(\bar{m})}\left(\left(m_6 - \frac{k}{a_S}\right)\theta_{bS}^{r}\right)\Lambda^{(\bar{n})}\left(\left(m_6 - \frac{k}{a_S}\right)\theta_{bS}^{\rho}\right) dk\right\}$$

Let $k + a_R q_S = u$, then

$$\bar{K}_{SS}^{(\bar{m},\bar{n})} = -\frac{N_S}{4\pi\rho_f U^2 a_S r_{RO}} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} \sum_{\substack{m_6=-\infty \\ m_6=q_S+\ell_6 N_S}}^{\infty}$$

$$\cdot \left\{g_6(0) - \frac{i}{\pi}\int_0^{\infty}[g_6(u) - g_6(-u)]\frac{du}{u}\right\}$$

where

$$g_6(u) = \left(a_S u - a_S a_R q_S + \frac{m_6}{r_S^2}\right)\left(a_S u - a_S a_R q_S + \frac{m_6}{\rho_S^2}\right) \tag{59}$$

$$\cdot\, I_{m_6}(|u - a_R q_S|\rho_S) K_{m_6}(|u - a_R q_S| r_S)$$

$$\cdot\, I^{(\bar{m})}\left(\left(m_6 + \frac{a_R}{a_S} q_S - \frac{u}{a_S}\right)\theta^r_{bS}\right)\Lambda^{(\bar{n})}\left(\left(m_6 + \frac{a_R}{a_S} q_S - \frac{u}{a_S}\right)\theta^\rho_{bS}\right)$$

Evaluation of the integrable singularity of K_{SS} at u=0 is shown in Appendix H.

7) Kernel Function K_{RD}

When the control point is on the duct and the loading point is on the rotor the nondimensionalized induced velocity normal to the duct, the first integral of Eq.(3), is shown in Reference 5 to be equivalent to

$$I_7 = \sum_{\lambda_7=0}^{\infty} \int_0^{\pi}\int_{\rho_R} L_R^{(\lambda_7)}(\rho_R, \theta_{RO}) e^{i\lambda_7 \Omega t} K_{RD} \frac{\sqrt{1+a_R^2\rho_R^2}}{a_R\rho_R} \sin\theta_\alpha d\theta_\alpha d\rho_R \tag{60}$$

where

$$K_{RD} = -\frac{1}{4\pi\rho_f U^2} \sum_{n=1}^{N_R} \lim_{\substack{\xi_R \to \theta_{RO}/a_R \\ r_D \to R_D}} \frac{\partial}{\partial r_D} \cdot \frac{\rho_R}{\sqrt{1+a_R^2\rho_R^2}} \left(a_R \frac{\partial}{\partial \xi_R} - \frac{1}{\rho_R^2}\frac{\partial}{\partial\theta_{RO}}\right)$$

$$\cdot \int_{-\infty}^{x_D-\xi_R} \frac{e^{i\lambda_7[a_R(\tau - x_D + \xi_R) - \bar{\theta}_{Rn}]}}{R_{RD}} d\tau$$

$$R_{RD} = \left\{\tau^2 + r_D^2 + \rho_R^2 - 2r_D\rho_R\cos\left[\theta_{RO} - \varphi_D - \Omega t + \bar{\theta}_{Rn} - a_R\left(\tau - x_D + \xi_R\right)\right]\right\}$$

On substituting

$$\frac{1}{R_{RD}} = \frac{1}{\pi}\sum_{m_7=-\infty}^{\infty} e^{im_7\beta} \int_{-\infty}^{\infty} I_{m_7}(|k|\rho_R) K_{m_7}(|k| r_D) e^{i\tau k} dk \tag{61}$$

where $\beta = \theta_{RO} - \varphi_D - \Omega t + \bar{\theta}_{Rn} - a_R(\tau - x_D + \xi_D)$, and $\rho_R < r_D$

the kernel becomes

$$K_{RD} = -\frac{1}{4\pi^2\rho_f U^2}\sum_{n=1}^{N_R}\lim_{\substack{\xi_R\to\theta_{RO}/a_R\\ r_D\to R_D}}\frac{\partial}{\partial r_D}\frac{\rho_R}{\sqrt{1+a_R^2\rho_R^2}}\left(a_R\frac{\partial}{\partial\xi_R}-\frac{1}{\rho_R^2}\frac{\partial}{\partial\theta_{RO}}\right)$$

$$\cdot\sum_{m_7=-\infty}^{\infty} e^{im_7(\theta_{RO}-\varphi_D-\Omega t)}\, e^{-i(\lambda_7-m_7)\bar{\theta}_{Rn}}\, e^{-i(\lambda_7-m_7)a_R(x_D-\xi_R)}$$

$$\cdot\int_{-\infty}^{x_D-\xi_R} e^{i(\lambda_7-m_\lambda)a\tau}\int_{-\infty}^{\infty} I_{m_7}(|k|)\rho_R)K_{m_7}(|k|r_D)e^{i\tau k}dk\,d\tau \qquad (62)$$

The n-summation yields

$$\sum_{n=1}^{N_R} e^{-i(\lambda_7-m_7)\bar{\theta}_{Rn}} = \begin{cases} N_R & \text{for } \lambda_7-m_7=\ell_7N_R,\quad \ell_7=0,\pm1,\pm2,\dots \\ 0 & \text{for all other values}\end{cases} \qquad (63)$$

From the time t relationship of Eq.(3), $\lambda_7-m_7 = \ell_7N_R = \lambda_8 = \lambda_9 = q_D$ where q_D is the order of the frequency in the second integral of Eq.(3). The integral I_7 can be written as a single infinite series

$$I_7 = \sum_{\lambda_7=0}^{\infty}\int_0^{\pi}\int_{\rho_R} L_R^{(\lambda_7)}(\rho_R,\theta_{RO})e^{i\ell_7N_R\Omega t}\,\bar{K}_{RD}^{(m_7)}\sin\theta_\alpha d\theta_\alpha d\rho_R \qquad (64)$$

evaluated at $m_7 = \lambda_7-\ell_7N_R$, and after the τ-integration and taking derivatives and limits, the modified kernel is

$$\bar{K}_{RD}^{(m_7)} = -\frac{N_Re^{-im_7\varphi_D}}{4\pi^2\rho_fU^2a_Rr_{RO}}\Bigg\{i\pi a_R|\ell_7N_R|\left[a_R^2(\ell_7N_R)-\frac{m_7}{\rho_R^2}\right]e^{-i\ell_7N_Ra_Rx_D}$$

$$e^{i\lambda_7\theta_{RO}}I_{m_7}\left(a_R|\ell_7N_R|\rho_R\right)K'_{m_7}\left(a_R|\ell_7N_R|R_D\right)$$

$$-\int_{-\infty}^{\infty}\left(a_Rk+\frac{m_7}{\rho_R^2}\right)\frac{|k|\,I_{m_7}\left(|k|\rho_R\right)K'_{m_7}\left(|k|R_D\right)}{k+a_R\ell_7N_R}e^{ikx_D}e^{i\left(m_7-\frac{k}{a_R}\right)\theta_{RO}}dk\Bigg\}$$

(65)

where

$$K'_{m_7}(z) = \frac{\partial K_{m_7}(z)}{\partial z} = -\frac{1}{2}\left[K_{m_7-1}(z) + K_{m_7+1}(z)\right]$$

and all linear dimensions within the braces and a_R are now fractions of rotor radius r_{RO}.

If the Birnbaum modes are assumed for the chordwise loading, i.e.,

$$L_R^{(\lambda_7)}(\rho_R,\theta_{RO}) = \frac{1}{\pi}\left\{L_R^{(\lambda_7,1)}(\rho_R)\cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\infty} L_R^{(\lambda_7,\bar{n})}(\rho_R)\sin(\bar{n}-1)\theta_\alpha\right\} \tag{66}$$

where $L_R^{(\lambda_7,\bar{n})}(\rho_R)$ are the spanwise loading coefficients, then Eq.(64) can be written as

$$I_7 = \sum_{\lambda_7=0}^{\infty} e^{i\ell_7 N_R \Omega t} \int_{\rho_R} \sum_{\bar{n}=1} L_R^{(\lambda_7,\bar{n})}(\rho_R)\bar{K}_{RD}^{(m_7,\bar{n})}\, d\rho_R \tag{67}$$

where $\bar{K}_{RD}^{(m_7,\bar{n})}$ is the modified kernel after the θ_α-integration.

Thus,

$$\bar{K}_{RD}^{(m_7,\bar{n})} = -\frac{N_R e^{-im_7\varphi_D}}{4\pi^2\rho_f U^2 a_R r_{RO}}\left\{i\pi a_R|\ell_7 N_R|\left[a_R^2(\ell_7 N_R) - \frac{m_7}{\rho_R^2}\right] e^{-i\ell_7 N_R a_R x_D}\right.$$

$$\cdot\, I_{m_7}(a_R|\ell_7 N_R|\rho_R)K'_{m_7}(a_R|\ell_7 N_R|R_D)e^{i\lambda_7\sigma_R}\,\Lambda^{(\bar{n})}(\lambda_7\theta_{bR})$$

$$\left. - e^{im_7\sigma_R}\int_{-\infty}^{\infty}\left(a_R k + \frac{m_7}{\rho_R^2}\right)\frac{|k|\, I_{m_7}\left(|k|\rho_R\right)K'_{m_7}\left(|k|R_D\right)}{k + a_R\ell_7 N_R} e^{ik(x_D-\sigma_R/a_R)}\Lambda^{(\bar{n})}((m_7-k/a_R)\theta_{bR})\, dk\right\} \tag{68}$$

Letting $u = k + a_R\ell_7 N_R$, it can be shown that

$$\bar{K}_{RD}^{(m_7,\bar{n})} = - \frac{N_R\, e^{-im_7\varphi_D}}{4\pi^2 \rho_f U^2 a_R r_{RO}}\; e^{i\lambda_7\sigma_R}\; e^{-i\ell_7 N_R a_R x_D}$$

$$\cdot \left\{ i\pi g_7(0) + \int_o^\infty \left[g_7(u) - g_7(-u) \right] \frac{du}{u} \right\} \tag{68a}$$

where
$$g_7(u) = \left[-a_R u + a_R^2 \ell_7 N_R - \frac{m_7}{\rho_R^2} \right] \cdot \left| u - a_R \ell_7 N_R \right|$$

$$\cdot\, I_{m_7}(\left| u - a_R\ell_7 N_R \right| \rho_R) K'_{m_7}(\left| u - a_R \ell_7 N_R \right| R_D)$$

$$\cdot\, e^{iu(x_D - \sigma_R/a_R)}\, \Lambda^{(\bar{n})}\left(\left(\lambda_7 - \frac{u}{a_R}\right)\theta_{bR}\right)$$

Since $\rho_R < R_D$, there is no singularity in the original kernel Eq.(60). The expansion of the inverse Descartes distance introduces an integrable Cauchy-type singularity.

Considering the L-H side of Eq.(3) in steady or unsteady case, certain relations will exist between λ_7, λ_8, and λ_9 and m_7, m_8, and m_9. These will be discussed later.

8) Kernel Function K_{DD}

When both control and loading points are on the duct, the nondimensionalized velocity normal to the duct at the control point is[5]

$$I_8 = \sum_{\lambda_8=0}^{\infty} \iint_{S_D} L_D^{(\lambda_8)}(\xi_D,\rho_D,\theta_D) e^{i\lambda_8\Omega t} K_{DD}(x_D,r_D,\varphi_D;\xi_D,\rho_D,\theta_D;\lambda_8)\, dS_D$$

$$= \sum_{\lambda_8=0}^{\infty} \int_o^{2\pi} \int_{2C_D} L_D^{(\lambda_8)} e^{i\lambda_8\Omega t} K_{DD}\, d\theta_D\, d\xi_D \tag{69}$$

where $L_D^{(\lambda_8)}$ = duct loading in lb/ft (i.e., $L_D^{\lambda_8}(\xi_D)$)

and

$$K_{DD} = -\frac{1}{4\pi\rho_f U^2} \lim_{\substack{r_D \\ \rho_D}\}\to R_D} \frac{\partial}{\partial r_D}\frac{\partial}{\partial \rho_D} \int_{-\infty}^{x_D-\xi_D} \frac{e^{i\lambda_8 a_R(\tau - x_D + \xi_D)}}{R_{DD}} d\tau$$

$$R_{DD} = \left\{\tau^2 + r_D^2 + \rho_D^2 - 2r_D\rho_D\cos(\theta_D - \varphi_D)\right\}^{\frac{1}{2}}$$

The loading can be expressed as before in a Fourier series

$$L_D^{(\lambda_8)}(\xi_D, \rho_D, \theta_D) = \sum_{\mu=-\infty}^{\infty} L_D^{(\lambda_8,\mu)}(\xi_D) e^{-i\mu\theta_D} \tag{70}$$

at $\rho_D = R_D$, and

$$\frac{1}{R_{DD}} = \frac{1}{\pi} \sum_{m_8=-\infty}^{\infty} e^{im_8(\theta_D - \varphi_D)} \int_{-\infty}^{\infty} I_{m_8}(|k|\rho_D) K_{m_8}(|k| r_D) e^{i\tau k} dk \tag{71}$$

Then the θ_D-integration involves

$$\int_0^{2\pi} e^{i(m_8-\mu)\theta_D} d\theta_D = \begin{cases} 2\pi & \text{for } m_8 = \mu \\ 0 & \text{for all other values} \end{cases} \tag{72}$$

Since $\lambda_8 = \ell_7 N_R \geq 0$, Eq.(69) becomes

$$I_8 = \sum_{\ell_7=0}^{\infty} e^{i\ell_7 N_R \Omega t} \int_{2C_D} \sum_{m_8=-\infty}^{\infty} L_D^{(\ell_7 N_R, m_8)}(\xi_D) K_{DD}^{(m_8)} d\xi_D \tag{73}$$

where

$$K_{DD}^{(m_8)} = -\frac{2}{4\pi\rho_f U^2} \lim_{\substack{r_D \\ \rho_D}\}\to R_D} \frac{\partial}{\partial r_D}\frac{\partial}{\partial \rho_D} e^{-im_8\varphi_D} e^{-i\ell_7 N_R a_R(x_D - \xi_D)}$$

$$\cdot \int_{-\infty}^{x_D-\xi_D} e^{i\ell_7 N_R a_R \tau} \int_{-\infty}^{\infty} I_{m_8}(|k|\rho_D) K_{m_8}(|k| r_D) e^{i\tau k} dk d\tau$$

After the τ-integration and the successive derivatives with respect to ρ_D and r_D, and nondimensionalizing the linear dimensions with respect to r_{RO}, the kernel becomes

$$K_{DD}^{(m_8)} = -\frac{e^{-im_8\varphi_D}}{4\pi\rho_f U^2 r_{RO}} \Big\{ -\frac{\pi}{2} a_R^2 \ell_7^2 N_R^2 \, e^{-ia_R\ell_7 N_R(x_D-\xi_D)} \cdot \Big[I_{m_8-1}(a_R\ell_7N_RR_D) + I_{m_8+1}(a_R\ell_7N_RR_D) \Big] \cdot \Big[K_{m_8-1}(a_R\ell_7N_RR_D) + K_{m_8+1}(a_R\ell_7N_RR_D) \Big]$$
$$+\frac{i}{2}\int_{-\infty}^{\infty} \frac{k^2\Big[I_{m_8-1}(|k|R_D)+I_{m_8+1}(|k|R_D)\Big]\Big[K_{m_8-1}(|k|R_D)+K_{m_8+1}(|k|R_D)\Big]e^{ik(x_D-\xi_D)}\,dk}{k+a_R\ell_7N_R} \tag{74}$$

Examination of the original integral reveals that it is singular since R_{DD} can go to zero when $x_D = \xi_D$ and $\rho_D = r_D = R_D$. The singularity is of the Hadamard-type (see Reference 2) whose principal value can be obtained.[5] Furthermore, the expansion scheme for the reciprocal of R_{DD} has introduced a Cauchy-type singularity in the k-integration.

The peripheral integration over φ_D and the duct chordwise integrations over θ_α and φ_α (using the mode shape expansion of the loading L_D and applying the generalized lift operator) will be done later after the last integral of Eq.(3) is derived.

9) Kernel Function K_{SD}

When the control point is on the duct and the loading point is on the stator, the nondimensional induced velocity normal to the duct is (cf. Ref.5 for K_{RD})

$$I_9 = \sum_{\lambda_9=0}^{\infty} \int_0^{\pi}\int_{\rho_S} L_S^{(\lambda_9)}(\rho_S,\theta_{SO})e^{i\lambda_9\Omega t} K'_{SD} \frac{\sqrt{1+a_S^2\rho_S^2}}{a_S\rho_S} \sin\theta_\alpha d\theta_\alpha d\rho_S \tag{75}$$

where

$$K'_{SD} = -\frac{1}{4\pi\rho_f U^2} \sum_{n=1}^{N_S} \lim_{\substack{\xi_S \to \theta_{0S}/a_S + \epsilon_S \\ r_D \to R_D}} \frac{\partial}{\partial r_D} \frac{\partial}{\partial n_S} \int_{-\infty}^{x_D - \xi_S} \frac{e^{i\lambda_g[a_R(\tau - x_D + \xi_S) - \bar{\theta}_{Sn}]}}{R_{SD}} d\tau$$

$$R_{SD} = \left\{\tau^2 + r_D^2 + \rho_S^2 - 2r_D\rho_S \cos[-\theta_{S0} - \varphi_D - \bar{\theta}_{Sn}]\right\}^{\frac{1}{2}}$$

on substituting

$$\frac{1}{R_{SD}} = \frac{1}{\pi} \sum_{m_g=-\infty}^{\infty} e^{im_g\beta_{SD}} \int_{-\infty}^{\infty} I_{m_g}(|k|\rho_S) K_{m_g}(|k|r_D) e^{i\tau k} dk \tag{76}$$

where $\beta_{SD} = -\theta_{S0} - \varphi_D - \bar{\theta}_{Sn}$, it is seen that

$$\sum_{n=1}^{N_S} e^{-i(\lambda_g + m_g)\bar{\theta}_{Sn}} = \begin{cases} N_S & \text{for } (\lambda_g + m_g) = \ell_g N_S, \ \ell_g = 0, \pm 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{77}$$

Also from the time relationship of Eq.(3),

$$\lambda_g = q_D \quad . \tag{78}$$

After the τ-integration, the derivatives and limits are taken. Then if the Birnbaum modes are assumed for the chordwise loading on the stator, I_g can be written for each q_D as

$$I_g = e^{iq_D\Omega t} \int_{\rho_S} \sum_{\bar{n}=1} L_S^{(q_D, \bar{n})}(\rho_S) \bar{K}_{SD}^{(\bar{n})} d\rho_S \tag{79}$$

where $\bar{K}_{SD}^{(\bar{n})}$ is the modified kernel after the θ_α-integration

$$\bar{K}_{SD}^{(\bar{n})} = -\frac{N_S}{4\pi\rho_f U^2 a_S r_{RO}} \sum_{\substack{m_9=-\infty \\ m_9=\ell_9 N_S - q_D}}^{\infty} e^{-im_9\varphi_D} e^{-im_9\sigma_S}$$

$$\left\{ i a_R q_D \left(a_S a_R q_D + \frac{m_9}{\rho_S^2} \right) e^{-i a_R q_D \left(x_D - \epsilon_S - \frac{\sigma_S}{a_S} \right)} \Lambda^{(\bar{n})} \left(\left(\frac{a_R}{a_S} q_D - m_9 \right) \theta_{bS} \right) \right.$$

$$\cdot I_{m_9}(a_R q_D \rho_S) K'_{m_9}(a_R q_D R_D)$$

$$\left. - \frac{1}{\pi} \int_{-\infty}^{\infty} |k| \left(a_S k - \frac{m_9}{\rho_S^2} \right) \frac{e^{ik\left(x_D - \epsilon - \frac{\sigma_S}{a_S}\right)} \Lambda^{(\bar{n})} \left(\left(-\frac{k}{a_S} - m_9 \right) \theta_{bS} \right) I_{m_9}(|k| \rho_S) K'_{m_9}(|k| R_D) \, dk}{k + a_R q_D} \right\} \tag{80}$$

The expansion of the inverse Descartes distance introduces an integrable Cauchy-type singularity. There is no other singularity since $\rho_S < R_D$.

Let $u = k + a_R q_D$

$$\bar{K}_{SD}^{(m,\bar{n})} = -\frac{N_S}{4\pi\rho_f U^2 a_S r_{RO}} \sum_{\substack{m_9=-\infty \\ m_9=\ell_9 N_S - q_D}}^{\infty} e^{-im_9\varphi_D} e^{-im_9\sigma_S} e^{-i a_R q_D \left(x_D - \epsilon_S - \frac{\sigma_S}{a_S} \right)}$$

$$\cdot \left\{ -i \, g_9(0) - \frac{1}{\pi} \int_0^{\infty} \frac{g_9(u) - g_9(-u)}{u} \, du \right\} \tag{81}$$

where

$$g_9(u) = |u - a_R q_D| \left[a_S u - a_S a_R q_D - \frac{m_9}{\rho_S^2} \right] e^{iu\left(x_D - \epsilon_S - \frac{\sigma_S}{a_S}\right)}$$

$$\cdot \Lambda^{(\bar{n})} \left(\left(-m_9 + \frac{a_R}{a_S} q_D - \frac{u}{a_S} \right) \theta_{bS} \right) I_{m_9}\left(|u - a_R q_D| \rho_S \right) K'_{m_9}\left(|u - a_R q_D| R_D \right)$$

and

$$K'_m(z) = \frac{\partial K_m(z)}{\partial z} = -\frac{1}{2} \left[K_{m-1}(z) + K_{m+1}(z) \right]$$

SOLUTION OF THE SIMULTANEOUS INTEGRAL EQUATIONS

1) Auxiliary Analysis of the Third Equation of the System

Relating the three integrals I_7, I_8, and I_9, for each value of $\ell_7=\ell$

$$I_7 = \sum_{\lambda_7=-\infty}^{\infty} E_{\lambda_7}\, e^{i\ell N_R\Omega t} \int_{\rho_R} \sum_{\bar{n}=1} L_R^{(\lambda_7,\bar{n})}\, e^{-i(\lambda_7-\ell N_R)\varphi_D}\, \bar{K}'^{(\lambda_7-\ell N_R,\bar{n})}_{RD}\, d\rho_R$$

where

$$E_{\lambda_7} = \begin{cases} 0, & \lambda_7 < 0 \\ 1, & \lambda_7 \geq 0 \end{cases}$$

$$I_8 = \sum_{m_8=-\infty}^{\infty} e^{i\ell N_R\Omega t} \int_{2C_D} L_D^{(\ell N_R,m_8)}\, e^{-im_8\varphi_D}\, K'^{(m_8)}_{DD}\, d\xi_D$$

$$I_9 = \sum_{m_9=-\infty}^{\infty} e^{i\ell N_R\Omega t} \int_{\rho_S} \sum_{\bar{n}=1} L_S^{(\ell N_R,\bar{n})}\, e^{-im_9\varphi_D}\, \bar{K}'^{(m_9=\ell_9 N_S-\ell N_R,\bar{n})}_{SD}\, d\rho_S$$

The φ_D-exponential factors have been detached from the kernels and the remainders are designated by primes. From the known onset velocities (see W_D in an earlier section), Eq.(3) can now be written as

$$\begin{aligned} I_7+I_8+I_9 = \sum_{m=-\infty}^{\infty} \Big\{ & E_m \sum_{\bar{n}=1} \int_{\rho_R} L_R^{(m,\bar{n})}\, e^{-i(m-\ell N_R)\varphi_D}\, \bar{K}'^{(m-\ell N_R,\bar{n})}_{RD}\, d\rho_R \\ & + \int_{2C_D} L_D^{(\ell N_R,m)}\, e^{-im\varphi_D}\, K'^{(m)}_{DD}\, d\xi_D \\ & + \sum_{\bar{n}=1} \int_{\rho_S} L_S^{(\ell N_R,\bar{n})}\, e^{-im\varphi_D}\, \bar{K}'^{(m=\ell_9 N_S-\ell N_R,\bar{n})}_{SD}\, d\rho_S \Big\}\, e^{i\ell N_R\Omega t} \end{aligned}$$

$$= \begin{cases} \bar{W}_{R_tD}^{(\ell N_R,\bar{m})}\, e^{i\ell N_R\varphi_D}\, e^{i\ell N_R\Omega t} & \text{for } \ell\neq 0 \quad (82a) \\ \bar{W}_D^{(0,\bar{m})} = \alpha I^{(\bar{m})}(0) + \bar{W}_{DC}^{(0,\bar{m})} + \bar{W}_{S_tD}^{(0,\bar{m})} + W_{R_tD}^{(0,\bar{m})} & \text{for } \ell=0 \quad (82b) \end{cases}$$

(A) $\ell \neq 0$

Applying the φ_D integral operator to both sides of Equation (82a)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\nu\varphi_D}(I_7+I_8+I_9)\,d\varphi_D = \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{W}_{R_tD}^{(\ell N_R,\bar{m})}\; e^{i(\nu+\ell N_R)\varphi_D}\; e^{i\ell N_R\Omega t}\,d\varphi_D$$

Now

$$\int_{-\pi}^{\pi} e^{(\nu-m+\ell N_R)\varphi_D}\,d\varphi_D = \begin{cases} 2\pi & \text{for } \nu = m-\ell N_R \\ 0 & \text{for } \nu \neq m-\ell N_R \end{cases}$$

$$\int_{-\pi}^{\pi} e^{i(\nu-m)\varphi_D}\,d\varphi_D = \begin{cases} 2\pi & \text{for } \nu = m \\ 0 & \text{for } \nu \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} e^{i(\nu+\ell N_R)\varphi_D}\,d\varphi_D = \begin{cases} 2\pi & \text{for } \nu = -\ell N_R \\ 0 & \text{for } \nu \neq -\ell N_R \end{cases}$$

Then for a non-trivial solution for all ν

$$E_{\nu+\ell N_R}\sum_{\bar{n}=1}\int_{\rho_R} L_R^{(\nu+\ell N_R,\bar{n})}\,\bar{K}_{RD}'^{(\nu,\bar{n})}\,d\rho_R + \int_{2C_D} L_D^{(\ell N_R,\nu)}\,K_{DD}'^{(\nu)}\,d\xi_D$$

$$+\sum_{\bar{n}=1}\int_{\rho_S} L_S^{(\ell N_R,\bar{n})}\,\bar{K}_{SD}'^{(\nu=\ell_9 N_S-\ell N_R,\bar{n})}\,d\rho_S = \begin{cases} \bar{W}_{R_tD}^{(\ell N_R,\bar{m})} & \text{for } \nu = -\ell N_R \quad (83a) \\ 0 & \text{for } \nu \neq -\ell N_R \quad (83b) \end{cases}$$

Comparing this third surface integral equation with the first integral equation, it is seen that $L_R^{(\nu+\ell N_R,\bar{n})}$ is limited to the values $L_R^{(q_R,\bar{n})}$ and $L_D^{(\ell N_R,\nu)}$ is limited to $L_D^{(\lambda_2,m_2)}$ where $m_2 = q_R - \lambda_2$. Therefore

$$\nu = m_2 = q_R - \ell N_R \quad .$$

From this

$$\ell_9 N_S = q_R \geq 0 \quad .$$

Thus (83a) applies when $q_R=0$ and (83b) when $q_R\neq 0$, i.e., $q_R = \ell' N_S > 0$

$$\int_{2C_D} L_D^{(\ell N_R,\nu)}\,K_{DD}'^{(\nu)}\,d\xi_D = -\sum_{\bar{n}=1}\int_{\rho_R} L_R^{(q_R,\bar{n})}\,\bar{K}_{RD}'^{(\nu,\bar{n})}\,d\rho_R$$

$$-\sum_{\bar{n}=1}\int_{\rho_S} L_S^{(\ell N_R,\bar{n})}\,\bar{K}_{SD}'^{(\nu,\bar{n})}\,d\rho_S \qquad (84)$$

For the solution of Eq.(84) each term of the infinite ν-series is taken separately and $\nu = q_R - \ell N_R$.

By analogy with I_2 and I_5, the chordwise integration over ξ_D is written as

$$\int_{2C_D} L_D^{(\ell N_R,\nu)} K'^{(\nu)}_{DD}\, d\xi_D = \sum_{\bar{n}=1} \bar{A}^{(\ell N_R,\nu,\bar{n})} \bar{K}_{DD}^{(\nu,\bar{n})}$$

where

$$\bar{K}_{DD}^{(\nu,\bar{n})} = \frac{1}{\pi}\int_0^{\pi} \Theta(\bar{n}) K'^{(\nu)}_{DD} \sin\theta_\alpha d\theta_\alpha \quad .$$

Now the generalized lift operators are applied to the third surface integral equation.

$$\frac{1}{\pi}\int_0^{\pi} \Phi(\bar{m}) \int_{2C_D} L_D^{(\ell N_R,\nu)} K'^{(\nu)}_{DD}\, d\xi_D d\varphi_\alpha = \sum_{\bar{m}=1} \sum_{\bar{n}=1} \bar{A}^{(\ell N_R,\nu,\bar{n})} \bar{K}_{DD}^{(\nu,\bar{m},\bar{n})} \tag{85}$$

Then with the relations

$$\xi_D = \epsilon_D - C_D \cos\theta_\alpha$$

$$x_D = \epsilon_D - C_D \cos\varphi_\alpha$$

the kernel becomes

$$\bar{K}_{DD}^{(\nu,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U^2 r_{RO}} \left\{ \frac{\pi}{2} a_R^2 \ell^2 N_R^2 \left[I_{\nu-1}(a_R \ell N_R R_D) + I_{\nu+1}(a_R \ell N_R R_D)\right] \right.$$

$$\cdot \left[K_{\nu-1}(a_R \ell N_R R_D) + K_{\nu+1}(a_R \ell N_R R_D)\right] I^{(\bar{m})}(a_R \ell N_R C_D) \Lambda^{(\bar{n})}(a_R \ell N_R C_D)$$

$$\left. - \frac{i}{2}\int_{-\infty}^{\infty} \frac{k^2\left[I_{\nu-1}(|k|R_D) + I_{\nu+1}(|k|R_D)\right]\left[K_{\nu-1}(|k|R_D) + K_{\nu+1}(|k|R_D)\right] I^{(\bar{m})}(-kC_D)\Lambda^{(\bar{n})}(-kC_D)\, dk}{k + a_R \ell N_R} \right. \tag{86}$$

and letting $u = k + a_R \ell N_R$

$$\bar{K}_{DD}^{(\nu,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U^2 r_{RO}} \left\{ \frac{\pi}{2} g_8(0) - \frac{i}{2}\int_0^{\infty} [g_8(u) - g_8(-u)] \frac{du}{u} \right\}$$

where

$$g_8(u) = (u-a_R \ell N_R)^2 \Big[I_{\nu-1}(|u-a_R\ell N_R|R_D) + I_{\nu+1}(|u-a_R\ell N_R|R_D) \Big]$$

$$\cdot \Big[K_{\nu-1}(|u-a_r\ell N_R|R_D) + K_{\nu+1}(|u-a_R\ell N_R|R_D) \, I^{(\bar{m})}((-u+a_R\ell N_R)C_D)$$

$$\cdot \Lambda^{(\bar{n})}((-u + a_R\ell N_R)C_D) \tag{86a}$$

For the evaluation of the finite part of the integrable singularity of Eq.(86), see Appendix J.

On applying the lift operators to the first integral on the R-H side of Eq.(84)

$$\frac{1}{\pi}\int_0^\pi \Phi(\bar{m})\Big[- \sum_{\bar{n}=1} \int_{\rho_R} L_R^{(q_R,\bar{n})} \; \bar{K}'^{(\nu,\bar{n})}_{RD} \, d\rho_R\Big] d\varphi_\alpha$$

$$= - \sum_{\bar{m}=1} \sum_{\bar{n}=1} \int_{\rho_R} L_R^{(q_R,\bar{n})} \; \bar{K}_{RD}^{(\nu,\bar{m},\bar{n})} \, d\rho_R$$

where

$$\bar{K}_{RD}^{(\nu,\bar{m},\bar{n})} = + \frac{N_R e^{i\nu\sigma_R}}{4\pi\rho_f U^2 r_{RO}} \Big\{ -i|\ell N_R|\Big[a_R^2 \, \ell N_R - \frac{\nu}{\rho_R^2}\Big] e^{-ia_R\ell N_R(\epsilon_D - \frac{\sigma_R}{a_R})}$$

$$\cdot I_\nu(a_R\ell N_R\rho_R)K'_\nu(a_R\ell N_R R_D) \; \Lambda^{(\bar{n})}((\nu+\ell N_R)\theta_{bR}) \, I^{(\bar{m})}(a_R\ell N_R C_D)$$

$$+ \frac{1}{a_R\pi}\int_{-\infty}^{\infty}\Big(a_R k + \frac{\nu}{\rho_R^2}\Big) \, |k| \, I_\nu(|k|\rho_R) K'_\nu(|k|R_D) \, e^{ik(\epsilon_D - \sigma_R/a_R)}$$

$$\cdot \frac{\Lambda^{(\bar{n})}\Big(\Big(\nu - \frac{k}{a_R}\Big)\theta_{bR}\Big) I^{(\bar{m})}\Big((-kC_D)\Big)}{k + a_R\ell N_R} \, dk \Big\} \tag{87}$$

or

$$\bar{K}_{RD}^{(\nu,\bar{m},\bar{n})} = -\frac{N_R e^{i\nu\sigma_R}}{4\pi^2 \rho_f U^2 a_R r_{RO}} e^{-ia_R \ell N_R (\epsilon_D - \sigma_R/a_R)}$$

$$\cdot \left\{ i\pi g_7(0) + \int_0^\infty \left[g_7(u) - g_7(-u) \right] \frac{du}{u} \right\}$$

and

$$g_7(u) = \left[-a_R u + a_R^2 \ell N_R - \frac{\nu}{\rho_R^2} \right] \cdot \left| u - a_R \ell N_R \right|$$

$$\cdot I_\nu(|u - a_R \ell N_R| \rho_R)\, K_\nu'(|u - a_R \ell N_R| R_D)$$

$$\cdot e^{iu(\epsilon_D - \sigma_R/a_R)} \Lambda^{(\bar{n})}\left(\left(\nu + \ell N_R - \frac{u}{a_R}\right)\theta_{bR}\right) I^{(\bar{m})}\left(\left(-u + a_R \ell N_R\right) c_D\right)$$

(87a)

The integral term of Eq.(87) has an integrable singularity, the finite part of which is evaluated in Appendix I.

For the second integral on the R-H side of Eq.(84)

$$\frac{1}{\pi}\int_0^\pi \Phi(\bar{m})\left[-\sum_{\bar{n}=1} \int_{\rho_S} L_S^{(\ell N_R,\bar{n})} \bar{K}_{SD}^{\prime(\nu,\bar{n})} d\rho_S\right] d\varphi_\alpha$$

$$= -\sum_{\bar{m}=1} \sum_{\bar{n}=1} \int_{\rho_S} L_S^{(\ell N_R,\bar{n})} \bar{K}_{SD}^{(\nu,\bar{m},\bar{n})} d\rho_S$$

where

$$\bar{K}_{SD}^{(\nu,\bar{m},\bar{n})} = + \frac{N_S e^{-i\nu\sigma_S}}{4\pi\rho_f U^2 r_{RO}} \left\{ -i \frac{a_R}{a_S} |\ell N_R| \left(a_S a_R \ell N_R + \frac{\nu}{\rho_S^2}\right) e^{-i a_R \ell N_R (\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S})} \right.$$

$$\cdot I_\nu(a_R \ell N_R \rho_S) K_\nu'(a_R \ell N_R R_D) \; \Lambda^{(\bar{n})}\left(\left(-\nu + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}(a_R \ell N_R C_D)$$

$$+ \frac{1}{a_S \pi} \int_{-\infty}^{\infty} \left(a_S k - \frac{\nu}{\rho_S^2}\right)|k| e^{ik(\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S})} I_\nu(|k|\rho_S) K_\nu'(|k| R_D)$$

$$\left. \cdot \frac{\Lambda^{(\bar{n})}\left(\left(-\nu - \frac{k}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}\left((-kC_D)\right)}{k + a_R \ell N_R} dk \right\} \tag{88}$$

or

$$\bar{K}_{SD}^{(\nu,\bar{m},\bar{n})} = - \frac{N_S e^{-i\nu\sigma_S}}{4\pi\rho_f U^2 a_S r_{RO}} e^{-i a_R \ell N_R (\epsilon_D - \epsilon_S - \sigma_S/a_S)}$$

$$\cdot \left\{ -i\, g_9(0) - \frac{1}{\pi} \int_0^\infty \frac{g_9(u) - g_9(-u)}{u} du \right\}$$

and

$$g_9(u) = |u - a_R \ell N_R| \left[a_S u - a_S a_R \ell N_R - \frac{\nu}{\rho_S^2}\right] e^{iu(\epsilon_D - \epsilon_S - \sigma_S/a_S)}$$

$$\cdot I_\nu(|u - a_R \ell N_R| \rho_S) K_\nu'(|u - a_R \ell N_R| R_D)$$

$$\cdot \Lambda^{(\bar{n})}\left(\left(-\nu + \frac{a_R \ell N_R}{a_S} - \frac{u}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}\left(\left(-u + a_R \ell N_R\right) C_D\right) \tag{88a}$$

Equation (84), the third surface integral equation for $q_R \neq 0$, becomes

$$\sum_{\bar{m}=1} \sum_{\bar{n}=1} \bar{A}^{(\ell N_R,\nu,\bar{n})} \bar{K}_{DD}^{(\nu,\bar{m},\bar{n})} = - \sum_{\bar{m}=1} \sum_{\bar{n}=1} \int_{\rho_R} L_R^{(q_R,\bar{n})} \bar{K}_{RD}^{(\nu,\bar{m},\bar{n})} d\rho_R$$

$$- \sum_{\bar{m}=1} \sum_{\bar{n}=1} \int_{\rho_S} L_S^{(\ell N_R,\bar{n})} \bar{K}_{SD}^{(\nu,\bar{m},\bar{n})} \tag{89}$$

where $\bar{K}_{DD}^{(\nu,\bar{m},\bar{n})}$ is given by Eq.(86); $\bar{K}_{RD}^{(\nu,\bar{m},\bar{n})}$ is given by Eq.(87), and $\bar{K}_{SD}^{(\nu,\bar{m},\bar{n})}$ is given by Eq.(88), for $\nu = q_R - \ell N_R$.

When $q_R=0$, $\ell\neq 0$, Equation (83a) is

$$\sum_{\bar{n}=1} \int_{\rho_R} \cdot L_R^{(0,\bar{n})} \bar{K}_{RD}^{\prime(-\ell N_R,\bar{n})} d\rho_R + \int_{2C_D} L_D^{(\ell N_R,-\ell N_R)} K_{DD}^{\prime(-\ell N_R)} d\xi_D$$

$$-\sum_{\bar{n}=1} \int_{\rho_S} L_S^{(\ell N_R,\bar{n})} \bar{K}_{SD}^{\prime(-\ell N_R,\bar{n})} d\rho_S = \bar{W}_{R_t D}^{(\ell N_R,\bar{m})}$$

which becomes (see References 5 and 6)

$$\bar{A}^{(\ell N_R,-\ell N_R,\bar{n})} \bar{K}_{DD}^{(-\ell N_R,\bar{m},\bar{n})} = W_{R_t D}^{(\ell N_R,\bar{m})}$$

$$-\int_{\rho_R} \left\{ L_R^{(0,\bar{n})} \bar{K}_{RD}^{(-\ell N_R,\bar{m},\bar{n})} + \text{conjugate} \left[L_R^{(0,\bar{n})} \bar{K}_{RD}^{(\ell N_R,\bar{m},\bar{n})} \right] \right\} d\rho_R$$

$$-\int_{\rho_S} L_S^{(\ell N_R,\bar{n})} \bar{K}_{SD}^{(-\ell N_R,\bar{m},\bar{n})} d\rho_S \qquad (90)$$

(B) $\ell = 0$

It can be shown that for $q_R=0$ and $\ell=0$, and for each $\bar{m}$ and $\bar{n}$

$$\bar{A}^{(0,0,\bar{n})} \bar{K}_{DD}^{(0,\bar{m},\bar{n})} = \bar{W}_D^{(0,\bar{m})} - \int_{\rho_S} L_S^{(0,\bar{n})} \bar{K}_{SD}^{(0,\bar{m},\bar{n})} d\rho_S$$

$$- \int_{\rho_R} L_R^{(0,\bar{n})} \bar{K}_{RD}^{(0,\bar{m},\bar{n})} d\rho_R \qquad (91)$$

2) <u>Formal Solution of the System of Integral Equations</u>

The thr.. integral equations are solved by an iterative procedure. At $q_R = \ell' N_S$, $\ell'=0,+1,+2, \ldots$, for given order $\bar{m}$ of lift operator mode and $\bar{n}$ of chordwise loading modes, Equation (1) will be

$$\bar{W}_R^{(q_R,\bar{m})}(r_R) = \int_{\rho_R} L_R^{(q_R,\bar{n})}(\rho_R)\left[\bar{K}_{RR}^{(\bar{m},\bar{n})}(\text{Eq.}21)\right]d\rho_R$$

$$+\sum_{\ell=0}^{\infty} \bar{A}^{(\ell N_R,\nu,\bar{n})}\left[\bar{K}_{DR}^{(\nu,\bar{m},\bar{n})}(\text{Eq.}31)\right]$$

$$+\sum_{\ell=0}^{\infty} \int_{\rho_S} L_S^{(\ell N_R,\bar{n})}(\rho_S)\left[\bar{K}_{SR}^{(\nu,\bar{m},\bar{n})}(\text{Eq.}37)\right]d\rho_S \qquad (92)$$

$\nu = q_R - \ell N_R$, $\ell=0,+1,+2,\ldots$

$q_R = \ell' N_S$, $\ell' = 0,+1,+2$

Equation (2) will be at $q_S=\ell N_R$, $\ell=0,+1,+2,\ldots$, for given q_R

$$\bar{W}_S^{(\ell N_R,\bar{m})}(r_S) = \sum_{q_R} \int_{\rho_R} L_R^{(q_R,\bar{n})}(\rho_R)\left[\bar{K}_{RS}^{(\nu,\bar{m},\bar{n})}(\text{Eq.}43,\ m_4=\nu)\right]d\rho_R$$

$$+\bar{A}^{(\ell N_R,\nu,\bar{n})}\left[\bar{K}_{DS}^{(\nu,\bar{m},\bar{n})}(\text{Eq.}52,\ m_5=\nu)\right]$$

$$+\int_{\rho_S} L_S^{(\ell N_R,\bar{n})}(\rho_S)\left[\bar{K}_{SS}^{(\bar{m},\bar{n})}(\text{Eq.}59)\right]d\rho_S \qquad (93)$$

Equation (3) will be when $q_R \neq 0$, whatever ℓ

$$\bar{A}^{(\ell N_R,\nu,\bar{n})}\left[\bar{K}_{DD}^{(\nu,\bar{m},\bar{n})}(\text{Eq.}86)\right] = -\int_{\rho_R} L_R^{(q_R,\bar{n})}\left[\bar{K}_{RD}^{(\nu,\bar{m},\bar{n})}(\text{Eq.}87)\right]d\rho_R$$

$$-\int_{\rho_S} L_S^{(\ell N_R,\bar{n})}\left[\bar{K}_{SD}^{(\nu,\bar{m},\bar{n})}(\text{Eq.}88)\right]d\rho_S$$

(94a)

When $q_R=0$ and $\ell=0$

$$\bar{A}^{(0,0,\bar{n})}\bar{K}_{DD}^{(0,\bar{m},\bar{n})} = \bar{W}_D^{(0,\bar{m})} - \int_{\rho_R} L_R^{(0,\bar{n})}\bar{K}_{RD}^{(0,\bar{m},\bar{n})} d\rho_R$$

$$- \int_{\rho_S} L_S^{(0,\bar{n})}\bar{K}_{SD}^{(0,\bar{m},\bar{n})} d\rho_S \qquad (94b)$$

When $q_R=0$ and $\ell\neq 0$

$$\bar{A}^{(\ell N_R,-\ell N_R,\bar{n})}\bar{K}_{DD}^{(-\ell N_R,\bar{m},\bar{n})} = \bar{W}_{R_tD}^{(\ell N_R,\bar{m})} - \int_{\rho_S} L_S^{(\ell N_R,\bar{n})}\bar{K}_{SD}^{(-\ell N_R,\bar{m},\bar{n})} d\rho_S$$

$$-\int_{\rho_R}\left\{L_R^{(0,\bar{n})}\bar{K}_{RD}^{(-\ell N_R,\bar{m},\bar{n})} + \text{conj } L_R^{(0,\bar{n})}\bar{K}_{RD}^{(\ell N_R,\bar{m},\bar{n})}\right\} d\rho_R$$

(94c)

3) Iteration Procedure

As a first step, it is assumed that rotor and duct have no effect on the stator loading. Note that $L_{S0}^{(\ell N_R,\bar{n})}$ is obtained for $\nu = q_R - \ell N_R$, $\ell=0,1$, $m_3 = m_4 = m_5 = \nu$.

First iteration

a$_1$) $L_{S0}^{(0,\bar{n})}(\rho_S) = \left[\bar{K}_{SS}^{(\bar{m},\bar{n})}\text{(Eq.59 for } \ell=0)\right]^{-1}\cdot\left[\bar{W}_S^{(0,\bar{m})}(r_S)\text{(Eq.12)}\right]$ for all ρ_S

b$_1$) $L_{S0}^{(N_R,\bar{n})}(\rho_S) = \left[\bar{K}_{SS}^{(\bar{m},\bar{n})}\text{(Eq.59 for } \ell=1)\right]^{-1}\cdot\left[\bar{W}_S^{(N_R,\bar{m})}(r_S)\text{(Eq.15)}\right]$ for all ρ_S

c$_1$) Then assuming that the duct has no effect on the rotor loading, $L_{R0}^{(q_R,\bar{n})}$ is obtained for all q_R's:

$$L_{R0}^{(q_R,\bar{n})}(\rho_R) = \left[K_{RR}^{(\bar{m},\bar{n})}(q_R)\text{(Eq.21)}\right]^{-1}\cdot\left\{\left[\bar{W}_R^{(q_R,\bar{m})}(r_R)\text{(Eq.4 when } q_R=0\text{,Eq.(6) when } q_R\neq 0)\right] - (\Delta\rho_S)\sum_{\rho 1}^{\rho F}\left[L_{S0}^{(0,\bar{n})}(\rho_S)\bar{K}_{SR}^{(q_R,\bar{m},\bar{n})}\text{(Eq.37 for } \ell=0)+\right.\right.$$

[Cont'd]

$$+ L_{SO}^{(N_R,\bar{n})}(\rho_S)\bar{K}_{SR}^{(q_R-N_R,\bar{m},\bar{n})}(\text{Eq.37 for } \ell=1)\Big]\Big\}$$

d$_1$) The loading on the duct is obtained in the presence of both stator and rotor.

$$\bar{A}_o^{(0,q_R,\bar{n})} = \bar{K}_{DD}^{(q_R,\bar{m},\bar{n})}(\text{Eq.86 for } \nu = q_R,\ \ell = 0)\Big]^{-1}\cdot$$

$$\left\{\begin{bmatrix}\bar{W}_D^{(0,\bar{m})}(q_R=0,\text{Eq.16})\\ 0(q_R\neq 0)\end{bmatrix} - (\Delta\rho_R)\sum_{\rho 1}^{\rho F} L_{RO}^{(q_R,\bar{n})}(\rho_R)\bar{K}_{RD}^{(q_R,\bar{m},\bar{n})}(\text{Eq.87},\ell=0)\right.$$

$$\left. -(\Delta\rho_S)\sum_{\rho 1}^{\rho F} L_{SO}^{(0,\bar{n})}(\rho_S)\bar{K}_{SD}^{(q_R,\bar{m},\bar{n})}(\text{Eq.88 for } \nu=q_R,\ \ell=0)\right\}$$

e$_1$) <u>for $q_R = 0$</u>

$$\bar{A}_o^{(N_R,-N_R,\bar{n})} = \left[\bar{K}_{DD}^{(-N_R,\bar{m},\bar{n})}(\text{Eq. 86 for } \nu=-N_R,\ \ell=1)\right]^{-1}$$

$$\cdot\left\{\left\{\left[\bar{W}_{R_tD}^{(N_R,\bar{m})}(\text{Eq.20 for } \ell=1)\right] - (\Delta\rho_R)\sum_{\rho 1}^{\rho F}\left\{L_{RO}^{(0,\bar{n})}\bar{K}_{RD}^{(-N_R,\bar{m},\bar{n})}(\text{Eq.87},\ \nu=-N_R,\ \ell=1)\right.\right.\right.$$

$$\left. + \text{conj}\left[L_{RO}^{(0,\bar{n})}\bar{K}_{RD}^{(N_R,\bar{m},\bar{n})}(\text{Eq.87 for } \nu=N_R,\ \ell=-1)\right]\right\}$$

$$\left.\left. -(\Delta\rho_S)\sum_{\rho 1}^{\rho F} L_{SO}^{(N_R,\bar{n})}(\rho_S)\bar{K}_{SD}^{(-N_R,\bar{m},\bar{n})}(\text{Eq.88 for } \nu=-N_R,\ \ell=1)\right\}\right\}$$

<u>for $q_R \neq 0$</u>

$$\bar{A}_o^{(N_R,q_R-N_R,\bar{n})} = \left[\bar{K}_{DD}^{(q_R-N_R,\bar{m},\bar{n})}(\text{Eq.86 for } \nu = q_R-N_R,\ \ell=1)\right]^{-1}\cdot$$

$$\cdot\left\{-(\Delta\rho_R)\sum_{\rho 1}^{\rho F} L_{RO}^{(q_R,\bar{n})}(\rho_R)\bar{K}_{RD}^{(q_R-N_R,\bar{m},\bar{n})}(\text{Eq.87 for } \nu = q_R-N_R,\ \ell=1)\right.$$

$$\left. -(\Delta\rho_S)\sum_{\rho 1}^{\rho F} L_{SO}^{(N_R,\bar{n})}(\rho_S)\bar{K}_{SD}^{(q_R-N_R,\bar{m},\bar{n})}(\text{Eq.88 for } \nu = q_R-N_R,\ \ell=1)\right\}$$

Second Iteration

$$\text{a}_2)\quad L_{S1}^{(0,\bar{n})}(\rho_S) = \left[\bar{K}_{SS}^{(\bar{m},\bar{n})}\,(\text{Eq.59 for } \ell{=}0)\right]^{-1} \cdot \Big\{\bar{W}_S^{(0,\bar{m})}(r_S)\,(\text{Eq.12})$$

$$- \sum_{q_R} (\Delta\rho_R) \sum_{\rho 1}^{\rho F} L_{R0}^{(q_R,\bar{n})}(\rho_R)\bar{K}_{RS}^{(q_R,\bar{m},\bar{n})}\,(\text{Eq.43, } \nu = q_R,\ \ell{=}0)$$

$$- \bar{A}_o^{(0,q_R,\bar{n})}\bar{K}_{DS}^{(q_R,\bar{m},\bar{n})}\,(\text{Eq.50, } \nu = q_R,\ \ell{=}0)\Big\}$$

$$\text{b}_2)\quad L_{S1}^{(N_R,\bar{n})}(\rho_S) = \left[\bar{K}_{SS}^{(\bar{m},\bar{n})}\,(\text{Eq.59 for } \ell{=}1)\right]^{-1} \cdot \Big\{\bar{W}_S^{(N_R,\bar{m})}(r_S)\ (\text{Eq.15})$$

$$- \sum_{q_R} (\Delta\rho_R) \sum_{\rho 1}^{\rho R} L_{R0}^{(q_R,\bar{n})}(\rho_R)\bar{K}_{RS}^{(q_R-N_R,\bar{m},\bar{n})}\,(\text{Eq.43, } \nu = q_R-N_R)$$

$$- \bar{A}_o^{(N_R,q_R-N_R,\bar{n})}\bar{K}_{DS}^{(q_R-N_R,\bar{m},\bar{n})}\,(\text{Eq.50, } \nu = q_R-N_R,\ \ell{=}1)\Big\}$$

$$\text{c}_2)\quad L_{R1}^{(q_R,\bar{n})}(\rho_R) = \left[\bar{K}_{RR}^{(\bar{m},\bar{n})}(q_R)\ (\text{Eq.21})\right]^{-1} \cdot$$

$$\cdot \Big\{\Big[\bar{W}_R^{(q_R,\bar{m})}(r_R)\ (\text{Eq.4 when } q_R{=}0,\ \text{Eq.11 when } q_R \neq 0)\Big]$$

$$-(\Delta\rho_S) \sum_{\rho 1}^{\rho F}\Big[L_{S1}^{(0,\bar{n})}(\rho_S)\bar{K}_{SR}^{(q_R,\bar{m},\bar{n})}\,(\text{Eq.37 for } \ell{=}0)$$

$$+ L_{S1}^{(N_R,\bar{n})}(\rho_S)\bar{K}_{SR}^{(q_R-N_R,\bar{m},\bar{n})}(\text{Eq.37 for } \ell{=}1)\Big]$$

$$- \Big[\bar{A}_o^{(0,q_R,\bar{n})}\bar{K}_{DR}^{(q_R,\bar{m},\bar{n})}\,(\text{Eq.31, } \nu = q_R,\ \ell{=}0)$$

$$+ \bar{A}_o^{(N_R,q_R-N_R,\bar{n})}\bar{K}_{DR}^{(q_R-N_R,\bar{m},\bar{n})}\,(\text{Eq.31, } \nu = q_R-N_R,\ \ell{=}1)\Big]\Big\}$$

$$\text{d}_2)\quad \bar{A}_1^{(0,q_R,\bar{n})} = \left[\bar{K}_{DD}^{(q_R,\bar{m},\bar{n})}(\text{Eq.86 for } \nu = q_R,\ \ell{=}0)\right]^{-1}$$

$$\cdot \left\{ \begin{bmatrix} \bar{W}_D^{(0,\bar{m})}\ (q_R{=}0,\ \text{Eq.16}) \\ 0\ (q_R{\neq}0) \end{bmatrix} - (\Delta\rho_R) \sum_{\rho 1}^{\rho F} L_{R1}^{(q_R,\bar{n})}(\rho_R)\bar{K}_{RD}^{(q_R,\bar{m},\bar{n})}\ (\text{Eq.87, } \ell{=}0)\right.$$

[Cont'd]

$$-(\Delta\rho_S)\sum_{\rho 1}^{\rho F} L_{S1}^{(0,\bar{n})}(\rho_S)\bar{K}_{SD}^{(q_R,\bar{m},\bar{n})}\text{(Eq.88, }\nu = q_R,\ \ell=0)\Big\}$$

e$_2$) <u>for $q_R = 0$</u>

$$\bar{A}_1^{(N_R,-N_R,\bar{n})} = \left[\bar{K}_{DD}^{(-N_R,\bar{m},\bar{n})}\text{(Eq.86 for }\nu = -N_R,\ \ell=1)\right]^{-1}$$

$$\cdot\Big\{\Big\{\left[\bar{W}_{R_tD}^{(N_R,\bar{m})}\text{(Eq.20 for }\ell=1)\right]-(\Delta\rho_R)\sum_{\rho 1}^{\rho F}\Big\{L_{R1}^{(0,\bar{n})}(\rho_R)\bar{K}_{RD}^{(-N_R,\bar{m},\bar{n})}$$

$$\text{(Eq.87, }\nu=-N_R,\ \ell=1)$$

$$+\ \text{conj}\left[L_{R1}^{(0,\bar{n})}\bar{K}_{RD}^{(N_R,\bar{m},\bar{n})}\text{(Eq.87 for }\nu = +N_R,\ \ell=-1)\right]$$

$$-(\Delta\rho_S)\sum_{\rho 1}^{\rho F} L_{S1}^{(N_R,\bar{n})}(\rho_S)\bar{K}_{SD}^{(-N_R,\bar{m},\bar{n})}\text{(Eq.88 for }\nu = -N_R,\ \ell = 1)\Big\}\Big\}$$

<u>for $q_R \neq 0$</u>

$$\bar{A}_1^{(N_R,q_R-N_R,\bar{n})} = \left[\bar{K}_{DD}^{(q_R-N_R,\bar{m},\bar{n})}\text{(Eq.86 for }\nu = q_R-N_R,\ \ell = 1)\right]^{-1}$$

$$\cdot\Big\{-(\Delta\rho_R)\sum_{\rho 1}^{\rho F} L_{R1}^{(q_R,\bar{n})}(\rho_R)\bar{K}_{RD}^{(q_R-N_R,\bar{m},\bar{n})}\text{(Eq.87 for }\nu = q_R-N_R,\ \ell=1)$$

$$-(\Delta\rho_S)\sum_{\rho 1}^{\rho F} L_{S1}^{(N_R,\bar{n})}(\rho_S)\bar{K}_{SD}^{(q_R-N_R,\bar{m},\bar{n})}\text{(Eq.88 for }\nu=q_R-N_R,\ \ell=1)\Big\}$$

LOADING DISTRIBUTIONS

The final values of rotor and stator loadings $L_R^{(q_R,\bar{n})}(r_R)$ and $L_S^{(\ell N_R,\bar{n})}(r_S)$ obtained in the iteration are used to determine the blade loading distributions of the respective surfaces.

The chordwise distributions over the rotor and stator blades are given respectively at each radial position and at any designated frequency by

(a) For the Rotor

$$L_R^{(q_R)}(r_R,\theta_{0R}) = \frac{1}{\pi}\left[L_R^{(q_R,1)}(r_R)\cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\bar{n}max} L_R^{(q_R,\bar{n})}(r_R)\sin(\bar{n}-1)\theta_\alpha\right] \qquad (95a)$$

where q_R = any rotor shaft-frequency and use has been made of the trigonometric transformation $x_R=-\theta_{bR}\cos\theta_\alpha$, θ_{bR} = subtended angle of the projected semichord in radians; and

(b) For the Stator

$$L_S^{(\ell N_R)}(r_S,\theta_{0S}) = \frac{1}{\pi}\left[L_S^{(\ell N_R,1)}(r_S)\cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\bar{n}max} L_S^{(\ell N_R,\bar{n})}(r_S)\sin(\bar{n}-1)\theta_\alpha\right] \qquad (95b)$$

where ℓN_R=stator frequencies for ℓ=0 and ℓ=1; and $x_S=-\theta_{bS}\cos\theta_\alpha$, θ_{bS} = subtended angle of the projected semichord of stator in radians.

The corresponding spanwise loading distributions (after integrating over the chord) are given by

$$L_R^{(q_R)}(r_R) = \int_0^\pi L_R^{(q_R)}(r_R,\theta_{0R})\sin\theta_\alpha d\theta_\alpha$$

$$= L_R^{(q_R,1)}(r_R) + \frac{1}{2} L_R^{(q_R,2)}(r_R) \qquad (96a)$$

and

$$L_S^{(\ell N_R)}(r_S) = \int_0^\pi L_S^{(\ell N_R)}(r_S,\theta_{0S})\sin\theta_\alpha d\theta_\alpha$$

$$= L_S^{(\ell N_R,1)}(r_S) + \frac{1}{2} L_S^{(\ell N_R,2)}(r_S) \qquad (96b)$$

From proven relations,[5,6] the loading distribution on the duct is

$$L_D^{(\ell N_R)}(x_D,\theta_D) = \sum_{\substack{\nu=-\infty \\ \nu=q_R-\ell N_R}}^{\infty} L_D^{(\ell N_R,\nu)}(x_D)\, e^{-i\nu\theta_D}$$

$$= \sum_{\substack{\nu=-\infty \\ \nu=q_R-\ell N_R}}^{\infty} \frac{e^{-i\nu\theta_D}}{\pi C_D} \left\{ \bar{A}^{(\ell N_R,\nu,1)} \cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\bar{n}\,max} \bar{A}^{(\ell N_R,\nu,\bar{n})} \sin(\bar{n}-1)\theta_\alpha \right\}$$

(97a)

$\ell = 0, 1, \ldots,$ *where the* $\bar{A}$'s *are the final values obtained in the iteration.* The superscript ℓN_R refers to the frequency of the duct loading which is zero or a multiple of blade frequency, ν refers to the order of the circumferential mode and $\bar{n}$ to the order of the chordwise mode.

After integrating around the circumference, the chordwise distribution of the duct loading is

$$L_{Dx}^{(\ell N_R)}(x_D) = \left[\int_0^{2\pi} L_D^{(\ell N_R)}(x_D,\theta_D)\, d\theta_D\right] \sin\alpha$$

$$= \frac{2}{C_D} \left\{ \bar{A}^{(\ell N_R,0,1)} \cot\frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\bar{n}\,max} \bar{A}^{(\ell N_R,0,\bar{n})} \sin(\bar{n}-1)\theta_\alpha \right\} \sin\alpha$$

(97b)

since the only non-zero result occurs at $\nu = 0$, $q_R = \ell N_R$.

HYDRODYNAMIC FORCES AND MOMENTS

A) Rotor-Generated Forces and Moments

The principal components of the rotor-induced forces and moments are listed below. (See Figure 4.)

Forces: F_x = thrust (x-direction)

F_y and F_z = horizontal and vertical components, respectively, of the bearing forces

Moments: Q_x = torque about the x-axis

Q_y and Q_z = bending moments about the y- and z-axis, respectively

The elementary forces and moments of the various components can be determined by resolving the loading force $L_R^{(q_R)}(r_R)$ acting on an elementary radial strip, normal to the strip, and taking the corresponding moments about any axis. The forces acting on a strip at radius r_R of the N_R-bladed rotor will be given by[1]

$$\Delta F_x^{(R)} = \sum_{n=1}^{N_R} L_R^{(q_R)}(r_R)\, e^{iq_R(\Omega t+\bar{\theta}_n)} \cos\theta_P^R(r)\, \Delta r_R$$

$$\Delta F_y^{(R)} = \sum_{n=1}^{N_R} L_R^{(q_R)}(r_R)\, e^{iq_R(\Omega t+\bar{\theta}_n)} \sin\theta_P^R(r)\, \cos(\Omega t+\varphi_{RO}+\bar{\theta}_n)\, \Delta r_R$$

$$\Delta F_z^{(R)} = \sum_{n=1}^{N_R} L_R^{(q_R)}(r_R)\, e^{iq_R(\Omega t+\bar{\theta}_n)} \sin\theta_P^R(r)\, \sin(\Omega t+\varphi_{RO}+\bar{\theta}_n)\, \Delta r_R$$

where $\bar{\theta}_P^R(r)$ is the geometric pitch angle of the rotor in radians.

Since

$$\sum_{n=1}^{N_R} e^{iq_R\bar{\theta}_n} = \begin{cases} N_R & \text{when } q_R = \ell N_R,\ \ell = 0,1,2,\ldots \\ 0 & \text{when } q_R \neq \ell N_R \end{cases}$$

and

$$\sum_{n=1}^{N_R} e^{(q_R \pm 1)\bar{\theta}_n} = \begin{cases} N_R & \text{when } q_R = \ell N_R \mp 1 \\ 0 & \text{when } q_R \neq 1 \end{cases}$$

Reference 1 shows that the total forces at frequency ℓN_R acting on the N_R-bladed rotor will be given by

$$F_x^{(R)} = \mathrm{Re}\left\{N_R r_{RO}\, e^{i\ell N_R \Omega t} \int_0^1 L_R^{(\ell N_R)}(r_R) \cos\theta_P^R(r_R)\, dr_R\right\} \tag{98}$$

$$F_y^{(R)} = \mathrm{Re}\left\{\frac{N_R r_{RO}}{2}\, e^{i\ell N_R \Omega t} \int_0^1 \sum_{\bar{n}=1} \left[L_R^{(\ell N_R - 1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(-\theta_{bR}^r) + L_R^{(\ell N_R + 1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(\theta_{bR}^r)\right] \sin\theta_P^R(r_R)\, dr_R\right\} \tag{99}$$

and

$$F_z^{(R)} = -\mathrm{Re}\left\{\frac{N_R r_{RO}}{2i}\, e^{i\ell N_R \Omega t} \int_0^1 \sum_{\bar{n}=1} \left[L_R^{(\ell N_R - 1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(-\theta_{bR}^r) - L_R^{(\ell N_R + 1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(\theta_{bR}^r)\right] \sin\theta_P^R(r_R)\, dr_R\right\} \tag{100}$$

The moments are determined by:

$$Q_x^{(R)} = -\mathrm{Re}\left\{N_R r_{RO}^2\, e^{i\ell N_R \Omega t} \int_0^1 L^{(\ell N_R)}(r_R) \sin\theta_P^R(r_R)\, r_R\, dr_R\right\} \tag{101}$$

$$\begin{aligned} Q_y^{(R)} = \mathrm{Re}\Bigg\{\Bigg\{\frac{N_R r_{RO}^2}{2}\, e^{i\ell N_R \Omega t} \int_0^1 \Bigg\{\sum_{\bar{n}=1} &\left[L_R^{(\ell N_R - 1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(-\theta_{bR}^r)\right. \\ &+ \left. L_R^{(\ell N_R + 1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(\theta_{bR}^r)\right] \cos\theta_P^R(r_R) + \sum_{\bar{n}=1}\left[L_R^{(\ell N_R - 1,\bar{n})}(r_R)\Lambda_1^{(\bar{n})}(-\theta_{bR}^r)\right. \\ &- \left. L_R^{(\ell N_R + 1,\bar{n})}(r_R)\Lambda_1^{(\bar{n})}(\theta_{bR}^r)\right](i\theta_{bR}^r)\sin\theta_P^R(r_R)\tan\theta_P^R(r_R)\Bigg\}\, r_R\, dr_R\Bigg\}\Bigg\} \end{aligned} \tag{102}$$

and

and

$$Q_z^{(R)} = -\mathrm{Re}\left\{\left\{\frac{N_R r_{RO}^2}{2i}\, e^{i\ell N_R \varphi t} \int_0^1 \left\{\sum_{\bar{n}=1} \left[L_R^{(\ell N_R-1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(-\theta_{bR}^r) - L_R^{(\ell N_R+1,\bar{n})}(r_R)\Lambda^{(\bar{n})}(\theta_{bR}^r)\right]\right.\right.\right.$$

$$\cos\theta_P^R(r_R) + \sum_{\bar{n}=1} \left[L_R^{(\ell N_R-1,\bar{n})}(r_R)\Lambda_1^{(\bar{n})}(-\theta_{bR}^r) + L_R^{(\ell N_R+1,\bar{n})}(r_R)\Lambda_1^{(\bar{n})}(\theta_{bR}^r)\right]$$

$$\left.\left.\left.(i\theta_{bR}^r)\sin\theta_P^R(r_R)\tan\theta_P^R(r_R)\right\} r_R dr_R \right\}\right\} \tag{103}$$

where $\Lambda^{(\bar{n})}(\cdots)$ and $\Lambda_1^{(\bar{n})}(\cdots)$ are given in Appendix A.

Thus the rotor-generated transverse forces and bending moments are evaluated from rotor loadings associated with wake harmonics at frequencies adjacent to blade frequency, i.e., at $q_R=\ell N_R \pm 1$, whereas the thrust and torque are determined by the loading at blade frequency. The steady-state thrust and torque are determined at zero frequency. The corresponding mean transverse forces and bending moment would be determined at first shaft frequency; in this case, $L_R^{(-1)}(r) = 0$ and only the second terms $L_R^{(1)}(r)$ of Eqs.(99), (100), (102), and (103) are present.

However, in the case of the pump-jet system $L_R^{(q_R,\bar{n})}$ is determined only when $q_R=\ell' N_S$, $\ell' = 0, 1, 2, \ldots$. Hence, thrust and torque will exist only at $\ell N_R = \ell' N_S$ (steady-state when $\ell = \ell' = 0$ and vibratory when $\ell = N_S$, $\ell' = N_R$) and transverse forces and bending moments only in the event that $\ell' N_S = \ell N_R \pm 1$.

For example, if $N_R = 5$ and $N_S = 7$, thrust and torque will exist at $q_R = \ell' N_S = \ell N_R$ equal to 0 (in the steady state) and equal to $mN_R N_S$, integer multiples of blade-crossing frequency. Side forces and moments will occur at $\ell' = 2$, $\ell = 3$, so that $\ell' N_S = \ell N_R - 1$ or $14 = 15-1$ and at $\ell' = 3$, $\ell = 4$, so that $\ell' N_S = \ell N_R + 1$ or $21 = 20+1$.

B) Stator-Generated Forces and Moments

In a similar fashion the elementary forces acting on a strip at radius r_S of the N_S-bladed stator can be shown to be

$$\Delta F_x^{(S)} = \sum_{n=1}^{N_S} L_S^{(\ell N_R)}(r_S)e^{i\ell N_R(\Omega t+\bar{\theta}_n)}\cos\theta_p^S(r)\Delta r_S$$

with

$$\sum_{n=1}^{N_S} e^{i\ell N_R\bar{\theta}_n} = \begin{cases} N_S & \text{when } \ell N_R = \ell' N_S \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{n=1}^{N_S} e^{i(\ell N_R\pm 1)\bar{\theta}_n} = \begin{cases} N_S & \text{when } \ell N_R = \ell' N_S \mp 1 \\ 0 & \text{otherwise} \end{cases}$$

The total forces and moments generated on an N_S-bladed stator at frequencies ℓN_R will be

$$F_x^{(S)} = \mathrm{Re}\left\{N_S r_{RO}(r'_{SO})e^{i\ell N_R\Omega t}\int_0^1 L_S^{(\ell N_R)}(r_S)\cos\theta_p^S(r_S)dr_S\right\} \tag{104}$$

$$F_y^{(S)} = \mathrm{Re}\left\{\frac{N_S}{2} r_{RO}(r'_{SO})e^{i\ell N_R\Omega t}\int_0^1 \sum_{\bar{n}=1}\left[L_S^{(\ell N_R-1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(-\theta_{bS}^r)+L_S^{(\ell N_R+1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(\theta_{bS}^r)\right]\right.$$
$$\left.\sin\theta_p^S(r_S)dr_S\right\} \tag{105}$$

$$F_z^{(S)} = -\mathrm{Re}\left\{\frac{N_S}{2i} r_{RO}(r'_{SO})e^{i\ell N_R\Omega t}\int_0^1 \sum_{\bar{n}=1}\left[L_S^{(\ell N_R-1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(-\theta_S^r)-L_S^{(\ell N_R+1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(\theta_{bS}^r)\right]\right.$$
$$\left.\sin\theta_p^S(r_S)dr_S\right\} \tag{106}$$

The moments are determined by:

$$Q_x^{(S)} = \mathrm{Re}\left\{N_S r_{RO}^2 (r'_{SO})^2 e^{i\ell N_R \Omega t} \int_0^1 L_S^{\ell N}(r_S)\sin\theta_P^S(r_S)\, r_S\, dr_S\right\} \tag{107}$$

$$Q_y^{(S)} = \mathrm{Re}\left\{\left\{\frac{N_S}{2} r_{RO}^2 (r'_{SO})^2 e^{i\ell N_R \Omega t} \int_0^1 \left\{\sum_{\bar{n}=1}\left[L_S^{(\ell N_R - 1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(-\theta_{bS}^r)\right.\right.\right.\right.$$
$$\left. + L_S^{(\ell N_R+1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(\theta_{bS}^r)\right]\cos\theta_P^S(r_S) + \sum_{\bar{n}=1}\left[L_S^{(\ell N_R-1,\bar{n})}(r_S)\Lambda_1^{(\bar{n})}(-\theta_{bS}^r)\right.$$
$$\left.\left.\left.\left. - L_S^{(\ell N_R+1,\bar{n})}(r_S)\Lambda_1^{(\bar{n})}(\theta_{bS}^r)\right](i\theta_{bS}^r)\sin\theta_P^S(r_S)\tan\theta_P^S(r_S)\right\} r_S\, dr_S\right\}\right\} \tag{108}$$

$$Q_z^{(S)} = -\mathrm{Re}\left\{\left\{\frac{N_S}{2i} r_{RO}^2 (r'_{SO})^2 e^{i\ell N_R \Omega t} \int_0^1 \left\{\sum_{\bar{n}=1}\left[L_S^{(\ell N_R+1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(-\theta_{bS}^r)\right.\right.\right.\right.$$
$$\left. - L_S^{(\ell N_R+1,\bar{n})}(r_S)\Lambda^{(\bar{n})}(\theta_{bS}^r)\right]\cos\theta_P^S(r_S) + \sum_{\bar{n}=1}\left[L_S^{(\ell N_R-1,\bar{n})}(r_S)\Lambda_1^{(\bar{n})}(-\theta_{bS}^r)\right.$$
$$\left.\left.\left.\left. + L_S^{(\ell N_R+1,\bar{n})}(r_S)\Lambda_1^{(\bar{n})}(\theta_{bS}^r)\right](i\theta_{bS}^r)\sin\theta_P^S(r_S)\tan\theta_P^S(r_S)\right\} r_S\, dr_S\right\}\right\} \tag{109}$$

where

$$r'_{SO} = \frac{r_{SO}}{r_{RO}} \text{ (nondimensional with respect to the rotor radius).}$$

See comments on relation between ℓ and ℓ' in Section A.

C) Duct Forces and Moments

The axial component of the force acting on the duct at the frequency ℓN_R is given by,[6]

$$F_{Dx}^{(\ell N_R)} = r_o \int_{2C_D} L_{Dx}^{(\ell N_R)}(x_D)\, dx_D$$

or

$$F_{Dx}^{(\ell N_R)} = r_o\, C_D \int_0^\pi L_{Dx}^{(\ell N_R)}(x_D)\sin\theta_\alpha\, d\theta_\alpha \tag{110}$$

where x_D and C_D are nondimensionalized with respect to rotor radius r_{RO}. With $L_{Dx}^{(\ell N_R)}$ given by Eq. (97b), the axial force or thrust from the duct is

$$F_{Dx}^{(\ell N_R)} = -2r_o \left[\int_0^{\pi} \left\{ \bar{A}^{(\ell N_R,0,1)}(1+\cos\theta_\alpha) + \sum_{\bar{n}=2}^{\infty} \bar{A}^{(\ell N_R,0,\bar{n})} \sin(\bar{n}-1)\theta_\alpha \sin\theta_\alpha \right\} d\theta_\alpha \right] \sin\alpha$$

$$= -2\pi r_o \left\{ \bar{A}^{(\ell N_R,0,1)} + \frac{1}{2} \bar{A}^{(\ell N_R,0,2)} \right\} \sin\alpha \qquad (111)$$

The lateral components of the hydrodynamic force acting on the duct are derived as follows. The horizontal (y) component is

$$F_{Dy}^{(\ell N_R)} = r_o C_D \cos\alpha \int_0^{\pi} \left[\int_0^{2\pi} L_D^{(\ell N_R)}(x_D,\theta_D) \sin\theta_D d\theta_D \right] \sin\theta_\alpha d\theta_\alpha$$

and the vertical (z) component is

$$F_{Dz}^{(\ell N_R)} = r_o C_D \cos\alpha \int_0^{\pi} \left[\int_0^{2\pi} L_D^{(\ell N_R)}(x_D,\theta_D) \cos\theta_D d\theta_D \right] \sin\theta_\alpha d\theta_\alpha$$

where $L_D^{(\ell N_R)}(x\ ,\theta\ )$ is given by Eq.(97a). Since

$$\int_0^{2\pi} e^{-i\nu\theta_D} \sin\theta_D d\theta_D = \begin{cases} 0 & \text{for } \nu \neq \pm 1 \\ -i\pi & \text{for } \nu = +1 \\ +i\pi & \text{for } \nu = -1 \end{cases}$$

and

$$\int_0^{2\pi} e^{-i\nu\theta_D} \cos\theta_D d\theta_D = \begin{cases} 0 & \text{for } \nu \neq \pm 1 \\ \pi & \text{for } \nu = \pm 1 \end{cases}$$

the horizontal and vertical components of the force become, respectively,

$$F_{Dy}^{(\ell N_R)} = -i\pi r_o \cos\alpha \left\{ \left[\bar{A}^{(\ell N_R,1,1)} + \frac{1}{2} \bar{A}^{(\ell N_R,1,2)} \right] - \left[\bar{A}^{(\ell N_R,-1,1)} + \frac{1}{2} \bar{A}^{(\ell N_R,-1,2)} \right] \right\} \qquad (112)$$

and

$$F_{Dz}^{(\ell N_R)} = \pi r_o \cos\alpha \left\{ \left[\bar{A}^{(\ell N_R,1,1)} + \frac{1}{2} \bar{A}^{(\ell N_R,1,2)} \right] - \left[\bar{A}^{(\ell N_R,-1,1)} + \frac{1}{2} \bar{A}^{(\ell N_R,-1,2)} \right] \right\} \qquad (113)$$

The hydrodynamic moments about the y- and z-axes, respectively, are

$$M_{Dy}^{(\ell N_R)} = r_o^2\, C_D\cos\alpha \int_0^{\pi} \left[\int_0^{2\pi} L_D^{(\ell N_R)}(x_D,\theta_D)\cos\theta_D d\theta_D \right] (\epsilon_D - C_D\cos\theta_\alpha)\sin\theta_\alpha d\theta_\alpha$$

and

$$M_{Dz}^{(\ell N_R)} = r_o^2\, C_D\cos\alpha \int_0^{\pi} \left[\int_0^{2\pi} L_D^{(\ell N_R)}(x_D,\theta_D)\sin\theta_D d\theta_D \right] (\epsilon_D - C_D\cos\theta_\alpha)\sin\theta_\alpha d\theta_\alpha$$

where $\epsilon_D - C_D \cos\theta_\alpha$ is the nondimensionalized moment arm. On integrating, these become

$$M_{Dy}^{(\ell N_R)} = \pi r_o^2 \cos\alpha \Big\{ \epsilon_D \Big[\bar{A}^{(\ell N_R,1,1)} + \frac{1}{2}\bar{A}^{(\ell N_R,1,2)} + \bar{A}^{(\ell N_R,-1,1)} + \frac{1}{2}\bar{A}^{(\ell N_R,-1,2)} \Big]$$
$$- C_D \Big[\frac{1}{2}\bar{A}^{(\ell N_R,1,1)} + \frac{1}{4}\bar{A}^{(\ell N_R,1,3)} + \frac{1}{2}\bar{A}^{(\ell N_R,-1,1)} + \frac{1}{4}\bar{A}^{(\ell N_R,-1,3)} \Big] \Big\} \tag{114}$$

$$M_{Dz}^{(\ell N_R)} = -i\pi r_o^2 \cos\alpha \Big\{ \epsilon_D \Big[\bar{A}^{(\ell N_R,1,1)} + \frac{1}{2}\bar{A}^{(\ell N_R,1,2)} - \bar{A}^{(\ell N_R,-1,1)} - \frac{1}{2}\bar{A}^{(\ell N_R,-1,2)} \Big]$$
$$- C_D \Big[\frac{1}{2}\bar{A}^{(\ell N_R,1,1)} + \frac{1}{4}\bar{A}^{(\ell N_R,1,3)} - \frac{1}{2}\bar{A}^{(\ell N_R,-1,1)} - \frac{1}{4}\bar{A}^{(\ell N_R,-1,3)} \Big] \Big\} \tag{115}$$

When $\ell = 0$

$$F_{Dy}^{(0)} = \mathrm{Re}\Big\{ -i\pi r_o \cos\alpha \Big[\bar{A}^{(0,1,1)} + \frac{1}{2}\bar{A}^{(0,1,2)} \Big] \Big\}$$

$$F_{Dz}^{(0)} = \mathrm{Re}\Big\{ +\pi r_o \cos\alpha \Big[\bar{A}^{(0,1,1)} + \frac{1}{2}\bar{A}^{(0,1,2)} \Big] \Big\}$$

$$M_{Dy}^{(0)} = \mathrm{Re}\Big\{ \pi r_o^2 \cos\alpha \Big\} \Big\{ \epsilon_D \Big[\bar{A}^{(0,1,1)} + \frac{1}{2}\bar{A}^{(0,1,2)} \Big] - C_D \Big[\frac{1}{2}\bar{A}^{(0,1,1)} + \frac{1}{4}\bar{A}^{(0,1,3)} \Big] \Big\}$$

$$M_{Dz}^{(0)} = \mathrm{Re}\Big\{ -i\pi r_o^2 \cos\alpha \Big\} \Big\{ \epsilon_D \Big[\bar{A}^{(0,1,1)} + \frac{1}{2}\bar{A}^{(0,1,2)} \Big] - C_D \Big[\frac{1}{2}\bar{A}^{(0,1,1)} + \frac{1}{4}\bar{A}^{(0,1,3)} \Big] \Big\} \tag{116}$$

Note that the second index of $\bar{A}$ is equal to $q_R - \ell N_R$. See comments under Section A as to limitations.

When $\ell = 1$

$$F_{Dy}^{(N_R)} = -i\pi r_o \cos\alpha \left\{ \left[\bar{A}^{(N_R,q-N_R=1,1)} + \frac{1}{2}\bar{A}^{(N_R,q-N_R=1,2)} \right] - \left[\bar{A}^{(N_R,q-N_R=-1,1)} + \frac{1}{2}\bar{A}^{(N_R,q-N_R=-1,2)} \right] \right\}$$

$$F_{Dz}^{(N_R)} = \pi r_o \cos\alpha \left\{ \left[\bar{A}^{(N_R,q-N_R=1,1)} + \frac{1}{2}\bar{A}^{(N_R,q-N_R=1,2)} \right] + \left[\bar{A}^{(N_R,q-N_R=-1,1)} + \frac{1}{2}\bar{A}^{(N_R,q-N_R=-1,2)} \right] \right\}$$

$$M_{Dy}^{(N_R)} = \pi r_o^2 \cos\alpha \left\{ \epsilon_D \left[\bar{A}^{(N_R,q,-N_R=1,1)} + \frac{1}{2}\bar{A}^{(N_R,1,2)} + \bar{A}^{(N_R,-1,1)} + \frac{1}{2}\bar{A}^{(N_R,-1,2)} \right] - C_D \left[\frac{\bar{A}^{(N_R,1,1)}}{2} + \frac{\bar{A}^{(N_R,1,3)}}{4} + \frac{\bar{A}^{(N_R,-1,1)}}{2} + \frac{\bar{A}^{(N_R,-1,3)}}{4} \right] \right\}$$

$$M_{Dz}^{(N_R)} = -\pi r_o^2 \cos\alpha \left\{ \epsilon_D \left[\bar{A}^{(N_R,1,1)} + \frac{1}{2}\bar{A}^{(N_R,1,2)} - \frac{1}{2}\bar{A}^{(N_R,-1,1)} - \frac{1}{2}\bar{A}^{(N_R,-1,2)} \right] - C_D \left[\frac{1}{2}\bar{A}^{(N_R,1,1)} + \frac{1}{4}\bar{A}^{(N_R,1,3)} - \frac{1}{2}\bar{A}^{(N_R,-1,1)} - \frac{1}{4}\bar{A}^{(N_R,-1,3)} \right] \right\}$$

(117

When $q-N_R = 1$ $\quad q = N_R+1$

When $q-N_R = -1$ $\quad q = N_R-1$

SUMMARY

A theory has been developed in treating the "Pump-Jet" propulsive unit comprised of stator, rotor, and enshrouding nozzle by taking into account accurate geometry, realistic flow conditions and hydrodynamic interactions between all the lifting surfaces of finite thicknesses of the system. The system is immersed in a non-uniform flow field of an ideal incompressible fluid.

The unsteady lifting surface theory has been utilized throughout the analysis and a numerical solution has been outlined using an iteration procedure guided by physical considerations.

Expressions have been developed for the various loadings on the interacting lifting surfaces and for the corresponding resulting forces and moments evaluated at the proper frequencies.

The analysis has been brought to the point where the suggested numerical procedure can be coded. The treatment of numerical difficulties, such as singularities, has also been studied and expressions for their finite contributions have been determined (see Appendices C-K). This numerical procedure is at present being used in developing a computing program which is adapted to the CDC-6600 or Cyber 176 high-speed digital computer. The various components of the evolved analysis are being coded for arbitrary values of time-dependent and space-dependent frequencies and other parameters as the theory indicates. Then by combining these components at the proper frequencies as the iterative procedure requires, the corresponding loading of all interacting surfaces will be determined. This part of the synthesis remains to be completed and tested for a realistic pump-jet configuration, details of which have not yet been provided by the proper Agency.

Until this program is completed and systematic calculations are made, no conclusions can be drawn as to the relative merits of this propulsive configuration as compared with a single screw. Nor can judgment be made as to the relative importance of the stator-rotor-duct components or on the effect of various parameters, such as number of blades, distance between

stator and rotor and their relative locations with respect to the duct, blade area ratio and pitch angles, on the steady state and vibratory forces and moments.

The present study is considered to be a complete reporting requirement of the theoretical analysis of the pump-jet propulsive device. The numerical coding when completed will be considered as a supplement of this report.

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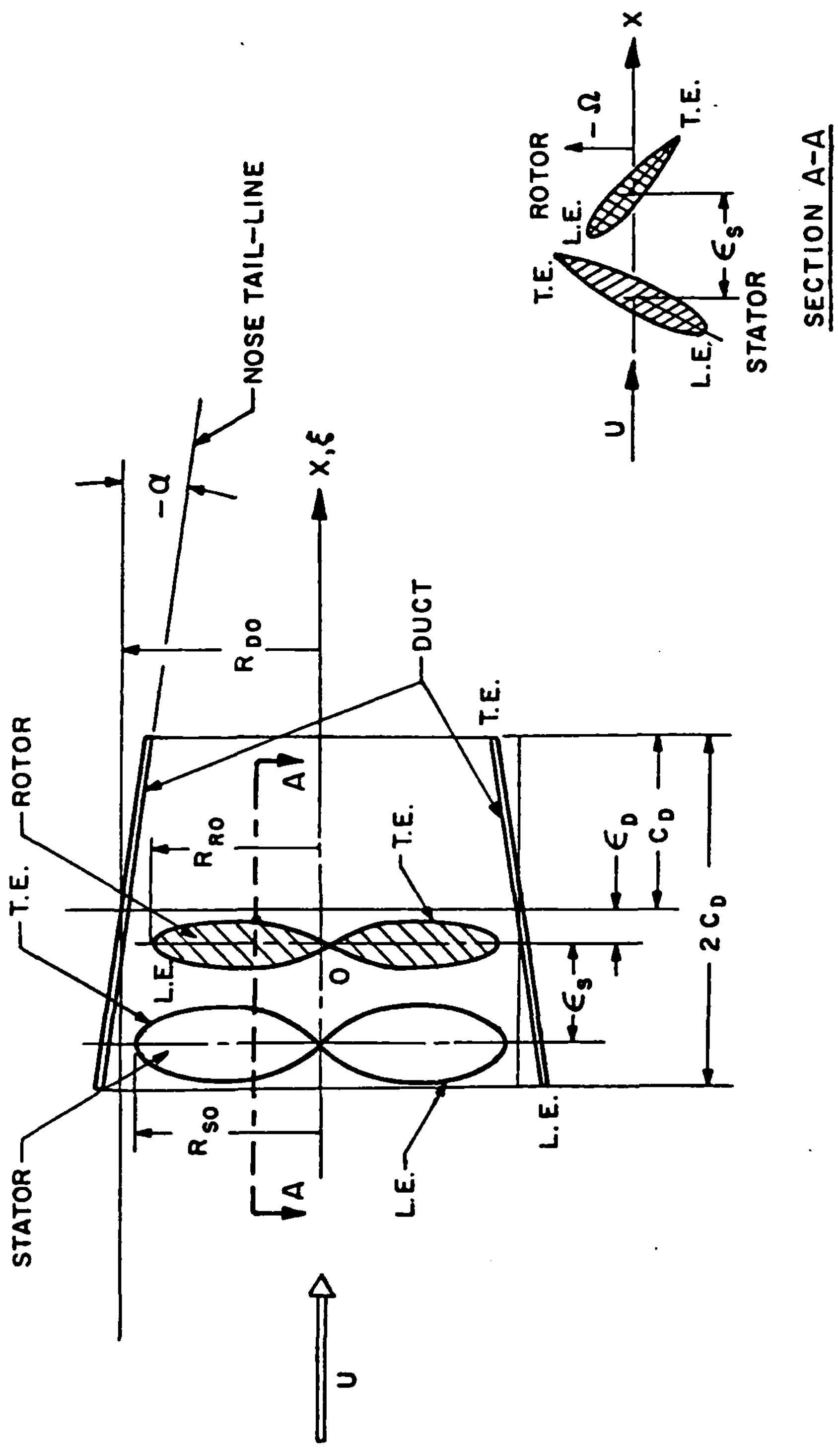


FIG. 1. STATOR—ROTOR—DUCT ARRANGEMENT

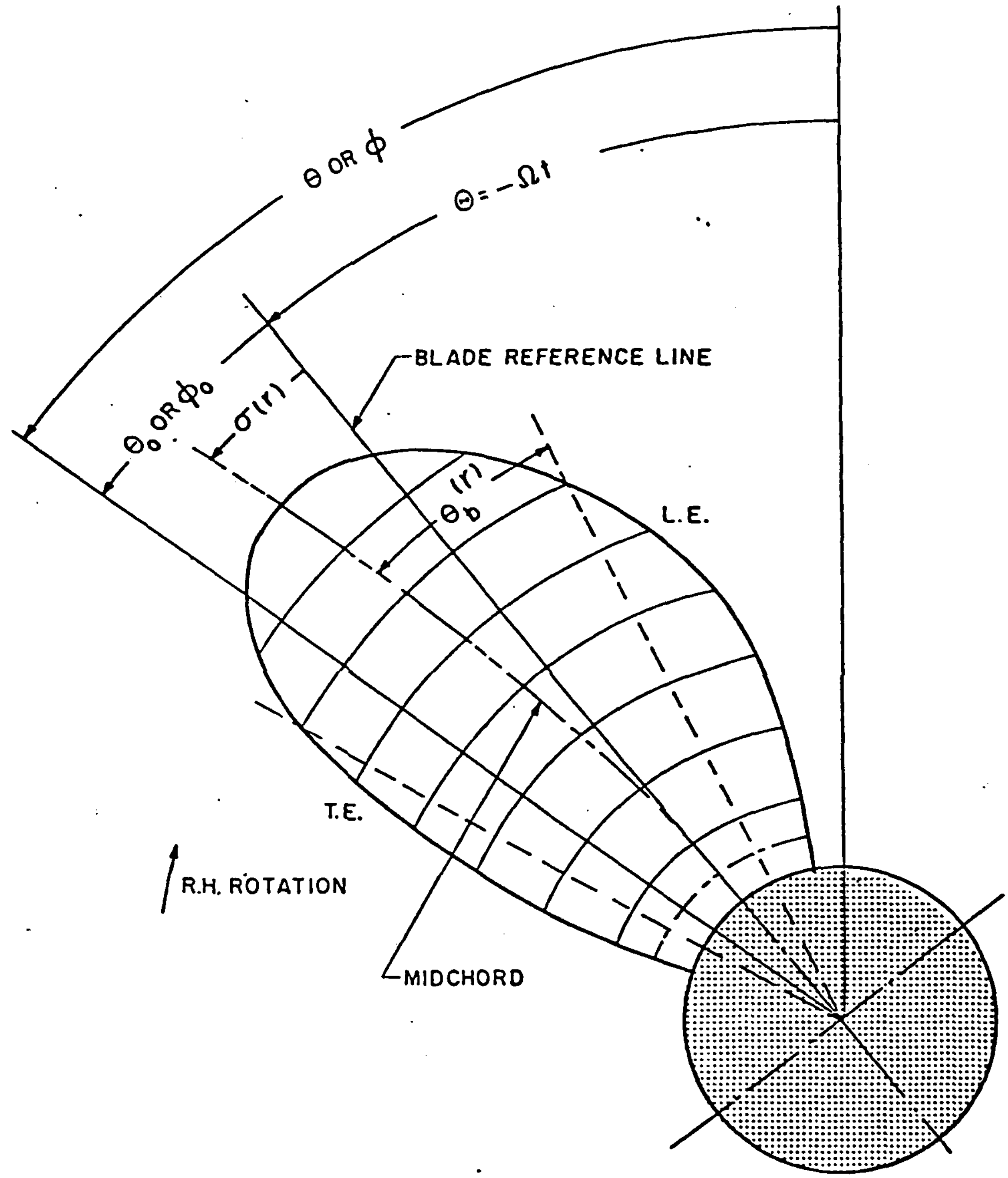
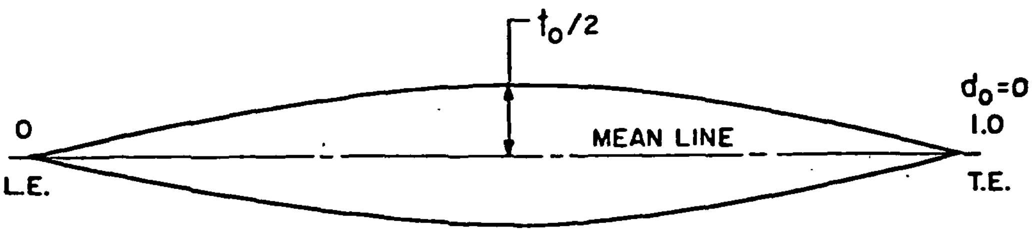


FIG. 2. DEFINITIONS OF ANGULAR MEASURES

NOTE: THE BLADE REFERENCE LINE IS THAT CONNECTING THE SHAFT CENTER WITH THE MIDPOINT OF THE CHORD AT THE HUB

LENTICULAR CROSS-SECTION



MODIFIED LENTICULAR CROSS-SECTION

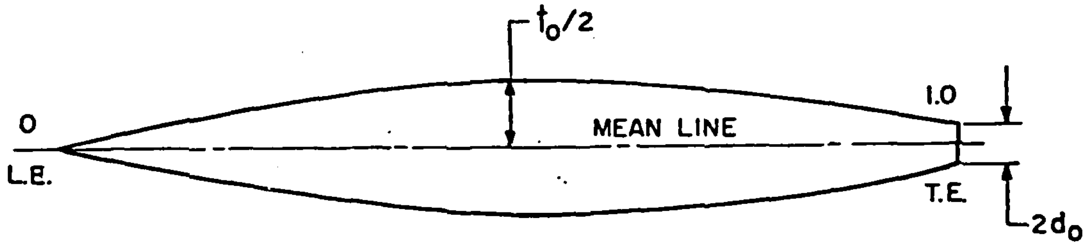


FIG. 3. THICKNESS DISTRIBUTION APPROXIMATION

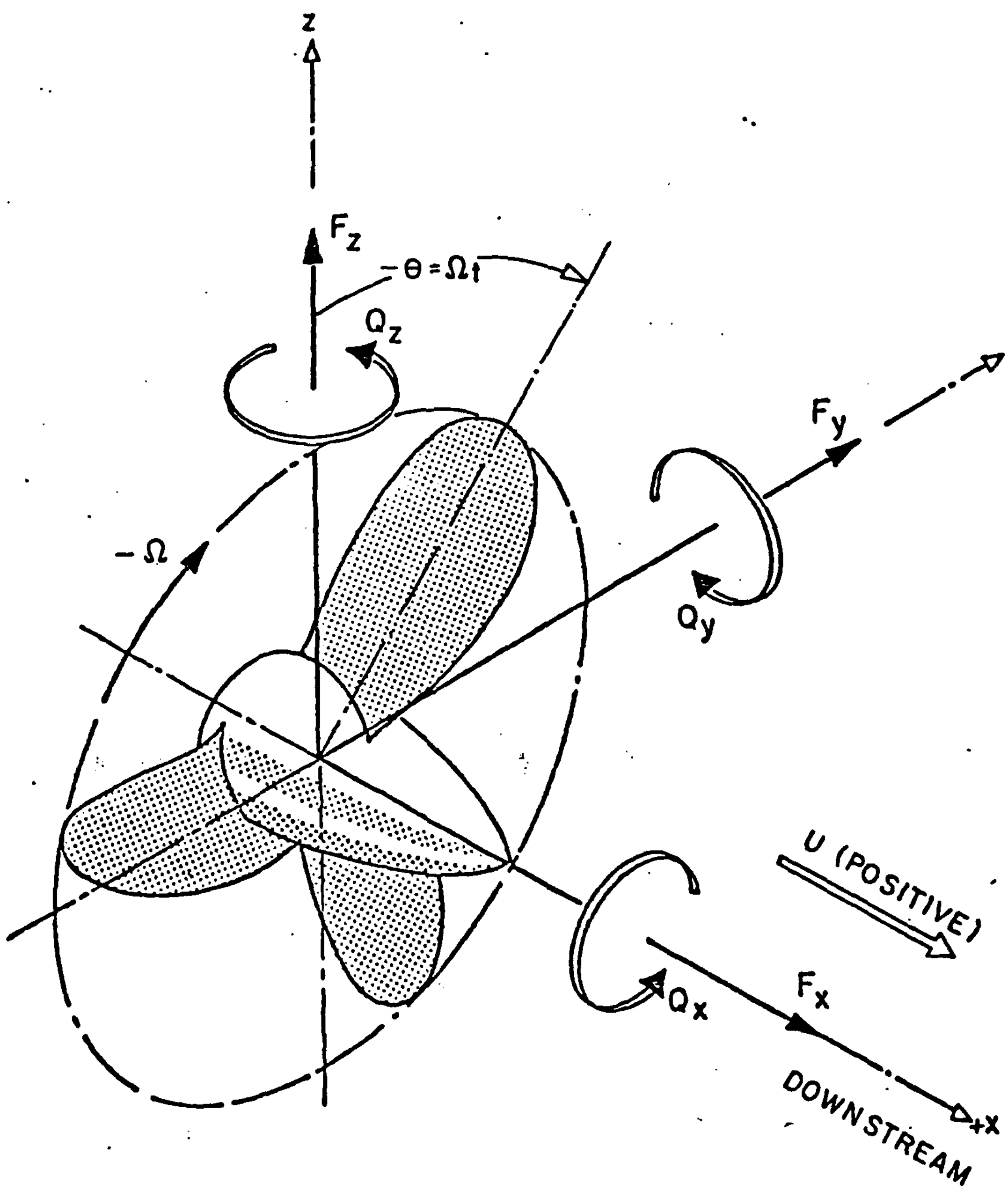


FIG. 4. RESOLUTION OF FORCES AND MOMENTS

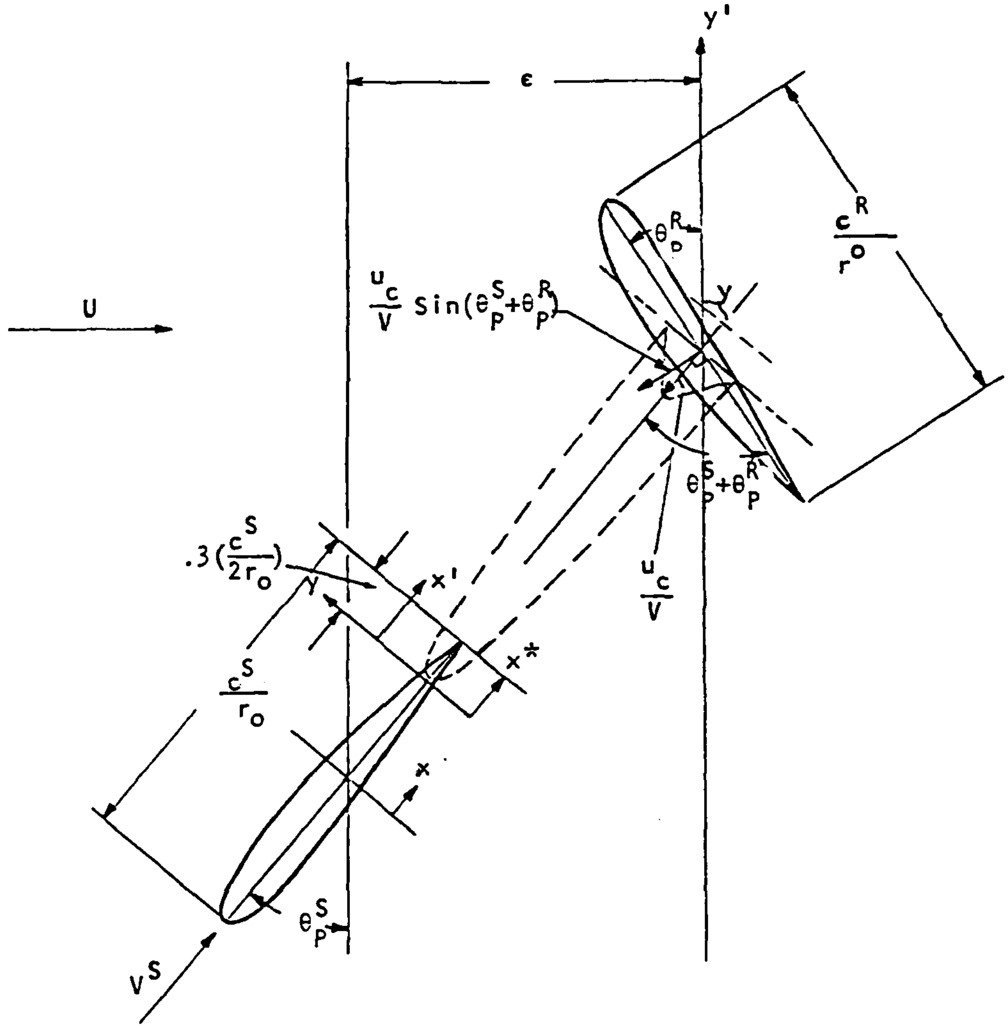


Fig. 5: Expanded view of two propeller blades at a particular radial position, r_S.

APPENDIX A

EVALUATION OF THE θ_α- AND φ_α-INTEGRALS

I. $$I^{(\bar{m})}(x) = \frac{1}{\pi}\int_0^\pi \Phi(\bar{m}) e^{ix\cos\varphi} d\varphi \qquad \text{(A-1)}$$

where for $\bar{m}=1$

$$I^{(1)}(x) = \frac{1}{\pi}\int_0^\pi (1-\cos\varphi) e^{ix\cos\varphi} d\varphi = J_o(x) - iJ_1(x)$$

for $\bar{m} = 2$

$$I^{(2)}(x) = \frac{1}{\pi}\int_0^\pi (1+2\cos\varphi) e^{ix\cos\varphi} d\varphi = J_o(x) + i2J_1(x)$$

and for $\bar{m} > 2$

$$I^{(\bar{m})}(x) = \frac{1}{\pi}\int_0^\pi \cos(\bar{m}-1)\varphi \; e^{ix\cos\varphi} d\varphi = i^{\bar{m}-1} J_{\bar{m}-1}(x)$$

where

$J_n(x)$ is the Bessel function of the first kind.

II. $$\Lambda^{(\bar{n})}(y) = \frac{1}{\pi}\int_0^\pi \Theta(\bar{n}) e^{-iy\cos\theta} \sin\theta d\theta \qquad \text{(A-2)}$$

where for $\bar{n}=1$

$$\Lambda^{(1)}(y) = \frac{1}{\pi}\int_0^\pi \cot\frac{\theta}{2} \sin\theta \; e^{-iy\cos\theta} d\theta = J_o(y) - iJ_1(y)$$

and for $\bar{n} > 1$

$$\Lambda^{(\bar{n})}(y) = \frac{1}{\pi}\int_0^\pi \sin(\bar{n}-1)\theta \; \sin\theta \; e^{-iy\cos\theta} d\theta$$

$$= \frac{(-i)^{\bar{n}-2}}{2}\left[J_{\bar{n}-2}(y) + J_{\bar{n}}(y)\right]$$

III. To evaluate

$$I_1^{(\bar{m})}(x) = \frac{1}{\pi}\int_0^{\pi} \Phi(\bar{m})\, e^{ix\cos\varphi}\cos\varphi\, d\varphi$$

$$\Lambda_1^{(\bar{n})}(y) = \frac{1}{\pi}\int_0^{\pi} \Theta(\bar{n})\sin\theta\cos\theta\, e^{-iy\cos\theta}\, d\theta$$

a) For $\bar{m}=1$

$$I_1^{(1)}(x) = \frac{-1}{2}\left[J_o(x) - J_2(x)\right] + iJ_1(x)$$

for $\bar{m}=2$

$$I_1^{(2)}(x) = \left[J_o(x) - J_2(x)\right] + iJ_1(x)$$

and for $\bar{m} > 2$

$$I_1^{(\bar{m})}(x) = \frac{i^{\bar{m}-2}}{2}\left[- J_{\bar{m}}(x) + J_{\bar{m}-2}(x)\right]$$

b) For $\bar{n}=1$

$$\Lambda_1^{(1)}(y) = \frac{1}{2}\left[J_o(y) - J_2(y)\right] - iJ_1(y)$$

and for $\bar{n} > 1$

$$\Lambda_1^{(\bar{n})}(y) = \frac{(-i)^{\bar{n}+1}}{4}\left[J_{\bar{n}-3}(y) - J_{\bar{n}+1}(y)\right]$$

APPENDIX B

EFFECT OF BLADE THICKNESS OF ROTOR ON THE VELOCITY FIELD OF THE STATOR

The thickness distribution of a blade section is represented by a source-sink distribution assumed to be smeared over a projection of the section in the rotor plane. The velocity potential due to the rotor blade thickness at a point (x_S', r_S, φ_S) on the stator is given by

$$\Phi_S(x_S', r_S, \varphi_S; t)_{R_t} = -\frac{1}{4\pi} \sum_{n=1}^{N_R} \int_{-\theta_{bR}}^{\theta_{bR}} \int_{\rho_R} \frac{M(\xi_R, \rho_R, \theta_{RO})}{R_{RS}} \frac{\sqrt{1+a_R^2\rho_R^2}}{a_R\rho_R} \rho_R d\rho_R d\theta_{RO} \tag{B-1}$$

where $M(\xi_R, \rho_R, \theta_{RO}) = 2U \dfrac{\partial f(\xi_R, \rho_R, \theta_{RO})}{\partial \xi_R}$ the source strength density determined in accordance with the "thin body" approach,

$f(\xi_R, \rho_R, \theta_{RO})$ = thickness distribution over <u>one</u> side of the blade section at radial distance ρ_R in the rotor plane

$$R_{RS} = \left\{ \left(x_S' - \xi_R\right)^2 + r_S^2 + \rho_R^2 - 2 r_S \rho_R \cos\left[\theta_{RO} + \varphi_{SO} - \Omega t + \bar{\theta}_{Rn}\right] \right\}^{\frac{1}{2}}$$

$$x_S' = \varphi_{SO}/a_S + \epsilon_S = (\sigma_S - \theta_{bS}\cos\varphi_\alpha)/a_S + \epsilon_S \quad , \quad 0 \le \varphi_\alpha \le \pi$$

$$\xi_R = \theta_{RO}/a_R = (\sigma_R - \theta_{bR}\cos\theta_\alpha)/a_R \quad , \quad 0 \le \theta_\alpha \le \pi$$

$$\bar{\theta}_{Rn} = \left(\frac{2\pi}{N_R}\right)(n-1) \quad , \quad n=1,2,\ldots,N_R$$

Since $\dfrac{\partial f}{\partial \xi_R} = \dfrac{a_R}{\theta_{bR}\sin\theta_\alpha} \dfrac{\partial f}{\partial \theta_\alpha}$, Eq.(B-1) can be reduced to

$$(\Phi_S)_{R_t} = -\frac{U}{2\pi} \sum_{n=1}^{N_R} \int_0^{\pi} \int_{\rho_R} \frac{\partial f(\rho_R, \theta_\alpha)}{\partial \theta_\alpha} \frac{\sqrt{1+a_R^2\rho_R^2}}{R_{RS}} d\rho_R d\theta_\alpha \tag{B-2}$$

The thickness distribution $f(\rho_R,\theta_\alpha)$ will be approximated by a lenticular section, i.e.,

$$f(\rho_R,\theta_\alpha) \approx \frac{\tau(\rho_R)}{2} \sin^2\theta_\alpha$$

$$\approx \frac{t_o(\rho_R)}{c} \rho_R\theta_{bR}\sin^2\theta_\alpha \qquad (B-3)$$

where τ is maximum thickness in the projected plane

$\frac{t_o}{c}$ is ratio of maximum thickness to chord of the expanded section

$\rho_R\theta_{bR}$ is projected semichord

Therefore

$$\frac{\partial f}{\partial\theta_\alpha} \approx 2 \frac{t_o}{c} (\rho_R) \cdot \rho_R\theta_{bR}\sin\theta_\alpha\cos\theta_\alpha \qquad (B-4)$$

The nondimensional velocity normal to the blades of the stator due to the velocity potential $(\Phi_S)_{R_t}$ is

$$(W_S)_{R_t} = -\frac{1}{U}\frac{\partial}{\partial n_S'}\left(\Phi_S\right)_{R_t} = -\frac{r_S}{U\sqrt{1+a_S^2 r_S^2}}\left(a\frac{\partial}{\partial x_S'} - \frac{1}{r_S^2}\frac{\partial}{\partial\varphi_{SO}}\right)\left(\Phi_S\right)_{R_t} \qquad (B-5)$$

Substituting Eq.(B-4) into (B-2), and (B-2) into (B-5) and, in addition, expanding the reciprocal of the Descartes distance R_{RS} as

$$\frac{1}{R_{RS}} = \frac{1}{\pi}\sum_{m=-\infty}^{\infty} e^{im\beta}\int_{-\infty}^{\infty} (IK)_m\, e^{i(x_S'-\xi_R)k}\,dk \qquad (B-6)$$

where $\beta = \theta_{RO} + \varphi_{SO} - \Omega t + \bar{\bar{\theta}}_{Rn}$

and $$(IK)_m = \begin{cases} I_m(|k|\rho_R)K_m(|k|r_S) & \text{for } \rho_R < r_S \\ I_m(|k|r_S)K_m(|k|\rho_R) & \text{for } \rho_R > r_S \end{cases}$$

yields

$$\left(W_S\right)_{R_t} = \frac{r_S}{\pi^2\sqrt{1+a_S^2 r_S^2}} \left(a_S \frac{\partial}{\partial x_S'} - \frac{1}{r_S^2}\frac{\partial}{\partial \varphi_{SO}}\right) \sum_{n=1}^{N_R} \int_0^{\pi}\int_{\rho_R} \frac{t_o}{c}(\rho_R)\,\rho_R\,\theta_{bR}\sqrt{1+a_R^2\rho_R^2}\,\sin\theta_\alpha\cos\theta_\alpha$$

$$\cdot \sum_{m=-\infty}^{\infty} e^{im(\theta_{RO}+\varphi_{SO}-\Omega t+\bar{\theta}_{Rn})} \int_{-\infty}^{\infty}(IK)_m\, e^{i(x_S'-\xi_R)k}\, dk\, d\rho_R\, d\theta_\alpha \qquad \text{(B-7)}$$

With $\sum_{n=1}^{N_R} e^{im\bar{\theta}_{Rn}} = \begin{cases} N_R & \text{if } m=\ell N_R,\ \ell=0,\pm1,\pm2,\ldots \\ 0 & \text{otherwise} \end{cases}$

and $e^{im\theta_{RO}} = e^{-im\theta_{bR}\cos\theta_\alpha}\, e^{im\sigma_R}$ and $e^{im\varphi_{SO}} = e^{-im\theta_{bS}\cos\varphi_\alpha}\, e^{im\sigma_S}$

and taking the derivatives with respect to x_S' and φ_{SO}

$$\left(W_S\right)_{R_t} = \frac{N_R r_S}{\pi^2\sqrt{1+a_S^2 r_S^2}} \sum_{\substack{m=-\infty \\ m=\ell N_R}}^{\infty} e^{-im\theta_{bS}\cos\varphi_\alpha}\, e^{im\sigma_S}\, e^{-im\Omega t}$$

$$\cdot \int_0^{\pi}\int_{\rho_R} \frac{t_o}{c}(\rho_R)\,\rho_R\,\theta_{bR}\sqrt{1+a_R^2\rho_R^2}\, e^{im\sigma_R}\, e^{-im\theta_{bR}\cos\theta_\alpha}\sin\theta_\alpha\cos\theta_\alpha$$

$$\cdot \int_{-\infty}^{\infty}\left(ia_S k - \frac{im}{r_S^2}\right)(IK)_m\, e^{i(x_S'-\xi_R)k}\, dk\, d\rho_R\, d\theta_\alpha \qquad \text{(B-8)}$$

On substituting the values for x_S' and ξ_R given before

$$\left(W_S\right)_{R_t} = \frac{iN_R r_S}{\pi^2\sqrt{1+a_S^2 r_S^2}} \sum_{\substack{m=-\infty \\ m=\ell N_R}}^{\infty} e^{-im\theta_{bS}\cos\varphi_\alpha}\, e^{im\sigma_S}\, e^{-im\Omega t}$$

$$\cdot \int_0^{\pi}\int_{\rho_R} \frac{t_o}{c}(\rho_R)\,\rho_R\,\theta_{bR}\sqrt{1+a_R^2\rho_R^2} \int_{-\infty}^{\infty}\left(a_S k - \frac{m}{r_S^2}\right)(IK)_m$$

$$\cdot\, e^{-i\,k/a_S\,\theta_{bS}\cos\varphi_\alpha}\, e^{ik(\sigma_S/a_S+\epsilon_S)}\, e^{+i(m-k/a_R)\sigma_R}\, e^{-i(m-k/a_R)\theta_{bR}\cos\theta_\alpha}$$

$$\cdot\, \sin\theta_\alpha\cos\theta_\alpha\, dk\, d\rho_R\, d\theta_\alpha \qquad \text{(B-9)}$$

The θ_α-integral involves

$$\int_0^\pi e^{i\left(\frac{k}{a_R}-m\right)\theta_{bR}\cos\theta_\alpha} \sin\theta_\alpha \cos\theta_\alpha d\theta_\alpha$$

Let $u=k-a_R m$ in (B-9)

$$\int_0^\pi e^{i\frac{u}{a_R}\theta_{bR}\cos\theta_\alpha} \sin\theta_\alpha \cos\theta_\alpha d\theta_\alpha = \frac{i2a_R^2}{\theta_{bR}^2} F(u,\rho_R) \tag{B-10}$$

where

$$F(u,\rho_R) = \left[\frac{\sin(u\ \theta_{bR}/a_R) - (u\ \theta_{bR}/a_R)\cos(u\ \theta_{bR}/a_R)}{u^2}\right] .$$

Then

$$\left(W_S\right)_{R_t} = \frac{-2N_R r_S a_R^2}{\pi^2\sqrt{1+a_S^2 r_S^2}} \sum_{\substack{m=-\infty \\ m=\ell N_R}}^{\infty} e^{-im\Omega t}\ e^{im\left(1+\frac{a_R}{a_S}\right)\sigma_S}\ e^{ima_R\epsilon_S}$$

$$\cdot \int_{\rho_R} \frac{t_o}{c}(\rho_R) \frac{\rho_R}{\theta_{bR}} \sqrt{1+a_R^2\rho_R^2} \int_{-\infty}^{\infty} \left(a_S u + a_S a_R m - \frac{m}{r_S^2}\right)(IK)_m F(u,\rho_R)$$

$$\cdot\ e^{iu\left(\frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R} + \epsilon_S\right)} \cdot e^{-i\left(\frac{u}{a_S} + \left(1+\frac{a_R}{a_S}\right)m\right)\theta_{bS}\cos\varphi_\alpha} du d\rho_R \tag{B-11}$$

where

$$(IK)_m = I_m(|u+a_R m|\rho_R)K_m(|u+a_R m|r_S) \quad \text{for } \rho_R < r_S \ .$$

On applying the general'zed lift operator, the nondimensional velocity becomes, for each lift operator mode $\bar{m}$,

$$\bar{W}_{R_tS}^{(\bar{m})}(r_S) = \frac{-2a_R^2 r_S N_R}{\pi^2\sqrt{1+a_S^2 r_S^2}} \sum_{\substack{m=-\infty \\ m=\ell N_R}}^{\infty} e^{-im\Omega t}\ e^{im\left(\sigma_S\left(1+\frac{a_R}{a_S}\right) + a_R\epsilon_S\right)}$$

$$\cdot \int_{\rho_R} \frac{t_o}{c}(\rho_R) \frac{\rho_R}{\theta_{bR}} \sqrt{1+a_R^2\rho_R^2} \int_{-\infty}^{\infty} \left(a_S u + a_S a_R m - \frac{m}{r_S^2}\right)(IK)_m \cdot F(u;\rho_R)$$

[Cont'd]

$$\cdot\ e^{iu\left(\frac{\sigma_S}{a_S}-\frac{\sigma_R}{a_R}+\epsilon_S\right)}\cdot I^{(\bar{m})}\left(\left(-\frac{u}{a_S}-\left(1+\frac{a_R}{a_R}\right)m\ \theta_{bS}\right)du\,d\rho_R \qquad \text{(B-12)}$$

On changing the doubly infinite u-integral to an integral from 0 to $+\infty$, (B-12) can be written as

$$\bar{W}^{(\bar{m})}_{R_tS}(r_S) = \frac{-2a_R^2 r_S N_R}{\pi^2\sqrt{1+a_S^2 r_S^2}} \sum_{\substack{m=-\infty\\ m=\ell N_R}}^{\infty} e^{-im\Omega t}\, e^{im\left(\sigma_S\left(1+\frac{a_R}{a_S}\right)+a_R\epsilon_S\right)}$$

$$\cdot\int_{\rho_R}\frac{t_o}{c}(\rho_R)\,\frac{\rho_R}{\theta_{bR}}\sqrt{1+a_R^2\rho_R^2}\int_0^\infty F(u,\rho_R)$$

$$\cdot\Big\{I_m(|u+a_Rm|\rho_R)K_m(|u+a_Rm|r_S)\left(a_Su+a_Sa_Rm-\frac{m}{r_S^2}\right)$$

$$\cdot\ I^{(\bar{m})}\left(\left(-\frac{u}{a_S}-\left(1+\frac{a_R}{a_S}\right)m\right)\theta_{bS}\right)e^{iu\left(\frac{\sigma_S}{a_S}-\frac{\sigma_R}{a_R}+\epsilon_S\right)}$$

$$+\ I_m(|u-a_Rm|\rho_R)K_m(|u-a_Rm|r_S)\left(a_Su-a_Sa_Rm+\frac{m}{r_S^2}\right)$$

$$\cdot\ I^{(\bar{m})}\left(\left(\frac{u}{a_S}-\left(1+\frac{a_R}{a_S}\right)m\right)\theta_{bS}\right)e^{-iu\left(\frac{\sigma_S}{a_S}-\frac{\sigma_R}{a_R}+\epsilon_S\right)}\Big\}du\,d\rho_R \qquad \text{(B-13)}$$

The integrand of (B-13) is zero when u is zero since $F(0,\rho_R)=0$.

In the steady state condition $m=\ell=0$, the velocity on the stator due to rotor blade thickness can be shown to be

$$\bar{W}^{(0,\bar{m})}_{R_tS}(r_S) = \frac{-4a_R^2 a_S r_S N_R}{\pi^2\sqrt{1+a_S^2r_S^2}}\int_{\rho_R}\frac{t_o}{c}(\rho_R)\,\frac{\rho_R}{\theta_{bR}}\sqrt{1+a_R^2\rho_R^2}$$

$$\cdot\int_0^\infty u\,F(u,\rho_R)\,I_o(u\rho_R)K_o(ur_S)\cdot R\cdot P\cdot\left\{e^{-iu\left(\frac{\sigma_S}{a_S}-\frac{\sigma_R}{a_R}+\epsilon_S\right)} I^{(\bar{m})}\left(\frac{u}{a_S}\theta_{bS}\right)\right\}du\,d\rho_R$$

for $\rho_R < r_S$. (B-14)

In the unsteady case, $m = \ell N_R$, $\ell = +1,+2, \ldots$

$$\bar{W}_{R_t S}^{(\ell N_R, \bar{m})}(r_S) = \frac{-4a_R^2 r_S N_R}{\pi^2 \sqrt{1+a_S^2 r_S^2}} \, e^{i \ell N_R \left[\sigma_S \left(1+\frac{a_R}{a_S}\right) + a_R \epsilon_S\right]}$$

$$\cdot \int_{\rho_R} \frac{t_o}{c}(\rho_R) \frac{\rho_R}{\theta_{bR}} \sqrt{1+a_R^2 \rho_R^2} \int_0^\infty F(u,\rho_R) \left[G_2(u) - G_2(-u)\right] du \, d\rho_R$$

where

$$G_2(u) = I_{\ell N_R}(|u+a_R \ell N_R| \rho_R) K_{\ell N_R}(|u+a_R \ell N_R| r_S)$$

$$\cdot \left[a_S u + \ell N_R \left(a_S a_R - \frac{1}{r_S^2}\right)\right] e^{iu\left(\frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R} + \epsilon_S\right)}$$

$$\cdot I^{(\bar{m})}\left(\left(-\ell N_R \left(1+\frac{a_R}{a_S}\right) - \frac{u}{a_S}\right)\theta_{bS}\right) \qquad \text{(B-15)}$$

APPENDIX C

Evaluation of Singularity of $\bar{K}_{RR}$ as $u \to 0$*

The integral of Eq.(21) is of the form

$$\int_0^\infty \frac{g(\lambda) - g(-\lambda)}{\lambda} \tag{C-1}$$

where

$$g(\lambda) = I_m(|\lambda + a\ell N|\rho)\, K_m(|\lambda + a\ell N| r)\, B_{\bar{m},\bar{n}}(\lambda) e^{i\frac{\lambda}{a}\Delta\sigma} \quad \text{for } \rho < r$$

$$B_{\bar{m},\bar{n}}(\lambda) = (a\lambda + a^2\ell N + \frac{m}{r^2})(a\lambda + a^2\ell N + \frac{m}{\rho^2})\, I^{(\bar{m})}((q - \frac{\lambda}{a})\theta_b^r)\ \Lambda^{(\bar{n})}((q - \frac{\lambda}{a})\theta_b^\rho)$$

$$m = q + \ell N$$

By L'Hospital's rule the integrand at $\lambda = 0$ becomes

$$\lim_{\lambda \to 0} \frac{g(\lambda) - g(-\lambda)}{\lambda} = \left[\frac{\partial g(\lambda)}{\partial \lambda} - \frac{\partial g(-\lambda)}{\partial \lambda}\right]_{\lambda=0} \tag{C-2}$$

It is obvious that

$$B_{\bar{m},n}(\lambda)\Big|_{\lambda=0} = B_{\bar{m},\bar{n}}(-\lambda)\Big|_{\lambda=0}$$

$$\left[I_m(|\lambda + a\ell N|\rho) K_m(|\lambda + a\ell N| r)\right]_{\lambda=0} = \left[I_m(|-\lambda + a\ell N|\rho) K_m(|-\lambda + a\ell N| r)\right]_{\lambda=0}$$

and

$$e^{i\frac{\lambda}{a}\Delta\sigma}\Big|_{\lambda=0} = e^{-i\frac{\lambda}{a}\Delta\sigma}\Big|_{\lambda=0}$$

Then

*The development is taken from Reference 2.

$$\left[\frac{\partial g(\lambda)}{\partial\lambda} - \frac{\partial g(-\lambda)}{\partial\lambda}\right]_{\lambda=0} = 2i\,\frac{\Delta\sigma}{a}\,(IK)_m\Big|_{\lambda=0} B_{\bar{m},\bar{n}}(0)$$

$$+ (IK)_m\Big|_{\lambda=0}\left[\frac{\partial B_{\bar{m},\bar{n}}(\lambda)}{\partial\lambda} - \frac{\partial B_{\bar{m},\bar{n}}(-\lambda)}{\partial\lambda}\right]_{\lambda=0}$$

$$+ B_{\bar{m},\bar{n}}(0)\left\{\frac{\partial\left[I_m(|\lambda+a\ell N|\rho)K_m(|\lambda+a\ell N|r)\right]}{\partial\lambda} - \frac{\partial\left[I_m(|-\lambda+a\ell N|\rho)K_m(|-\lambda+a\ell N|r)\right]}{\partial\lambda}\right\}_{\lambda=0} \qquad \text{(C-3)}$$

Here $(IK)_m\Big|_{\lambda=0} = I_m(|a\ell N|\rho)K_m(|a\ell N|r)$ for $\rho \le r$ (C-4)

$$\frac{-\partial B_{\bar{m},\bar{n}}(-\lambda)}{\partial\lambda}\Bigg|_{\lambda=0} = \frac{+\partial B_{\bar{m},\bar{n}}(\lambda)}{\partial\lambda}\Bigg|_{\lambda=0} = a\left(2a^2\ell N + \frac{m}{r^2} + \frac{m}{\rho^2}\right) I^{(\bar{m})}(q\theta_b^r)\Lambda^{(\bar{n})}(q\theta_b^\rho)$$

$$+ \frac{i}{a}\left(a^2\ell N + \frac{m}{r^2}\right)\left(a^2\ell N + \frac{m}{\rho^2}\right)\left[-\theta_b^r I_1^{(\bar{m})}(q\theta_b^r)\Lambda^{(\bar{n})}(q\theta_b^\rho) + \theta_b^\rho I^{(\bar{m})}(q\theta_b^r)\Lambda_1^{(\bar{n})}(q\theta_b^\rho)\right]$$

(C-5)

and $I_1^{(\bar{m})}(x)$ and $\Lambda_1^{(\bar{n})}(x)$ are as defined in Appendix A.

The third term of (C-3) is treated as follows:

a) <u>For $\lambda = 0+$ and $a\ell N > 0$</u>

$$I_m(|\lambda+a\ell N|\rho)K_m(|\lambda+a\ell N|r) = I_m((\lambda+a\ell N)\rho)K_m((\lambda+a\ell N)r)$$

and

$$I_m(|-\lambda+a\ell N|\rho)K_m(|-\lambda+a\ell N|r) = I_m((a\ell N-\lambda)\rho)K_m((a\ell N-\lambda)r)$$

so that the third term of (C-3) becomes

$$2B_{\bar{m},\bar{n}}(0)\,\frac{\partial\left[I_m((\lambda+|a\ell N|)\rho)K_m((\lambda+|a\ell N|)r)\right]}{\partial\lambda}\Bigg|_{\lambda=0} \quad (\text{for } \rho \le r)$$

[Cont'd]

$$= 2B_{\bar{m},\bar{n}}(0)\left\{\frac{\rho}{2} K_m(|a\ell N|r)\left[I_{m-1}(|a\ell N|\rho) + I_{m+1}(|a\ell N|\rho)\right]\right.$$

$$\left. - \frac{r}{2} I_m(|a\ell N|\rho)\left[K_{m-1}(|a\ell N|r) + K_{m+1}(|a\ell N|r)\right]\right\} \qquad \text{(C-6)}$$

(Note that for $\rho \geq r$, ρ and r are interchanged in Eqs. C-3, C-4 and C-6.)

b) For λ = 0+ and $a\ell N < 0$

$$I_m(|\lambda+a\ell N|\rho)K_m(|\lambda+a\ell N|r) = I_m((|a\ell N|-\lambda)\rho)K_m((|a\ell N|-\lambda)r)$$

and

$$I_m(|-\lambda+a\ell N|\rho)K_m(|-\lambda+a\ell N|r) = I_m((|a\ell N|+\lambda)\rho)K_m((|a\ell N|+\lambda)r)$$

The third term of C-3 then becomes

$$- 2B_{\bar{m},\bar{n}}(0) \left.\frac{\partial\left[I_m((\lambda+|a\ell N|)\rho)K_m((\lambda+|a\ell N|)r)\right]}{\partial\lambda}\right|_{\lambda=0} \text{ for } \rho \leq r \qquad \text{(C-7)}$$

Therefore Eq. (C-3) can be written as

$$\left[\frac{\partial g(\lambda) - \partial(g-\lambda)}{\partial\lambda}\right]_{\lambda=0} = 2i\,\frac{\Delta\sigma}{a}\,(IK)_m\Big|_{\lambda=0}\, B_{\bar{m},\bar{n}}(0)$$

$$+ 2(IK)_m\Big|_{\lambda=0} \left.\frac{\partial B_{\bar{m},\bar{n}}(\lambda)}{\partial\lambda}\right|_{\lambda=0} \pm 2B_{\bar{m},\bar{n}}(0) \left.\frac{\partial(IK)_m}{\partial\lambda}\right|_{\lambda=0} \qquad \text{(C-8)}$$

where $(IK)_m\Big|_{\lambda=0}$ is given in (C-4)

$$B_{\bar{m},\bar{n}}(0) = (a^2\ell N+ \frac{m}{r^2})(a^2\ell N+ \frac{m}{\rho^2})\,I^{(\bar{m})}(q\theta_b^r)\Lambda^{(\bar{n})}(q\theta_b^\rho)$$

$\left.\frac{\partial B_{\bar{m},\bar{n}}(\lambda)}{\partial\lambda}\right|_{\lambda=0}$ is given in (C-5)

$\left.\frac{\partial(IK)_m}{\partial\lambda}\right|_{\lambda=0}$ is given in (C-6)

and the upper sign is taken when $\ell > 0$ and the lower sign when $\ell < 0$.

When $\ell = m = q = 0$, by the limiting process, it is easily shown that

$$\lim_{\ell \to 0} B_{\bar{m},\bar{n}}(0) \to \lim_{\ell \to 0} \ell^2 \to 0$$

$$\lim_{\ell \to 0} B_{\bar{m},\bar{n}}(0) \left. \frac{\partial (IK)_o}{\partial \lambda} \right|_{\lambda=0} \to \lim_{\ell \to 0} \frac{\ell^2}{\ell} \to 0$$

$$\lim_{\ell \to 0} (IK)_o \Big|_{\lambda=0} \cdot B_{\bar{m},\bar{n}}(0) \to \lim_{\ell \to 0} \ell^2 \log \ell \to 0$$

$$\lim_{\ell \to 0} (IK)_o \Big|_{\lambda=0} \left. \frac{\partial B_{\bar{m},\bar{n}}(\lambda)}{\partial \lambda} \right|_{\lambda=0} \to \lim_{\ell \to 0} (\ell + \ell^2) \log \ell \to 0$$

Hence when $\ell = m = q = 0$

$$\lim_{\lambda \to 0} \frac{g(\lambda) - g(-\lambda)}{\lambda} \to 0 \tag{C-9}$$

When $\ell = 0$ but $m = q \neq 0$, it is easily shown that

$$\lim_{\ell \to 0} (IK)_m \Big|_{\lambda=0} = \begin{cases} \dfrac{1}{2|m|} \left(\dfrac{\rho}{r}\right)^{|m|} & \text{for } \rho \leq r \\ \dfrac{1}{2|m|} \left(\dfrac{r}{\rho}\right)^{|m|} & \text{for } \rho \geq r \end{cases}$$

$$\lim_{\ell \to 0} \left. \frac{\partial (IK)_m}{\partial \lambda} \right|_{\lambda=0} \to 0$$

Hence for $\ell = 0$, $m = q \neq 0$

$$\begin{aligned} \lim_{\lambda \to 0} \frac{g(\lambda) - g(-\lambda)}{\lambda} \to 2 & \left\{ \lim_{\ell \to 0} (IK)_m \Big|_{\lambda=0} \right\} \\ & \cdot \left\{ I^{(\bar{m})}(q\theta_b^r) \Lambda^{(\bar{n})}(q\theta_b^\rho) \left[i \frac{\Delta\sigma}{a} \frac{m^2}{r^2 \rho^2} + am \left(\frac{1}{r^2} + \frac{1}{\rho^2} \right) \right] \right. \\ & \left. - \frac{im^2}{ar^2 \rho^2} \left[\theta_b^r I_1^{(\bar{m})}(q\theta_b^r) \Lambda^{(\bar{n})}(q\theta_b^\rho) - \theta_b^\rho I^{(\bar{m})}(q\theta_b^r) \Lambda_1^{(\bar{n})}(q\theta_b^\rho) \right] \right\} \end{aligned} \tag{C-10}$$

When $q = 0$ and Eq. (C-9) is used for the kernel functions, the value of the integrand at $\lambda = 0$ is zero for $m = 0$. For $m \neq 0$ it can be easily shown that the integrand at $\lambda = 0$ when $q = 0$, $\bar{m} = \ell N$ is

$$i \frac{4m^2}{a}\left(a^2 + \frac{1}{r^2}\right)\left(a^2 + \frac{1}{\rho^2}\right) I_m(am\rho) K_m(amr)$$

$$\cdot \left\{\Delta\sigma I^{(\bar{m})}(0)\Lambda^{(\bar{n})}(0) - \theta_b^r I_1^{(\bar{m})}(0)\Lambda^{(\bar{n})}(0) + \theta_b^\rho I^{(\bar{m})}(0)\Lambda_1^{(\bar{n})}(0)\right\} \quad \text{(C-11)}$$

APPENDIX D

Evaluation of the Singular k-Integral of $\bar{K}_{DR}$

The integral term of Eq. (31) can be written*

$$I = \int_{-\infty}^{\infty} \frac{F(k)\,dk}{k + a\ell N} \tag{D-1}$$

where $F(k) = (ak + \frac{m}{r_R^2})\,|k|\, I_m(|k|\, r_R)\left[K_{m-1}(|k|\, R_D) + K_{m+1}(|k|\, R_D)\right]$

$$\cdot\; I^{(\bar{m})}((m - \tfrac{k}{a})\theta_b)\; \Lambda^{(\bar{n})}(-kC_D)e^{-ik(\epsilon_D - \sigma/a)}$$

This integral exists in the sense of a Cauchy principal value. Therefore

$$I = \int_{-\infty}^{\infty} \frac{F(k) - F(-a\ell N)}{k + a\ell N)}\,dk + F(-a\ell N)\int_{-\infty}^{\infty} \frac{dk}{k + a\ell N}$$

$$= \int_{-\infty}^{\infty} \frac{F(k) - F(-a\ell N)}{k + a\ell N}\,dk \tag{D-2}$$

where $F(-a\ell N) = (-a^2\ell N + \frac{m}{r_R^2})(a\ell N) I_m(a\ell N r_R)\left[K_{m-1}(a\ell N R_D) + K_{m+1}(a\ell N R_D)\right]$

$$\cdot\; I^{(\bar{m})}(q\theta_b)\; \Lambda^{(\bar{n})}(a\ell N C_D)e^{ia\ell N(\epsilon_D - \sigma/a)}$$

and $-F(-a\ell N)$ is equivalent to $(\frac{i}{\pi})$ times the closed term of Eq. (31).

For large $|k| \geq |M|$, $|M| > a\ell N$

$$F(k) \approx (ak + \frac{m}{r^2})\,|k|\,\frac{e^{|k|r}}{\sqrt{2\pi|k|r}}\;\frac{2e^{-|k|R}}{\sqrt{2|k|R/\pi}}\;\; I^{(\bar{m})}(-\tfrac{k}{a}\theta_b)\;\Lambda^{(\bar{n})}(-kC)e^{-ik(\epsilon - \sigma/a)}$$

$$\approx (ak + \frac{m}{r^2})\,\frac{e^{-|k|(R-r)}}{\sqrt{rR}}\;\; I^{(\bar{m})}(-\tfrac{k}{a}\,\theta_b)\;\Lambda^{(\bar{n})}\;(-kC)e^{-ik(\epsilon - \sigma/a)}$$

*$a\ell N = a_R \ell\, N_R$ throughout Appendix D. The development is taken from Reference 5.

The factor

$$(ak + \frac{m}{r^2})\ e^{-|k|(R-r)} \to 0 \text{ as } k \to \infty$$

since $R > r$. The product of the other factors also tends to 0 as k becomes large. Therefore

$$I \approx \int_{-M}^{M} \frac{F(k) - F(-a\ell N)}{k + a\ N}\, dk \ - F(-a\ell N) \left[\int_{-\infty}^{-M} + \int_{M}^{\infty}\right] \frac{dk}{k+a\ell N} \tag{D-3}$$

Since

$$\left[\int_{-\infty}^{-M} + \int_{M}^{\infty}\right] \frac{dk}{k+a\ell N} = -2a\ell N \int_{M}^{\infty} \frac{dk}{k^2-a^2\ell^2N^2} = \log\left(\frac{M - a\ell N}{M + a\ell N}\right)$$

$$I \approx \int_{-M}^{M} \frac{F(k) - F(-a\ell N)}{k + a\ell N} dk - F(-a\ell N) \log\left(\frac{M - a\ell N}{M + a\ell N}\right)$$

Therefore

$$\bar{K}_{DR}^{(m,\bar{m},\bar{n})} \approx \frac{1}{4\pi\rho_f U r_o} \frac{r_R}{\sqrt{1+a^2 r_R^2}}\ e^{-im\sigma}$$

$$\cdot \left\{ i\pi a\ell N(-a^2\ell N + \frac{m}{r_R^2})\ I_m(a\ell N r_R) \left[K_{m-1}(a\ell NR_D) + K_{m+1}(a\ell NR_D)\right]\right.$$

$$\cdot\ e^{ia\ell N(\epsilon_D - \sigma/a)}\ I^{(\bar{m})}(q\theta_b)\ \Lambda^{(\bar{n})}(a\ell NC_D) \left[1 + \frac{i}{\pi} \log \frac{M-a\ell N}{M+a\ell N}\right]$$

$$\left. + \int_{-M}^{M} \frac{F(k) - F(a\ell N)}{k + a\ell N}\, dk \right\} \tag{D-4}$$

The singularity in the k-integral

The integral in (D-4) can be rewritten as

$$I_k = \int_{o}^{M} \frac{F'(k) - F'(a\ell N)}{(k+a\ell N)\ (k-a\ell N)}\, dk \tag{D-5}$$

where

$$F'(k) = kI_m(kr_R)\left[K_{m-1}(kR_D) + K_{m+1}(kR_D)\right]$$

$$\cdot \left\{(ak + \frac{m}{r_R^2})(k-a\ell N)\ I^{(\bar{m})}((m - \frac{k}{a})\theta_b)\ \Lambda^{(\bar{n})}(-kC_D)e^{-ik(\epsilon_D-\sigma/a)}\right.$$

$$\left. + \ (ak - \frac{m}{r_R^2})(k+a\ell N)\ I^{(\bar{m})}((m + \frac{k}{a})\theta_b)\ \Lambda^{(\bar{n})}(kC_D)\ e^{ik(\epsilon_D-\sigma/a)}\right\}$$

and

$$F'(a\ell N) = 2a^2\ell^2N^2 I_m(a\ell Nr_R)\left[K_{m-1}(a\ell NR_D) + K_{m+1}(a\ell NR_D)\right]$$

$$\cdot\ (a^2\ell N - \frac{m}{r_R^2})\ I^{(\bar{m})}(q\theta_b)\ \Lambda^{(\bar{n})}(a\ell NC_D)e^{ia\ell N(\epsilon_D-\sigma/a)}$$

At the singularity

$$\lim_{k\to a\ell N}\left\{\frac{F'(k) - F'(a\ell N)}{(k+a\ell N)(k-a\ell N)}\right\} = \left.\frac{\partial F'(k)}{\partial k}\right|_{k=a\ell N} \div 2a\ell N \qquad \text{(D-6)}$$

It is easily shown that (D-6) equals

$$I^{(\bar{m})}(q\theta_b)\ \Lambda^{(\bar{n})}(a\ell NC_D)e^{ia\ell N(\epsilon_D\ -\ \sigma/a)}$$

$$\cdot \left\{\left[\tfrac{1}{2}(5a^2\ell N - \frac{3m}{r_R^2}) + ia\ell N(\epsilon_D - \frac{\sigma}{a})(a^2\ell N - \frac{m}{r_R^2})\right] I_m(a\ell Nr_R)\left[-2K_m'(a\ell NR_D)\right]\right.$$

$$+\ a\ell Nr_R(a^2\ell N - \frac{m}{r^2})\ \ I_m'(a\ell Nr_R)\ \left[-2K_m'(a\ell NR_D)\right]$$

$$\left. +\ a\ell NR_D(a^2\ell N - \frac{m}{r_R^2})\ \ I_m(a\ell Nr_R)\ \left[K_{m-1}'(a\ell NR_D) + K_{m+1}'(a\ell NR_D)\right]\right\}$$

$$+ I_m(a\ell N r_R)\left[-2K_m'(a\ell N R_D)\right]$$

$$\cdot \left\{\frac{1}{2}\left(a^2\ell N + \frac{m}{r_R^2}\right) I^{(\bar{m})}((q-2\ell N)\theta_b)\Lambda^{(\bar{n})}(-a\ell N C_D) e^{-ia\ell N(\epsilon_D - \sigma/a)}\right.$$

$$\left. + ia\ell N\left(a^2\ell N - \frac{m}{r_R^2}\right) e^{ia\ell N(\epsilon_D - \sigma/a)}\left[-\frac{\theta_b}{a} I_1^{(\bar{m})}(q\theta_b)\Lambda^{(\bar{n})}(a\ell N C_D) - C_D I^{(\bar{m})}(q\theta_b)\Lambda_1^{(\bar{n})}(a\ell N C_D)\right]\right\} \qquad \text{(D-7)}$$

where $I_\nu'(z) = \dfrac{\partial I_\nu(z)}{\partial z}$, $K_\nu'(z) = \dfrac{\partial K_\nu(z)}{\partial z}$

and $I_1^{(\bar{m})}(x)$ and $\Lambda_1^{(\bar{n})}(x)$ are given in Appendix A.

When $\ell = 0$ $(m=q)$ and $k \to 0$

$$\left.\frac{\partial F'(k)}{\partial k}\right|_{k=a\ell N} \div 2a\ell N = \frac{(r_R)^q}{(R_D)^{q+1}} f(\bar{m},\bar{n}) \qquad \text{(D-8)}$$

where

$$f(\bar{m},\bar{n}) = 2aI^{(\bar{m})}(q\theta_b)\Lambda^{(\bar{n})}(0) - i\frac{m}{r_R^2}\left[\epsilon_D - \frac{\sigma}{a}\right] I^{(\bar{m})}(q\theta_b)\Lambda^{(\bar{n})}(0)$$

$$+ i\frac{m}{r_R^2}\left[C_D I^{(\bar{m})}(q\theta_b)\Lambda_1^{(\bar{n})}(0)\right.$$

$$\left. - \frac{\theta_b}{a} I_1^{(\bar{m})}(q\theta_b)\Lambda^{(\bar{n})}(0)\right]$$

When $m = q = 0$, $\ell = 0$, $k = 0$, the integrand is equal to zero.

APPENDIX E

Evaluation of Singularity of $\bar{K}_{SR}$ When $u \to 0$

The singularity of $\bar{K}_{SR}$ (see Eq.37) can be studied in a fashion similar to that used in Appendix C by making use of L'Hospital's Rule.

The $\bar{K}_{SR}$ singularity at $u=0$ is obtained through the limiting process:

$$\lim_{u\to 0} \frac{g_3(u) - g_3(-u)}{u} = \left[\frac{\partial g_3(u)}{\partial u} - \frac{\partial g_3(-u)}{\partial u} \right]_{u=0} \tag{E-1}$$

with

$$q_S = \ell N_R \quad , \; m_3 = q_R - \ell N_R$$

$$g_3(u) = I_{m_3}\left(|u-a_R q_S|\rho_S\right) K_{m_3}\left(|u-a_R q_S| r_R\right) \cdot B_{\bar{m},\bar{n}}(u)\, e^{-iu\left(\epsilon_S - \frac{\sigma_R}{a_R} - \frac{\sigma_S}{a_S}\right)}$$

$$\text{for} \quad \rho_S < r_R$$

where

$$B_{\bar{m},\bar{n}}(u) = \left(a_S u - a_S a_R q_S - \frac{m_3}{\rho_S^2}\right)\left(a_R u - a_R^2 q_S + \frac{m_3}{r_R^2}\right)$$

$$\cdot \Lambda^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S} q_S - \frac{u}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}\left(\left(m_3 + q_S - \frac{u}{a_R}\right)\theta_{bR}\right)$$

$$\left[\frac{\partial g_3(u)}{\partial u} - \frac{\partial g_3(-u)}{\partial u}\right]_{u=0} = -2i\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)\left[(IK)_{m_3}\right]_{u=0} B_{\bar{m},\bar{n}}(0)$$

$$+ (IK)_{m_3}\Big|_{u=0} \left[\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u} - \frac{\partial B_{\bar{m},\bar{n}}(-u)}{\partial u}\right]_{u=0}$$

$$+ B_{\bar{m},\bar{n}}(0)\left[\frac{\partial I_{m_3}(|u-a_R q_S|\rho_S) K_{m_3}(|u-a_R q_S| r_R)}{\partial u}\right.$$

$$\left. - \frac{\partial I_{m_3}(|-u-a_R q_S|\rho_S) K_{m_3}(|-u-a_R q_S| r_R)}{\partial u}\right]_{u=0} \tag{E-2}$$

$$(IK)_{m_3}\Big|_{u=0} = I_{m_3}(a_R q_S \rho_S)\,K_{m_3}(a_R q_S r_R)$$

$$= I_{m_3}(a_R \ell N_R \rho_S)\,K_{m_3}(a_R \ell N_R r_R)$$

for $\rho_S < N_R$ (otherwise, ρ_S and r_R are interchanged) (E-3)

$$(q_S = \ell N_R)$$

$$B_{\bar{m},\bar{n}}(0) = \left(a_S a_R q_S + \frac{m_3}{\rho_S^2}\right)\left(a_R^2 q_S - \frac{m_3}{r_R^2}\right)\Lambda^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}\right) I^{(\bar{m})}\left((m_3+q_S)\,\theta_{bR}\right) \tag{E-4}$$

$$\left[\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u} - \frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u}\right]_{u=0} = 2\,\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u}\Big|_{u=0}$$

$$= 2\Bigg\{\left[a_S\left(-a_R^2 q_S + \frac{m_3}{r_R^2}\right) + a_R\left(-a_S a_R q_S - \frac{m_3}{\rho_S^2}\right)\right]\Lambda^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}\right) I^{(\bar{m})}\left((m_3+q_S)\,\theta_{bR}\right)$$

$$+\left(-a_S a_R q_S - \frac{m_3}{\rho_S^2}\right)\left(-a_R^2 q_S + \frac{m_3}{r_R^2}\right)\Bigg[-i\,\frac{\theta_{bR}}{a_R}\,I_1^{(\bar{m})}\left((m_3+q_S)\,\theta_{bR}\right)\Lambda^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}\right)$$

$$+\,i\,\frac{\theta_{bS}}{a_S}\,I^{(\bar{m})}\left((m_3+q_S)\,\theta_{bR}\right)\Lambda_1^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}\right)\Bigg]\Bigg\}$$

$$= 2\Bigg\{\left(a_S\,\frac{m_3}{r_R^2} - \frac{a_R m_3}{\rho_S^2} - 2a_S a_R^2 q_S\right) I^{(\bar{m})}\left((m_3+q_S)\,\theta_{bR}\right)\Lambda^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}\right)$$

$$+\,i\left(a_S a_R q_S + \frac{m_3}{\rho_S^2}\right)\left(a_R^2 q_S - \frac{m_3}{r_R^2}\right)\Bigg[-\frac{\theta_{bR}}{a_R}\,I_1^{(\bar{m})}\left((m_3+q_S)\,\theta_{bR}\right)\Lambda^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}\right)$$

$$+\,\frac{\theta_{bS}}{a_S}\,I^{(\bar{m})}\left((m_3+q_S)\,\theta_{bR}\right)\Lambda_1^{(\bar{n})}\left(\left(-m_3 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}\right)\Bigg]\Bigg\} \tag{E-5}$$

$$\frac{\partial}{\partial u}\Big[I_{m_3}\left(|u - a_R q_S|\,\rho_S\right)K_{m_3}\left(|u - a_R q_S|\,r_R\right)\Big] - \frac{\partial}{\partial u}\Big[I_{m_3}\left(|-u - a_R q_S|\,\rho_S\right)K_{m_3}\left(|-u - a_R q_S|\,r_R\right)\Big]\Big|_{u=0}$$

$$= -2\Big\{\frac{\rho_S}{2}\,K_{m_3}(a_R q_S r_R)\Big[I_{m_3-1}(a_R q_S \rho_S) + I_{m_3+1}(a_R q_S \rho_S)\Big] -$$

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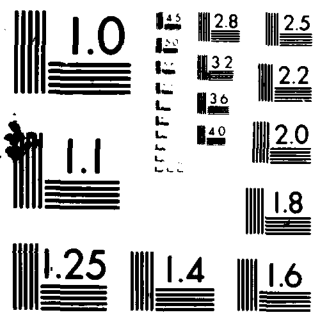

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$$- \frac{r_R}{2} I_{m_3}(a_R q_S \rho_S)\Big[K_{m_3-1}(a_R q_S r_R) + K_{m_3+1}(a_R q_S r_R)\Big]\Big\} \qquad \text{(E-6)}$$

for $\rho_S < r_R$. For $\rho_S > r_R$, ρ_S and r_R are interchanged in Eq.(E-6) above.

Then Eq.(E-1) yields

$$\left[\frac{\partial g_3(u)}{\partial u} - \frac{\partial g_3(-u)}{\partial u}\right]_{u=0} =$$

$$-2i\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)\Big\{I_m(a_R \ell N_R \rho_S) K_m(a_R \ell N_R r_R)\Big\}$$

$$\cdot \left\{\left(a_S a_R \ell N_R + \frac{m}{\rho_S^2}\right)\left(a_R^2 \ell N_R - \frac{m}{r_R^2}\right)\Lambda^{(\bar n)}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\right.$$

$$\left. \cdot I^{(\bar m)}\left(\left(m + \ell N_R\right)\theta_{bR}\right)\right\}$$

$$+2I_m(a_R \ell N_R \rho_S) K_m(a_R \ell N_R r_R)\left\{\frac{a_S m}{r_R^2} - \frac{a_R m}{\rho_S^2} - 2a_S a_R^2 \ell N_R\right) I^{(\bar m)}\left((m + \ell N_R)\theta_{bR}\right)$$

$$\cdot \Lambda^{(\bar n)}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)$$

$$+ i\left(a_S a_R \ell N_R + \frac{m}{\rho_S^2}\right)\left(a_R^2 \ell N_R - \frac{m}{r_R^2}\right)$$

$$\cdot \left[-\frac{\theta_{bR}}{a_R} I_1^{(\bar m)}\left((m+\ell N_R)\theta_{bR}\right)\Lambda^{(\bar n)}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\right.$$

$$\left. + \frac{\theta_{bS}}{a_S} I^{(\bar m)}\left((m+\ell N_R)\theta_{bR}\right)\Lambda_1^{(\bar n)}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\right]\right\}$$

$$-I^{(\bar m)}\left((m+\ell N_R)\theta_{bR}\right)\Lambda^{(\bar n)}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\left(a_S a_R \ell N_R + \frac{m}{\rho_S^2}\right)\left(a_R^2 \ell N_R - \frac{m}{r_R^2}\right)$$

$$\cdot \Big\{\rho_S K_m(a_R \ell N_R r_R)\Big[I_{m-1}(a_R \ell N_R \rho_S) + I_{m+1}(a_R \ell N_R \rho_S)\Big]$$

$$-r_R I_m(a_R \ell N_R \rho_S)\Big[K_{m-1}(a_R \ell N_R r_R) + K_{m+1}(a_R \ell N_R r_R)\Big]\Big\}$$

(E-7)

When $\ell=0$, then the integrand at $u=0$ becomes

$$\left(\frac{\rho_S}{r_R}\right)^m \left\{-i\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)\left(-\frac{m}{r_R^2\rho_S^2}\right)\Lambda^{(\bar{n})}(-m\theta_{bS})\,I^{(\bar{m})}(m\theta_{bR}) + \right.$$

$$+\left(\frac{a_S}{r_R^2} - \frac{a_R}{\rho_S^2}\right)I^{(\bar{m})}(m\theta_{bR})\Lambda^{(\bar{n})}(-m\theta_{bS})$$

$$\left. + i\left(-\frac{m}{\rho_S^2 r_R^2}\right)\left[-\frac{\theta_{bR}}{a_R}I_1^{(\bar{m})}\left(m\theta_{bR}\right)\Lambda^{(\bar{n})}(-m\theta_{bS}) + \frac{\theta_{bS}}{a_S}I^{(\bar{m})}(m\theta_{bR})\Lambda_1^{(\bar{n})}(-m\theta_{bS})\right]\right\}$$

(E-8)

When $\ell=0$ and $m=0$, the integrand is zero.

APPENDIX F

Evaluation of the Singular Part of $\bar{K}_{RS}$ at u=0

The singularity of Eq.(43) is determined in a similar fashion as before by performing the limiting process of the following expression:

$$\lim_{u\to 0}\left[\frac{g_4(u) - g_4(-u)}{u}\right] = \left[\frac{\partial g_4(u)}{\partial u} - \frac{\partial g_4(-u)}{\partial u}\right]_{u=0}$$

$$g_4(u) = I_{m_4}(|u - a_R q_S|\rho_R)K_{m_4}(|u - a_R q_S| r_S)$$

$$\cdot\, B_{\bar{m},\bar{n}}(u)\, e^{iu(\epsilon_S + \sigma_S/a_S - \sigma_R/a_R)} \qquad (\text{for } \rho_R < r_S) \tag{F-1}$$

where

$$B_{\bar{m},\bar{n}}(u) = \left(a_S u - a_S a_R q_S - \frac{m_4}{r_S^2}\right)\left(a_R u - a_R^2 q_S + \frac{m_4}{\rho_R^2}\right)$$

$$\cdot\, \Lambda^{(\bar{n})}\left(\left(m_4 + q_S - \frac{u}{a_R}\right)\theta_{bR}\right) I^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S - \frac{u}{a_S}\right)\theta_{bS}\right)$$

$$m_4 = q_R - \ell N_R \;\; (m_4 = \lambda_4 - q_S,\; q_S = \ell N_R,\; \lambda_4 = q_R)$$

$$g_4(u)\big|_{u=0} = g_4(-u)\big|_{u=0}$$

$$\left[\frac{\partial g_4(u)}{\partial u} - \frac{\partial g_4(-u)}{\partial u}\right]_{u=0} = 2i\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)\left[(IK)_{m_4}\right]_{u=0} B_{\bar{m},\bar{n}}(0)$$

$$+ (IK)_{m_4}\Big|_{u=0}\left[\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u} - \frac{\partial B_{\bar{m},\bar{n}}(-u)}{\partial u}\right]_{u=0}$$

$$+ B_{\bar{m},\bar{n}}(0)\left[\frac{\partial I_{m_4}(|u - a_R q_S|\rho_R)K_{m_4}(|u - a_R q_S| r_S)}{\partial u}\right.$$

$$\left. - \frac{\partial I_{m_4}(|-u - a_R q_S|\rho_R)K_{m_4}(|-u - a_R q_S| r_S)}{\partial u}\right]_{u=0} \tag{F-2}$$

$$(IK)_{m_4}\Big|_{u=0} = I_{m_4}(a_R q_S \rho_R) K_{m_4}(a_R q_S r_S)$$

$$= I_{m_4}(a_R \ell N_R \rho_R) K_{m_4}(a_R \ell N_R r_S) \qquad \text{for} \quad \rho_R < r_S \qquad \text{(F-3)}$$

$$B_{\bar{m},\bar{n}}(0) = \left(a_S a_R q_S + \frac{m_4}{r_S^2}\right)\left(a_R^2 q_S - \frac{m_4}{\rho_R^2}\right) \Lambda^{(\bar{n})}\Big((m_4 + q_S)\theta_{bR}\Big) I^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)$$

(F-4)

$$\frac{\partial B_{\bar{m},\bar{n}}(-u)}{\partial u}\Big|_{u=0} = \frac{-\partial B_{\bar{m},\bar{n}}(u)}{\partial u}\Big|_{u=0}$$

$$\left[\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u} - \frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u}\right]_{u=0} = 2\,\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u}$$

$$= 2\Big\{ a_S \left(-a_R^2 q_S + \frac{m_4}{\rho_R^2}\right)\Lambda^{(\bar{n})}\Big((m_4 + q_S)\theta_{bR}\Big) I^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)$$

$$+ a_R\left(-a_S a_R q_S - \frac{m_4}{r_S^2}\right)\Lambda^{(\bar{n})}\Big(\quad '' \quad \theta_{bR}\Big) I^{(\bar{m})}\Big(\Big(\quad '' \quad \Big)\theta_{bS}\Big)$$

$$+\left(-a_S a_R q_S - \frac{m_4}{r_S^2}\right)\left(-a_R^2 q_S + \frac{m_4}{\rho_R^2}\right)\Big[-i\,\frac{\theta_{bS}}{a_S}\, I_1^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)\Lambda^{(\bar{n})}\Big((m_4 + q_S)\theta_{bR}\Big)$$

$$+ i\,\frac{\theta_{bR}}{a_R}\, I^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)\Lambda_1^{(\bar{n})}\Big((m_4 + q_S)\theta_{bR}\Big)\Big]\Big\}$$

$$= 2\Big[\left(a_S \frac{m_4}{\rho_R^2} - a_R \frac{m_4}{r_S^2} - 2 a_S a_R^2 q_S\right) I^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)\Lambda^{(\bar{n})}\Big((m_4 + q_S)\theta_{bR}\Big)$$

$$+ i\left(a_S a_R q_S + \frac{m_4}{r_S^2}\right)\left(a_R^2 q_S - \frac{m_4}{\rho_R^2}\right)\Big[-\frac{\theta_{bS}}{a_S}\, I_1^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)\Lambda^{(\bar{n})}\Big((m_4 + q_S)\theta_{bR}\Big)$$

$$+ \frac{\theta_{bR}}{a_R}\, I^{(\bar{m})}\left(\left(-m_4 + \frac{a_R}{a_S} q_S\right)\theta_{bS}\right)\Lambda_1^{(\bar{n})}\Big((m_4 + q_S)\theta_{bR}\Big)\Big]\Big\}$$

(F-5)

$$\left\{ \frac{\partial}{\partial u} \left[I_{m_4}(|u - a_R q_S| \rho_R) K_{m_4}(|u - a_R q_S| r_s) \right] - \right.$$

$$\left. - \frac{\partial}{\partial u} \left[I_{m_4}(|-u-a_S q_S| \rho_R) K_{m_4}(|-u-a_R q_S| r_S) \right] \right\}_{u=0}$$

$$= -2 \left\{ \frac{\rho_R}{2} K_{m_4}(a_R q_S r_S) \left[I_{m_4-1}(a_R q_S \rho_R) + I_{m_4+1}(a_R q_S \rho_R) \right] \right.$$

$$\left. - \frac{r_S}{2} I_{m_4}(a_R q_S \rho_R) \left[K_{m_4-1}(a_R q_S r_S) + K_{m_4+1}(a_R q_S r_S) \right] \right\}$$

$$\text{for} \quad \rho_R < r_S \qquad \text{(F-6)}$$

For $\rho_R < r_S$, ρ_R and r_S are interchanged in (F-6) above. (See singularity of K_{SS}.)

Then Eq.(F-2) yields (for $\ell \neq 0 \quad m \neq 0$)

$$\left[\frac{\partial g_4(u)}{\partial u} - \frac{\partial g_4(-u)}{\partial u} \right]_{u=0} =$$

$$2i\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right) I_m(a_R \ell N_R \rho_R) K_m(a_R \ell N_R r_S)$$

$$\cdot \left[a_S a_R \ell N_R + \frac{m}{r_S^2} \right] \left[a_R^2 \ell N_R - \frac{m}{\rho_R^2} \right] \Lambda^{(\bar{n})}\left((m + \ell N_R)\theta_{bR}\right) I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)$$

$$+ 2 I_m(a_R \ell N_R \rho_R) K_m(a_R \ell N_R r_S)$$

$$\cdot \left\{ \left[a_S \frac{m}{\rho_R^2} - a_R \frac{m}{r_S^2} - 2 a_S a_R \ell N_R \right] \Lambda^{(\bar{n})}\left((m+\ell N_R)\theta_{bR}\right) I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) \right.$$

$$+ i\left(a_S a_R \ell N_R + \frac{m}{r_S^2}\right)\left(a_R^2 \ell N_R - \frac{m}{\rho_R^2}\right)\left[- \frac{\theta_{bS}}{a_S} \Lambda^{(\bar{n})}\left((m+\ell N_R)\theta_{bR}\right) I_1^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) \right.$$

$$\left.\left. + \frac{\theta_{bR}}{a_R} \Lambda_1^{(\bar{n})}\left((m+\ell N_R)\theta_{bR}\right) I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) \right] \right\}$$

$$-\left(a_S a_R \ell N_R + \frac{m}{r_S^2}\right)\left(a_R^2 \ell N_R - \frac{m}{\rho_R^2}\right) \Lambda^{(\bar{n})}\left((m+\ell N_R)\theta_{bR}\right) I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)$$

[Cont'd]

$$\cdot \left\{\rho_R K_m(a_R \ell N_R r_S)\left[I_{m-1}(a_R \ell N_R \rho_R) + I_{m+1}(a_R \ell N_R \rho_R)\right]\right.$$

$$\left. -r_S I_m(a_R \ell N_R \rho_R)\left[K_{m-1}(a_R \ell N_R r_S) + K_{m+1}(a_R \ell N_R r_S)\right]\right\} \qquad \text{(F-7)}$$

When $\ell = 0$, the integrand at $u=0$ becomes (for $m \neq 0$)

$$\left(\frac{\rho_R}{r_S}\right)^m \left\{- \frac{im}{r_S^2 \rho_R^2}\left(\epsilon_S + \frac{\sigma_S}{a_S} - \frac{\sigma_R}{a_R}\right)\Lambda^{(\bar{n})}(m\theta_{bR}) I^{(\bar{m})}(-m\theta_{bS})\right.$$

$$+\left(\frac{a_S}{\rho_R^2} - \frac{a_R}{r_S^2}\right)\Lambda^{(\bar{n})}(m\theta_{bR}) I^{(\bar{m})}(-m\theta_{bS}) +$$

$$\left. + \frac{im}{r_S^2 \rho_R^2}\left[+ \frac{\theta_{bS}}{a_S}\Lambda^{(\bar{n})}(m\theta_{bR}) I_1^{(\bar{m})}(-m\theta_{bS}) - \frac{\theta_{bR}}{a_R}\Lambda_1^{(\bar{n})}(m\theta_{bR}) I^{(\bar{m})}(-m\theta_{bS})\right]\right\}$$

and when $m=0$, $\ell=0$, the integrand at $u=0$ becomes zero.

APPENDIX G

Evaluation of Singularity of $\bar{K}_{DS}$ at u=0

Equation (52) has an integrable singularity at $k=-a_R q_S$. The integral term of Eq.(52) can be written as

$$I = \int_{-\infty}^{\infty} \frac{F(k)\,dk}{k+a_R \ell N_R} = \int_{-\infty}^{\infty} \frac{F(k) - F(-a_R \ell N_R)}{k+a_R \ell N_R}\,dk \qquad \text{(G-1)}$$

(see Appendix D) where

$$F(k) = \left(a_S k - \frac{m}{r_S^2}\right)|k|\, I_m(|k| r_S)\left[K_{m-1}(|k|R_D) + K_{m+1}(|k|R_D)\right]$$
$$\cdot\, I^{(\bar{m})}\left(\left(-m - \frac{k}{a_S}\right)\theta_{bS}\right)\Lambda^{(\bar{n})}(-kC_D)\, e^{-ik(\epsilon_D-\epsilon_S - \sigma_S/a_S)}$$

$$F(-a_R\ell N_R) = \left(-a_S a_R \ell N_R - \frac{m}{r_S^2}\right)(a_R \ell N_R)\, I_m(a_R \ell N_R r_S)\left[K_{m-1}(a_R \ell N_R R_D) + K_{m+1}(a_R \ell N_R R_D)\right]$$
$$\cdot\, I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\Lambda^{(\bar{n})}\left(a_R \ell N_R C_D\right) e^{+i a_R \ell N_R(\epsilon_D-\epsilon_S - \sigma_S/a_S)}$$

and $-F(-a_R \ell N_R) = \frac{i}{\pi}$ times the closed term of Eq.(52)

For large $|k| \geq |M|$, $|M| > a_R \ell N_R$

$$F(k) \approx \left(a_S k - \frac{m}{r_S^2}\right)|k|\, \frac{e^{|k| r_S}}{\sqrt{2\pi |k| r_S}}\, \frac{2e^{-|k|R_D}}{\sqrt{2|k|R_D/\pi}}\, I^{(\bar{m})}\left(-\frac{k}{a_S}\theta_{bS}\right)\Lambda^{(\bar{n})}(-kC_D)\, e^{-ik(\epsilon_D-\epsilon_S - \sigma_S/a_S)}$$
$$\approx \left(a_S k - \frac{m}{r_S^2}\right) \frac{e^{-|k|(R_D - r_S)}}{\sqrt{r_S R_D}}\, I^{(\bar{m})}\left(-\frac{k}{a_S}\theta_{bS}\right)\Lambda^{(\bar{n})}(-kC_D)\, e^{-ik(\epsilon_D-\epsilon_S - \sigma_S/a_S)}$$

which tends to zero as $k \to \infty$. Therefore,

$$I \approx \int_{-M}^{M} \frac{F(k) - F(-a_R \ell N_R)}{k+a_R \ell N_R}\,dk - F(-a_R \ell N_R)\left[\int_{-\infty}^{-M} + \int_{M}^{\infty}\right]\frac{dk}{k+a_R \ell N_R} \qquad \text{(G-2)}$$

Since

$$\left[\int_{-\infty}^{-M} + \int_{+M}^{\infty}\right] \frac{dk}{k+a_R \ell N_R} = -2a_R \ell N_R \int_{M}^{\infty} \frac{dk}{k^2 - a_R^2 \ell^2 N_R^2} = \log \frac{(M-a_R \ell N_R)}{(M+a_R \ell N_R)}$$

$$I \approx \int_{-M}^{M} \frac{F(k) - F(-a_R \ell N_R)}{k+a_R \ell N_R} dk - F(-a_R \ell N_R) \log \left(\frac{M-a_R \ell N_R}{M+a_R \ell N_R}\right)$$

Therefore

$$\bar{K}_{DS}^{(m,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U^2 r_{RO}} \frac{r_S}{\sqrt{1+a_S^2 r_S^2}} e^{im\sigma_S}$$

$$\cdot\left\{-i\pi a_R \ell N_R \left(a_S a_R \ell N_R + \frac{m}{r_S^2}\right) e^{ia_R \ell N_R (\epsilon_D - \epsilon_S - \sigma_S/a_S)} I_m(a_R \ell N_R r_S)\right.$$

$$\cdot\left[K_{m-1}(a_R \ell N_R R_D) + K_{m+1}(a_R \ell N_R R_D)\right] I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) \Lambda^{(\bar{n})}(a_R \ell N_R C_D)$$

$$\left.\cdot\left[1 + \frac{i}{\pi} \log \left(\frac{M-a_R \ell N_R}{M+a_R \ell N_R}\right)\right] + \int_{-M}^{M} \frac{F(k) - F(-a_R \ell N_R)}{k+a_R \ell N_R} dk\right\} \tag{G-3}$$

The integral in Eq.(G-3) can be rewritten as

$$I_k = \int_{0}^{M} \frac{F'(k) - F'(a_R \ell N_R)}{k^2 - a_R^2 \ell^2 N_R^2} dk \tag{G-4}$$

where

$$F'(k) = (k - a_R \ell N_R) F(k) - (k+a_R \ell N_R) F(-k)$$

and

$$F'(a_R \ell N_R) = -2a_R \ell N_R \; F(-a_R \ell N_R)$$

At the singularity

$$\lim_{k \to a_R \ell N_R} \left\{\frac{F'(k) - F'(a_R \ell N_R)}{(k+a_R \ell N_R)(k-a_R \ell N_R)}\right\} = \left.\frac{\partial F'(k)}{\partial k}\right|_{k=a_R \ell N_R} \div 2a_R \ell N_R \tag{G-5}$$

with

$$F'(k) = k I_m(k r_S)\left[K_{m-1}(k R_D) + K_{m+1}(k R_D)\right]$$

[Cont'd]

$$\cdot\left\{\left(a_S k - \frac{m}{r_S^2}\right)\left(k - a_R \ell N_R\right) I^{(\bar{m})}\left(\left(-m - \frac{k}{a_S}\right)\theta_{bS}\right)\Lambda^{(\bar{n})}(-kC_D)\, e^{-ik(\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right.$$

$$\left. + \left(a_S k + \frac{m}{r_S^2}\right)\left(k + a_R \ell N_R\right) I^{(\bar{m})}\left(\left(-m + \frac{k}{a_S}\right)\theta_{bS}\right)\Lambda^{(\bar{n})}(kC_D)\, e^{ik(\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right\}$$

After some lengthy manipulations, Eq.(G-5) becomes

fo. $\ell \neq 0 \quad m \neq 0$

$$\frac{1}{2a_R \ell N_R}\frac{\partial F'(k)}{\partial k} = I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\Lambda^{(\bar{n})}(a_R \ell N_R C_D)\, e^{ia_R \ell N_R(\epsilon_D - \epsilon_S - \sigma_S/a_S)}$$

$$\cdot\left\{ I_m(a_R \ell N_R r_S)\left[-2K_m'(a_R \ell N_R R_D)\right]\left[\frac{5}{2} a_S a_R \ell N_R + \frac{3}{2}\frac{m}{r_S^2} + i a_R \ell N_R\left(\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S}\right)\left(a_S a_R \ell N_R + \frac{m}{r_S^2}\right)\right]\right.$$

$$+ I_m'(a_R \ell N_R r_S)\left[-2K_m'(a_R \ell N_R R_D)\right](a_R \ell N_R r_S)\left(a_S a_R \ell N_R + \frac{m}{r_S^2}\right)$$

$$\left. + I_m(a_R \ell N_R r_S)\left[K_{m-1}'(a_R \ell N_R R_D) + K_{m+1}'(a_R \ell N_R R_D)\right](a_R \ell N_R R_D)\left(a_S a_R \ell N_R + \frac{m}{r_S^2}\right)\right\}$$

$$+ I_m(a_R \ell N_R r_S)\left[-2K_m'(a_R \ell N_R R_D)\right]\cdot\left\{\left(\frac{1}{2} a_S a_R \ell N_R - \frac{1}{2}\frac{m}{r_S^2}\right) I^{(\bar{m})}\left(\left(-m - \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\right.$$

$$\left. \cdot\, \Lambda^{(\bar{n})}(-a_R \ell N_R C_D)\, e^{-ia_R \ell N_R(\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right\}$$

$$+ I_m(a_R \ell N_R r_S)\left[-2K_m'(a_R \ell N_R R_D)\right](i a_R \ell N_R)\left(a_S a_R \ell N_R + \frac{m}{r_S^2}\right) e^{ia_R \ell N_R(\epsilon_D - \epsilon_S - \sigma_S/a_S)}$$

$$\cdot\left\{\frac{\theta_{bS}}{a_S} I_1^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\Lambda^{(\bar{n})}(a_R \ell N_R C_D) - C_D I^{(\bar{m})}\left(\left(-m + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\Lambda_1^{(\bar{n})}(a_R \ell N_R C_D)\right\} \tag{G-6}$$

for $\ell = 0 \quad m \neq 0$

$$\frac{(r_S)^m}{(R_D)^{m+1}}\left\{\left[2a_S + i\frac{m}{r_S^2}\left(\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S}\right)\right] I^{(\bar{m})}(-m\theta_{bS})\Lambda^{(\bar{n})}(0)\right.$$

$$\left. - i\frac{m}{r_S^2}\left[C_D I^{(\bar{m})}(-m\theta_{bS})\Lambda_1^{(\bar{n})}(0) - \frac{\theta_{bS}}{a_S} I_1^{(\bar{m})}(-m\theta_{bS})\Lambda^{(\bar{n})}(0)\right]\right\} \tag{G-7}$$

When $m=0$, $k=0$, $\ell=0$, the integrand is equal to zero.

APPENDIX H

Evaluation of Singularity of $\bar{K}_{SS}$ at u=0

The singularity of $\bar{K}_{SS}$ (see Eq.59) at u=0 is evaluated by means of L'Hospital's rule

$$\lim_{u\to 0} \frac{g_6(u) - g_6(-u)}{u} = \left[\frac{\partial g_6(u)}{\partial u} - \frac{\partial g_6(-u)}{\partial u}\right]_{u=0} \tag{H-1}$$

$$g_6(u) = I_{m_6}(|u-a_R q_S|\rho_S)K_{m_6}(|u-a_R q_S| r_S)\cdot B_{\bar{m},\bar{n}}(u) \qquad (\text{for } \rho_S < r_S)$$

Here $q_S = \underset{\ell\geq 0}{\ell N_R}$ $\quad m_6 = q_S + \ell_6 N_S \quad (\ell_6 = 0, \pm 1, \pm 2, \ldots)$, and

$$B_{\bar{m},\bar{n}}(u) = \left(a_S u - a_S a_R q_S + \frac{m_6}{r_S^2}\right)\left(a_S u - a_S a_R q_S + \frac{m_6}{\rho_S^2}\right)$$

$$\cdot I^{(\bar{m})}\left(\left(m_6 + \frac{a_R}{a_S} q_S - \frac{u}{a_S}\right)\theta^r_{bS}\right)\Lambda^{(\bar{n})}\left(\left(m_6 + \frac{a_R}{a_S} q_S - \frac{u}{a_S}\right)\theta^\rho_{bS}\right)$$

It is obvious that

$$+g_6(u)\Big|_{u=0} = +g_6(-u)\Big|_{u=0}$$

$$\left[\frac{\partial g_6(u)}{\partial u} - \frac{\partial g_6(-u)}{\partial u}\right]_{u=0} = (IK)_{m_6}\Big|_{u=0}\left[\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u} - \frac{\partial B_{\bar{m},\bar{n}}(-u)}{\partial u}\right]_{u=0}$$

$$+B_{\bar{m},\bar{n}}(0)\left\{\frac{\partial}{\partial u}\left[I_m\left(|u-a_R \ell N_R|\rho_S\right)K_m\left(|u-a_R \ell N_R| r_S\right)\right]\right.$$

$$\left. - \frac{\partial}{\partial u}\left[I_m\left(|-u-a_R \ell N_R|\rho_S\right)K_m\left(|-u-a_R \ell N_R| r_S\right)\right]\right\}_{u=0} \tag{H-2}$$

$$(IK)_{m_6}\Big|_{u=0} = I_{m_6}(a_R \ell N_R \rho_S)K_{m_6}(a_R \ell N_R r_S) = I_{m_6}(a_R q_S \rho_S)K_{m_6}(a_R q_S r_S) \quad \text{for } \rho_S < r_S \tag{H-3}$$

$$B_{\bar{m},\bar{n}}(0) = \left(a_S a_R \ell N_R - \frac{m_6}{r_S^2}\right)\left(a_S a_R \ell N_R - \frac{m_6}{\rho_S^2}\right) I^{(\bar{m})}\left(\left(m_6 + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}^r\right)$$

$$\cdot\ \Lambda^{(\bar{n})}\left(\left(m_6 + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}^\rho\right) \qquad \text{(H-4)}$$

$$m_6 = q_S + \ell_6 N_S\ ,\ m_6 = \ell N_R + \ell_6 N_S\ ,\quad \begin{array}{l}\ell\ = 0,+1,+2,\ \dots\\ \ell_6 = 0,\pm 1,\pm 2,\ \dots\end{array}$$

$$\left.\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u}\right|_{u=0} = a_S\left(-a_S a_R q_S + \frac{m_6}{\rho_S^2}\right) I^{(\bar{m})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^r\right)\Lambda^{(\bar{n})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^\rho\right)$$

$$+\ a_S\left(-a_S a_R q_S + \frac{m_6}{r_S^2}\right) I^{(\bar{m})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^r\right)\Lambda^{(\bar{n})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^\rho\right)$$

$$+\ \left(-a_S a_R q_S + \frac{m_6}{r_S^2}\right)\left(-a_S a_R q_S + \frac{m_6}{\rho_S^2}\right)$$

$$\cdot\ \left[-i\ \frac{\theta_{bS}^r}{a_S}\ I_1^{(\bar{m})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^r\right)\right] \Lambda^{(\bar{n})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^\rho\right)$$

$$+\ \left(-a_S a_R q_S + \frac{m_6}{r_S^2}\right)\left(-a_S a_R q_S + \frac{m_6}{\rho_S^2}\right) I^{(\bar{m})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^r\right)$$

$$\cdot\ \left[+i\ \frac{\theta_{bS}^\rho}{a_S}\ \Lambda_1^{(\bar{n})}\left(\left(m_6 + \frac{a_R}{a_S}\,q_S\right)\theta_{bS}^\rho\right)\right]$$

(See Appendix B of Reference 2.)

$$\left.\frac{\partial B_{\bar{m},\bar{n}}(-u)}{\partial u}\right|_{u=0} = -\left.\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u}\right|_{u=0}$$

Hence

$$\left[\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u} - \frac{\partial B_{\bar{m},\bar{n}}(-u)}{\partial u}\right]_{u=0} = 2\left.\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u}\right|_{u=0}$$

Thus

$$\left[\frac{\partial B_{\bar{m},\bar{n}}(u)}{\partial u} - \frac{\partial B_{\bar{m},\bar{n}}(-u)}{\partial u}\right]_{u=0} = 2a_S\left(\frac{m_6}{\rho_S^2} + \frac{m_6}{r_S^2} - 2a_S a_R q_S\right) I^{(\bar{m})}\left(z\theta_{bS}^r\right)\Lambda^{(\bar{n})}\left(z\theta_{bS}^\rho\right)$$

[Cont'd]

$$+\frac{i2}{a_S}\left(\frac{m_6}{\rho_S^2} - a_S a_R q_S\right)\left(\frac{m_6}{r_S^2} - a_S a_R q_S\right)\Big[-\theta_{bS}^{r} I_1^{(\bar{m})}(z\theta_{bS}^{r})\Lambda^{(\bar{n})}(z\theta_{bS}^{\rho})$$

$$+\theta_{bS}^{\rho} I^{(\bar{m})}(z\theta_{bS}^{r})\Lambda^{(\bar{n})}(z\theta_{bS}^{\rho})\Big] \qquad \text{(H-5)}$$

where $\quad z = m_6 + \dfrac{a_R}{a_S} q_S \quad .$

The first term of Eq.(H-2) is therefore given by the product of (H-3) and (H-5).

The second term of Eq.(H-2) is treated as follows:

Since $u=0+$ and $a_R q_S > 0$

$$I_{m_6}(|u - a_R q_S|\rho_S)K_{m_6}(|u - a_R q_S|r_S) =$$

$$I_{m_6}((a_R q_S - u)\rho_S)K_{m_6}((a_R q_S - u)r_S)$$

and

$$I_{m_6}(|-u- a_R q_S|\rho_S)K_{m_6}(|-u- a_R q_S|r_S) =$$

$$I_{m_6}((a_R q_S + u)\rho_S)K_{m_6}((a_R q_S + u)r_S) \text{ for } \rho_S < r_S$$

The second term of Eq.(H-2) then becomes

$$-2B_{\bar{m},\bar{n}}(0)\left\{\frac{\rho_S}{2} K_{m_6}(a_R q_S r_S)\Big[I_{m_6-1}(a_R q_S \rho_S) + I_{m_6+1}(a_R q_S \rho_S)\Big]\right.$$

$$\left.-\frac{r_S}{2} I_{m_6}(a_R q_S \rho_S)\Big[K_{m_6-1}(a_R q_S r_S) + K_{m_6+1}(a_R q_S r_S)\Big]\right\} \qquad \text{(H-6)}$$

Thus the singularity of K_{SS} at $u=0$, when $\ell\neq 0$ $m_6\neq 0$, is given by

$$\left[\frac{\partial g_6(u)}{\partial u} - \frac{\partial g_6(-u)}{\partial u}\right]_{u=0} =$$

$$= I_{m_6}(a_R q_S \rho_S)K_{m_6}(a_R q_S r_S)\left\{2a_S\left(\frac{m_6}{\rho_S^2} + \frac{m_6}{r_S^2} - 2a_S a_R q_S\right) I^{(\bar{m})}(z\theta_{bS}^{r})\Lambda^{(\bar{n})}(z\theta_{bS}^{\rho})\right.$$

$$+\frac{i2}{a_S}\left(\frac{m_6}{\rho_S^2} - a_S a_R q_S\right)\left(\frac{m_6}{r_S^2} - a_S a_R q_S\right)\Big[-\theta_{bS}^{r} I_1^{(\bar{m})}(z\theta_{bS}^{r})\Lambda^{(\bar{n})}(z\theta_{bS}^{\rho})$$

[Cont'd]

$$+ \theta^{\rho}_{bS} I^{(\bar{m})}(z\theta^{r}_{bS}) \Lambda^{(\bar{n})}_{1}(z\theta^{\rho}_{bS}) \Big] \Big\}$$

$$-2\left(a_S a_R q_S - \frac{m_6}{r_S^2}\right)\left(a_S a_R q_S - \frac{m_6}{\rho_S^2}\right) I^{(\bar{m})}(z\theta^{r}_{bS}) \Lambda^{(\bar{n})}(z\theta^{\rho}_{bS}) \cdot$$

$$\cdot \Big\{ \frac{\rho_S}{2} K_{m_6}(a_R q_S r_S) \Big[I_{m_6-1}(a_R q_S \rho_S) + I_{m_6+1}(a_R q_S \rho_S) \Big]$$

$$- \frac{r_S}{2} I_{m_6}(a_R q_S \rho_S) \Big[K_{m_6-1}(a_R q_S r_S) + K_{m_6+1}(a_R q_S r_S) \Big] \Big\}$$

$$z = m_6 + \frac{a_R}{a_S} q_S \quad . \tag{H-7}$$

<u>When $\ell=0 \;\; m\neq 0$</u> , i.e., $q_S=0 \;\; m_6=\ell_6 N_S \neq 0$

$$\left[\frac{\partial g_6(u)}{\partial u} - \frac{\partial g_6(-u)}{\partial u}\right]_{u=0} = \frac{1}{2}\left(\frac{\rho_S}{r_S}\right)^{|m|} \Big\{ 2a_S \left(\frac{1}{\rho_S^2} + \frac{1}{r_S^2}\right) I^{(\bar{m})}(m\theta^{r}_{bS}) \Lambda^{(\bar{n})}(m\theta^{\rho}_{bS}) \text{ (sign m)}$$

$$+ \frac{i2}{a_S} \frac{|m|}{\rho_S^2 r_S^2} \Big[-\theta^{r}_{bS} I^{(\bar{m})}_{1}(m\theta^{r}_{bS}) \Lambda^{(\bar{n})}(m\theta^{\rho}_{bS}) + \theta^{\rho}_{bS} I^{(\bar{m})}(m\theta^{r}_{bS}) \Lambda^{(\bar{n})}_{1}(m\theta^{\rho}_{bS}) \Big] \Big\} \tag{H-8}$$

<u>When $\ell=0 \;\; m=0$</u> , it can be shown that

$$\left[\frac{\partial g_6(u)}{\partial u} - \frac{\partial g_6(-u)}{\partial u}\right]_{u=0} = 0 \tag{H-9}$$

APPENDIX I

Evaluation of the Singular k-Integral of $\overline{K}_{RD}$*

The k-integral of Eq. (87) can be written as (see Appendix D)

$$I = \int_{-\infty}^{\infty} \frac{G(k) - G(-a\ell N)}{k + a\ell N}\, dk \tag{I-1}$$

where subscripts are omitted and

$$G(k) = -\frac{1}{2\pi a}\left(ak + \frac{\nu}{\rho_R^2}\right) |k|\, I_\nu(|k|\rho_R) \left[K_{\nu-1}(|k| R_D) + K_{\nu+1}(|k| R_D)\right]$$

$$\cdot\ \Lambda^{(\bar{n})}\left(\left(\nu - \frac{k}{a}\right)\theta_b\right) I^{(\bar{m})}(-kC_D)\, e^{ik(\epsilon_D - \sigma/a)}$$

and

$$G(-a\ell N) = -\frac{i}{\pi}\left\{\text{closed term of Eq. (87)}\right\}$$

It can be shown (following Appendix D) that for large $|k| \geq |M| > a\ell N$, $G(k)$ is approximately zero. Therefore

$$I \approx \int_{-M}^{M} \frac{G(k) - G(-a\ell N)}{k + a\ell N}\, dk - G(-a\ell N)\log\left(\frac{M - a\ell N}{M + a\ell N}\right) \tag{I-2}$$

and

$$\overline{K}_{RD}^{(\nu,\bar{m},\bar{n})} \approx \frac{Ne^{i\nu\sigma}}{4\pi\rho_f U_{RO}^2}\left\{+\frac{i\ell N}{2}\left(a^2\ell N - \frac{\nu}{\rho_R^2}\right) I_\nu(a\ell N\rho_R)\left[K_{\nu-1}(a\ell NR_D) + K_{\nu+1}(a\ell NR_D)\right]\right.$$

$$\cdot\ \Lambda^{(\bar{n})}((\ell N+\nu)\theta_b)\, I^{(\bar{m})}(a\ell NC_D)\, e^{-ia\ell N(\epsilon_D - \sigma/a)}\left[1 + \frac{i}{\pi}\log\left(\frac{M - a\ell N}{M + a\ell N}\right)\right]$$

$$\left. + \int_{-M}^{M} \frac{G(k) - G(-a\ell N)}{k + a\ell N}\, dk\right\} \tag{I-3}$$

The singularity in the k-integral

The integral in (I-3) can be rewritten as

---

*The development is taken from Reference 5.

$$I_k = \int_0^M \frac{G'(k) - G'(a\ell N)}{(k+a\ell N)(k-a\ell N)}\, dk \tag{I-4}$$

where

$$G'(k) = -\frac{k}{2\pi a}\, I_\nu(k\rho_R)\left[K_{\nu-1}(kR_D) + K_{\nu+1}(kR_D)\right]$$

$$\cdot \left\{(ak + \frac{\nu}{\rho_R^2})(k-a\ell N)\ \Lambda^{(\bar{n})}((\nu - \frac{k}{a})\theta_b)\ I^{(\bar{m})}\ (-kC_D)e^{ik(\epsilon_D-\sigma/a)}\right.$$

$$\left. + \ (ak - \frac{\nu}{\rho_R^2})(k+a\ell N)\ \Lambda^{(\bar{n})}((\nu + \frac{k}{a})\theta_b)\, I^{(\bar{m})}(kC_D)e^{-ik(\epsilon_D-\sigma/a)}\right\}$$

and

$$G'(a\ell N) = -\frac{a\ell^2 N^2}{\pi}\ I_\nu(a\ell N\rho_R)\left[K_{\nu-1}(a\ell NR_D) + K_{\nu+1}(a\ell NR_D)\right]$$

$$\cdot\ (a^2\ell N - \frac{\nu}{\rho_R^2})\Lambda^{(\bar{n})}((\nu+\ell N)\theta_b)\ I^{(\bar{m})}(a\ell NC_D)e^{-ia\ell N(\epsilon_D-\sigma/a)}$$

At the singularity $k = a\ell N$ the integrand is

$$\lim_{k\to a\ell N}\left\{\frac{G'(k) - G'(a\ell N)}{(k+a\ell N)\ (k-a\ell N)}\right. = \left.\frac{\partial G'(k)}{\partial k}\right|_{k = a\ell N} \div\ 2a\ell N \tag{I-5}$$

It can be readily shown that (I-5) is equal to

$$\frac{1}{\pi}\ I^{(\bar{m})}(a\ell NC_D)\ \Lambda^{(\bar{n})}\ ((\nu + \ell N)\theta_b)e^{-ia\ell N(\epsilon_D-\sigma/a)}$$

$$\cdot\left\{\left[\frac{5}{2}a\ell N - \frac{3}{2}\frac{\nu}{a\rho_R^2} - ia\ell N(\epsilon_D - \frac{\sigma}{a})(a\ell N - \frac{\nu}{a\rho_R^2})\right]I_\nu(a\ell N\rho_R)K'_\nu(a\ell NR_D)\right.$$

$$+\ a\ell N\rho_R(a\ell N - \frac{\nu}{a\rho_R^2})\ I'_\nu(a\ell N\rho_R)\ K'_\nu(a\ell NR_D)$$

$$+ a\ell NR_D\left(a\ell N - \frac{\nu^2}{a\rho_R}\right) I_\nu(a\ell N\rho_p)\ K''_\nu(a\ell NR_D)\Big\}$$

$$+ \frac{1}{\pi}\ I'_\nu(a\ell N\rho_R)\ K'_\nu(a\ell NR_D)$$

$$\cdot \Big\{ \frac{1}{2}\ \left(a\ell N + \frac{\nu^2}{a\rho_R^2}\right) I^{(\bar{m})}\ (-a\ell NC_D)\ \Lambda^{(\bar{n})}((\nu-\ell N)\theta_b)e^{ia\ell N(\epsilon_D-\sigma/a)}$$

$$+ ia\ell Ne^{-ia\ell N(\epsilon_D-\sigma/a)}\left(a\ell N - \frac{\nu^2}{a\rho_R^2}\right)\Big[C_D I_1^{(\bar{m})}(a\ell NC_D)\ \Lambda^{(\bar{n})}((\nu+\ell N)\theta_b)$$

$$- \frac{\theta_b}{a}\ I^{(\bar{m})}(a\ell NC_D)\ \Lambda_1^{(\bar{n})}((\nu+\ell N)\theta_b)\Big]\Big\} \qquad (1\text{-}6)$$

where $I'_\nu(z) = \dfrac{\partial I_\nu(z)}{\partial z}$

$$K'_\nu(z) = \frac{\partial K_\nu(z)}{\partial z}$$

and

$$K''_\nu(z) = \frac{\partial^2 K_\nu(z)}{\partial z^2}$$

$I_1{}^{(\bar{m})}(x)$ and $\Lambda_1{}^{(\bar{n})}(x)$ are as defined in Appendix A.

When $k = \ell = 0$, it can be shown that the integrand is

$$\frac{(\rho_R)^\nu}{\pi(R_D)^{\nu+1}}\ g(\bar{m},\bar{n}) \qquad (1\text{-}7)$$

where

$$g(\bar{m},\bar{n}) = -\Lambda^{(\bar{n})}(\nu\theta_b)\left\{\left[1 + i\,\frac{\nu}{2a\rho_R^2}\left(\epsilon_D - \frac{\sigma}{a}\right)\right] I^{(\bar{m})}(0) - i\,\frac{\nu}{2a\rho_R^2}\, c_D I_1^{(\bar{m})}(0)\right\}$$

$$-i\,\frac{\nu}{2a\rho_R^2}\,\frac{\theta_b}{a}\,\Lambda_1^{(\bar{n})}(\nu\theta_b)\, I^{(\bar{m})}(0) \quad .$$

When $\nu=0$, $\ell=0$, $k=0$ the integrand is equal to zero.

APPENDIX J

Evaluation of the Singular k-integral of $\overline{K}_{DD}$*

The integral term of Eq. (86) is

$$I = -\frac{i}{2}\int_{-\infty}^{\infty} \frac{f(k)}{k+a\ell N}\, dk \qquad \text{(J-1)}$$

where $f(k) = k^2 \left[I_{\nu-1}(|k|R_D) + I_{\nu+1}(|k|R_D)\right]\left[K_{\nu-1}(|k|R_D) + K_{\nu+1}(|k|R_D)\right]$

$$\cdot\ I^{(\bar{m})}(-kC_D)\Lambda^{(\bar{n})}(-kC_D)$$

This integral exists only in the sense of a Cauchy principal value. If it is rewritten as

$$I = -\frac{i}{2}\left\{ \int_{-\infty}^{\infty} \frac{f(k) - f(-a\ell N)}{k + a\ell N}\, dk + f(-a\ell N)\int_{-\infty}^{\infty} \frac{dk}{k+a\ell N} \right\}$$

it can be shown that

$$I = -\frac{i}{2}\int_{-\infty}^{\infty} \frac{f(k) - f(-a\ell N)}{k + a\ell N}\, dk \qquad \text{(J-2)}$$

For large $|k| \geq |M|$, $|M| > a\ell N$

$$f(k) \approx 4k^2 \frac{e^{|k|R}}{\sqrt{2\pi|k|R}} \frac{e^{-|k|R}}{\sqrt{2|k|R/\pi}}\ I^{(\bar{m})}(-kC)\ \Lambda^{(\bar{n})}(-kC)$$

$$\approx \frac{2|k|}{R}\ I^{(\bar{m})}(-kC)\ \Lambda^{(\bar{n})}(-kC) \qquad \text{(J-3)}$$

For various $\bar{m}$ and $\bar{n}$ and large k, the approximate values of the I and Λ functions are tabulated below.

*The development is taken from Reference 5.

Table J-1

$\bar{m},\bar{n}$	$I^{(\bar{m})}(\mp\lvert k\rvert C)/\sqrt{\frac{2}{\pi\lvert k\rvert C}}$	$\Lambda^{(\bar{n})}(\mp\lvert k\rvert C)/\sqrt{\frac{2}{\pi\lvert k\rvert C}}$
1	$\cos(\lvert k\rvert C-\frac{\pi}{4}) \pm i\sin(\lvert k\rvert C-\frac{\pi}{4})$	$\cos(\lvert k\rvert C-\frac{\pi}{4}) \pm i\sin(\lvert k\rvert C-\frac{\pi}{4})$
2	$\cos(\lvert k\rvert C-\frac{\pi}{4}) \mp 2i\sin(\lvert k\rvert C-\frac{\pi}{4})$	$\frac{1}{2}\left[\cos(\lvert k\rvert C-\frac{\pi}{4}) - \cos(\lvert k\rvert C-\frac{\pi}{4})\right] = 0$
3,5,7...	$\cos(\lvert k\rvert C-\frac{\pi}{4})$	$\pm\frac{i}{2}\left[\sin(\lvert k\rvert C-\frac{\pi}{4}) - \sin(\lvert k\rvert C-\frac{\pi}{4})\right] = 0$
4,6,8...	$\mp i\sin(\lvert k\rvert C-\frac{\pi}{4})$	0

Thus, for k large, f(k) is nonzero only when $\bar{n} = 1$. The values for $f(\pm\lvert k\rvert)$ are given below.

Table J-2

$\bar{n}$	$\bar{m}$	$f(\pm\lvert k\rvert)$
1	1	$\mp i\frac{4}{\pi RC}e^{\pm i2\lvert k\rvert C}$
1	2	$\frac{4}{\pi RC}\left[\frac{3}{2} \pm \frac{i}{2}e^{\pm i2\lvert k\rvert C}\right]$
1	3,5,7...	$\frac{4}{\pi RC}\left[\frac{1}{2} \mp \frac{i}{2}e^{\pm i2\lvert k\rvert C}\right]$
1	4,6,8...	$\frac{4}{\pi RC}\left[\frac{1}{2} \pm \frac{i}{2}e^{\pm i2\lvert k\rvert C}\right]$
>1	all	0

Equation (J-2) is now rewritten as

$$I = -\frac{i}{2}\int_{-M}^{M}\frac{f(k) - f(-a\ell N)}{k + a\ell N}\,dk$$

[Cont'd]

$$- \frac{i}{2}\left[\int_{-\infty}^{-M} + \int_{M}^{\infty}\right] \frac{f(k) - f(-a\ell n)}{k + a\ell N}\, dk$$

$$= - \frac{i}{2}\int_{-M}^{M} \frac{f(k) - f(-a\ell N)}{k + a\ell N}\, dk - \frac{i}{2}\left[\int_{-\infty}^{-M} + \int_{M}^{\infty}\right] \frac{f(k)}{k+a\ell N}\, dk$$

$$+ \frac{i}{2}\, f(-a\ell N)\, \log\left(\frac{M - a\ell N}{M + a\ell N}\right) \tag{J-4}$$

where in the second term $f(k)$ is given by Table J-2 and where

$$f(-a\ell N) = a^2\ell^2N^2\left[I_{\nu-1}(a\ell NR_D) + I_{\nu+1}(a\ell NR_D)\right]$$

$$\cdot \left[K_{\nu-1}(a\ell NR_D) + K_{\nu+1}(a\ell NR_D)\right] I^{(\bar{m})}(a\ell NC_D)\, \Lambda^{(\bar{n})}(a\ell NC_D)$$

In the case $\bar{n} > 1$ where $f(k) \to 0$ as $k \to \infty$, the kernel becomes

$$\bar{K}_{DD}^{(\nu,\bar{m},\bar{n})} = \frac{1}{4\pi\rho_f U r_o}\left\{\frac{\pi}{2}\, a^2\ell^2N^2\left[I_{\nu-1}(a\ell NR_D)+I_{\nu+1}(a\ell NR_D)\right]\left[K_{\nu-1}(a\ell NR_D)+K_{\nu+1}(a\ell NR_D)\right]\right.$$

$$\cdot\, I^{(\bar{m})}(a\ell NC_D)\Lambda^{(\bar{n})}(a\ell NC_D)\left[1 + \frac{i}{\pi}\log\left(\frac{M-a\ell N}{M+a\ell N}\right)\right]$$

$$\left. - \frac{i}{2}\int_{-M}^{M} \frac{f(k) - f(-a\ell N)}{k + a\, N}\, dk\right\} \tag{J-5}$$

When $\bar{n} = 1$, there are additional terms which may involve

$$\text{const.}\left\{\int_{-\infty}^{-M} + \int_{M}^{\infty}\right] \frac{dk}{k+a\ell N} = \text{const.}\left[\log\left(\frac{M-a\ell N}{M+a\ell N}\right)\right]$$

and

$$\text{const.}\left[\int_{-\infty}^{-M} + \int_{M}^{\infty}\right]\frac{g(k)\,dk}{k+a\ell N} = \text{const.}\left\{ -\int_{M}^{\infty}\frac{g(-k)\,dk}{k-a\ell N} + \int_{M}^{\infty}\frac{g(k)\,dk}{k+a\ell N}\right.$$

$$\text{where } g(-k) = \begin{cases} +e^{-i2kC}, & \bar{m} \text{ odd} \\ -e^{-i2kC}, & \bar{m} \text{ even}\end{cases}$$

$$g(k) = \begin{cases} -e^{i2kC}, & \bar{m} \text{ odd} \\ +e^{i2kC}, & \bar{m} \text{ even}\end{cases}$$

The integrals are evaluated below. By means of the substitution $\lambda = 2C(k+a\ell N)$

$$\int_{M}^{\infty}\frac{e^{i2Ck}dk}{k+a\ell N} = e^{-i2Ca\ell N}\int_{2C(M+a\ell N)}^{\infty}\frac{e^{i\lambda}d\lambda}{\lambda}$$

$$= e^{-i2Ca\ell N}\left\{-\,Ci\left[2C(M+a\ell N)\right] - i\,si\left[2C(M+a\ell N)\right]\right\}$$

$$\text{where } Ci(x) = -\int_{x}^{\infty}\frac{\cos\lambda\, d\lambda}{\lambda} \approx \frac{\sin x}{x} \quad \text{for } x \gg 1$$

$$si(x) = -\int_{x}^{\infty}\frac{\sin\lambda\, d\lambda}{\lambda} \approx \frac{-\cos x}{x} \quad \text{for } x \gg 1$$

(See Jahnke and Emde: Tables of Functions, Dover Publications, New York, 1945.)

Therefore

$$\int_{M}^{\infty}\frac{e^{i2Ck}dk}{k+a\ell N} \approx e^{-i2Ca\ell N}\left\{\frac{-\sin\left[2C(M+a\ell N)\right] + i\cos\left[2C(M+a\ell N)\right]}{2C(M+a\ell N)}\right\}$$

$$\approx \frac{i\,e^{i2CM}}{2C(M+a\ell N)} \tag{J-6}$$

Similarly, if $\lambda = 2C(k-a\ell N)$, it can be shown that

$$\int_M^\infty \frac{e^{-i2Ck}}{k-a\ell N}\,dk = e^{-i2Ca\ell N}\int_{2C(M-a\ell N)}^\infty \frac{e^{-i\lambda}}{\lambda}\,d\lambda \approx \frac{-ie^{-i2CM}}{2C(M-a\ell N)} \qquad \text{(J-7)}$$

For $\bar{n} = 1$ and varying $\bar{m}$ the terms to be added within the brace of Eq. (J-5) are listed below.

Table J-3

$\bar{n}$	$\bar{m}$	$4\pi\rho_f U r_o \bar{K}_{DD}$ (additional)
1	1	$+\frac{i}{\pi RC^2}\left[\frac{e^{-i2CM}}{M-a\ell N} - \frac{e^{i2CM}}{M+a\ell N}\right]$
	2	$-\frac{i}{2\pi RC^2}\left[\frac{e^{-i2CM}}{M-a\ell N} - \frac{e^{i2CM}}{M+a\ell N}\right] - \frac{i3}{\pi RC}\log\left(\frac{M-a\ell N}{M+a\ell N}\right)$
	3,5,7...	$+\frac{i}{2\pi RC^2}\left[\frac{e^{-i2CM}}{M-a\ell N} - \frac{e^{i2CM}}{M+a\ell N}\right] - \frac{i}{\pi RC}\log\left(\frac{M-a\ell N}{M+a\ell N}\right)$
	4,6,8...	$-\frac{i}{2\pi RC^2}\left[\frac{e^{-i2CM}}{M-a\ell N} - \frac{e^{i2CM}}{M+a\ell N}\right] - \frac{i}{\pi RC}\log\left(\frac{M-a\ell N}{M+a\ell N}\right)$

The singularity in the k-integral

The k-integral in (J-5) can be rewritten as

$$I_k = -\frac{i}{2}\int_o^M \frac{f'(k) - f'(a\ell N)}{(k-a\ell N)(k+a\ell N)}\,dk \qquad \text{(J-8)}$$

where

$$f'(k) = 2ik^2 I_\nu'(kR_D)K_\nu'(kR_D)$$

$$\cdot\left\{(k-a\ell N)\,I^{(\bar{m})}(-kC_D)\,\Lambda^{(\bar{n})}(-kC_D) - (k+a\ell N)\,I^{(\bar{m})}(kC_D)\Lambda^{(\bar{n})}(kC_D)\right\}$$

and

$$f'(a\ell N) = -4i(a\ell N)^3 \; I_\nu'(a\ell NR_D)K_\nu'(a\ell NR_D)\, I^{(\bar{m})}(a\ell NC_D)\; \Lambda^{(\bar{n})}(a\ell NC_D)$$

It can be easily proved that

$$\left.\frac{\partial f'(k)}{\partial k}\right|_{k=a\ell N} \div 2a\ell N =$$

$$-2i\Big\{2a\ell N I_\nu'(a\ell NR_D)K_\nu'(a\ell NR_D)+a^2\ell^2N^2R_D \; I_\nu''(a\ell NR_D)K_\nu'(a\ell NR_D)$$

$$+ a^2\ell^2N^2R_D I_\nu'(a\ell NR_D)K_\nu''(a\ell NR_D)\Big\}\Big\{I^{(\bar{m})}(a\ell NC_D)\Lambda^{(\bar{n})}(a\ell NC_D)\Big\}$$

$$+ ia\ell N\Big\{I_\nu'(a\ell NR_D)K_\nu'(a\ell NR_D)\Big\}$$

$$\cdot \Big\{I^{(\bar{m})}(-a\ell NC_D)\Lambda^{(n)}(-a\ell NC_D) - I^{(\bar{m})}(a\ell NC_D)\Lambda^{(\bar{n})}(a\ell NC_D)$$

$$+ i2a\ell NC_D\Big[I^{(\bar{m})}(a\ell NC_D)\Lambda_1^{(\bar{n})}(a\ell NC_D) - I_1^{(\bar{m})}(a\ell NC_D)\Lambda^{(\bar{n})}(a\ell NC_D)\Big]\Big\}$$

(J-9)

When $\ell = 0$

$$\left.\frac{\partial f(k)}{\partial k}\right|_{k=a\ell N} \div 2a\ell N = \frac{-2\nu C_D}{R_D^2}\Big[I_1^{(\bar{m})}(0)\Lambda^{(\bar{n})}(0) - I^{(\bar{m})}(0)\Lambda_1^{(\bar{n})}(0)\Big] \qquad \text{(J-10)}$$

where (see Appendix A)

$$I^{(\bar{m})}(0) = \begin{cases} 1 \text{ for } \bar{m} = 1,2 \\ 0 \text{ for } \bar{m} > 2 \end{cases} \qquad I_1^{(\bar{m})}(0) = \begin{cases} -\tfrac{1}{2} \text{ for } \bar{m} = 1 \\ 1 \text{ for } \bar{m} = 2 \\ 0 \text{ for } \bar{m} > 2 \end{cases}$$

$$\Lambda^{(\bar{n})}(0) = \begin{cases} 1 \text{ for } \bar{n} = 1 \\ \tfrac{1}{2} \text{ for } \bar{n} = 2 \\ 0 \text{ for } \bar{n} > 2 \end{cases} \qquad \Lambda_1^{(\bar{m})}(0) = \begin{cases} \tfrac{1}{2} \text{ for } \bar{n} = 1 \\ \tfrac{1}{4} \text{ for } \bar{n} = 3 \\ 0 \text{ for all other } \bar{n} \end{cases}$$

APPENDIX K

Evaluation of the Singular k-integral of $\bar{K}_{SD}$

The k-integral of Eq.(88) can be written as

$$I = \int_{-\infty}^{\infty} \frac{G(k) - G(-a_R \ell N_R)}{k+a_R \ell N_R}\, dk \tag{K-1}$$

$$\approx \int_{-M}^{M} \frac{G(k) - G(-a_R \ell N_R)}{k+a_R \ell N_R}\, dk - G(-a\ \ell N_R)\log\left(\frac{M - a_R \ell N_R}{M + a_R \ell N_R}\right) \tag{K-2}$$

where

$$G(k) = -\frac{1}{2\pi a_S}\left(a_s k - \frac{\nu}{\rho_S^2}\right)|k|\ I_\nu(|k|\rho_S)\left[K_{\nu-1}(|k|R_D) + K_{\nu+1}(|k|R_D)\right]$$

$$\cdot\ \Lambda^{(\bar{n})}\left(\left(-\nu - \frac{k}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}(-kC_D)\, e^{ik(\epsilon_D - \epsilon_S - \sigma_S/a_S)}$$

and $\quad G(-a_R \ell N_R) = -\frac{i}{\pi}\{\text{closed term of Eq.(88)}\}$.

Therefore

$$\bar{K}_{SD}^{(\nu,\bar{m},\bar{n})} \approx \frac{N_S e^{-i\nu\sigma_S}}{4\pi\rho_f U^2 r_{RO}} \left\{ -i\ \frac{a_R}{a_S}\ \frac{\ell N_R}{2}\left(a_S a_R \ell N_R + \frac{\nu}{\rho_S^2}\right) e^{-ia_R \ell N_R(\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right.$$

$$\cdot\ I_\nu(a_R \ell N_R \rho_S)\left[K_{\nu-1}(a_R \ell N_R R_D) + K_{\nu+1}(a_R \ell N_R R_D)\right]$$

$$\cdot\ \Lambda^{(\bar{n})}\left(\left(-\nu + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}(a_R \ell N_R C_D)$$

$$\cdot\ \left[1 + \frac{i}{\pi}\log\left(\frac{M - a_R \ell N_R}{M + a_R \ell N_R}\right)\right]$$

$$\left. + \int_{-M}^{M} \frac{G(k) - G(-a_R \ell N_R)}{k+a_R \ell N_R}\, dk \right\} \tag{K-3}$$

$$\int_{-M}^{M} \frac{G(k) - G(-a_R \ell N_R)}{k+a_R \ell N_R}\, dk = \int_{p}^{n} \frac{G'(k) - G'(a_R \ell N_R)}{(k+a_R \ell N_R)(k-a_R \ell N_R)}\, dk \qquad \text{(K-4)}$$

where

$$G'(k) = -\frac{k}{2\pi a_S} I_\nu(k\rho_S)\left[K_{\nu-1}(kR_D) + K_{\nu+1}(kR_D)\right]$$

$$\cdot\left\{\left(a_S k - \frac{\nu}{\rho_S^2}\right)(k-a_R \ell N_R)\Lambda^{(\bar{n})}\left(\left(-\nu - \frac{k}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}(-kC_D)\, e^{ik(\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right.$$

$$\left.+\left(a_S k + \frac{\nu}{\rho_S^2}\right)(k+a_R \ell N_R)\Lambda^{(\bar{n})}\left(\left(-\nu + \frac{k}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}(kC_D)\, e^{-ik(\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right\}$$

and

$$G'(a_R \ell N_R) = \frac{a_R^2 \ell^2 N_R^2}{a_S \pi} I_\nu(a_R \ell N_R \rho_S)\left[K_{\nu-1}(a_R \ell N_R R_D) + K_{\nu+1}(a_R \ell N_R R_D)\right]$$

$$\cdot\left\{\left(a_S a_R \ell N_R + \frac{\nu}{\rho_S^2}\right)\Lambda^{(\bar{n})}\left(\left(-\nu + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) I^{(\bar{m})}(a_R \ell N_R C_D)\, e^{-i a_R \ell N_R(\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right\}$$

$$\lim_{k \to a_R \ell N_R} \left\{ \frac{G'(k) - G'(a_R \ell N_R)}{(k+a_R \ell N_R)(k-a_R \ell N_R)} \right\} = \left.\frac{\partial G'(k)}{\partial k}\right|_{k=a_R \ell N_R} \div 2\, a_R \ell N_R \qquad \text{(K-5)}$$

Equation (K-5) is equal to

$$\frac{1}{\pi} I^{(\bar{m})}(a_R \ell N_R C_D)\Lambda^{(\bar{n})}\left(\left(-\nu + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) e^{-i a_R \ell N_R(\epsilon_D - \epsilon_S - \sigma_S/a_S)}$$

$$\cdot\left\{\left[\frac{5}{2} a_R \ell N_R + \frac{3}{2}\frac{\nu}{a_S \rho_S^2} - i\, a_R \ell N_R\left(\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S}\right)\left(a_R \ell N_R + \frac{\nu}{a_S \rho_S^2}\right)\right]\right.$$

$$\cdot I_\nu(a_R \ell N_R \rho_S) K'_\nu(a_R \ell N_R R_D)$$

$$+ a_R \ell N_R\left(a_R \ell N_R + \frac{\nu}{a_S \rho_S^2}\right)\rho_S\, I'_\nu(a_R \ell N_R \rho_S) K'_\nu(a_R \ell N_R R_D)$$

$$\left. + a_R \ell N_R\left(a_R \ell N_R + \frac{\nu}{a_S \rho_S^2}\right) R_D\, I_\nu(a_R \ell N_R \rho_S) K''_\nu(a_R \ell N_R R_D)\right\}$$

[Cont'd]

$$+ \frac{1}{\pi} I_\nu(a_R \ell N_R \rho_S) K'_\nu(a_R \ell N_R R_D)$$

$$\cdot \left\{ \frac{1}{2}\left(a_R \ell N_R - \frac{\nu}{a_S \rho_S^2}\right) I^{(\bar{m})}(-a_R \ell N_R C_D) \Lambda^{(\bar{n})}\left(\left(-\nu - \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right) e^{+i a_R \ell N_R (\epsilon_D - \epsilon_S - \sigma_S/a_S)}\right.$$

$$+ i a_R \ell N_R e^{-i a_R \ell N_R (\epsilon_D - \epsilon_S - \sigma_S/a_S)} \left(a_R \ell N_R + \frac{\nu}{a_S \rho_S^2}\right)\left[C_D I_1^{(\bar{m})}(a_R \ell N_R C_D)\right.$$

$$\cdot \Lambda^{(\bar{n})}\left(\left(-\nu + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)$$

$$\left.\left. - \frac{\theta_{bS}}{a_S} I^{(\bar{m})}(a_R \ell N_R C_D) \Lambda_1^{(\bar{n})}\left(\left(-\nu + \frac{a_R \ell N_R}{a_S}\right)\theta_{bS}\right)\right]\right\} \tag{K-6}$$

where

$$I'_\nu(z) = \frac{\partial I_\nu(z)}{\partial z}$$

$$K'_\nu(z) = \frac{\partial K_\nu(z)}{\partial z}$$

$$K''_\nu(z) = \frac{\partial^2 K_\nu(z)}{\partial z^2}$$

$I_1^{(\bar{m})}(x)$ and $\Lambda_1^{(\bar{n})}(x)$ are given in Appendix A.

<u>When $k=\ell=0$</u> , it can be shown that the integrand is

$$\frac{1}{\pi} \frac{(\rho_S)^\nu}{(R_D)^{\nu+1}} g'(\bar{m},\bar{n})$$

where $g'(\bar{m},\bar{n}) =$

$$-\Lambda^{(\bar{n})}(-\nu\theta_{bS})\left\{\left[1 - i \frac{\nu}{2a_S\rho_S^2}\left(\epsilon_D - \epsilon_S - \frac{\sigma_S}{a_S}\right)\right] I^{(\bar{m})}(0)\right.$$

$$\left. + i \frac{\nu}{2a_S\rho_S^2} C_D I_1^{(\bar{m})}(0)\right\}$$

$$+ i \frac{\nu}{2a_S\rho^2}\left(\frac{\theta_{bS}}{a_S}\right)\Lambda_1^{(\bar{n})}(-\nu\theta_{bS}) I^{(\bar{m})}(0) \tag{K-7}$$

or $\quad g'(\bar{m},\bar{n}) =$

$$I^{(\bar{m})}(0)\Lambda^{(\bar{n})}(-\nu\theta_{bS})\left[-1 + i\,\frac{\nu}{2a_S\rho_S^2}\left(\epsilon_D-\epsilon_S - \frac{\sigma_S}{a_S}\right)\right]$$

$$- \frac{i}{2}\,\frac{\nu}{2a_S\rho_S^2}\left[C_D I_1^{(\bar{m})}(0)\Lambda^{(\bar{n})}(-\nu\theta_{bS}) - \frac{\theta_{bS}}{a_S}\,I^{(\bar{m})}(0)\Lambda_1^{(\bar{n})}(-\nu\theta_{bS})\right]$$

<u>When $\nu=0, \ell=0$</u>, the integrand is zero.

APPENDIX L

EFFECT OF RACE OF STATOR (S) ON ROTOR (R)

In the course of a re-examination of the theoretical development of Reference 9, it was found that the behavior of the velocity field for points inside the propeller race is quite different from that at any other point in the field around the propeller. The existing theory and program dealing with the propeller-induced velocity field have therefore been modified to include the region of the propeller race. The wake effect of the stator on the rotor, designated by $\Delta W_R/U$, has been developed and incorporated in the present program to be used whenever there is no available wake survey at the rotor plane in the presence of the hull and stator.

The W_R induced velocity at points on the right-handed after rotor by the presence of a "left-handed" forward stator is given by

$$\frac{W_R}{U}(x_R, r_R, \varphi_R, t) = -\frac{1}{4\pi\rho_f U^2}\sum_{n=1}^{N_S}\int_{\xi_S}\int_{\rho_S}\sum_{\lambda=0}\Delta p_S^{(\lambda)}(\xi_S, \rho_S, \theta_S)\, e^{+i\lambda\Omega_R t}$$

$$\cdot \frac{\partial}{\partial n_R'}\int_{-\infty}^{x_R} e^{-i\lambda[a_R(\tau'-x_R)-\bar{\theta}_{Sn}]}\left(a_S\frac{\partial}{d\xi_S} - \frac{1}{\rho_S^2}\frac{\partial}{\partial\theta_{S0}}\right)\left(\frac{1}{R_{SR}}\right)\rho_S\, d\rho_S\, d\xi_S\, d\tau' \qquad \text{(L-1)}$$

where $\lambda = \ell N_R$, $\ell = 0,1,2, \ldots,$

and $$\frac{\partial}{\partial n_R'} = \frac{r_R}{\sqrt{1+a_R^2 r_R^2}}\left(a_R\frac{\partial}{\partial x_R} - \frac{1}{r_R^2}\frac{\partial}{\partial\varphi_{R0}}\right)$$

Since

$$x_R = \frac{\varphi_{R0}}{a_R} \qquad \text{and} \qquad \frac{\partial}{\partial\varphi_{R0}} = \frac{1}{a_R}\frac{\partial}{\partial x_R}$$

$$\frac{\partial}{\partial n_R'} = \frac{1}{\sqrt{1+a_R^2 r_R^2}}\left(a_R r_R - \frac{1}{a_R r_R}\right)\frac{\partial}{\partial x_R}$$

In (L-1), $R_{SR}=\left\{(\tau'-\xi_S)^2+r_R^2+\rho_S^2-2r_R\rho_S\cos\left[\theta_{SO}+\varphi_{RO}-\Omega_R t+\bar{\theta}_{Sn}\right]\right\}^{\frac{1}{2}}$.

Let $\tau'-x_R=\tau$, and $\Theta=-\Omega_R t$. The τ-integral then yields

$$I_\tau=\int_{-\infty}^{0} e^{+i\lambda[a_R\tau-\bar{\theta}_{Sn}]}\left(a_S\frac{\partial}{\partial\xi_S}-\frac{1}{\rho_S^2}\frac{\partial}{\partial\theta_{SO}}\right)\left(\frac{1}{R'_{SR}}\right)\rho_S\,d\rho_S\,d\xi_S\,d\tau$$

where

$$R'_{SR}=\left\{(\tau+x_R-\xi_S)^2+r_R^2+\rho_S^2-2r_R\rho_S\cos[\theta_{SO}+\varphi_{RO}+\Theta+\bar{\theta}_{Sn}]\right\}^{\frac{1}{2}}$$

Then

$$\frac{\partial I_\tau}{\partial x_R}=\int_{-\infty}^{0} e^{+i\lambda(a_R\tau-\bar{\theta}_{Sn})}\left(a_S\frac{\partial^2}{\partial x_R\partial\xi_S}-\frac{1}{\rho_S^2}\frac{\partial^2}{\partial x_R\partial\theta_{SO}}\right)\left(\frac{1}{R_{SR}}\right)\rho_S\,d\rho_S\,d\xi_S\,d\tau$$

But
$$\frac{\partial^2}{\partial x_R\partial\xi_S}=-\frac{\partial^2}{\partial x_R^2}$$

Therefore

$$\frac{\partial I_\tau}{\partial x_R}=-\int_{-\infty}^{0} e^{+i\lambda(a_R\tau-\bar{\theta}_{Sn})}\left(a_S\frac{\partial^2}{\partial x_R^2}+\frac{1}{\rho_S^2}\frac{\partial^2}{\partial x_R\partial\theta_{SO}}\right)\left(\frac{1}{R_{SR}}\right)\rho_S\,d\rho_S\,d\xi_S\,d\tau \qquad \text{(L-2)}$$

Furthermore for points inside the propeller race, Laplace's equation written in cylindrical coordinates takes the form

$$\frac{\partial^2}{\partial x_R^2}\left(\frac{1}{R}\right)+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\frac{1}{R}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta_o^2}\left(\frac{1}{R}\right)=-\frac{4\pi}{\rho_S}\delta(\tau+x_R-\xi_S)\delta(r_R-\rho_S)\delta(\theta_{SO}+\varphi_{RO}+\Theta+\bar{\theta}_{Sn})$$

where $\delta(\;)$ is the Dirac delta function.

Thus whenever the field point coincides with the helices of the wake,

$$\frac{\partial^2}{\partial x_R^2}\left(\frac{1}{R}\right)=-\frac{4\pi}{\rho_S}\delta(\tau+x_R-\xi_S)\delta(r_R-\rho_S)\delta(\theta_{SO}+\varphi_{RO}+\Theta+\bar{\theta}_{Sn})-\left[\frac{1}{\rho_S}\frac{\partial}{\partial\rho_S}\left(\rho_S\frac{\partial}{\partial\rho_S}\right)\left(\frac{1}{R}\right)+\frac{1}{\rho_S^2}\frac{\partial^2}{\partial\theta_{SO}^2}\left(\frac{1}{R}\right)\right] \qquad \text{(L-3)}$$

The induction ΔW_R is the first term of $\frac{\partial^2}{\partial x_R^2}\left(\frac{1}{R}\right)$ (see Eq.L-3)

$$\frac{\Delta W_R^{(1)}}{U} = -\frac{1}{4\pi\rho_f U^2}\frac{1}{\sqrt{1+a_R^2 r_R^2}}\left(a_R r_R - \frac{1}{a_R r_R}\right)$$

$$\cdot\sum_{n=1}^{N_S}\left\{\int_{\xi_S}\int_{\rho_S}\sum_{\lambda=0}\Delta p_S^{(\lambda)}(\xi_S,\rho_S,\theta_S)e^{-i\lambda\Theta}\int_{-\infty}^{0} e^{+i\lambda(a_R\tau-\bar{\theta}_{Sn})}\rho_S d\rho_S d\xi_S\right.$$

$$\left.\cdot\frac{4\pi a_S}{\rho_S}\delta(\tau+x_R-\xi_S)\delta(r_R-\rho_S)\delta(\theta_{S0}+\varphi_{R0}+\Theta+\bar{\theta}_{Sn})d\tau\right\} \tag{L-4}$$

where since $\int_a^b f(x)\delta(x-c)dx = f(c)$ as long as the range a to b includes $x=c$

$$\{\ \} = \int_{\xi_S}\sum_{\lambda=0}\Delta p_S^{(\lambda)}(\xi_S,r_R,\theta_S)e^{-i\lambda\Theta}\int_{-\infty}^{0} e^{+i\lambda(a_R\tau-\bar{\theta}_{Sn})} r_R d\xi_S$$

$$\cdot\frac{4\pi a_S}{r_R}\delta(\tau+x_R-\xi_S)\delta(\theta_{S0}+\varphi_{R0}+\Theta+\bar{\theta}_{Sn})d\tau$$

$$= \int_{\xi_S}\sum_{\lambda=0}\Delta p_S^{(\lambda)}(\xi_S,r_R,\theta_S)e^{-i\lambda[a_R(x_R-\xi_S)+\bar{\theta}_{Sn}]} r_R\frac{4\pi a_S}{r_R}$$

$$\cdot\,\delta(\theta_{S0}+\varphi_{R0}+\Theta+\bar{\theta}_{Sn})d\xi_S\ .$$

But $\theta_{S0} = \sigma_S - \theta_{bS}\cos\theta_\alpha = a_S(\xi_S-\epsilon_S)$

$a_S d\xi_S = \theta_{bS}\sin\theta_\alpha d\theta_\alpha$

$\varphi_{R0} = a_R x_R$

$L_S = \Delta p_S\cdot r_R\theta_{bS}$

Therefore

$$\{\ \} = 4\pi\int_0^\pi\sum_{\lambda=0}L_S^{(\lambda)}(r_R,\theta_\alpha)e^{-i\lambda\Theta}e^{-i\lambda(\varphi_{R0}-\frac{a_R}{a_S}\theta_{S0}-a_R\epsilon_S+\bar{\theta}_{Sn})}\frac{1}{r_R}$$

$$\cdot\,\delta(\theta_{S0}+\varphi_{R0}+\Theta+\bar{\theta}_{Sn})\sin\theta_\alpha d\theta_\alpha$$

The induction can be expressed in a Fourier series expansion as

$$\frac{\Delta W_R^{(1)}}{U} = \sum_{n=-\infty}^{\infty} c_n e^{in\Theta} \quad , \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Delta W_R^{(1)}}{U} e^{-in\Theta} d\Theta$$

Let $\Theta=\theta'$. Then

$$\{\ \} = \sum_{n=-\infty}^{\infty} \frac{2e^{in\theta'}}{r_R} \int_{\theta_\alpha=0}^{\pi} \int_{\theta'=-\pi}^{\pi} \sum_{\lambda} L_S^{(\lambda)}(r_R,\theta_\alpha) e^{-i(\lambda+n)\theta'}$$

$$\cdot\, e^{-i\lambda\left(\varphi_{R0} - \frac{a_R}{a_S}\theta_{S0} - a_R\epsilon_S + \bar{\theta}_{Sn}\right)}$$

$$\cdot\, \delta(\theta_{S0} + \varphi_{R0} + \bar{\theta}_{Sn} + \theta')\sin\theta_\alpha d\theta_\alpha d\theta'$$

$$= \sum_{n=-\infty}^{\infty} \frac{2e^{in\theta'}}{r_R} \int_{\theta_\alpha=0}^{\pi} \sum_{\lambda} L_S^{(\lambda)} e^{i(\lambda+n)(\theta_{S0}+\varphi_{R0}+\bar{\theta}_{Sn})} e^{-i\lambda\left(\varphi_{R0} - \frac{a_R}{a_S}\theta_{S0} - a_R\epsilon_S + \bar{\theta}_{Sn}\right)}$$

$$\cdot \sin\theta_\alpha d\theta_\alpha$$

$$= \sum_{n=-\infty}^{\infty} \frac{2e^{in\theta'}}{r_R} \int_{0}^{\pi} \sum_{\lambda} L_S^{(\lambda)} e^{in(\theta_{S0}+\varphi_{R0}+\bar{\theta}_{Sn})} e^{i\lambda\left[\left(1+\frac{a_R}{a_S}\right)\theta_{S0} + a_R\epsilon_S\right]} \sin\theta_\alpha d\theta_\alpha$$

Since

$$\sum_{n=1}^{N_S} e^{in\bar{\theta}_{Sn}} = \begin{cases} N_S & \text{when } n = \ell_1 N_S,\ \ell_1 = 0, \pm 1, \pm 2, \ldots\ . \\ 0 & \text{otherwise} \end{cases}$$

Equation (L-4) becomes

$$\frac{\Delta W_R^{(1)}}{U} = -\frac{N_S}{2\pi\rho_f U^2} \frac{1}{\sqrt{1+a_R^2 r_R^2}} \left(a_R r_R - \frac{1}{a_R r_R}\right) \frac{1}{r_R} \sum_{\substack{n=-\infty \\ n=\ell_1 N_S}}^{\infty} e^{in\Theta}$$

$$\cdot \int_0^{\pi} \sum_{\lambda=0}^{\infty} L_S^{(\lambda)}(r_R,\theta_\alpha) e^{in\varphi_{R0}} e^{i(\lambda+n)\theta_{S0}} e^{i\lambda a_R\left(\frac{\theta_{S0}}{a_S} + \epsilon_S\right)} \sin\theta_\alpha d\theta_\alpha \qquad \text{(L-5)}$$

With $\varphi_{R0} = \sigma_R - \theta_{bR}\cos\varphi_\alpha$

$\theta_{S0} = \sigma_S - \theta_{bS}\cos\theta_\alpha$

and assuming

$$L_S^{(\lambda)}(r_R,\theta_\alpha) = \frac{1}{\pi}\sum_{\bar{n}=1} L_S^{(\lambda,\bar{n})}(r_R)\Theta(\bar{n})$$

where $\Theta(\bar{n})$ represents the Birnbaum chordwise modes, then the integral part of (L-5) becomes

$$I_{\theta_\alpha} = \sum_{\bar{n}=1}^{\max\bar{n}} \sum_{\lambda=0} L_S^{(\lambda,\bar{n})}(r_R) e^{i\lambda a_R \epsilon_S} e^{in\sigma_R} e^{-in\theta_{bR}\cos\varphi_\alpha} e^{i(\lambda+n)\sigma_S} e^{i\lambda\frac{a_R}{a_S}\sigma_S}$$

$$\cdot\ \Lambda^{(\bar{n})}\left(\left(n+\lambda\left(1+\frac{a_R}{a_S}\right)\right)\theta_{bS}\right) \quad . \qquad \text{(L-6)}$$

Taking the lift operator at each $\bar{m}$-order and nondimensionalizing with respect to r_o(rotor radius), Eq.(L-5) can be expressed as

$$\left(\frac{\Delta W_R^{(n,\bar{m})}(r_R)}{U}\right)_1 = \frac{-N_S}{2\pi\rho_f U^2 r_o}\frac{1}{\sqrt{1+a_R^2 r_R^2}}\left(a_R r_R - \frac{1}{a_R r_R}\right)\frac{1}{r_R}$$

$$\cdot \sum_{n=-\infty}^{\infty} e^{in\Theta} e^{in(\sigma_R+\sigma_S)}\, I^{(\bar{m})}(-n\theta_{bR})$$

$$\cdot \sum_{\bar{n}=1}\sum_{\lambda=0} L_S^{(\lambda,\bar{n})}(r_R) e^{i\lambda a_R\epsilon_S} e^{i\lambda\left(1+\frac{a_R}{a_S}\right)\sigma_S} \Lambda^{(\bar{n})}\left(\left(n+\lambda\left(1+\frac{a_R}{a_S}\right)\right)\theta_{bS}\right) \qquad \text{(L-7)}$$

where $n = \ell_1 N_S$, $\ell_1 = 0, \pm1, \pm2, \ldots$.
$\lambda = \ell N_R$, $\ell = 0, +1, +2, \ldots$.

(It can be shown by a similar approach that the second term on the right-hand side of Eq.(L-3) does not contribute to $\partial^2/\partial x_R^2(\frac{1}{R})$.)

In the steady-state condition, ℓ_1=0, and retaining only the ℓ=0 and 1 terms (i.e., λ=0 and λ=N_R)), Eq.(L-7) becomes

$$\frac{\Delta W_R^{(0,\bar{m})}(r_R)}{U} = -\frac{N_S}{2\pi\rho_f U^2 r_o}\frac{1}{\sqrt{1+a_R^2 r_R^2}}\left(a_R r_R - \frac{1}{a_R r_R}\right)\frac{1}{r_R}\, I^{(\bar{m})}(0)\sum_{\bar{n}=1}\left\{L_S^{(0,\bar{n})}(r_R)\Lambda^{(\bar{n})}(0) + \right.$$

$$\left. L_S^{(N_R,\bar{n})}(r_R) e^{iN_R a_R \epsilon_S} e^{iN_R\left(1+\frac{a_R}{a_S}\right)\sigma_S} \Lambda^{(\bar{n})}\left(N_R\left(1+\frac{a_R}{a_S}\right)\theta_{bS}\right)\right\} \qquad \text{(L-8)}$$

APPENDIX M

THE VISCOUS WAKE OF THE STATOR

In a pump-jet propulsive system the rotor, being located in the race (wake) of the stator, operates in a real fluid and hence should include both the potential and viscous effects. In the absence of wake measurements in the plane of the rotor when the stator is in place, it is necessary to take this into account theoretically. The potential contribution has already been dealt with in Appendix L. The effect of the viscous wake is approximately considered by the Kemp-Sears method described in Reference 10.

The configuration of viscous wakes of propeller blades is approximated from single airfoil experiments. The unsteady force-and-moment on a downstream blade passing through such wakes is then calculated on the basis of the theory of isolated thin airfoil in nonuniform flow. The same approach has been adapted to the unsteady lifting surface theory.

Silverstein, Katzoff, and Bullivant,[11] have shown that the half-width of the wake, Y, may be calculated from the following formula

$$Y = 0.68\sqrt{2}\, C_D^{\frac{1}{2}}\, c(x/c - 0.7)^{\frac{1}{2}} \tag{M-1}$$

where

c = airfoil half-chord

x = distance measured along the wake axis (free-stream direction) rearward from the center of the airfoil

C_D = the airfoil profile-drag coefficient

NOTE: C_D will be calculated according to Hoerner's method.[12]

For convenience, a new coordinate x^* along the wake axis is introduced in Eq.(M-1):

$$x^* = x - 0.7c \tag{M-2}$$

Kemp and Sears[10] have shown that in terms of x^* the wake half-width and the velocity at the center become

$$Y = 0.68\sqrt{2}\, c(C_D x^*/c)^{\frac{1}{2}} \qquad \text{(M-3)}$$

$$u_c/V = -(2.42 C_D^{\frac{1}{2}})/(x^*/c + 0.3) \qquad \text{(M-4)}$$

and that the velocity profile to be used is

$$\frac{u}{u_c} = \exp\left[-\pi\left(\frac{y}{Y}\right)^2\right] \qquad \text{(M-5)}$$

Since the propeller blade moves along a line oblique to the x (or x^*) axis, it is convenient to introduce oblique coordinates x', y' as shown in Figure 5. The relation between x^*, y and x', y' is given by

$$x^* = x' - y'\cos\theta_p^S \quad , \quad y = y'\sin\theta_p^S \qquad \text{(M-6)}$$

(The superscripts S and R refer to stator and rotor blades, respectively.)

Since the wake is narrow in the region of interest, see Figure 5, y'/x' is small in the wake itself, and one may write, approximately,

$$x^* \approx x' \quad , \quad y \approx y'\sin\theta_p^S \qquad \text{(M-7)}$$

Then the wake half-width and centerline velocity are as follows:

$$\frac{Y}{r_o} = 0.68\left[C_D^S\left(\frac{x^*}{r_o}\right)\left(\frac{c^S}{r_o}\right)\right]^{\frac{1}{2}} \qquad \text{(M-8)}$$

$$\frac{u_c}{V^S} = -\left(2.42\sqrt{C_D^S}\right)\Big/\left(\frac{x'}{c^S} + 0.3\right) \qquad \text{(M-9)}$$

where c^S is the total chord length of the stator.

The velocity profile from Eq.(M-5) is now

$$\frac{u}{u_c} = \exp\left[-\pi\left(\frac{\sin\theta_p^S}{Y}\right)^2 y'^2\right] \qquad \text{(M-10)}$$

and

$$y' = r\left(\theta - \frac{\omega\epsilon}{U} - \Upsilon\right)$$

$$\therefore \quad \frac{u}{u_c} = \exp\left[-\pi\left(\frac{r\sin\theta_p^S}{Y}\right)^2\left(\theta - \frac{\omega\epsilon}{U} - \Upsilon\right)^2\right] \tag{M-11}$$

where

θ — angular coordinate of the stator

Υ — angular coordinate of the rotor

r — radial position

Equation (M-11) can be expanded in a Fourier series in terms of $(\theta-\Upsilon)$

$$\frac{u}{u_c} = \sum_n \left(a_n \cos n(\theta-\Upsilon) + b_n \sin n(\theta-\Upsilon)\right) \tag{M-12}$$

or

$$\frac{u}{u_c} = \sum_n (a_n \cos n\varphi + b_n \sin n\varphi) \tag{M-13}$$

where

$$\varphi = \theta - \Upsilon \tag{M-14}$$

$$a_n = \frac{N_R}{2\pi}\int_0^{2\pi}\left(\frac{u}{u_c}\right)\cos n\varphi\, d\varphi \quad (N_R = \text{no. of blades of rotor}) \tag{M-15}$$

$$b_n = \frac{N_R}{2\pi}\int_0^{2\pi}\left(\frac{u}{u_c}\right)\sin n\varphi\, d\varphi \tag{M-16}$$

The velocity, u_c, is in the direction of x^*, which makes an angle $(\theta_p^S+\theta_p^R)$ with the after propeller blade so that the component giving upwash at the blade is

$$\frac{u_c^n}{U} = \frac{u_c}{V^S}\cdot\frac{V^S}{U}\sin(\theta_p^S+\theta_p^R) \tag{M-17}$$

and since

$$\frac{V^S}{U} \approx \frac{1}{\sin\theta_p^S}$$

then from Eqs.(M-9) and (M-17),

$$\frac{u_c^{\ n}}{U} = -\frac{\left(2.42\sqrt{C_D^S}\right)}{\left(\frac{x'}{c^S} + 0.3\right)} \cdot \frac{1}{\sin\theta_p^S} \sin\left(\theta_p^R + \theta_p^S\right) \qquad \text{(M-18)}$$

where

$$\frac{x'}{c^S} = \frac{c^R}{c^S}\left(\frac{\epsilon}{c^R} \csc \ \theta_p^S + \frac{x^R}{c^R} \cdot \frac{V^S}{V^R}\right) - 0.7 \qquad \text{(M-19)}$$

Choose $\frac{x^R}{c^R} = 0$, which means the point is at the mid-chord of the rotor blade. Then

$$\frac{x'}{c^S} = \frac{\frac{\epsilon}{Y_o}}{\left(\frac{c^S}{Y_o}\right)} \csc \ \theta_p^S - 0.7 \qquad \text{(M-20)}$$

The viscous wake, then, can be expressed in the following form:

$$\frac{u^{(q)}}{U} = \frac{u_c^{\ n}}{U} \left(a_n \cos n\varphi + b_n \sin n\varphi\right) \qquad \text{(M-21)}$$

where

$$q = 2n \qquad \text{(M-22)}$$

$$\varphi = \theta - Y = 2\theta \qquad \text{(M-23)}$$

The left-hand side due to unsteady wake in the PPEXACT (Propeller-propeller Exact) program (Reference 1) is, in lift operator form,

$$\frac{\tilde{W}^{(q,\bar{m})}}{U}(r) = \frac{u^{(q)}}{U}(r) \ e^{-iq\sigma^r} \ I^{(\bar{m})}(q\theta_b^r) \qquad \text{(M-24)}$$

where

$$I^{(\bar{m})}(q\theta_b^r) = \frac{1}{\pi}\int_0^{\pi} \Phi(\bar{m}) \ e^{iq\theta_b^r \cos\varphi_\alpha} \ d\varphi_\alpha \qquad \text{(M-25)}$$

$$\Phi(1) = 1 - \cos\varphi_\alpha$$

$$\Phi(2) = 1 + 2\cos\varphi_\alpha$$

$$\Phi(\bar{m}) = \cos(\bar{m}-1)\varphi_\alpha \qquad \text{for } \bar{m} > 2$$

Thus, the resulting unsteady force and moment or unsteady side force and moment, at the specified blade frequency, can be determined as in the PPEXACT program. These viscous effects are then superposed on the results from the potential flow.

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