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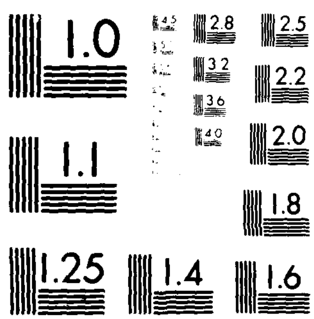

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AN ESTIMATION THEORY FOR DIFFERENTIAL EQUATIONS AND OTHER PROBLEMS, WITH APPLICATIONS

Final Technical Report

by

Johann Schröder

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The research described here is concerned with various methods to estimate solutions of equations, in particular differential equations. These methods can be used to obtain information on the qualitative behavior of solutions, and they can also be applied for numerical estimates. This report provides a survey on the results obtained.

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AN ESTIMATION THEORY FOR DIFFERENTIAL EQUATIONS AND OTHER PROBLEMS, WITH APPLICATIONS

Abstract

The research described here is concerned with various methods to estimate solutions of equations, in particular differential equations. These methods can be used to obtain information on the qualitative behavior of solutions, and they can also be applied for numerical estimates. This report provides a survey on the results obtained.

Table of contents

1. General survey

The research reported herein is concerned with estimates for solutions of equations, and related topics. The estimates are often combined with existence statements. A series of applications is investigated. Particular attention is given to numerical applications.

The equations considered are described by operators M of the following type:

(1) matrices, and nonlinear functions in $\mathbb{R}^n$.

(2) Ordinary differential operators of the second order, together with boundary operators.

(3) Ordinary differential operators of higher order, together with boundary operators.

(4) Ordinary differential operators of the first order, together with an initial condition.

(5) Vector-valued ordinary differential operators of the first and second order.

(6) Vector-valued elliptic differential operators of the second order, together with boundary operators.

(7) Vector-valued parabolic differential operators, together with boundary operators and initial conditions.

(8) Abstract differential operators of the first order.

(9) More general abstract operators.

Thus the quantities v to be estimated may be vectors, real-valued functions, vector-valued functions, functions with values in an abstract space, or more generally, elements of an abstract space. The estimates (inclusion properties) which are proved state that v belongs to a certain set K . In particular, we consider inclusion properties such that $v \in K$ is equivalent to one of the following relations (a) through (e').

Estimates for abstract elements v (with applications to matrices, functions, etc.)

(a) $\varphi \le v \le \psi$ (two-sided bounds) .

(a') $0 \le v$ (positivity) .

(b) $v \in K$ (K subset of an abstract space with certain properties) .

Estimates for vector-valued functions v :

(c) $\|v(x)\| \le \psi(x)$ (pointwise norm bounds, $\| \ \| = \langle \ , \ \rangle^{\frac{1}{2}}$) .

(d) $v(x) \in \psi(x)G$ (shape-invariant bounds, $\psi(x) \in \mathbb{R}^1$, $G \in \mathbb{R}^n$).

(d') $v(x) \in G$ (invariance statements) .

(d") $v(x) \in \psi_\ell(x)G_\ell$ $(\ell=1,2,\ldots,N)$ (generalization of (d)) .

(In these estimates the variable $x \in \mathbb{R}^m$ is to be replaced by $(x,t) \in \mathbb{R}^{m+1}$ for parabolic problems.)

Estimates for functions v with values in a Banach space:

(e) $v(t) \in \psi(t)G$ (G a subset of a Banach space X , in particular, $X = C_0^n(\bar{\Omega})$) .

(e') $\int_\Omega v^T(x,t)\, H(x)\, v(x,t)\, dx \le \psi^2(t)$ (special case of (e))

The basic statements proved are of two different types (U) and (E) :

(U) Range-domain implications:

$$Mv \in C \Rightarrow v \in K \quad . \tag{1.1}$$

Here the estimate $v \in K$ of the unknown quantity v is derived from certain properties of the known image Mv of v under an operator M . For example, if M is a differential operator, the term $Mv \in C$, in general, represents a differential inequality, or a set of differential inequalities. Hence, we are concerned here, in particular, with a theory of differential inequalities.

Statements of the form (1.1) can be used to obtain estimates for all solutions of a given equation $Mv = r$:

$$r \in C \Rightarrow v \in K \quad \text{for each solution of} \quad Mv = r \quad . \tag{1.2}$$

From this result, one can often derive the uniqueness of the solution. This, however, is not the main object in proving statements of type (U) .

(E) Existence and inclusion statements:

$$r \in C \Rightarrow \text{there exists a solution } v \in K \text{ of } Mv = r \quad . \tag{1.3}$$

In addition, modifications of (1.1) are considered, where certain properties of v (such as positivity) are assumed to be known. Then we have an implication of the form

$$(Mv \in C\ ,\ v \in K^{o}) \Rightarrow v \in K\ . \tag{1.4}$$

Most of the results are contained in the monograph "Operator Inequalities" [1] and the papers [2,3,4,5]. Certain numerical results have not yet been published.

This report is divided into three chapters. Chapter I gives a brief survey of the contents of "Operator Inequalities". This book is mainly concerned with estimates of type (a), (a'), (c). In addition, methods for obtaining results on more general estimates (b) are described. The operators treated are abstract operators (9), matrices (1) and ordinary differential operators (2) through (5). Chapter II of this report describes the results on estimates (d), (d'), (d"), (e), (e') for vector-valued differential operators (5), (6), (7) and abstract differential operators (8), obtained in [2,3,4,5]. Chapter III is concerned with numerical applications. The appendix yields a detailed table of contents of [1].

I. OPERATOR INEQUALITIES

The book [1] is concerned with inequalities that are described by operators or, briefly, with operator inequalities. As the title suggests, abstract terms are used in developing the theory and methods. Abstract results, however, are not considered as ends in themselves, but as means to obtain results for concrete problems.

The book concentrates on matrices and (scalar-valued and vector-valued) ordinary differential operators. Because of size-limitations on the book, partial differential operators essentially had to be omitted. We point out, however, that most of the results of type (U) for ordinary differential operators can be carried over without any essential difficulty to elliptic-parabolic partial differential operators of the second order.

More generally, a main purpose of the book is to provide methods of various kinds which can also be used for problems other than those treated in the book.

The Sections 2 through 5 below are concerned with Chapters II through V of [1], respectively.

The book contains more than 330 references, which will not be listed here.

2. Inverse-positive linear operators

The second chapter of [1] provides a unified theory of linear operators M in an ordered linear space R such that for $v \in R$

$$Mv \geq 0 \Rightarrow v \geq 0 \quad \text{(inverse-positivity)} \quad .$$

This implication is of type (1.1) where $Mv \in C$ is equivalent to $Mv \geq 0$. The basic result is the monotonicity theorem (Theorem 1.2).

From this theory most of the known results on M-matrices and inverse-positive second order differential operators are derived, as well as many new ones. The abstract formulation used here allows one, for example, to recognize the common properties of these two different operator classes. This leads to a theory of abstract M-operators. Certain general methods for proving inverse-positivity are applied, for example, to differential operators of higher order.

Inverse-positivity (strict inverse-positivity etc.) is related to a series of other theories, and it can be used in many applications. Here, we can only indicate briefly some of the topics which are treated in [1]:

Eigenvalue theory for M-matrices, second order differential operators and M-operators, in particular, the Perron-Frobenius theory and generalizations.

Convergence theory for iterative procedures involving matrices or abstract operators.

Oscillation theory for second order differential operators.

Boundary maximum principles.

Convergence theory for difference methods for boundary value problems.

Error estimation for approximate solutions of initial value problems and boundary value problems.

Let us mention some typical statements of the corresponding theories, which, of course, hold only under certain assumptions which we cannot formulate here:

"M is inverse-positive if and only if each real eigenvalue of M is positive" (see Corollary 2.14b and Theorems 3.24, 4.19).

"The iterative procedure for $Mv = r$ converges if and only if M is inverse-positive" (see Theorems 2.19, 4.20).

"M on $[0,1]$ is inverse-positive if and only if $[0,1]$ is an interval of non-oscillation" (see Theorem 3.22).

"The difference method converges if the differential operator is inverse-positive" (see Proposition 5.5).

These statements show that each of the sufficient properties for inverse-positivity which are proved in Chapter II can also be used in other theories and applications.

3. Two-sided bounds for second order differential operators

Chapter III of [1] treats the theory of two-sided estimates $\varphi \le v \le \psi$ for solutions v of two-point boundary value problems of the second order. (Initial value problems of the first order are included as a special case.)

First we describe a theory on inverse-monotone differential operators M , which have the property that

$$M\varphi \le Mv \le M\psi \Rightarrow \varphi \le v \le \psi \quad .$$

Here, $\le$ denotes a pointwise inequalitiy, and M is an operator of the form

$$Mu(x) = \begin{cases} -a(x)u''(x) + f(x,u(x),u'(x)) & \text{for} \quad 0 < x < 1 \\ g_0(u(0),u'(0)) & \text{for} \quad x = 0 \\ g_1(u(1),u'(1)) & \text{for} \quad x = 1 \end{cases}$$

Secondly, we treat a generalized inverse-monotonicity, where the pointwise differential inequalities $M\varphi \le Mv \le M\psi$ are replaced by weak differential inequalities. It is shown, for example, that a differential operator which is monotone-definite, i.e. monotone in the sense of Browder-Minty, is also inverse-monotone (see Theorem 3.9).

Furthermore, Chapter III of [1] deals with the theory of comparison functions, where statements of the following form are proved:

$$M\varphi \le 0 \le M\psi \Rightarrow \begin{cases} \text{there exists a solution } v \text{ of} \\ Mv = 0 \text{ with } \varphi \le v \le \psi \end{cases},$$

and, in addition, estimates $\Phi \le v' \le \Psi$ for the derivative of the solution are obtained. The theory is derived in a way which makes consequent use of the theory of inverse-monotone operators.

Finally, the theory of generalized inverse-monotonicity is used to develop a theory of weak comparison functions, where the pointwise differential inequalities $M\varphi \le 0 \le M\psi$ are replaced by weak differential inequalities.

The examples treated include singular perturbation problems, bifurcation problems, existence proofs for positive solutions and numerical error estimates.

In developing the theory, we explain (several) methods of proof and show also, how these methods can be applied to other problems, such as problems with periodic boundary conditions and certain singular boundary value problems.

4. An estimation theory, range-domain implications

Chapter IV of [1] describes some general principles which can be used to obtain sufficient conditions for statements of the form (1.1):

$$Mv \in C \Rightarrow v \in K , \tag{4.1}$$

where C and K are subsets of certain linear spaces R and S, respectively. The results can be used to obtain sufficient conditions on M such that (4.1) holds for given sets C and K. They may also be used, however, to find a set C such that (4.1) holds, if M and K are given.

The main tool is a continuity principle which can be applied in the following way. One constructs a family of sets K_λ $(0 \le \lambda < \gamma \le \infty)$ with $K_0 = K$ and $K_\lambda \neq K$ for $\lambda > 0$ such that there exists a minimal λ with $v \in K_\lambda$ and that for $\lambda > 0$ the element v belongs to the boundary (or some other distinguished subset) Γ_λ of K_λ. Then, from some general properties of the elements $v \in \Gamma_\lambda$, one deduces certain relations for Mv. If these relations contradict $Mv \in C$, the statement (4.1) is proved.

This continuity principle is here applied to obtain results on range-domain implications of the following type:

Inverse-monotonicity: $Mv \le Mw \Rightarrow v \le w$; (4.2)

Two-sided bounds (with $Mu = H(u,u)$) :

$$\left.\begin{array}{l} H(\varphi,h) \le Mv \le H(\psi,h) \\ \text{for all } h \text{ with } \varphi \le h \le \psi \end{array}\right\} \Rightarrow \varphi \le v \le \psi \tag{4.3}$$

In the theory on these two properties for operators in ordered spaces we use sets

$$K_\lambda = \{u \in R : u \le w + z_\lambda\} \qquad (0 \le \lambda < \infty)$$

and

$$K_\lambda = \{u \in R : \varphi - \bar{z}_\lambda \le u \le \psi + z_\lambda\} \qquad (0 \le \lambda < \infty) \quad ,$$

respectively, where z_λ and $\bar{z}_\lambda$ denote elements in R. The results on inverse-monotonicity are applied to nonlinear operators in $\mathbb{R}^n$ and vector-valued functional differential operators of the second order.

For initial value problems of the first order a second approach is described, where the sets K_λ are defined differently.

The object of this chapter is not only to derive sufficient conditions for special statements of the form (4.1), but also to provide methods which can be used for operators and problems not treated here. The derivation of results for infinite systems of differential equations and abstract differential equations in this chapter are to be considered as further examples for the application of these methods. (See also Section 5.)

Chapter IV of [1] contains also a special theory on statements of the form (4.1) for linear (and concave) operators. This theory is applied to obtain statements of the form

$$-\Phi \le Mv \le \Psi \Rightarrow \varphi \le v \le \psi$$

for second order linear differential operators which are not inverse-positive.

5. Estimation and existence theory for vector-valued differential operators

Chapter V of [1] is mainly concerned with vector-valued differential operators M of the form

$$Mu(x) = \begin{cases} -a(x)u''(x) + b(x)u'(x) + f(x,u(x),u'(x)) & \text{for} \quad 0 < x < 1 \\ -\alpha^0 u'(0) + f^0(u(0)) & \text{for} \quad x = 0 \\ \alpha^1 u'(1) + f^1(u(1)) & \text{for} \quad x = 1 \end{cases}$$

with $u(x) = (u_i(x)) \in \mathbb{R}^n$ and diagonal matrices $a(x)$, $b(x)$, α^0, α^1. Part of the theory is also formulated for functional-differential operators, where, for example, $f(x,u(x),u'(x))$ is replaced by $g(x,u(x),u'(x),u,u')$. Again, operators of the first order can often be treated as a special case.

Operators as described above are inverse-monotone only under very restrictive assumptions. In general, one requires the following two conditions on the components f_i of f:

(i) M is weakly coupled, i.e., f_i does not depend on any $u_k'(x)$ with $k \neq i$.

(ii) M is quasi-monotone, i.e., f_i is an antitone function of each $u_k(x)$ with $k \neq i$ (antitone = non-increasing).

If both these conditions are satisfied, the theory of inverse-monotonicity and the theory of comparison functions for scalar-valued operators can be generalized to vector-valued operators, without major modifications. The question arises what kind of results can be obtained without requiring both these conditions.

Chapter V of [1] contains results of type (U) and (E) with estimates of the form

(a) $\varphi \le v \le \psi$, i.e.,

$\varphi_i(x) \le v_i(x) \le \psi_i(x)$ for all indices i and all $x \in [0,1]$,

and

(c) $\|v(x)\| \le \psi(x)$ for $0 \le x \le 1$,

ψ real-valued, $\| \ \| = \langle \ , \ \rangle^{\frac{1}{2}}$, $\langle \ , \ \rangle$ an inner product in $\mathbb{R}^n$.

First, these estimates are derived from differential inequalities involving v (results of type (U)). In the case (a) one obtains 2n coupled differential inequalities, which, in addition, contain two parameters $h,k \in \mathbb{R}^n$. For example, for a fixed $x \in (0,1)$, the differential inequality for ψ_1 assumes the form

$$(Mv)_1(x) \le - a_{11}(x)\psi_1''(x) + b_{11}(x)\psi_1(x) + f_1(x,h,k) \ . \tag{5.1}$$

This inequalitiy has to be satisfied for all $h,k \in \mathbb{R}^n$ such that

$$h = (h_i) \ , \quad h_1 = \psi_1(x) \ , \ \varphi_i(x) \le h_i \le \psi_i(x) \ ,$$

$$k = (k_i) \ , \quad k_1 = \psi_1'(x) \ .$$

The occurrence of k leads one to require assumption (i) , in general. If (i) is satisfied, the parameter k can be completely eliminated. Then, the corresponding theory can be derived from the abstract theory on property (4.3). Assumption (ii) need not be required. If (ii) is also satisfied, then h in (5.1) may be replaced by $\psi(x)$, so that no parameter occurs.

For estimates (c) one obtains a differential inequality for ψ which again contains two parameters $q,\eta \in \mathbb{R}^n$:

$$- \psi''(x) + \langle q,q\rangle\psi(x) + \langle \eta, f(x,\psi(x)\eta, \ \psi'(x)\eta + \psi(x)q)\rangle \ge \langle \eta, Mv(x)\rangle$$

$$\text{for} \quad 0 < x < 1 \ , \ \langle \eta,\eta\rangle = 1 \ , \ \langle \eta,q\rangle = 0 \ .$$

Here none of the assumptions (i), (ii) need to be required.

There are other essential differences between the theories of the estimates (a) and (c). For example, if $f(x,y) = Cy$ with a matrix C , certain eigenvalues of matrices related to C play an important role; in case (a) the smallest eigenvalue of the matrix $B = (b_{ij})$, $b_{ii} = c_{ii}$, $b_{ij} = - |c_{ij}|$; in case (c) with $\langle y,y\rangle = y^T y$ the smallest eigenvalue of $\frac{1}{2}(C+C^T)$.

The methods used to prove these results on range-domain implications can also be applied to estimates different from (a), (c).

In addition to these results, Chapter V yields a theory on the existence of solutions v which satisfy estimates of type (a) or (c). Here, differential inequalities similar to those described above occur.

II. SHAPE-INVARIANT BOUNDS AND GENERALIZATIONS

In this chapter we describe results on estimates for differential operators which were presented in [2-5]. These estimates are more general than estimates by two-sided bounds and pointwise norm bounds. On the other hand, the sufficient conditions derived are still comparibly simple. In order to obtain such estimates, one essentially has to construct one (or several) real-valued functions satisfying certain differential inequalities.

The papers [2] and [3] treat ordinary differential operators and elliptic-parabolic partial differential operators, respectively. [4] describes a different approach to parabolic operators, which yields somewhat stronger (and additional) statements. [5] is a brief survey with additional results.

Here, we shall describe the basic ideas and results of the theory by considering certain elliptic and parabolic operators of a special form. The papers mentioned treat more general differential operators. Moreover, we explained in [5], that the results can easily be generalized to certain functional-differential operators. Furthermore, the methods used in this theory may also be applied in proving more general inclusion properties.

6. Description of the operators and estimates considered

In this report on the results in [2,3,4,5] we restrict ourselves to considering certain special classes of elliptic and parabolic differential operators.

The elliptic operators discussed in Section 7 have the form

$$Mu(x) = -D(x)\,\Delta u(x) + f(x,u(x),u_x(x)) \quad \text{for} \quad x \in \Omega \qquad (6.1)$$

$$Mu(x) = -\alpha(x)\,\frac{\partial u}{\partial \nu}(x) + B(x)\,u(x) \quad \text{for} \quad x \in \partial\Omega \qquad (6.2)$$

where Ω is a bounded domain in $\mathbb{R}^m$, $x = (x_k)$, $u(x) = (u_i(x)) \in \mathbb{R}^n$, $u_x(x) = (\partial u_i/\partial x_k)(x) \in \mathbb{R}^{n,m}$, $\partial/\partial\nu$ denotes the interior normal derivative (or some other directional derivative into Ω), $D(x) = (d_i(x)\,\delta_{ij}) \in \mathbb{R}^{n,n}$, $d_i(x) \geq 0$, $f = (f_i) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n,m} \to \mathbb{R}^n$, $\alpha(x) = (\alpha_i(x)\,\delta_{ij}) \in \mathbb{R}^{n,n}$, $\alpha_i(x) \geq 0$, $B(x) = (b_{ik}(x)) \in \mathbb{R}^{n,n}$, and $\mathbb{R}^{n,m}$ denotes the set of real $n \times m$ matrices.

In the following sections we shall often use a simpler notation, where the independent variable is omitted in some places. For example, we may write (6.1) in the form

$$Mu = -D\,\Delta u + f(x,u,u_x) \quad \text{on the set} \quad \Omega \quad .$$

We point this out because such relations will be combined with side conditions, where the independent variable cannot be omitted.

For operators (6.1), (6.2) we consider estimates of the form

$$v(x) \in \psi(x)G \quad \text{for} \quad x \in \bar{\Omega} \tag{6.3}$$

$$v(x) \in \psi_\ell(x)G_\ell \quad (\ell=1,2,\ldots,N) \quad \text{for} \quad x \in \bar{\Omega} \; . \tag{6.4}$$

In (6.3), G denotes a given closed set in $\mathbb{R}^n$, $v \in R = S^n$ is the unknown function to be estimated, $\psi \in S$ a function to be constructed, and $S = C_1(\bar{\Omega}) \cap C_2(\Omega)$ (for simplicity).

In order to obtain practicable conditions, we assume that the set $G \subset \mathbb{R}^n$ is described by a real-valued function W on $\mathbb{R}^n$, such that

$$y \in G \Leftrightarrow W(y) \leq 1 \quad .$$

This function has to satisfy certain smoothness conditions and other assumptions, which imply that G is star-shaped with respect to O and that G has a smooth boundary $\Gamma = \{y : W(y) = 1\}$. It is also required that the normal vector $W'(y)$ at a boundary point y of G must not be orthogonal to y , i.e.,

$$\omega(y) := W'(y)y > 0 \quad \text{for} \quad W(y) = 1 \quad . \tag{6.5}$$

The estimate (6.1) is equivalent to

$$V(v(x)) \leq \psi(x) \quad \text{for} \quad x \in \bar{\Omega} \quad ,$$

where V is the Minkowski functional of G , which, however, need not be known (see [2]).

For constant $\psi(x) \equiv \psi_0$ one obtains results on invariant sets. For variable $\psi(x)$, the size of the set $\psi(x)G$ depends on x , but its "shape" remains invariant. We speak of shape-invariant bounds.

For estimates (6.3) analogous assumptions are made: the sets G_ℓ are described by functions W_ℓ , etc.

The parabolic operators M considered in Section 8.1 have the form

$$Mu(x,t) = u_t(x,t) - D(x)\ \Delta u(x,t) + f(x,t,u(x),u_x(x)) \quad \text{for } x \in \Omega\ ,\ 0 < t \le T\ , \tag{6.6}$$

$$Mu(x,t) = -\ \alpha(x)\ \frac{\partial u}{\partial \nu}(x,t) + B(x)\ u(x,t) \quad \text{for } x \in \Omega\ ,\ 0 < t \le T\ , \tag{6.7}$$

$$Mu(x,0) = u(x,0) - r(x) \quad \text{for } x \in \Omega\ . \tag{6.8}$$

Here, the notation is essentially the same as above. (In particular, $\Omega \subset \mathbb{R}^m$ is bounded, Δ and $\partial/\partial\nu$ are differential operators with respect to x .)

We prove estimates of the form

$$v(x,t) \in \phi(x,t)G \quad \text{for } x \in \overline{\Omega}\ ,\ 0 \le t \le T \tag{6.9}$$

and

$$v(x,t) \in \phi_\ell(x,t)G_\ell \ (\ell=1,2,\dots,n) \quad \text{for } x \in \overline{\Omega}\ ,\ 0 \le t \le T\ . \tag{6.10}$$

Here, G and G_ℓ are sets as above, $v \in R = S^n$, $\phi \in S$, $\phi_\ell \in S$, where now S is the set of all real-valued functions on $\overline{\Omega} \times [0,T]$ which are continuously differentiable with respect to t and twice continuously differentiable with respect to x on $\overline{\Omega} \times (0,T]$. (Of course, these smoothness conditions can be weakened.)

In Section 8.2 estimates of the form

$$v(t) \in \phi(t)G \quad \text{for } 0 \le t \le T \tag{6.11}$$

are treated, where

$v(t) \in C_0^n(\overline{\Omega})$ is defined by $(v(t))(x) = v(x,t)$,

$C_0^n(\overline{\Omega}) = (C_0(\overline{\Omega}))^n$ is the set of continuous functions on $\overline{\Omega}$ with values in $\mathbb{R}^n$,

$\phi \in C_0[0,T] \cap C_1(0,T]$,

$G \subset C_0^n(\overline{\Omega})$ satisfies conditions similar to those described above for $G \subset \mathbb{R}^n$.

In the basic results for estimates (6.11) of $v \in R$ the boundary terms (6.7) do not occur. However, in verifying the assumptions, in general, certain boundary conditions are needed for all components v_i for which the coefficient d_i does not vanish.

In particular, we discuss estimates of the form

$$\left(\int_\Omega v^T(x,t)\ H(x)\ v(x,t)\ dx\right)^{\frac{1}{2}} \le \psi(t) \qquad (0 \le t \le T) \tag{6.12}$$

where $H(x) = (h_i(x)\ \delta_{ij})$ with $h_i \in C_0(\bar\Omega)$, $h_i(x) \ge 0$ on $\bar\Omega$. This estimate is equivalent to (6.11) with

$$G = \{y \in C_0^n(\bar\Omega) : W(y) \le 1\}$$
$$W(y) = \langle y,y\rangle \ , \ \langle y,\eta\rangle = \int_\Omega y^T(x)\ H(x)\ \eta(x)\ dx \ . \tag{6.13}$$

In particular, one may choose $H(x) = I$ or $H(x) = (d^{-1}(x)\ \delta_{ik})$, assuming in the latter case that all coefficients d_i are strictly positive.

The results in [4] on estimates (6.11) for parabolic operators are derived from a theory on corresponding estimates for a first order differential operator in a Banach space:

$$\mathcal{M}\, v(t) = \begin{cases} v'(t) - F(t,v(t)) & \text{for } 0 < t \le T \\ v(0) - r & \text{for } t = 0 \end{cases}$$

These abstract results will not be described here.

Analogously, estimates on abstract differential operators of the second order can sometimes be used to derive estimates for elliptic differential operators (see [5]).

The various estimates described here require different assumptions on the form of the matrices $D(x)$ and $\alpha(x)$ and on the coupling of the function f, where both types of assumptions are related to each other (see the discussion below).

In [4] we proved also estimates which are obtained when the sets G and G_ℓ in (6.9), (6.10) and (6.11) are replaced by its interior. The corresponding theory is similar, but somewhat simpler.

As mentioned above, the estimates proved yield invariance statements, by choosing constant functions ψ (ψ_ℓ). Invariance statements (of type (U) and (E)) have been proved for elliptic operators in [6-12], for parabolic operators in [7,8,9,13-16], for abstract differential operators in [17-20]. In [21] certain estimates are proved which can be considered as special cases of (6.11) with $\psi(t) \equiv$ const. . The paper [22] treats estimates (6.12) with $H(x) \equiv I$ and $\psi(t) = \exp(-\varkappa t)$. In [23] estimates $v(t) \in G_t$ are derived. For further references see [2-4].

7. Elliptic operators

7.1 Shape-invariant bounds

For deriving estimates (6.3) we have to assume that all functions d_i are equal and all functions α_i are equal:

$$d_i = d \quad , \quad \alpha_i = \alpha \quad (i=1,2,\ldots,n) \quad . \tag{7.1}$$

We shall explain below, that this inconvenient assumption is not necessary for more general estimates (6.4).

First, we consider the following

<u>Special case:</u> G is convex, f does not depend on u_x . (7.2)

Here the theory leads us to differential inequalities for ψ , which involve a parameter $\eta \in \mathbb{R}^n$. These inequalities have the following form (recall (6.5)):

$$- \omega(\eta)\ d\ \Delta\psi \ + \ W'(\eta) f(x,\psi\eta) \ \geq \ W'(\eta) Mv \tag{7.3}$$

on the set of all (x,η) with $x \in \Omega$, $W(\eta) = 1$.

$$- \omega(\eta)\ \alpha \frac{\partial\psi}{\partial\nu} \ + \ W'(\eta) B\eta\ \psi \ \geq \ W'(\eta) Mv \tag{7.4}$$

on the set of all (x,η) with $x \in \partial\Omega$, $W(\eta) = 1$.

Inequalities of this type occur in results of both types (U) and (E) , which we have proved for this special case.

The <u>results of type (U)</u> are derived by means of the continuity principle (Section 4) using a family of functions $\psi_\lambda = \psi + z_\lambda$ $(0 \leq \lambda < \infty)$. One obtains differential inequalities for all these functions ψ_λ . In general, one can choose functions z_λ of the special form $z_\lambda = \lambda z$ with a strictly positive function z . Also, one may split each differential inequality for ψ_λ into a condition for ψ and a second condition involving z_λ . Then, for $z_\lambda = \lambda z$ one obtains the differential inequalities for ψ described above and, in addition, the following inequalities:

$$- \omega(\eta)\ d\ \Delta z \ + \ W'(\eta)[f(x,(\psi+\lambda z)\eta) - f(x,\psi\eta)] > 0$$

on the set of all (λ,x,η) such that $\lambda > 0$, $x \in \Omega$ and

$$W(\eta) = 1 \quad , \quad v(x) = (\psi(x) + \lambda z(x)\eta \quad ; \tag{7.5}$$

$$- \omega(\eta)\ \alpha \frac{\partial z}{\partial \nu} + W'(\eta)B\eta\ z > 0$$

on the set of all (x,η) with $x \in \partial\Omega$, $W(\eta) = 1$.

The idea is to prove the existence of suitable functions z for a certain class of operators M . Then, for each operator M in this class the following statement holds:

<u>If the inequalities</u> (7.3), (7.4) <u>are satisfied, then</u> $v(x) \in \psi(x)G$ <u>for</u> $x \in \overline{\Omega}$. <u>In particular, each solution</u> v <u>of</u> $Mu = 0$ <u>satisfies this estimate, if</u> (7.3), (7.4) <u>hold with</u> Mv <u>replaced by</u> 0 .

In [3] the application of these results is explained in detail, in particular, the use of side conditions on v such as the one in (7.5).

<u>Results of type (E)</u> *have been proved for the special case* (7.2) under additional assumptions on d , α , B , f , $\partial\Omega$, such as $d(x) \geq d_o > 0$ and smoothness conditions. For this case, the statements have the following form:

<u>If the inequalities</u> (7.3), (7.4) <u>are satisfied with</u> Mv <u>replaced by</u> 0 , <u>then the boundary value problem</u> $Mu = 0$ <u>has a solution</u> v <u>such that</u> $v(x) \in \psi(x)G$ <u>for</u> $x \in \overline{\Omega}$.

For the general case, where f may depend on u_x and G need not be convex, we have proved results of type (U) . Here, the differential inequalities on Ω become more complicated. In particular, these inequalities involve two parameters $\eta \in \mathbb{R}^n$ and $q \in \mathbb{R}^{n,m}$. For example, the condition for ψ which generalizes (7.3) has the following form, where $Q(x,\eta,q)$ is the trace of the $m{\times}m$ matrix $d(x)q^T W''(\eta)q$:

$$- \omega(\eta)\ d\ \Delta\psi + Q(x,\eta,q)\psi + W'(\eta)f(x,\psi\eta,\psi q+\eta\psi_x) \geq W'(\eta)Mv \tag{7.6}$$

on the set of all (x,η,q) with

$$x \in \Omega \ , \quad W(\eta) = 1 \quad , \ W'(\eta)q = 0 \quad .$$

Observe that the parameter q is not bounded, so that assumptions of this type may impose <u>strong conditions on the dependence of</u> $f(x,u,u_x)$ on u_x . We shall make some brief remarks concerning conditions on the coupling of f with respect to u_x .

If the set G is "strictly convex" and $d(x) \geq d_o > 0$, then the term $Q(x,\eta,q)$ in (7.6) is strictly positive (for $W'(\eta)q = 0$) and $W(y)$ depends on all components y_i $(i=1,2,\ldots,n)$. Thus, (7.6) constitutes a certain quadratic growth condition on f as a function of u_x , but f may be strongly coupled.

If W depends only on some of the variables y_i , say y_i with $i \in J$, then $Q(x,\eta,q)$ does not depend on elements q_{jk} of q with $j \notin J$. Consequently, the condition (7.6) cannot be satisfied for all q (with $W'(\eta)q = 0$), if f depends on any derivative $\partial u_j/\partial x_k$ with $j \notin I$, except for very special nonlinear functions f .

In particular, we have investigated linear operators M with Dirichlet boundary terms for the case $W(y) = y^T y$. These statements on linear operators contain results on boundary maximum principles, more general than those of [24,25,26].

7.2 Estimates $v \in \cap\, \psi_\ell G_\ell$

For estimates of the more general form (6.4), where G_ℓ is described by a function W_ℓ , condition (7.1) can be replaced by the following weaker

Assumption Λ : If there exists an ℓ such that $W_\ell(y)$ depends on y_i and y_j , then

$$d_i = d_j \quad \text{and} \quad \alpha_i = \alpha_j \quad .$$

Example 1: If all d_i are pairwise different, then each W_ℓ must depend on one variable y_j only. This is the case, for instance, if

$$N = 2n \; , \; W_i(y) = y_i \quad , \quad W_{n+i}(y) = -\, y_i \quad (i=1,2,\ldots,n) \quad .$$

The corresponding statement (6.4) is an estimate by two-sided bounds.

Example 2: In order to obtain an estimate

$$(v_1^2(x) + v_2^2(x))^{\frac{1}{2}} \leq \psi_1(x) \quad , \quad (v_3^2(x) + v_4^2(x))^{\frac{1}{2}} \leq \psi_2(x)$$

in the case $n = 4$, one may choose

$$W_1(y) = y_1^2 + y_2^2 \quad , \quad W_2(y) = y_3^2 + y_4^2 \quad .$$

Here, we need to require that $d_1 = d_2$, $d_3 = d_4$.

Thus, the choice of functions W_ℓ which do not depend on all y_i, has certain advantages; but it has disadvantages, too. Such a choice, in general, requires stronger conditions on the coupling of $f(x,u,u_x)$ with respect to u_x .

In Example 1, each component f_i of f must not depend on any derivative of any function u_j with $j \neq i$, except for very special nonlinear functions f_i . In Example 2, we must, in general, assume that f_1 and f_2 do not depend on any derivative of u_3 and u_4, and f_3 and f_4 do not depend on any derivative of u_1 and u_2 .

For a better understanding of the conditions required see the remarks concerning the coupling of f which follow formula (7.6). For the more general estimates considered here one obtains differential inequalities for the functions ϕ_ℓ analogous to (7.6), with W replaced by W_ℓ, etc..

We proved in [3] results of type (U) and (E) for the special case (7.2), and results of type (U) for the general case.

As an example, we derived in [3] existence statements and numerical error estimates for the steady state of a simple reaction-diffusion problem, where

$$K_x := \cap_{\ell=1}^{4} \phi_\ell(x)G_\ell \quad \text{is a non-rectangular quadrangle} .$$

This choice of the sets G_ℓ is natural because of the special structure of such problems.

8. Parabolic operators

8.1 Shape-invariant bounds

The theory in Section 7 on results of type (U) can also be applied to parabolic operators, as discussed in [3]. One obtains somewhat stronger (and additional) statements, however, by the approach described in [4]. We shall present some of the pertinent results in more detail, since they have not been published in this form. We use the definitions and notation in Section 6, and assume in this section that (7.1) holds.

Theorem A . The estimate

$$v(x,t) \in \phi(x,t)G \quad \text{for } x \in \Omega ,\ 0 < t \leq T \tag{8.1}$$

holds under the following assumptions (I) , (II) :

(I) Suppose that

(i) $\omega(\eta)(\phi_t - d\,\Delta\phi) + Q(x,\eta,q)\phi + W'(\eta)f(x,t,\phi\eta,\phi q+\eta\phi_x)$
$$\geq W'(\eta)Mv$$

on the set of all (x,t,η,q) with

$$x \in \Omega \quad , \quad 0 < t \leq T \quad , \quad W(\eta) = 1 \quad , \quad W'(\eta)q = 0 \quad ; \qquad (8.2)$$

(ii) $\omega(\eta)\phi_\nu + W'(\eta)B\eta\,\phi > W'(\eta)Mv$

on the set of all (x,t,η) with

$$x \in \partial\Omega \quad , \quad 0 < t \leq T \quad , \quad W(\eta) = 1 \quad ; \qquad (8.3)$$

(iii) $v(x,0) \in \phi(x,0)G$ for $x \in \overline{\Omega}$.

(II) Suppose that there exists a function $z \in S$ such that $z(x,t) > 0$ on $\overline{\Omega} \times [0,T]$ and

(i) $\omega(\eta)(z_t - d\,\Delta z) + Q(x,\eta,q)z$
$$+ \frac{1}{\lambda} W'(\eta)[f(x,t,v,v_x) - f(x,t,v-\lambda z\eta,v_x-\lambda zq-\lambda\eta z_x)] > 0$$

on the set of all (x,t,λ,η,q) which satisfy (8.2), $0 < \lambda \leq \varepsilon$ and

$$v(x) - \lambda z(x)\eta = \phi(x) \quad , \quad v(x) - \lambda z(x)q - \lambda\eta z_x(x) = \phi_x(x) \qquad (8.4)$$

(ii) $\omega(\eta)z_\nu + W'(\eta)B\eta\, z > 0$

on the set of all (x,t,η) which satisfy (8.3).

Assumption (II) can be verified under rather mild conditions. For example, if G is bounded, the following statements (a) , (b) hold, where

$$q,p \in \mathbb{R}^{n,m} \quad \text{and} \quad ||p|| = (\Sigma_{ik}\ (p_{ik})^2)^{\frac{1}{2}} :$$

(a) Assumption (II)(i) is satisfied, if there exist constants $\mu_1 > 0$, $c_0 > 0$, c_1 , c_2 , such that

$$\omega(\eta) \geq \mu_1 \quad ,$$

$$Q(x,\eta,q) \geq c_0||q||^2 \quad ,$$

$$W'(\eta)[f(x,v,v_x) - f(x,v-\alpha\eta,v_x-p)] \geq c_1\alpha - c_2||p|| \qquad (8.5)$$

on the set of all (x,t,η,q,α,p) which satisfy (8.2), $\alpha \geq 0$ and

$$v(x) - \alpha\eta = \psi(x)\eta \quad , \quad v_x(x) - p = \eta\psi_x(x) \quad , \tag{8.6}$$

<u>and if, in addition, there exist a function</u> $\varphi \in C_2(\bar{\Omega})$ <u>and a constant</u> $\varkappa$ <u>such that</u>

$$\varphi > 0 \text{ on } \bar{\Omega} \, , \; - d\,\Delta\varphi + \varkappa\varphi \geq 0 \text{ on } \Omega \, .$$

(b) <u>Assumption</u> (II)(ii) <u>is satisfied, if</u>

$$\alpha \frac{\partial\varphi}{\partial\nu} + W'(\eta)B\eta\, \varphi > 0$$

<u>on the set of all</u> (x,η) <u>with</u> $x \in \partial\Omega$, $W(\eta) = 1$.

(One verifies by formal calculations, that under the assumptions made above, the function $z(x,t) = \varphi(x)\exp(Nt)$ satisfies (II)(i) and (ii) , respectively, if N is sufficiently large.)

<u>Remarks.</u> 1) Assumption (8.5) essentially is a <u>local Lipschitz condition on f</u> , due to the side condition (8.6). Observe also that the constants c_1 and c_2 , which may depend on v , need not be known numerically.

2) In the case of a Dirichlet boundary operator

$$Mv(x,t) = v(x,t) \quad \text{for } x \in \partial\Omega \, , \; 0 < t \leq T \quad ,$$

one may choose $\varphi(x) \equiv 1$. In may other cases a function of the form $\varphi(x) = r_0^2 - r^2$ can be used, where r is the distance of x from some point in Ω and r_0 is sufficiently large.

3) If Assumption (II) is satisfied, then the estimate (8.1) follows from the conditions on ψ described in Assumption (I) . Assumption (I)(i) constitutes conditions on the dependence of $f(x,t,u,u_x)$ on u_x , as described for elliptic operators.

4) In [4] we explained in more detail, in which way the results can be used for an <u>error estimation</u>, if $f(x,t,u,u_x)$ satisfies a certain <u>quadratic growth condition</u> with respect to u_x . (See Corollary 1.c in [4] and the text succeeding this corollary.) In particular, one can obtain in this way estimates (8.1) with

$$\psi(x,t) = \vartheta(x) + \varphi(x)\exp(-\rho t) \quad ,$$

which yield statements on the <u>stability of solutions</u>.

In some of the results in [4], we did not use the side conditions (8.6). For this reason, we formulated there an assumption which is obtained from (8.5), when $c_1\alpha - c_2\|p\|$ is replaced by

$c_1\alpha - c_2\|p\| - c_3\|p\|^2$. (See Corollary 1b.)

8.2 More general estimates

In [4] we described also results of type (U) on estimates

$$v(x,t) \in \psi_\ell(x,t)G_\ell \quad (\ell=1,2,\ldots,N) \tag{8.7}$$

for parabolic operators. For such estimates condition (7.1) can be replaced by Assumption (Λ) in Section 7.2.

In addition, [4] contains results of type (U) on estimates of the form (6.11):

$$v(t) \in \psi(t)G \quad (0 \le t \le T) \quad , \tag{8.8}$$

where (6.12) is a special case. These estimates have the following important advantage to estimates of the form (8.1) or (8.7):

The coefficients d_i (and α_i) may all be different and, at the same time, there is no restriction on the coupling of f .

The basic results on estimates (8.8) contain conditions on certain function z (or a family of functions z_λ) which are analogous to those in Theorem A . We shall not describe this general theory here. Instead, we formulate a more special result on estimates (6.12).

Here, we define $\langle\ ,\ \rangle$ as in (6.13) and use abbreviations such as

$$\langle \eta, f(v,v_x)\rangle (t) = \int_\Omega \eta^T(x) f(x,t,v(x,t),v_x(x,t))\, dx$$

Moreover, for sufficiently smooth $\eta \in C_0^n(\bar\Omega)$ let

$$I(\eta) = \int_\Omega \eta^T(x)\ H(x)(-\ D(x)\ \Delta\eta(x))\ dx \tag{8.9}$$

and denote by $\|\eta_x\|$ some semi-norm of the matrix η_x (see the example below).

Theorem B. The estimate

$$\left(\int_\Omega v^T(x,t)\ H(x)\ v(x,t)\ dx\right)^{\frac{1}{2}} \le \psi(t) \quad (0 \le t \le T)$$

holds under the following assumptions (i), (ii), (iii):

(i) Suppose there exist constants $\delta > 0$, μ_0 , μ_1 , $\mu_2 > 0$,

c_1 , $c_2 > 0$ such that

$$I(\eta) \geq \mu_2 \|\eta_x\|^2 + \mu_1 \|\eta_x\| + \mu_0 \tag{8.10}$$

and

$$\langle \eta, f(v,v_x) - f(v-\alpha\eta, v_x-\alpha\eta_x)\rangle (t) \geq c_1\alpha - c_2\alpha\|\eta_x\| \tag{8.11}$$

for all

$$\alpha \in (0,\delta] \quad , \; t \in (0,T] \quad , \; \eta \in C_0^n(\bar{\Omega}) \tag{8.12}$$

which satisfy

$$\langle \eta,\eta \rangle = 1 \quad \text{and} \; v(x,t) = (\psi(t) + \alpha)\eta(x) \quad \text{for} \quad x \in \bar{\Omega} \; . \tag{8.13}$$

(ii) Suppose that

$$\psi_t(t) + I(\eta)\psi(t) + \langle \eta, f(\psi\eta,\psi\eta_x)\rangle (t) \geq \langle \eta, Mv\rangle (t) \tag{8.14}$$

under the side conditions (8.12), (8.13).

(iii) $\langle v,v \rangle (0) \leq \psi^2(0)$.

Remarks. 1) Due to (8.13) the function η occuring in (i) and (ii) is sufficiently smooth.

2) Estimates of the form (8.10), in general, can be obtained, if v satisfies certain boundary condition. One uses partial integration and the eigenvalue theory of elliptic differential operators.

3) The assumption (8.11), (8.13) describes a certain local Lipschitz condition, which is satisfied for sufficiently smooth f .

4) In the differential inequalities (8.14) for ψ , one will use an estimate of the form (8.10) where, however, μ_2 need not satisfy $\mu_2 > 0$. If $\mu_2 > 0$, then (8.14) constitutes a certain quadratic growth condition on f as a function of u_x .

Example: Suppose that

$$\Omega = \{x \in \mathbb{R}^n : |x_i| < 1 \quad \text{for all} \quad i\} \; , \; H(x) = D(x) = I \; ,$$

and

$$v(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega \; .$$

Then

$$I(\eta) = \|\eta_x\|^2 \quad \text{with} \quad \|\eta_x\| = \left(\int_\Omega \left|\frac{\partial \eta_i}{\partial x_k}(x)\right|^2 dx\right)^{\frac{1}{2}}$$

and $I(\eta) \geq \mu := 4m\pi^2$, so that

$$I(\eta) \geq s\,\|\eta_x\|^2 + (1-s)\mu \quad \text{for each} \quad s \in [0,1] \quad . \tag{8.15}$$

Thus, (8.10) holds, for instance, with $\mu_2 = 1$, $\mu_1 = \mu_0 = 0$. In (8.14) any estimate (8.15) can be used. If f does not depend on u_x , one will use $I(\eta) \geq \mu$.

III. NUMERICAL APPLICATIONS

The results described above have many applications in numerical analysis. For example, they can be used as tools to prove the convergence of iterative procedures or the convergence of difference methods for differential equations. This was described for linear problems in [1] (see Section 2 above). The application to error estimations is discussed in the following section.

9. Approximation methods and a posteriori error estimates for two-point boundary value problems

9.1 Error estimates

In principle, each of the estimation methods described in the previous sections can be applied to obtain a posteriori error bounds for approximate solutions of equations. (For the application of inverse-monotonicity see [27,28].) The question arises, whether one can construct algorithms of error estimation which on one hand yield sufficiently good error bounds and, on the other hand, do not require too much additional work. We investigated these possibilities for the case of (scalar-valued and vector-valued) two-point boundary value problems of the second order, as part of the research project described in this report. (For methods of error estimation for initial value problems see [29,30].) The work on boundary value problems has not yet been completed. Nevertheless, it seems worthwhile to report briefly on our experiences.

Let us first describe the general principle of error estimation.

For a given equation

$$\tilde{M}u = r$$

denote by

u^*	a solution,
w	an approximation solution,
$v = u^* - w$	the error of w .

The error v satisfies the equation

$$Mv = d$$

with the known

<u>defect</u> $d = d[w] = -\tilde{M}w + r$, and $Mv = \tilde{M}(w+v) - \tilde{M}w$.

After this reformulation of the given problem one tries to apply one of the estimation methods described in the previous sections, in order to obtain an inclusion statement $v \in K$ for the error.

Some first results on the error estimation for two-point boundary value problems are described in [1, Sections II.5.2 , III.6.3, and Example V.4.12]. The example mentioned is a vector-valued problem with a singular differential operator, and the estimate was one by pointwise norm bounds. Here, we shall restrict ourselves to considering scalar-valued equations of the form

$$-u''(x) + f(x,u(x)) = 0 \quad \text{for} \quad 0 < x < 1$$

with linear Sturm-Liouville boundary conditions, and estimates

$$|v(x)| = |u^*(x) - w(x)| \le \psi(x) \quad (0 \le x \le 1)$$

with $\psi \in C_2[0,1]$. Suppose, for simplicity, that the approximate solution $w \in C_2[0,1]$ satisfies the boundary conditions, and write

$$d(x) = d[w](x) = w''(x) - f(x,w(x)) \quad .$$

For continuous f (on $[0,1] \times \mathbb{R}$) one obtains the following statement:

<u>If</u> $\psi \ge 0$ <u>satisfies the homogeneous boundary conditions and if</u>

$$-\psi'' + f(x,w+\psi) - f(x,w) \ge d \quad \text{on} \quad (0,1)$$

<u>and</u>

$$-\psi'' + f(x,w) - f(x,w-\psi) \ge -d \quad \text{on} \quad (0,1) \quad ,$$

<u>then there exists a solution</u> $u^* \in C_2[0,1]$ <u>such that</u> $|u^* - w| \le \psi$ <u>on</u> $[0,1]$.

The application of this result poses essentially three problems:

1) The computation of an approximate solution w .

2) The estimation of the defect $d[w]$ on the entire interval $[0,1]$.

3) The construction of a function ψ , which solves the differential inequalities.

To 1: Approximation methods will be discussed in the succeeding section.

To 2: The defect $d(x)$ usually is a function of a very complicated form. One has to replace it by a simpler bound, say a constant Γ :

$$|d(x)| \leq \Gamma \quad (0 \leq x \leq 1) . \tag{9.1}$$

In the general case, the calculation of a good bound Γ , such that this inequality holds for all $x \in [0,1]$, is the most difficult and time-consuming part of the procedure.

There are cases, however, where such an estimate is comparibly easy. If $d(x)$ is a polynominal, then Γ can be obtained by calculating finitely many values $d(x_k)$. Here, one uses a result of [31]. In such cases, the estimate (9.1) often requires much less work than the computation of w .

This method of estimating the defect can also be used, if $[0,1]$ is divided into a few intervals and $d(x)$ is a polynomial on each of these subintervals.

More general cases are still being investigated.

To 3: In general, this is the easiest part of the procedure (see [1, Section III.6.1]).

9.2 Approximation methods

For carrying out the error estimation we need to calculate an approximate solution w , which is defined for all $x \in [0,1]$. There exist several methods and programs to obtain such approximations (see [32,33], for example).

In order to investigate the power of the estimation methods, we first used polynomial approximations. A series of approximation methods for obtaining such (global) polynomials were tested. We finally arrived at the following

Method P : (Collocation using integrated Legendre polynomials and Chebychev collocation points.) Write the polynomial $w(x)$ in the form

$$w(x) = \varphi_0(x) + \sum_{i=1}^{m} a_i \varphi_i(x) ,$$

where $\varphi_0(x)$ is linear and the functions $\varphi_i(x)$ are once (or twice) integrated Legendre polynomials (depending on the boundary conditions). Calculate the constans a_i such that

$$d[w](x_k) = 0 \qquad (k=1,2,\ldots,m) \quad ,$$

where x_k denote the zeros of the m-th Chebychev polynomial (transformed onto $[0,1]$).

We obtained excellent results (concerning accuracy and computing time) with method P for a series of problems whose solutions did not behave "too badly". For example, the problem

$$- u'' - [90 \operatorname{ctg} \tfrac{1}{6}\pi(1+x) + 60 \tan \tfrac{1}{6}\pi(1+x)]u' - 630 u = 0 \quad ,$$

$$u(0) = 0 \quad , \quad u(1) = 5$$

could be solved without any difficulty. The solution u^* increases very fast from $u^*(0) = 0$ to its maximal value ≈ 224 , which is attained at approximately $x = 0.022$.

For a series of problems of this type we compared this polynomial approximation with methods of spline approximation, using programs of de Boor [32] and Ascher et al. [33]. It turned out that for such problems of "medium difficulty" method P gave by far the best results (see [34]).

Of course, this does not mean, that one should use polynomials instead of spline functions in general differential equation solvers. For example, method P failed for certain singular perturbation problems with very small perturbation parameters, while the other methods still were applicable. Our results suggest, however, to construct programs for the approximation by piecewise polynomial functions where the polynomials in the subintervals can have different orders and very high orders. Then one may not need many subintervals, after one has found out, where the boundary layers, transitions layers etc. occur. Such a procedure would also simplify the a posteriori error estimation.

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Appendix: Contents of "Operator Inequalities"

Chapter III Two-sided bounds for second order differential equations