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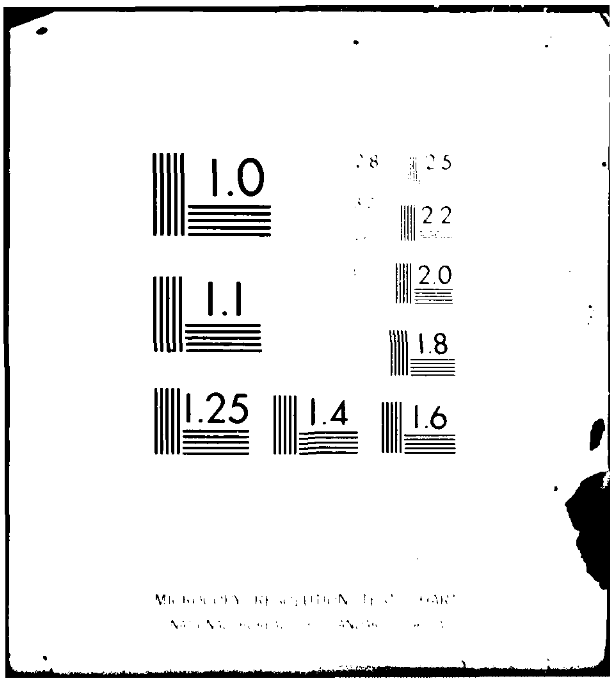
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MRC Technical Summary Report #2281

THE RIEMANN PROBLEM FOR THE SYSTEM

$$u_t + \sigma_x = 0 \quad \text{and}$$

$$(\sigma - f(u))_t + (\sigma - \mu f(u)) = 0$$

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THE RIEMANN PROBLEM FOR THE SYSTEM $u_t + \sigma_x = 0$ and $(\sigma - f(u))_t + (\sigma - \mu f(u)) = 0$

J. M. Greenberg[1,*] and Ling Hsiao[2]

Technical Summary Report #2281
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ABSTRACT

In this paper we study the Riemann Problem for a system of conservation laws which exhibit internal friction similar to that seen in viscoelastic solids of the maxwell type. The solutions we obtain have a single shock and a single contact discontinuity and off of these singular curves they are smooth. The results we obtain are two fold. First we show this problem is globally solvable in time; this requires precise a-priori estimates for the solution off of the singular curves. Secondly, we obtain asymptotic or large time information about the solution which guarantees that in a weak sense it converges to special traveling wave solutions of the equations with compatible data.

AMS (MOS) Subject Classifications: 35B40, 35B45, 35L65, 35L67, 73F99

Key Words: Shock waves, Riemann Problems, Viscoelasticity

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

Problems arising in continuum mechanics are often modelled by systems of conservation laws. For perfect materials such as an elastic solid or an ideal gas, these balance laws, when adjoined with the constitutive equations describing the material, lead to systems of nonlinear hyperbolic partial differential equations in which the characteristic speeds are dependent on the amplitudes of the motion. Such systems have the property that nonconstant disturbances are amplified and solutions which were initially smooth develop discontinuities in finite time. It is well-known that this loss of regularity can be prevented if viscous frictional forces are incorporated into the constitutive assumptions describing the material.

A different situation obtains when the constitutive assumptions of a perfect material are modified to account for long range memory effects but viscous forces are neglected. The following results are typical: (i) solutions generated by small amplitude, smooth data persist for all times and decay to a rest state as time proceeds to plus infinity, and (ii) solutions generated by large amplitude, smooth data develop discontinuities in finite time. In a word, these results show that for small data the damping generated by the memory effects competes favorably against the destabalizing mechanism generated by the nonlinearity in the system while for large disturbances the reverse is true.

It is also well-known that such memory like continua support nonequilibrium solutions called traveling waves. These solutions have a richer structure than their counterparts for perfect materials. In the latter case, such traveling waves are piecewise constant solutions separated by a shock wave advancing at constant speed. What is not known is whether these traveling wave solutions are stable for memory like continua.

The purpose of this paper is to study a model system consisting of a single conservation law adjoined to a constitutive equation of the memory type. This system supports traveling waves similar in structure to those seen in real systems of actual interest. What we are able to show is that the solution of this system with piecewise constant initial data (the Riemann Problem) converges, albeit in a weak sense, to the traveling waves. In the process of establishing this result a number of techniques are developed which should be helpful in examining more realistic systems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE RIEMANN PROBLEM FOR THE SYSTEM $u_t + \sigma_x = 0$ and $(\sigma - f(u))_t + (\sigma - \mu f(u)) = 0$

J. M. Greenberg[1,*] and Ling Hsiao[2]

1. Introduction

In this note we consider the system

$$u_t + \sigma_x = 0 \,, \quad \text{and} \tag{B}$$

$$(\sigma - f(u))_t + (\sigma - \mu f(u)) = 0 \tag{C}$$

where $0 < \mu < 1$ and f satisfies

$$f(0) = 0,\ 0 < f'(u), \ \text{and} \ 0 < f''(u),\ 0 \le u \,. \tag{1.1}$$

Other assumptions will be imposed on f as they are required. This system serves as a model for the equations of motion of a viscoelastic solid. In this analogy (B) represents the balance law and (C) the constitutive equation. In fact we take (C) to be the same as the equation representing a nonlinear-maxwell solid [1,2]. The relation of the system (B) and (C) to the equations of motion for a maxwell solid are the same as the relation of the Hopf equation, $u_t + \left(\frac{u^2}{2}\right)_x = 0$, to the equations of motion for an ideal gas; namely in certain situations the respective reduced equations approximately describe what is happening in the full system and, as importantly, they provide one with some intuition about the more complicated systems.

Our interest is in the Riemann Problem for (B) and (C); that is in solutions of (B) and (C) which assume the following data at $t = 0$:

$$(u,\sigma)(x,0) = \begin{cases} (u_-, \mu f(u_-)), & x < 0 \\ (0,0), & x > 0 \,. \end{cases} \tag{1.2}$$

Here, $u_- > 0$. This system is hyperbolic. It has one nondegenerate characteristic field

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which propagates with speed $f'(u)$ and one linearly degenerate field, namely the lines x = constant. In regions where (u,σ) is smooth, (B) and (C) are equivalent to

$$\sigma_t + f'(u)\sigma_x + (1-\mu)\sigma - \mu\phi = 0, \quad \text{and} \tag{σ}$$

$$\phi_t - (1-\mu)\sigma + \mu\phi = 0 , \tag{ϕ}$$

where ϕ, σ, and u are related by

$$\phi = f(u) - \sigma \quad \text{or equivalently} \quad u = f^{-1}(\sigma + \phi) . \tag{u}$$

The system (B) and (C) also supports traveling waves which assume the data $(u_-,\mu f(u_-))$ and $(0,0)$ at x equal minus and plus infinity respectively. Since these waves will play an important role in what follows we record their properties now. A traveling wave which meets the aforementioned boundary conditions is a solution of (B) and (C) which is a function of $\tau = t - x/c$ where

$$c = \frac{\mu f(u_-)}{u_-} . \tag{1.3}$$

The u component of this solution satisfies

$$\frac{d}{d\tau}(f(\tilde{u}) - c\tilde{u}) = (c\tilde{u} - \mu f(\tilde{u})) \tag{1.4}$$

and the limit relations

$$\lim_{\tau\to-\infty} \tilde{u}(\tau) = 0 \quad \text{and} \quad \lim_{\tau\to+\infty} \tilde{u}(\tau) = u_- . \tag{1.5}$$

σ and ϕ are obtained from $\tilde{u}$ by

$$\sigma(\tau) = c\tilde{u}(\tau) \quad \text{and} \quad \phi(\tau) = f(\tilde{u}(\tau)) - c\tilde{u}(\tau) . \tag{1.6}$$

The results for traveling waves are summarized below:

(a) When $c < f'(0)$ there is a unique (to within a translation), smooth, strictly increasing solution of (1.4) satisfying (1.5).

(b) When $c > f'(0)$ there is a nondecreasing weak solution of (1.4) which is unique to within a translation. The particular solution with a jump discontinuity at $\tau = 0$ has the following additional properties:

(i) $\tilde{u}(\tau) \equiv 0, \quad -\infty < \tau < 0$,

(ii) $\lim_{\tau\to 0^+} \tilde{u}(\tau) = u_*$ where $u_* > 0$ is the unique solution of

$$f(u_*) = cu_*, \quad \text{and} \tag{1.7}$$

(iii) $\tilde{u}(\tau)$ satisfies (1.4) on $(0,\infty)$ and meets the boundary condition (1.5) at plus infinity.

The global (in time) existence question for the Riemann Problem (B), (C), and (1.2) is easily resolved. One may either use a fractional step variant of Glimm's method [3] or the finite difference approach used by Greenberg [4] for a system of integro-differential equations of which (B) and (C) are a special case. Using an iteration scheme and the contraction mapping principle it is also relatively easy to show there is some time $T_{max} > 0$ such that u, σ, and ϕ have the following properties:

(i) $(u,\sigma,\phi) = (u_-,\mu f(u_-),(1-\mu)f(u_-))$ for $x < 0$ and $0 \le t < T_{max}$;

(ii) there is a $C^1(0,T_{max})$ curve $x = s(t)$ with $\frac{ds}{dt} > 0$ such that (u,σ,ϕ) is C^1 and satisfies (B), (C), (σ), (ϕ), and (u) on $0 < x < s(t)$ and $0 < t < T_{max}$;

(iii) $(u,\sigma,\phi) = (0,0,0)$, $x > s(t)$ and $0 \le t < T_{max}$;

(iv) the line $x = 0$, $0 < t < T_{max}$, is a contact discontinuity and the following limit relations obtain:

$$\lim_{x \to 0^+} (u,\sigma,\phi)(x,t) = (u_0(t),\mu f(u_-),\phi_0(t)) \tag{1.8}$$

where $u_0(\cdot)$ is defined implicitly by

$$f(u_0(t)) = \mu f(u_-)e^{-\mu t} + f(u_-)(1 - e^{-\mu t}) \tag{1.9}$$

and

$$\phi_0(t) = (1 - \mu)f(u_-)(1 - e^{-\mu t}) \; ; \tag{1.10}$$

(v) the curve $x = s(t)$, $0 \le t < T_{max}$, is an admissible shock wave and satisfies the Rankine-Hugoniot conditions for (B) and (C); namely the relations

$$\lim_{\substack{x \to s(t) \\ x < s(t)}} (u,\sigma,\phi)(x,t) = (U(t),f(U(t)),0) \; , \tag{1.11}$$

$$U(t) > 0^{(1} , \tag{1.12}$$

and

$$\frac{ds}{dt} = \frac{f(U(t))}{U(t)}, \quad \text{and} \quad s(0) = 0 ; \tag{1.13}$$

(vi) in the region $0 < x \leq s(t)$ and $0 \leq t < T_{max}$ the following inequalities obtain:

$$0 < u < u_-, \ 0 < \sigma \leq \mu f(u_-), \ \text{and} \ 0 < \varphi < (1 - \mu)f(u_-), \ \text{and} \tag{1.14}$$

$$0 < u_t, \ 0 < \sigma_t, \ 0 < \phi_t, \ u_x < 0, \ \sigma_x < 0, \ \text{and} \ \varphi_x < 0 . \tag{1.15}$$

One goal of this investigation as to establish conditions which guarantee that $T_\infty = +\infty$. Our results take two forms. When u_- is small we are able to obtain uniform bounds for the t and x derivatives of σ, ϕ, and u which are independent of T_{max} and depend only on the fact that $U(t) = \lim_{\substack{x \to s(t) \\ x < s(t)}} u(x,t)$ is positive. That $T_{max} = +\infty$ then follows from the fact that for u_- small $U(t)$ cannot vanish for any finite time t. When u_- is large we are also able to obtain uniform bounds for the t and x derivatives of σ, ϕ, and u which are independent of T_{max}. These bounds are predicated on the lower bound estimate $u(x,t) \geq \underline{u} > 0$, a fact which is true if f satisfies a certain technical condition (see (2.53)) and one which is met in the special case $f(u) = f'(0)u + k_0u^2/2$ when u_- is large.

Another goal is to obtain asymptotic information about the solution as t goes to infinity. One set of results apply when $c = \frac{\mu f(u_-)}{u_-} < f'(0)$ and the associated traveling wave is smooth. Here we show that $U(t) \overset{def}{=} \lim_{\substack{x \to s(t) \\ x < s(t)}} u(x,t)$ satisfies

$$\frac{dU}{dt} < 0 \quad \text{and} \quad \lim_{t \to \infty} U(t) \to 0 . \tag{1.16}$$

We also establish a connection between the solution of the Riemann Problem and the smooth traveling wave discussed earlier. Specifically, if we let $x = x(\alpha,t)$, $t \geq \tau(\alpha)$, be a

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[1] The constraint $U(t) > 0$ is simply the Lax entropy condition for this problem. It is easily checked that $U(0^+) = u_0(0) > 0$.

level line of u; i.e.

$$u(x(\alpha,t),t) = \alpha, \quad 0 < \alpha < u_- \quad \text{and} \quad t \geq \tau(\alpha) \tag{1.17}$$

where $\tau(\alpha) \geq 0$ is the first time that either $u_0(\tau(\alpha)) = \alpha$ or $U(\tau(\alpha)) = \alpha$, and

$$c_{AV}(t) = \frac{1}{(u_0(t) - U(t))} \int_{U(t)}^{u_0(t)} \frac{\partial x}{\partial t}(\alpha,t)d\alpha , \tag{1.18}$$

then we show that

$$\lim_{t\to\infty} c_{AV}(t) = c = \frac{\mu f(u_-)}{u_-} . \tag{1.19}$$

Equation (1.19) is a weak statement about the convergence of the solution of the Riemann Problem to a traveling wave; it states that the average speed of propagation of the level lines of u converge to the speed at which the traveling wave propagates.

When $c = \frac{\mu f(u_-)}{u_-} >> f'(0)$ our asymptotic results are of a different character. Here we show that

$$\tilde{u}(t - x/c) < u < u_-, \ c\tilde{u}(t - x/c) < \sigma(x,t) < \mu f(u_-), \ \text{and}$$

$$f(\tilde{u}(t - x/c)) - c\tilde{u}(t - x/c) < \phi(x,t) < (1 - \mu)f(u_-) \tag{1.20}$$

on $0 \leq x \leq ct$ and $t \geq 0$ and that the shock $x = s(t)$ satisfies

$$ct < s(t) < ct + \frac{1}{c(c - f'(0))} \int_{u_*}^{u_-} \frac{(u_- - u)(f'(u) - c)}{u\left(c - \frac{\mu f(u)}{u}\right)} du . \tag{1.21}$$

The function $\tilde{u}(\cdot)$ is the u component of the traveling wave defined earlier in item (b) on traveling wave solutions. It satisfies (1.4) on $\tau > 0$ and the initial condition (1.7). The inequalities (1.20) and (1.21) also guarantee that

$$\lim_{t\to\infty} (u,\sigma,\phi)(\lambda t,t) = \begin{cases} (u_-,\mu f(u_-),(1 - \mu)f(u_-)), & \lambda < c \\ (0,0,0), & \lambda > c . \end{cases} \tag{1.22}$$

2. A-Priori Estimates

A. General Remarks

In this section we develop a-priori bounds which guarantee that the upper bound for the interval of existence, T_{max}, is plus infinity.

We confine our attention to the characteristic equations

$$\sigma_t + f'(u)\sigma_x + (1 - \mu)\sigma - \mu\phi = 0 \, , \tag{σ}$$

$$\phi_t - (1 - \mu)\sigma + \mu\phi = 0 \, , \tag{ϕ}$$

and

$$u = f^{-1}(\sigma + \phi) \iff \sigma + \phi = f(u) \tag{u}$$

which we insist hold in $0 < x < s(t)$ and $0 < t < T_{max}$. σ also satisfies the boundary condition:

$$\sigma(0,t) = \mu f(u_-), \qquad 0 \le t < T_{max} \tag{BC}$$

and (σ,ϕ) obey the Rankine-Hugoniot conditions:

$$\left.\begin{array}{l} \lim\limits_{\substack{x \to s(t) \\ x < s(t)}} \sigma(x,t) \overset{\text{def}}{=} \Sigma(t) > 0 \quad \text{and} \quad \lim\limits_{\substack{x \to s(t) \\ x < s(t)}} \phi(x,t) = 0, \quad 0 < t < T_{max} \\ \\ \text{where} \\ \\ \dfrac{ds}{dt} = \Sigma(t)/f^{-1}(\Sigma(t)) \quad \text{and} \quad s(0) = 0 \, . \end{array}\right\} \tag{RH}$$

If (σ,ϕ) is a solution of (σ), (ϕ), (u), (BC), and (RH) it extends to a solution of the original Riemann Problem (B), (C) and (1.2) by the procedure:

$$(u,\sigma)(x,t) = \begin{cases} (u_-,\mu f(u_-)), & x < 0 \\ (0,0), & x > s(t) \text{ and } 0 < t < T_{max} \, . \end{cases}$$

In the sequel when we say (σ,ϕ) is a "solution" we mean it is a solution to (σ), (ϕ), (u), (BC), and (RH). Our first task is to develop some properties of the "solution" of this problem. These will depend upon the entropy constraint that $\Sigma(t) > 0$, $0 \le t < T_{max}$.

Theorem 2.1. Suppose (σ,ϕ) is $C\{T(x) \le t \le T_{max}$ and $0 \le x \le s(T_{max})\}$ and $C^1\{T(x) < t < T_{max}$ and $0 < x < s(T_{max})\}$ and is a "solution". Then

$$\sigma_t > 0, \quad T(x) \le t \le T_{max} \text{ and } 0 < x < s(T_{max}) \, , \tag{2.1}$$

and

$$\phi_t > 0,\ \sigma_x < 0,\ \text{ and }\ \phi_x < 0,\quad T(x) \le t \le T_{max} \text{ and } 0 \le x < s(T_{max}) . \quad (2.2)$$

The function $t = T(x)$ is the inverse of $x = s(t)$ and satisfies

$$\frac{dT}{dx} = \frac{f^{-1}(\Sigma(T(x)))}{\Sigma(T(x))} \text{ and } T(0) = 0 . \quad \square \quad (2.3)$$

<u>Proof.</u> We start with the observation that $(p,q) \stackrel{def}{=} (\sigma_t,\phi_t)$ satisfies

$$\left.\begin{aligned} p_t + f'(u)p_x - k(u)(p+q)^2 + (1-\mu)p - \mu q = 0 , \\ q_t - (1-\mu)p + \mu q = 0 , \end{aligned}\right\} \begin{aligned} &0 < x < s(t) , \\ &0 < t < T_{max} \end{aligned} \quad (2.4)$$

$$p(0,t) = 0 \text{ and } \lim_{\substack{x \to s(t) \\ x < s(t)}} q(x,t) = (1-\mu)\Sigma(t) > 0, \quad 0 < t < T_{max} . \quad (2.5)$$

Again, the $u = f^{-1}(\sigma + \phi)$ and $u \to k(u)$ is definea by

$$k(u) \stackrel{def}{=} f''(u)/(f'(u))^2 . \quad (2.6)$$

<u>Lemma 2.1.</u> Suppose $p(x,t) > 0$ for $T(x) < t < T_{max}$ and $0 < x < s(T_{max})$. Then $q(x,t) > 0$ for $T(x) \le t \le T_{max}$ and $0 \le x < s(T_{max})$. $\square$

<u>Proof.</u> The lemma follows from the identity

$$q(x,t) = (1-\mu)\Sigma(T(x))e^{-\mu(t-T(x))} + (1-\mu)\int_{T(x)}^{t} e^{-\mu(t-s)}p(x,s)ds$$

and the hypotheses on p and Σ. $\square$

<u>Lemma 2.2.</u> $p(x,t) > 0$ fo $T(x) \le t \le T_{max}$ and $0 < x < s(T_{max})$. $\square$

<u>Proof.</u> We start with the observation that

$$\begin{aligned} \lim_{x \to 0^+} (p_t + f'(u)p_x)(x,t) &= k(u_0(t))(1-\mu)^2\Sigma^2(0)e^{-2\mu t} + \mu(1-\mu)\Sigma(0)e^{-\mu t} \\ &= k(u_0(t))(1-\mu)^2\mu^2 f^2(u_-)e^{-2\mu t} + \mu^2(1-\mu)f(u_-)e^{-\mu t} > 0 \end{aligned}$$

and thus we are quaranteed that p is positive in some domain $T(x) \le t \le T_{max}$ and $0 < x \le \varepsilon$. We now assume the lemma is false. Then there is a first time, t_*, and point, x_*, with $T(x_*) \le t_* \le T_{max}$ and $0 < x_* < s(T_{max})$ such that $p(x_*,t_*) = 0$ and $(p_t + f'(u)p_x)(x_*,t_*) \le 0$. But, $(2.4)_1$ implies that

$$(p_t + f'(u)p_x)(x_*,t_*) = k(u(x_*,t_*))q^2(x_*,t_*) + \mu q(x_*,t_*)$$

and thus Lemma 2.1 implies that the right hand side of the last equation is positive. This is the desired contradiction and establishes the lemma. □

The identities

$$u = f^{-1}(\sigma + \phi) \quad \text{and} \quad f'(u)u_t = p + q \tag{2.7}$$

together with

$$f'(u) > 0, \quad p > 0, \quad \text{and} \quad q > 0 \tag{2.8}$$

imply that

$$-u_t = \sigma_x < 0, \quad T(x) \le t \le T_{max} \quad \text{and} \quad 0 \le x < s(T_{max}) . \tag{2.9}$$

If we now exploit the identities

$$\phi(x,t) = (1 - \mu) \int_{T(x)}^{t} e^{-\mu(t-s)}\sigma(x,s)ds ,$$

$$\phi_x = -(1 - \mu)T'(x)\Sigma(T(x))e^{-\mu(t-T(x))} + (1 - \mu) \int_{T(x)}^{t} e^{-\mu(t-s)}\sigma_x(x,s)ds ,$$

and

$$T'(x) = f^{-1}(\Sigma(T(x)))/\Sigma(T(x)) \quad \text{and} \quad \Sigma(T(x)) > 0, \quad 0 \le x < T_{max}$$

and (2.9) we obtain

$$\phi_x < 0, \quad T(x) \le t \le T_{max} \quad \text{and} \quad 0 \le x < s(T_{max}) . \tag{2.10}$$

This concludes the proof of Theorem 2.1. □

An immediate corollary to Theorem 2.1 is

Theorem 2.2. Under the conditions of Theorem 2.1 the following bounds prevail

$$0 < \sigma \le \mu f(u_-), \quad 0 \le \phi < (1 - \mu)f(u_-), \quad \text{and} \quad 0 < u < u_- \tag{2.11}$$

for $0 \le x \le s(t)$ and $0 \le t \le T_{max}$. □

B. The case where u_- is small.

We now focus on obtaining upper bounds for $(\sigma_t,\phi_t) = (p,q)$ and lower bounds for (σ_x,ϕ_x). These estimates will rely on the assumption that u_- is small. For simplicity we shall assume that $k(u) \overset{\text{def}}{=} f''(u)/(f'(u))^2$ satisfies

$$\frac{dk}{du}(u) < 0 . \tag{2.12}$$

This assumption guarantees that

$$k_0 \overset{\text{def}}{=} k(0) \geq k(u), \qquad 0 \leq u \leq u_- . \tag{2.13}$$

The assumption (2.12) is not essential; if we abandon it, the same estimates obtain provided we redefine k_0 by $k_0 \overset{\text{def}}{=} \max_{0 \leq u \leq u_-} k(u)$.

The inequalities $\sigma \leq \mu f(u_-)$ and $0 < \phi$ and the equation $q = \phi_t = (1 - \mu)\sigma - \mu\phi$ imply that

$$q \leq \mu(1 - \mu)f(u_-), \qquad 0 \leq x \leq s(t) \text{ and } 0 \leq t \leq T_{max} .$$

Equations $(2.4)_1$ and (2.5) in turn imply that

$$p_t + f'(u)p_x \leq k_0(p + \mu(1 - \mu)f(u_-))^2 - (1 - \mu)p + \mu^2(1 - \mu)f(u_-),$$
$$0 < x < s(t) \text{ and } 0 < t < T_{max} \tag{2.14}$$

and

$$p(0,t) = 0, \qquad 0 < t < T_{max} . \tag{2.15}$$

Moreover, if

$$f(u_-) \leq \frac{(1 - \mu)}{4\mu k_0} , \tag{2.16}$$

then (2.14) may be rewritten as

$$p_t + f'(u)p_x \leq k_0(p - p_-)(p - p_+), \qquad 0 < x < s(t) \text{ and } 0 < t < T_{max} \tag{2.17}$$

where

$$0 < p_- = \frac{(1 - \mu)}{2k_0} - \mu(1 - \mu)f(u_-) - \frac{1}{2k_0}\sqrt{(1 - \mu)^2 - 4k_0\mu(1 - \mu)f(u_-)} <$$

$$p_+ = \frac{(1 - \mu)}{2k_0} - \mu(1 - \mu)f(u_-) + \frac{1}{2k_0}\sqrt{(1 - \mu)^2 - 4k_0\mu(1 - \mu)f(u_-)} . \tag{2.18}$$

An immediate consequence of (2.15) and (2.17) and the bound $q \leq \mu(1 - \mu)f(u_-)$ is

<u>Theorem 2.3.</u> If u_- satisfies (2.16), then

$$\phi_t \leq \mu(1-\mu)f(u_-) \text{ and } \sigma_t \leq \frac{(1-\mu)}{2k_0} - \mu(1-\mu)f(u_-) - \frac{1}{2k_0}\sqrt{(1-\mu)^2 - 4k_0\mu(1-\mu)f(u_-)} . \tag{2.19}$$

Moreover, the identities

$$\sigma_x = -u_t = -(p + q)/f'(u) \tag{2.20}$$

and

$$\phi_x = -(1-\mu)f^{-1}(\Sigma(T(x)))e^{-\mu(t-T(x))} + (1-\mu)\int_{T(x)}^{t} e^{-\mu(t-s)}\sigma_x(x,s)ds \tag{2.21}$$

imply that

$$\sigma_x \geq \frac{1}{2k_0 f'(0)}\left(-(1-\mu) + \sqrt{(1-\mu)^2 - 4k_0\mu(1-\mu)f(u_-)}\right) \tag{2.22}$$

and

$$\begin{aligned}\phi_x \geq & -(1-\mu)u_- e^{-\mu(t-T(x))} \\ & - \frac{(1-\mu)}{2k_0\mu f'(0)}\left(1-\mu - \sqrt{(1-\mu)^2 - 4k_0\mu(1-\mu)f(u_-)}\right)(1-e^{-\mu(t-T(x))}) . \quad \square\end{aligned} \tag{2.23}$$

<u>Remark 1.</u> If we assume that f satisfies the additional hypothesis $\frac{d^3f}{du^3} \leq 0$, then we find that if (2.16) holds the following inequality must also prevail:

$$c = \frac{\mu f(u_-)}{u_-} < f'(0) . \tag{2.24}$$

Thus, if (2.16) holds and $\frac{d^3f}{du^3} \leq 0$ the traveling wave associated with the states $(u_-,\mu f(u_-))$ and $(0,0)$ at x equal minus and plus infinity respectively is the smooth one described in Section 1. $\square$

The results of Theorems 2.1-2.3 imply that if (2.16) holds, then the functions σ, ϕ, σ_t, ϕ_t, σ_x, and ϕ_x are uniformly bounded independently of T_{max} so long as $\Sigma(t) > 0$ for $0 \leq t < T_{max}$. Thus, if (2.16) holds, the only way for T_{max} to be finite is if Σ satisfies

$$\lim_{t \to T_{max}^-} \Sigma(t) = 0 . \tag{2.25}$$

<u>Theorem 2.4.</u> *The function* $\Sigma(t) \overset{\text{def}}{=} \lim_{\substack{x \to s(t) \\ x < s(t)}} \sigma(x,t)$ *cannot satisfy (2.25) for any finite* T_{max}. $\square$

<u>Proof.</u> Suppose the Theorem is false. Then, there is a first finite time, $T_{max} > 0$, such that $\Sigma(t) > 0$, $0 \leq t < T_{max}$, and (2.25) holds. To show this cannot happen we observe that

$$\frac{d\Sigma}{dt} = (\sigma_t + \dot{s}\sigma_x)(s(t),t) = -(1-\mu)\Sigma(t) + \sigma_x(s(t),t)\left(\frac{\Sigma(t)}{f^{-1}(\Sigma)} - f'(f^{-1}(\Sigma))\right) \tag{2.26}$$

where $\sigma_x(s(t),t) = \lim_{\substack{x \to s(t) \\ x < s(t)}} \sigma_x(x,t)$. The a-priori bound

$-\frac{1}{2k_0 f'(0)}\left(1 - \mu - \sqrt{(1-\mu)^2 - 4k_0\mu(1-\mu)f(u_-)}\right) \le \sigma_x < 0$, $0 \le x \le s(t)$ and $0 \le t \le T_{max}$, together with the uniqueness theorem for ordinary differential equations, applied to (2.26), imply that if $\lim_{t \to T_{max}^-} \Sigma(t) = 0$, then $\Sigma(t) \equiv 0$ for $0 \le t \le T_{max}$.

This contradicts the fact that $\lim_{t \to 0^+} \Sigma(t) = \mu f(u_-) > 0$ and establishes the Theorem. □

Summarizing the results of this subsection we obtain

Theorem 2.5. If u_- satisfies (2.16), then $T_{max} = +\infty$, and the bounds of Theorems 2.1-2.3 obtain. □

C. The case where u_- is large.

We now turn to the case where u_- is large. We start with a lower bound estimate for the shock, $x = s(t)$. This estimate is valid for all $u_- > 0$, but it is only useful when u_- is large.

Theorem 2.6. The shock curve $x = s(t)$, $0 \le t \le T_{max}$, satisfies

$$s(t) > ct \quad \text{where again} \quad c = \frac{\mu f(u_-)}{u_-}. \quad \square \tag{2.27}$$

Proof. The conservation law $u_t + \sigma_x = 0$, boundary condition $\sigma(0,t) = \mu f(u_-)$, and Rankine-Hugoniot condition (RH) imply that

$$\frac{d}{dt}\int_0^{s(t)} u(x,t)dx = \mu f(u_-) \quad \text{and} \quad \int_0^{s(t)} u(x,t)dx = \mu f(u_-)t \,. \tag{2.28}$$

The upper bound $u(x,t) < u_-$ for $0 \le x \le s(t)$ and $0 \le t \le T_{max}$ combines with $(2.28)_2$ to yield the desired result. □

In the remainder of this subsection we shall assume that

$$c \overset{\text{def}}{=} \frac{\mu f(u_-)}{u_-} > f'(0) . \tag{2.29}$$

Our next theorem provides lower bounds for σ, ϕ, and u on the domain $0 \le x \le ct$ and $0 \le t \le T_{max}$.

Theorem 2.7. Let (σ,ϕ,u) be a solution of (σ), (ϕ), (u), (BC), and (RH). Then the following lower bounds prevail:

$$\sigma(x,t) > c\tilde{u}(t - x/c),\ \phi(x,t) > f(\tilde{u}(t - x/c)) - c\tilde{u}(t - x/c)^{1)},$$

$$\text{and} \quad u(x,t) > \tilde{u}(t - x/c) \tag{2.30}$$

on the domain $0 \le x \le ct$ and $0 \le t \le T_{max}$. The function $\tilde{u}(\cdot)$ is the u component of the traveling wave described in Section 1 and satisfies

$$(f'(\tilde{u}) - c)\frac{d\tilde{u}}{d\tau} = (c\tilde{u} - \mu f(\tilde{u})), \quad \tau > 0, \quad \text{and} \quad \tilde{u}(0) = u_* > 0 , \tag{2.31}$$

where

$$\frac{f(u_*)}{u_*} = c = \frac{\mu f(u_-)}{u_-} , \tag{2.32}$$

and the triple

$$(\underline{\sigma},\underline{\phi},\underline{u})(x,t) = (c\tilde{u},f(\tilde{u}) - c\tilde{u},\tilde{u})(t - x/c) \tag{2.33}$$

is the unique solution of (σ), (ϕ), and (u) satisfying the boundary conditions

$$\underline{\sigma}(0,t) = c\tilde{u}(t) < \mu f(u_-) \quad \text{and} \quad \underline{\phi}(ct,t) = 0, \quad 0 < t . \quad \square \tag{2.34}$$

Remark 2. The function $\tilde{u}(\cdot)$ may be extended to the interval $[-\tau(c),\infty)$ as a solution of (2.31). The number $\tau(c) > 0$ is given by the quadrature formula

$$\tau(c) = \int_{\underline{u}(c)}^{u_*} \frac{(f'(u) - c)du}{(cu - \mu f(u))} \tag{2.35}$$

and $0 < \underline{u}(c)$ is the unique solution of

$$f'(\underline{u}(c)) = c = \frac{\mu f(u_-)}{u_-} . \quad \square \tag{2.36}$$

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1) The strict inequality for ϕ fails at $(x,t) = (0,0)$ where $\phi = 0$.

Remark 3. The triple $(\underline{\sigma},\underline{\phi},\underline{u})$ defined by (2.33) solves (σ), (ϕ), and (u) and satisfies the initial-boundary conditions

$$(\underline{\sigma},\underline{\phi})(x,0) = (c\tilde{u},f(\tilde{u}) - c\tilde{u})(-x/c), \quad 0 < x < c\tau(c) \tag{2.37}$$

and

$$\underline{\sigma}(0,t) = c\tilde{u}(t) \text{ and } \underline{\phi}(c(t + \tau(c)),t) = f(\underline{u}(c)) - c\underline{u}(c) < 0, \quad t > 0 . \tag{2.38}$$

Moreover, for any curve $x = \rho(t)$ satisfying $ct < \rho(t) < c(t + \tau(c))$ we have $\underline{\phi}(\rho(t),t) < 0$.

Proof of Theorem 2.7. We first observe that if (σ,ϕ,u) solve (σ), (ϕ), (u), (BC), and (RH), then the properties of $\tilde{u}(\cdot)$ and the fact that $ct < s(t)$ imply

$$\underline{\sigma}(0,t) = c\tilde{u}(t) < \sigma(0,t) = \mu f(u_-), \quad 0 \le t < T_{max} , \tag{2.39}$$

and

$$\underline{\phi}(ct,t) = 0 < \phi(ct,t), \quad 0 \le t < T_{max} . \tag{2.40}$$

Lemma 2.3. Suppose $\underline{\sigma} < \sigma$ on $0 \le x < ct$ and $0 \le t < T_{max}$. Then $\underline{\phi} < \phi$ on $0 \le x \le ct$ and $0 \le t \le T_{max}$. □

Proof. The lemma follows from (2.40) and the identities

$$\begin{aligned}\phi(x,t) &= \phi(x,x/c)e^{-\mu(t-x/c)} + (1 - \mu) \int_{x/c}^{t} e^{-\mu(t-s)}\sigma(x,s)ds \\ &> \phi(x,x/c)e^{-\mu(t-x/c)} + (1 - \mu) \int_{x/c}^{t} e^{-\mu(t-s)}\underline{\sigma}(x,s)ds ,\end{aligned} \tag{2.41}$$

and

$$\underline{\phi}(x,t) = (1 - \mu) \int_{x/c}^{t} e^{-\mu(t-s)}\underline{\sigma}(x,s)ds . \quad □ \tag{2.42}$$

The preceding lemma together with (2.39) imply

$$\underline{\phi}(0,t) < \phi(0,t), \quad t > 0 \tag{2.43}$$

and (2.39), (2.40), and (2.43) are sufficient to guarantee the Theorem is true in some neighborhood $\frac{x}{c} \le t \le T_{max}$ and $0 \le x \le \varepsilon$ with the caveat that $\phi = \underline{\phi} = 0$ at the origin. We now suppose the Theorem is false. Then there is a first time t_* and point

x_* with $\frac{x_*}{c} \leq t \leq T_{max}$ and $\varepsilon < x_* \leq cT_{max}$ such that $\sigma(x_*,t_*) = \underline{\sigma}(x_*,t_*)$. Moreover, at (x_*,t_*) the following inequalities hold:

$$\left(\frac{\partial}{\partial t} + f'(u)\frac{\partial}{\partial x}\right)(\sigma-\underline{\sigma})(x_*,t_*) \leq 0,\quad (\phi-\underline{\phi})(x_*,t_*) > 0, \quad \text{and} \quad (u-\underline{u})(x_*,t_*) > 0\,. \tag{2.44}$$

The latter two inequalities follow from Lemma 2.3 and the relations $u = f^{-1}(\sigma+\phi)$ and $\underline{u} = f^{-1}(\underline{\sigma}+\underline{\phi})$. A little algebra shows that

$$\left(\frac{\partial}{\partial t} + f'(u)\frac{\partial}{\partial x}\right)(\sigma-\underline{\sigma})(x_*,t_*) = \mu(\phi-\underline{\phi})(x_*,t_*) + \underline{\sigma}_x(x_*,t_*)(f'(\underline{u}) - f'(u))(x_*,t_*)$$

and this identity, together with (2.44), the hypothesis $f'' > 0$, and the fact that $\underline{\sigma}_x(x_*,t_*) = -c\frac{d\tilde{u}}{d\tau}(\tau = t_* - x/c) < 0$ yield the contradictory inequality $\left(\frac{\partial}{\partial t} + f'(u)\frac{\partial}{\partial x}\right)(\sigma-\underline{\sigma})(x_*,t_*) > 0$. This completes the proof of the Theorem. □

Our next task is to obtain an upper bound for the shock curve $x = s(t)$. We start with the identity

$$\int_0^{s(t)} u(x,t)dx = \mu f(u_-)t = \int_0^{ct} u_- dx \tag{2.45}$$

which we rewrite as

$$\int_{ct}^{s(t)} u(x,t)dx = \int_0^{ct} (u_- - u(x,t))dx\,. \tag{2.46}$$

If we now exploit the inequality $u_x < 0$ and $(2.30)_3$ we obtain

$$u(s(t),t)(s(t)-ct) < \int_{ct}^{s(t)} u(x,t)dx = \int_0^{ct} (u_- - u(x,t))dx < \frac{1}{c}\int_0^t (u_- - \tilde{u}(\tau))d\tau$$
$$= \frac{1}{c}\int_{u_*}^{\tilde{u}(\tau)} \frac{(u_- - u)(f'(u)-c)}{u(c-\mu f(u)/u)}du < \frac{1}{c}\int_{u_*}^{u_-} \frac{(u_- - u)(f'(u)-c)}{u(c-\mu f(u)/u)}du \tag{2.47}$$

where

$$u(s(t),t) = \lim_{\substack{x\to s(t)\\ x<s(t)}} u(x,t) = f^{-1}(\Sigma(t)) \quad \text{and} \quad c = \frac{f(u_*)}{u_*} = \frac{\mu f(u_-)}{u_-} \tag{2.48}$$

and $\tilde{u}(\cdot)$ is the function defined in (2.31). The Rankine-Hugoniot equation

$\frac{ds}{dt} = \frac{f(u(s(t),t))}{u(s(t),t)}$ together with (2.47) imply that $y(t) \overset{\text{def}}{=} s(t) - ct$ satisfies

$$\frac{dy}{dt} < -(c - f'(0)) + \left(\frac{f''_{max}}{2c} \int_{u_*}^{u_-} \frac{(u_- - u)(f'(u) - c)}{u(c - \mu f(u)/u)} du\right) \frac{1}{y} \tag{2.49}$$

and $y(0) = 0$. The number f''_{max} is given by $f''_{max} = \max_{0 \le u \le u_-} f''(u)$. An immediate consequence of (2.49) and $y(0) = 0$ is

Theorem 2.8. The shock curve $x = s(t)$ satisfies

$$s(t) < ct + c\delta(c), \qquad 0 \le t \le T_{max}, \tag{2.50}$$

where

$$\delta(c) = \frac{f''_{max}}{2(c - f'(0))c^2} \int_{u_*}^{u_-} \frac{(u_- - u)(f'(u) - c)}{u(c - \mu f(u)/u)} du . \quad \square \tag{2.51}$$

Remark 4. If we restrict our attention to the case where $f(u) = f'(0)u + \frac{k_0 u^2}{2}$ it is easily established that the numbers $\tau(c)$ of (2.35) and $\delta(c)$ of (2.52) satisfy

$$\frac{\tau(c)}{\delta(c)} = \frac{2(c-f'(0))c^2\left\{\left(1 - \frac{\mu}{2}\frac{(c-f'(0))}{(c-\mu f'(0))}\right)\log\left(\frac{(2-\mu)c-\mu f'(0)}{2(1-\mu)c}\right) - \frac{\mu(c-f'(0))}{2(c-\mu f'(0))}\log 2\right\}}{\left\{\frac{2c(1-\mu)}{\mu} + (c-f'(0))\log\left(\frac{\mu(c-f'(0))}{(c-\mu f'(0))}\right)\right\}} > 1 \tag{2.52}$$

uniformly on $f'(0) \ll c$ and $0 < 1 - \mu \ll 1$. $\square$

Motivated by the results of Remark 4 we shall make the following additional assumption

$$\frac{\tau(c)}{\delta(c)} > 1 \tag{2.53}$$

uniformly on $f'(0) \ll c$ and $0 < 1 - \mu \ll 1$. $\square$

Using the results of Remark 3 and the arguments used to establish Theorem 2.7, one easily obtains

Theorem 2.9. If (2.53) holds, then for $f'(0) \ll c = \frac{\mu f(u_-)}{u_-}$ the conclusion of Theorem 2.7 are valid on the domain $0 \le x \le s(t)$ and $0 \le t \le T_{max}$. $\square$

Our final task in this subsection is to obtain upper bounds for $p = \sigma_t$ and $q = \phi_t$ and lower bounds for σ_x and ϕ_x. Throughout, we shall assume that $k(u) \overset{\text{def}}{=} f''(u)/(f'(u))^2$ is decreasing on $u \ge 0$. We also assume that (2.53) holds and that $c > f'(0)$ is large enough so that the conclusions of Theorem 2.9 hold. Finally, we shall assume that

$$j(u) \overset{\text{def}}{=} \frac{uf'(u) - f(u)}{f(u)} > 0, \qquad 0 < u \tag{2.54}$$

satisfies

$$\frac{dj}{du}(u) > 0, \qquad 0 \le u \ . \tag{2.55}$$

Our goal is to show that subject to the above hypotheses, there is some $0 < \lambda < \mu$ such that the functions $p = \sigma_t$ and $q = \phi_t$ satisfy

$$p(x,t) = e^{-\lambda(t-T(x))}P(x,t), \quad \text{and} \qquad \tag{2.56}$$

$$q(x,t) = e^{-\lambda(t-T(x))}Q(x,t) \qquad \left.\right\} \begin{array}{l} T(x) \le t \le T_{max} \ , \\ 0 \le x \le s(T_{max}) \end{array} \tag{2.57}$$

where $P > 0$ and $Q > 0$ are bounded independently of T_{max}. Again, $t = T(x)$ is the inverse of the shock $x = s(t)$.

It is a relatively simple matter to show that if p and q are given by (2.56) and (2.57) and satisfy (2.4) and (2.5), then P and Q satisfy

$$P_t + f'(u)P_x \le k(u)e^{-\lambda(t-T(x))}(P + Q)^2 - (1 - \mu + \lambda j(u))P + \mu Q \ , \tag{2.58}$$

and

$$Q_t = (1 - \mu)P - (\mu - \lambda)Q \ , \tag{2.59}$$

in the domain $0 < x < s(t)$ and $0 < t < T_{max}$, and

$$P(0,t) = 0 \quad \text{and} \quad \lim_{\substack{x \to s(t) \\ x < s(t)}} Q(x,t) = (1 - \mu)\Sigma(t), \qquad 0 < t < T_{max} \ . \tag{2.60}$$

The results of Theorem 2.1 guarantee that P and Q are positive on $0 < x < s(t)$ and $0 < t < T_{max}$ while the results of Theorem 2.9 and (2.53) guarantee that

$$u(x,t) > \underline{u}(c) > 0, \qquad 0 \le x \le s(t) \quad \text{and} \quad 0 \le t \le T_{max} \tag{2.61}$$

where again $\underline{u}(c)$ is defined by

$$f'(\underline{u}(c)) = c \tag{2.62}$$

and $\lim_{c \to \infty} \underline{u}(c) = +\infty$. The inequality (2.61), when combined with the hypotheses

$$\frac{dk}{du} < 0 \quad \text{and} \quad \frac{dj}{du} > 0 \ ,$$

and the results $P > 0$ and $Q > 0$ for $0 < x < s(t)$ and $0 < t < T_{max}$ imply that P and Q satisfy the inequalities

$$P_t + f'(u)P_x < \underline{k}(P + Q)^2 - (1 - \mu + \lambda\underline{j})P + \mu Q \,, \tag{2.63}$$

$$Q_t = (1 - \mu)P - (\mu - \lambda)Q \,, \tag{2.64}$$

on the same domain. The numbers $\underline{k}$ and $\underline{j}$ are defined by

$$\underline{k} = k(\underline{u}(c)) \quad \text{and} \quad 0 < \underline{j} = j(\underline{u}(c)) \,. \tag{2.65}$$

Remark 5. If we restrict our attention to the case $f(u) = f'(0)u + k_0u^2/2$, then

$$k(u) = \frac{k_0}{(f'(0) + k_0u)^2}, \; j(u) = \frac{k_0u}{(2f'(0) + k_0u)}, \; \underline{u}(c) = \frac{(c - f'(0))}{k_0} \,,$$

and $\lim_{c\to\infty} (k(\underline{u}(c)), j(\underline{u}(c))) = (0,1)$. □

Motivated by the last remark we shall assume that $k(\cdot)$ and $j(\cdot)$ satisfy

$$\lim_{c\to\infty} (k(\underline{u}(c)), j(\underline{u}(c))) = (0,1) \,. \tag{2.67}$$

We now turn to the inequalities (2.63) and (2.64). Throughout, we assume that $0 < \lambda < \mu$. It is easily checked that the set $P \geq 0$ and $Q \geq 0$ such that $\underline{k}(P + Q)^2 - (1 + \lambda\underline{j} - \mu)P + \mu Q = 0$ is given by

$$Q = \hat{Q}(P) \overset{\text{def}}{=} -P + \frac{1}{2\underline{k}} \sqrt{\mu^2 + 4\underline{k}(1 + \lambda\underline{j})} - \mu, \quad 0 \leq P \leq (1 + \lambda\underline{j} - \mu) \tag{2.68}$$

and that

$$\underline{k}(P + Q)^2 - (1 + \lambda\underline{j} - \mu)P + \mu Q < 0, \; 0 \leq Q < \hat{Q}(P) \;\text{ and }\; 0 < P < (1 + \lambda\underline{j} - \mu) \,. \tag{2.69}$$

It is also a simple matter to verify that if $(\mu - \lambda)\underline{j} - (1 - \mu) > 0$, then the right hand sides of (2.63) and (2.64) vanish simultaneously at (P_{eq}, Q_{eq}) where

$$P_{eq} = \frac{\lambda(\mu - \lambda)((\mu - \lambda)\underline{j} - (1 - \mu))}{(1 - \lambda)^2\underline{k}} > 0 \,, \tag{2.70}$$

and

$$Q_{eq} = \frac{\lambda(1 - \mu)((\mu - \lambda)\underline{j} - (1 - \mu))}{(1 - \lambda)^2\underline{k}} > 0 \,. \tag{2.71}$$

We now choose λ so as to maximize $Q_{eq}(\lambda)$. The result is

$$\lambda_{max} = \frac{\mu\underline{j} - (1 - \mu)}{(2 - \mu)\underline{j} + 1 - \mu} = \frac{\mu(1 + \underline{j}) - 1}{(2 - \mu)\underline{j} + 1 - \mu} \,. \tag{2.72}$$

Substitution of (2.72) into (2.70) and (2.71) yields

$$P_{eq}(\mu,\underline{j}) \overset{\text{def}}{=} P_{eq}(\lambda_{max}) = \frac{(\mu\underline{j}+1+\mu)(\mu\underline{j} - (1-\mu)(2(2\mu-1) - (1-\underline{j})(2\mu-1+\mu(1+\underline{j}))))}{4((2-\mu)\underline{j} + (1-\mu))(1+\underline{j})^2\underline{k}}, \tag{2.73}$$

and

$$Q_{eq}(\mu,\underline{j}) \overset{\text{def}}{=} Q_{eq}(\lambda_{max}) = \frac{(\mu\underline{j} - (1-\mu))(2(2\mu-1) - (1-\underline{j})(2\mu-1+\mu(1+\underline{j})))}{4(1+\underline{j})^2\underline{k}}, \tag{2.74}$$

and these formulas, together with (2.67), imply that if $0 < 1 - \mu \ll 1$ and $f'(0) \ll c$, then $P_{eq}(\mu,\underline{j})$ and $Q_{eq}(\mu,\underline{j})$ are both positive; in fact we have

$$\lim_{\substack{c\to\infty \\ \mu\to 1^-}} (\underline{k}P_{eq}(\mu,\underline{j}),\underline{k}Q_{eq}(\mu,\underline{j})) = (\tfrac{3}{8}, \tfrac{1}{8}) . \tag{2.75}$$

Our final task is to obtain conditions which guarantee that $P < P_{eq}(\mu,\underline{j})$ and $Q < Q_{eq}(\mu,\underline{j})$ for $0 < x < s(t)$ and $0 < t < T_{max}$. The fact that $\lim_{\substack{x\to s(t)\\ x<s(t)}} \sigma(x,t) = \Sigma(t) \leq \mu f(u_-)$ for $0 < t < T_{max}$ together with the boundary condition $\lim_{\substack{x\to s(t)\\ x<s(t)}} Q(x,t) = (1 - \mu)\Sigma(t)$ guarantee that $Q < Q_{eq}(\mu,\underline{j})$ on $x = s(t)$ provided

$$\mu(1-\mu)f(u_-) < Q_{eq}(\mu,\underline{j}) \text{ or } \mu(1-\mu)\underline{k}f(u_-) < \frac{(\mu\underline{j}-(1-\mu))(2(2\mu-1)-(1-\underline{j})(2\mu-1+\mu(1+\underline{j})))}{4(1+\underline{j})^2}. \tag{2.76}$$

Moreover, the hypotheses on f, k, and j guarantee that for $c = f'(\underline{u}(c)) = \frac{\mu f(u_-)}{u_-}$ sufficiently large the last inequality holds provided $0 < 1 - \mu \ll 1$. With these preliminaries we are now in a position to prove

<u>Theorem 2.10.</u> Suppose the functions $k > 0$ and $j > 0$ satisfy $dk/du < 0$ and $dj/du > 0$ and that (2.67) holds. Suppose further that $f'(0) \ll c$ and $0 < 1 - \mu \ll 1$ are such that (i) (2.53) holds, (ii) that the functions $P_{eq}(\mu,\underline{j})$ and $Q_{eq}(\mu,\underline{j})$ defined in (2.73) and (2.74) are both positive, and (iii) that (2.76) holds. Then the functions P and Q defined in (2.56) and (2.57) with $\lambda = \frac{\mu\underline{j} - (1 - \mu)}{(2 - \mu)\underline{j} + (1 - \mu)} > 0$ satisfy

$$0 < P(x,t) < P_{eq}(\mu,\underline{j}) \quad \text{and} \quad 0 < Q < Q_{eq}(\mu,\underline{j}) \tag{2.77}$$

for $T(x) \leq t \leq T_{max}$ and $0 < x \leq s(T_{max})$. □

Proof. The lower bounds follow from the results of Theorem 2.1 and as previously observed Q satisfies the desired upper bound on the boundary $x = s(t)$, $0 \leq t \leq T_{max}$. The identity $\lim_{x\to 0^+} Q(x,t) = \mu(1-\mu)f(u_-)e^{-(\mu-\lambda)t}$ together with (2.76) and

$(\mu - \lambda) = \dfrac{2(1-\mu)\underline{j}}{(2-\mu)\underline{j} + 1 - \mu} > 0$ imply that $Q(0,t) < Q_{eq}$ for $0 \leq t \leq T_{max}$. The differential inequality (2.63), together with (2.64) and the results summarized in (2.68) and (2.69) imply that the upper bounds cannot fail in $T(x) \leq t \leq T_{max}$ and $0 < x \leq s(T_{max})$ and this concludes the proof. □

Lower bounds for σ_x and ϕ_x are readily obtainable from the upper bounds for σ_t and ϕ_t. The results are that if the hypotheses of Theorem 2.10 hold, then

$$\sigma_x \geq e^{-\lambda(t-T(x))} \frac{(P_{eq}(\mu,\underline{j}) + Q_{eq}(\mu,\underline{j}))}{f'(0)} \tag{2.78}$$

and

$$\phi_x \geq -(1-\mu)u_- e^{-\mu(t-T(x))} - \frac{(1-\mu)}{(\mu-\lambda)}(P_{eq}(\mu,\underline{j}) + Q_{eq}(\mu,\underline{j}))(e^{-\lambda(t-T(x))} - e^{-\mu(t-T(x))}) \tag{2.79}$$

for $T(x) \leq t \leq T_{max}$ and $0 \leq x \leq s(T_{max})$ with $\lambda = \dfrac{\mu\underline{j} - (1-\mu)}{(2-\mu)\underline{j} + (1-\mu)} > 0$ and

$(\mu - \lambda) = \dfrac{2(1-\mu)\underline{j}}{(2-\mu)\underline{j} + 1 - \mu} > 0$. Finally, since the bounds of Theorems 2.7 through 2.10 and the inequalities (2.78) and (2.79) are independent of T_{max}, we obtain

Theorem 2.11. If the hypotheses of Theorem 2.10 hold, then $T_{max} = +\infty$ and the bounds of Theorems 2.7 through 2.10 and the inequalities (2.78) and (2.79) hold with $T_{max} = +\infty$. □

This concludes Section 2.

3. Asymptotic Results

A. The case where u_- is small

In this section we shall assume that (2.12) and (2.13) hold. We shall also assume that u_- is small enough so that (2.16) is valid. Our goal is an asymptotic estimate for $U(t) \overset{\text{def}}{=} \lim_{\substack{x \to s(t) \\ x < s(t)}} u(x,t)$. The Rankine-Hugoniot conditions (1.11)-(1.13) imply that U satisfies

$$\frac{dU}{dt} = u_t(s(t),t)\left(1 - \frac{f(U)}{Uf'(U)}\right) - \frac{(1-\mu)f(U)}{f'(U)} \quad \text{and} \quad u(0) = u_0 , \tag{3.1}$$

where

$$u_t(s(t),t) = \lim_{\substack{x \to s(t) \\ x < s(t)}} u_t(x,t) \le \frac{1}{2k_0 f'(0)}\left((1-\mu) - \sqrt{(1-\mu)^2 - 4k_0\mu(1-\mu)f(u_-)}\right) , \tag{3.2}$$

and $0 < u_0 < u_-$ is the unique solution of

$$f(u_0) = \mu f(u_-) . \tag{3.3}$$

The fact that $k_0 f'(0) = f''(0)/f'(0)$ and the inequalities

$$(1-\mu) - \sqrt{(1-\mu)^2 - 4k_0\mu(1-\mu)f(u_-)} < 4k_0\mu f(u_-) < (1-\mu) \tag{3.4}$$

when combined with (2.16), (3.1), and (3.2) yield

$$\frac{dU}{dt} \le (1-\mu)\left(\frac{f'(0)}{2f''(0)}\left(1 - \frac{f(U)}{Uf'(U)}\right) - \frac{f(U)}{f'(U)}\right) \quad \text{and} \quad u(0) = u_0 . \tag{3.5}$$

The fact that

$$h(U) \overset{\text{def}}{=} \frac{f'(0)}{2f''(0)}\left(1 - \frac{f(U)}{Uf'(U)}\right) - \frac{f(U)}{f'(U)} \tag{3.6}$$

satisfies

$$h(0) = 0 \quad \text{and} \quad \frac{dh}{dU}(0) = -3/4 < 0$$

guarantees that if u_0 and hence u_- are small, then $U(t) \to 0$ as $t \to \infty$; in fact for u_0 small $\lim_{t \to \infty} e^{\lambda t} U(t) = 0$ for any $0 < \lambda < -3/4$. In the special case where $f(U) = f'(0)U + k_0 U^2/2$, it is easily verified that

$$h(U) = -\frac{(3f'(0)U + 2k_0 U^2)}{4(f'(0) + k_0 U)} < 0 \tag{3.7}$$

and thus no additional restrictions on u_0 are required.

We conclude this subsection by showing that when u_- is small (in particular small enough so that $U(t) \to 0$ as $t \to \infty$) the solution to (B), (C), and (1.2) converges, in a weak sense, to the traveling wave described in Section 1. Specifically we shall prove

<u>Theorem 3.1.</u> The average speed of propagation of the level lines of u converge to the speed of the traveling wave; that is the function

$$c_{AV}(t) = \frac{1}{u_0(t) - U(t)} \int_{U(t)}^{u_0(t)} \frac{\partial x}{\partial t}(\alpha,t)d\alpha \tag{3.8}$$

satisfies

$$\lim_{t\to\infty} c_{AV}(t) = \frac{\mu f(u_-)}{u_-} . \tag{3.9}$$

The function $u_0(t)$ is defined in (1.9). For numbers $0 < \alpha < u_-$, the curve $x = x(\alpha,t)$, $t \geq \tau(\alpha)$, is a level line of u; that is it satisfies $u(x(\alpha,t),t) = \alpha$. For $0 < \alpha < u_0(0)$, $\tau(\alpha)$ is defined by $U(\tau(\alpha)) = \alpha$ and for $u_0(0) \leq \alpha < u_-$, $\tau(\alpha)$ solves $u_0(\tau(\alpha)) = \alpha$. □

<u>Proof.</u> Our first task is to show that

$$c_{AV}(t) = \frac{\mu f(u_-) - f(U(t))}{u_0(t) - U(t)} . \tag{3.10}$$

To obtain (3.10) we note that the defining relation $u(x(\alpha,t),t) = \alpha$ implies that $u_x x_\alpha = 1$ and $u_t + x_t u_x = 0$. If we now define $\Sigma(\alpha,t) = \sigma(x(\alpha,t),t)$, we see that $\Sigma_\alpha(\alpha,t) = \sigma_x x_\alpha$ and this, combined with $u_t = -\sigma_x$ and $x_t = -u_t x_\alpha$, yields the conservation law:

$$x_t(\alpha,t) = \Sigma_\alpha(\alpha,t), \quad U(t) \leq \alpha \leq u_0(t) . \tag{3.11}$$

Equation (3.10) now follows from (3.8) and (3.11) and the limit relation, (3.9), is a consequence of (3.10) and $\lim_{t\to\infty} (U(t),u_0(t)) = (0,u_-)$. □

B. <u>The case where u_- is large</u>

To obtain the asymptotic result

$$\lim_{t\to\infty} (u,\sigma,\phi)(\lambda t,t) = \begin{cases} (u_-,\mu f(u_-),(1-\mu)f(u_-)), & \lambda < c = \dfrac{\mu f(u_-)}{u_-} \\ (0,0,0), & \lambda > c = \dfrac{\mu f(u_-)}{u_-} \end{cases} \tag{3.12}$$

all that is required are the lower bound estimates (2.30), the inequalities (2.27), (2.51) and (2.52), and the assumption that $T_{max} = +\infty$ (which we know is true if the hypotheses of Theorem 2.10 hold). When $\lambda > c$, the curve $x = \lambda t$ satisfies

$$s(t) < \lambda t \tag{3.13}$$

for times $t > \dfrac{f''_{max}}{2(\lambda - c)(c - f'(0))c} \displaystyle\int_{u_*}^{u_-} \dfrac{(u_- - u)(f'(u) - c)}{u(c - \mu f(u)/u)}\, du \overset{def}{=} T(\lambda,c)$. Thus

$(u,\sigma,\phi)(x,t) \equiv 0$ for $s(t) < x$ and $t > T(\lambda,c)$ establishes $(3.13)_2$. That $(3.13)_1$ holds follows from (2.30) and the fact that when $\lambda < c$

$$\lim_{t\to\infty} \tilde{u}(t - \frac{\lambda}{c} t) = \lim_{\tau\to\infty} \tilde{u}(\tau) = u_- . \tag{3.14}$$

C. Concluding remarks and open questions

As was mentioned earlier, we have succeeded in showing that when u_- is small, $c_{AV}(t)$ converges to the speed of the traveling wave. One would also like to know if this result is true when u_- is large. The result would follow from (3.10) provided we knew that when u_- was large, $U(t) \overset{def}{=} \lim_{\substack{x\to s(t)\\ x<s(t)}} u(x,t)$ converged to the number u_* (defined by $\dfrac{f(u_*)}{u_*} = \dfrac{\mu f(u_-)}{u_-}$) as t approached infinity. At the moment this is an open question (and is equivalent to establishing that $\lim_{t\to\infty} \dot{s}(t) = \dfrac{f(u_*)}{u_*}$).

Assuming for the moment that $\lim_{t\to\infty} U(t) = \begin{cases} u_*, & \mu f(u_-)/u_- > f'(0) \\ 0, & \mu f(u_-)/u_- < f'(0) \end{cases}$ and hence that $\lim_{t\to\infty} c_{AV}(t) = \dfrac{\mu f(u_-)}{u_-}$ we would also like to know whether $c(a,t) = x_t(a,t) = \Sigma_a(a,t)$ (see

(3.11)) satisfies $\lim_{t\to\infty} c(\alpha,t) = \dfrac{\mu f(u_-)}{u_-}$ for $\begin{Bmatrix} u_* < \alpha < u_- & \text{when} & \mu f(u_-)/u_- > f'(0) \\ 0 < \alpha < u_- & \text{when} & \mu f(u_-)/u_- < f'(0) \end{Bmatrix}$.

Finally, we end with a tantalizing calculation which would be sufficient to guarantee that $\lim_{t\to\infty} U(t) = u_*$ when $\dfrac{\mu f(u_-)}{u_-} > f'(0)$. The differential equation

$$\frac{dU}{dt} = u_t(s(t),t)\left(1 - \frac{f(U)}{Uf'(U)}\right) - (1-\mu)\,\frac{f(U)}{f'(U)}$$

together with the identity

$$c(U(t),t) = \frac{f'(U(t))u_t(s(t),t)}{u_t(s(t),t) + (1-\mu)U(t)}$$

implies that U satisfies

$$\frac{dU}{dt} = \frac{(1-\mu)(c(U(t),t)U(t) - f(U(t)))}{(f'(U(t)) - c(U(t),t))} \quad \text{and} \quad U(0) = u_0 \qquad (3.15)$$

where again $c(\alpha,t) \overset{\text{def}}{=} x_t(\alpha,t) = \Sigma_\alpha(\alpha,t)$ and u_0 is defined in (3.3). If we now knew that $c_{AV}(t) = \dfrac{\mu f(u_-) - f(U)}{u_0(t) - U(t)}$ satisfied

$$c_{AV}(t) < c(U(t),t)$$

then the inequality $u_0(t) < u_-$ and (3.15) would imply that U okeys the inequality

$$\frac{dU}{dt} > \frac{(1-\mu)(\mu f(u_-)U - f(U)u_-)}{(f'(U)(u_- - U) - (\mu f(u_-) - f(U)))} \quad \text{and} \quad U(0) = u_0\ . \qquad (3.16)$$

An immediate consequence of (3.16) is that $U(t) > u_*$ for all $t > 0$. At this point we would be able to conclude that $\lim_{t\to\infty} U(t) = u_*$. The result would follow from the upper and lower bounds for the shock curve (see (2.50)) and from the fact that $\frac{dU}{dt}$ is uniformly bounded on $[0,\infty)$.

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In this paper we study the Riemann Problem for a system of conservation laws which exhibit internal friction similar to that seen in viscoelastic solids of the maxwell type. The solutions we obtain have a single shock and a single contact discontinuity and off of these singular curves they are smooth. The results we obtain are two fold. First we show this problem is globally solvable in time; this requires precise a-priori estimates for the solution off of the singular curves. Secondly, we obtain asymptotic or large time information about the solution which guarantees that in a weak sense it converges to special traveling wave solutions of the equations with compatible data.

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