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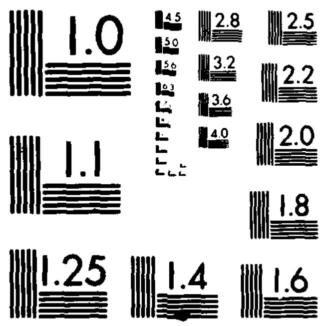

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The Sojourn Time
in a Three Node, Acyclic,
Jackson Queueing Network

by

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In this paper we consider the three node, acyclic, Jackson queueing network discussed by Simon and Foley (1980). We provide a solution to the sojourn time problem for a given customer. To the best of our knowledge, this is the first solution to the sojourn time problem in any acyclic Jackson network with overtaking.

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ABSTRACT

In this paper we consider the three node, acyclic, Jackson queueing network discussed by Simon and Foley (1980). We provide a solution to the sojourn time problem for a given customer. To the best of our knowledge, this is the first solution to the sojourn time problem in any acyclic Jackson network with overtaking.

Key words: Sojourn times
Acyclic Jackson Networks
Distribution of Sojourn Times.

1. Introduction and Background. A customer's sojourn time in a queueing network is the total amount of time the customer spends in the network. Equivalently, it is the sum of the customer's sojourn times at each of the queues it visits while in the network. The equilibrium or steady state sojourn time distribution in a queueing network is the sojourn time distribution of a customer who sees an equilibrium queue length distribution upon his arrival to the network. In this paper, we analyze the equilibrium sojourn time distribution in a three node Jackson network.

The network, which we will refer to as the three node network, consists of three single server queues each having exponential service times with parameter μ_i, $i = 1,2,3$, at queue i, FIFO queueing discipline and infinite queueing capacity. There is an exogenous arrival process to queue 1 with parameter λ. Customers departing queue 1 go, with probability p, to queue 2, and, with probability 1-p, to queue 3. Customers departing queue 2 go to queue 3 and customers departing queue 3 leave the network. This network is shown in Figure 1.

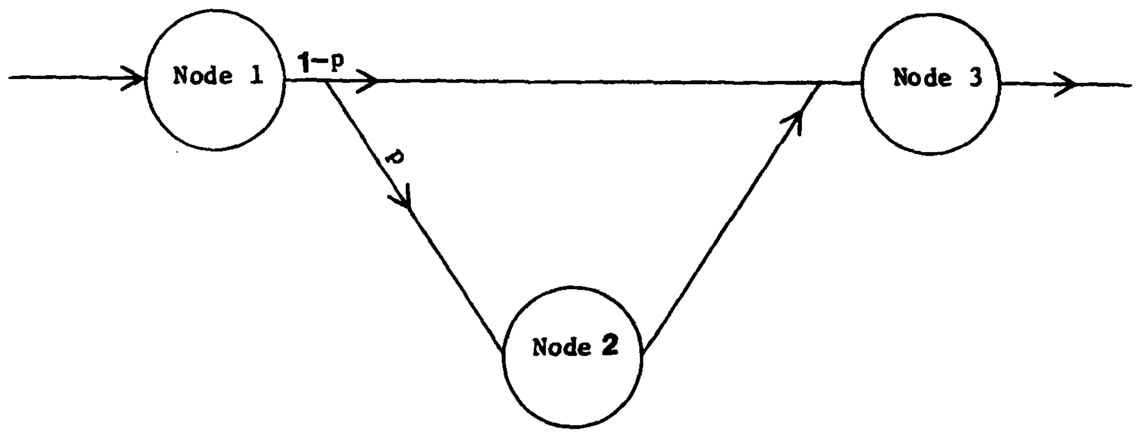

Figure 1

In the three node network a customer C's sojourn time is either the sum of C's sojourn times in queue 1, queue 2, and queue 3 if C takes the route from queue 1 to queue 2 to queue 3 or the sum of C's sojourn times in queue 1 and queue 3 if C takes the route from queue 1 to queue 3. Determining the equilibrium sojourn time is not trivial since if C takes the route from queue 1 to queue 2 to queue 3, C's sojourn times in queue 1 and queue 3 are dependent.

That the sojourn times of a customer in queue 1 and queue 3 are dependent given the customer goes to queue 2 was first observed by Mitrani (1979). He observed that if C's sojourn time in queue 1 was long enough to guarantee a large number of arrivals to queue 1, then while C is at queue 2 many of those customers at queue 1 could depart from queue 1 and go directly to queue 3. Hence, the expected number of customers at queue 3 upon C's arrival there given his long sojourn time at queue 1 is larger than the unconditioned expected queue length there. Hence, C's expected sojourn time at queue 3 given his long sojourn time at queue 1 is larger than C's unconditional sojourn time at queue 3. Simon and Foley (1979) formalized this argument.

In a queueing network a customer a is said to overtake a customer b if there exists a pair of queues A and B in the network such that a and b go from A to B, either directly or indirectly, such that a departs queue A before b but arrives to queue B after b. It is this overtaking which causes the dependence of C's sojourn times in queue 1 and queue 3. In fact Lemoine (1979) showed that in networks without overtaking the individual sojourn times are independent. Such overtaking occur, of course, when there are alternate paths between 2 nodes.

Kiessler (1980) in an simulation analysis of the three node network showed that even though C's sojourn times in queue 1 and queue 3 are dependent the correlation coefficient of the sojourn times between these queues was insignificant at the $\alpha = .05$ level. Further, the sojourn time distribution assuming the sojourn times in queue 1 and queue 3 to be independent was not significantly different from the actual distribution.

In this paper we will derive an expression for the equilibrium sojourn time distribution in the three node network. We consider the network's queue length as a stationary Markov process, i.e., a Markov process given it's equilibrium distribution. We start at an arbitrary arrival time to queue 1 and using a Palm distribution move this arrival to the origin. Finding the total sojourn time distribution can be reduced to finding the sojourn time in queue 3 given A, where A is the customer's sojourn times in queue 2 and queue 1 and that the customer takes the route from queue 1 to queue 2 to queue 3. This problem is reduced to finding the queue length distribution at queue 3 when the customer arrives there given A. Then we look at the queue length distribution at queue 3 when the customer arrives there given A and the queue length at queue 2 when the customer arrives to queue 2. Then we find the joint queue 1, queue 3 distribution when the customer arrives to queue 3 given A and the queue length at queue 2 when the customer arrives there. Working backwards the equilibrium sojourn time can be computed.

2. Formal Problem Statement. For the three queue network, let

$Q_i(t)$ = be the queue length in queue i at time t for $i = 1,2,3$;

$\underset{\sim}{Q}(t) = (Q_1(t), Q_2(t), Q_3(t))$.

From the comments made in section 1, $\{\underset{\sim}{Q}(t);\ t \in \mathbb{R}\}$ is a stationary Markov process in which a customer, C, arrives to the network at time 0 and

$$\Pr\{\underset{\sim}{Q}(0) = i_1 + 1, i_2, i_3\} = (1-\rho_1)\rho_1^{i_1}(1-\rho_2)\rho_2^{i_2}(1-\rho_3)\rho_3^{i_3} \tag{1}$$

where $\rho_1 = \lambda/\mu_1$, $\rho_2 = p\lambda/\mu_2$, and $\rho_3 = \lambda/\mu_3$, $i_1, i_2, i_3 \in \mathbb{N} = \{0,1,2,\ldots\}$. Note that the distribution Pr is actually the Palm probability of the stationary Markov process $\{\underset{\sim}{Q}(t);\ t \in \mathbb{R}\}$ embedded at arrival times to the network.

Let

S = C's sojourn time in the network;

S_i = C's sojourn time in queue i, $i = 1,2,3$;

$$R = \begin{cases} r_1 & \text{if C takes the route from queue 1 to queue 2, to queue 3} \\ r_2 & \text{if C takes the route from queue 1 to queue 3.} \end{cases}$$

Then C's sojourn time distribution is given by

$$\begin{aligned} \Pr\{S \le t\} &= p\Pr\{S \le t \mid R = r_1\} + (1-p)\Pr\{S \le t \mid R = r_2\} \\ &= p \int_0^t \int_0^{t-y} \Pr\{S_3 \le t-u-y \mid S_2 = u, S_1 = y, R = r_1\} \\ &\quad \cdot \Pr\{S_2 \in (u,u+du) \mid S_1 = y, R = r_1\} \cdot \Pr\{S_1 \in (y,y+dy) \mid R = r_1\} \\ &\quad + (1-p)\int_0^t \Pr\{S_3 \le t-y \mid S_2 = y, R = r_2\} \cdot \Pr\{S_2 \in (y,y+dy) \mid R = r_2\} \end{aligned} \tag{2}$$

3. Solution. From Simon and Foley (1979), we have

$$\Pr\{Q_2(y) = i_2 + 1 \mid S_1 = y, R = r_1\} = (1-\rho_2)\rho_2^{i_2},\ i_2 \in \mathbb{N}, \tag{3}$$

and

$$\Pr\{Q_3(y) = i_3 + 1 \mid S_1 = y, R = r_2\} = (1-\rho_3)\rho_3^{i_3},\ i_3 \in \mathbb{N}. \tag{4}$$

It follows from (1), (3), and (4) that

$$\Pr\{S_1 \in (y,y+dy)\,|\,R=r_1\} = (\mu_1-\lambda)e^{-(\mu_1-\lambda)y}\,dy, \quad i=1,2, \tag{5}$$

$$\Pr\{S_2 \in (u,u+du)\,|\,S_1=y,R=r_1\} = (\mu_2-p\lambda)e^{-(\mu_2-p\lambda)u}\,du, \tag{6}$$

and

$$\Pr\{S_3 \le t-y\,|\,S_1=y,R=r_2\} = 1-e^{-(\mu_3-\lambda)(t-y)}. \tag{7}$$

Hence, the only term not known in (2) is

$$\Pr\{S_3 \le t-u-y\,|\,S_2=u,S_1=y,R=r_1\}. \tag{8}$$

This is C's sojourn time in queue 3 given C's sojourn times in queues 1 and 2 and that C takes the route r_1. The remainder of this paper deals with the calculation of this quantity.

4. Calculation of $P\{S_3 \le t-u-y\,|\,S_2=u,S_1=y,R=r_1\}$. Note that

$$\Pr\{S_3 \le t-u-y\,|\,S_2=u,S_1=y,R=r_1\}$$

$$= \sum_{i=1}^{\infty} \Pr\{S_3 \le t-u-y\,|\,Q_3(u+y)=i,S_2=u,S_1=y,R=r_1\}$$

$$\cdot \Pr\{Q_3(u+y)=i\,|\,S_2=u,S_1=y,R=r_1\} \tag{9}$$

The first term on the right hand side of (9) is C's sojourn time at queue 3 given the queue length at queue 3 when C arrives there. Hence,

$$\Pr\{S_3 \le t-u-y\,|\,Q_3(u+y)=i,S_2=u,S_1=y,R=r_1\} = F^{*i}(t-u-y) \tag{10}$$

where

$$F(t-u-y) = 1-e^{-\mu_3(t-u-y)}$$

and F^{*i} is the ith fold convolution of F with itself.

Now

$$\Pr\{Q_3(u+y) = i \mid S_2 = u, S_1 = y, R = r_1\}$$

$$= \sum_{m=1}^{\infty} \Pr\{Q_3(u+y) = i \mid Q_2(y) = m, S_2 = u, S_1 = y, R = r_1\}$$

$$\cdot \Pr\{Q_2(u+y) = m \mid S_2 = u, S_1 = y, R = r_1\}. \tag{11}$$

The following lemma determines the second factor on the right hand side of (11).

<u>*Lemma 1*</u>. $\Pr\{Q_2(y) = m \mid S_2 = u, S_1 = y, R = r_1\} = \frac{(\lambda p y)^{m-1}}{(m-1)!} e^{-\lambda p y}.$

<u>Proof</u>. From (3) and (6) we have

$$\Pr\{Q_2(y) = m \mid S_2 = u, S_1 = y, R = r_1\}$$

$$= \frac{\Pr\{S_2 \;\varepsilon\; (u,u+du) \mid Q_2(y) = m, S_1 = y, R = r_1\}\Pr\{Q_2(y) = m \mid S_1 = y, R = r_1\}}{\Pr\{S_2 \;\varepsilon\; (u,u+du) \mid S_1 = y, R = r_1\}}$$

$$= \frac{\mu_2 \frac{(\mu_2 u)^{m-1}}{(m-1)!} e^{-\mu_2 u} du (1-\rho_2)\rho_2^{\,m-1}}{(\mu_2 - p\lambda) e^{-(\mu_2 - p\lambda)} du}$$

$$= \frac{(p\lambda u)^{m-1}}{(m-1)!} e^{-p\lambda u} . \qquad \square$$

Hence, all we need to determine is

$$\Pr\{Q_3(u+y) = i \mid Q_2(y) = m, S_2 = y, S_1 = y, R = r_1\}.$$

<u>Remark</u>. We will analyze the $(Q_1(v), Q_3(v);\ v \;\varepsilon\; (y,y+u])$ process at departure points from queue 2. The reason for analyzing this joint process is that to determine $Q_3(u+y)$ we need to know the departure process from queue 1 in the interval $(y,u+y]$. In order to determine the departure process from

queue 1 in this interval we need to know $Q_1(v)$ for $v \in (y, y+u]$. With this in mind we get

$$\Pr\{Q_3(u+y) = i \mid Q_2(y) = m, S_2 = u, S_1 = y, R = r_1\}$$

$$= \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \Pr\{Q_1(u+y) = i_1, Q_3(u+y) = i \mid$$

$$Q_2(y) = j_1, Q_3(y) = j_2, Q_2(y) = m, S_2 = u, S_1 = y, R = r_1\}$$

$$\cdot \Pr\{Q_1(y) = j_1, Q_3(y) = j_2 \mid Q_2(y) = m, S_2 = u, S_1 = y, R = r_1\}. \tag{12}$$

From Simon and Foley (1979), the second factor on the right hand side of (12) is

$$\frac{(\lambda y)^{j_1}}{j_1!} e^{-\lambda y}(1-\rho_3)\rho_3^{\,j_2}.$$

Define

Z_i = time from y until the ith departure from queue 2 after y for $i = 1, \ldots, m$.

Now,

$$\Pr\{Q_3(u+y)=i, Q_1(u+y)=i_1 \mid Q_1(y)=j_1, Q_3(y)=j_2, Q_2(y)=m, S_2=u, S_1=y, R=r_1\}$$

$$= \sum_{\ell_1, k_1} \cdots \sum_{\ell_{m-1}, k_{m-1}} \int_{z_1} \cdots \int_{z_{m-1}}$$

$$\Pr\{Q_3(y+u)=i, Q_1(u+y)=i_1, Z_m - Z_{m-1} \in (z_m - z_{m-1}, z_m - z_{m-1} + dz_m) \mid$$

$$Q_1(z_{m-1})=\ell_1, Q_3(z_{m-1})=k_1, Q_1(z_{m-2})=\ell_2, Q_3(z_{m-2})=k_2, \ldots,$$

$$Q_1(y)=j_1, Q_3(y)=j_2, Q_2(y)=m, Z_{m-1}=z_{m-1}, Z_{m-2}=z_{m-2}, \ldots, Z_1=z_1, S_2=u, S_1=y, R=r_1\}$$

$$\cdot \Pr\{Q_1(z_{m-1})=\ell_1, Q_3(z_{m-1})=k_1, Z_{m-1}-Z_{m-2} \in (z_{m-1}-z_{m-2}, z_{m-1}-z_{m-2}+dz_{m-1}) \mid$$

$$Q_1(z_{m-2})=\ell_2, Q_3(z_{m-2})=k_2, \ldots, Q_1(y)=j_1, Q_3(y)=j_2, Z_{m-2}=z_{m-2}, \ldots, Z_1=z_1,$$

$$Q_2(y)=m, S_2=u, S_1=y, R=r_1\}$$

$$\cdot \quad \ldots$$

$$\cdot \Pr\{Q_1(z_1)=\ell_{m-1}, Q_3(z_1)=k_{m-1}, Z_1 \in (z_1, z_1+dz_1) \mid Q_1(y)=j_1, Q_3(y)=j_2, Q_2(y)=m$$

$$S_2=u, S_1=y, R=r_1\}. \tag{13}$$

We can break each term in this product form down as follows. For $h = 1, \ldots, m-1$,

$$\Pr\{Q_1(z_{m-h})=\ell_{m-h}, Q_3(z_{m-h})=k_{m-h}, Z_{m-h}-Z_{m-h-1} \in (z_{m-h}-z_{m-h-1}, z_{m-h}-z_{m-h-1}+dz_{m-h})$$

$$\mid Q_1(z_{m-h-1})=\ell_{m-h-1}, Q_3(z_{m-h-1})=k_{m-h-1}, \ldots, Q_1(y)=j_1, Q_3(y)=j_2,$$

$$Z_{m-h-1}=z_{m-h-1}, \ldots, Z_1=z_1, Q_1(y)=j_1, Q_3(y)=j_2, Q_2(y)=m, S_2=u, S_1=y, R=r_1\}$$

$$= \Pr\{Q_1(z_{m-h})=\ell_{m-h}, Q_3(z_{m-h})=k_{m-h} \mid Z_{m-h}-Z_{m-h-1}=z_{m-1}-z_{m-h-1}, Q_1(z_{m-h-1})=\ell_{m-h-1},$$

$$Q_3(z_{m-h-1})=k_{m-h-1}, \ldots, Q_1(y)=j_1, Q_3(y)=j_2, Z_{m-h-1}=z_{m-h-1}, \ldots, Z_1=z_1,$$

$$Q_2(y)=m, S_2=u, S_1=y, R=r_1\}$$

$$\cdot \Pr\{Z_{m-h}-Z_{m-h-1} \in (z_{m-h}-z_{m-h-1}, z_{m-h}-z_{m-h-1}+dz_{m-h} \mid Q_1(z_{m-h-1})=\ell_{m-h-1},$$

$$Q_3(z_{m-h-1})=k_{m-h-1}, \ldots, Q_1(y)=j_1, Q_3(y)=j_2, Z_{m-h-1}=z_{m-h-1}, \ldots, Z_1=z_1,$$

$$Q_2(y)=m, S_2=u, S_1=y, R=r_1\}. \tag{14}$$

The following two lemmas interpret each factor on the right hand side of equation (14).

Lemma 2. For $h=1,\ldots,m$,

$$\Pr\{Q_1(z_{m-h})=\ell_{m-h},Q_3(z_{m-h})=k_{m-h}|Z_{m-h}-Z_{m-h-1}=z_{m-h}-z_{m-h-1},Q_1(z_{m-h-1})=\ell_{m-h-1},$$

$$Q_3(z_{m-h-1})=k_{m-h-1},\ldots,Q_1(y)=j_1,Q_3(y)=j_2,Z_{m-h-1}=z_{m-h-1},\ldots,Z_1=z_1,$$

$$Q_2(y)=m,S_2=u,S_1=y,R=r_1\}$$

$$=\Pr\{Q_1(z_{m-h})=\ell_{m-h},Q_3(z_{m-h})=k_{m-h}|Z_{m-h}-Z_{m-h-1}=z_{m-h}-z_{m-h-1},Q_1(z_{m-h-1})=\ell_{m-h-1}$$

$$Q_3(z_{m-h-1})=k_{m-h-1},Q_2(y)=m,S_2=u,S_1=y,R=r_1\}. \tag{15}$$

Proof. Note that

(a) $Q_1(z_{m-h})=Q_1(z_{m-h-1})$ + the number of arrivals to queue 1 in $(Z_{m-h-1},Z_{m-h}]$

- the number of departures from queue 1 in $(Z_{m-h-1},Z_{m-h}]$

and

(b) $Q_3(z_{m-h-1})=Q_3(z_{m-h})$ + the number of departures from queue 1 in $(Z_{m-h-1},Z_{m-h}]$ who go to queue 3 + the departure from queue 2 at Z_{m-h} - the number of departures from queue 3 in $(Z_{m-h-1},Z_{m-h}]$.

Since the arrival process to queue 1 is Poisson and the service times at queue 1 and queue 3 are exponential only $Q_1(z_{m-h-1}),Q_3(z_{m-h-1})$ and $Z_{m-h}-Z_{m-h-1}$ are needed to compute the probability of $Q_1(z_{m-h}),Q_3(z_{m-h})$. Hence, the result follows. □

In order to compute the right hand side of (15) we need to know the departure process from queue 1 in the interval $(Z_{m-h-1}Z_{m-h}]$.

Lemma 3.

$$\Pr\{Z_m - Z_{m-1} \in (z_m - z_{m-1}, z_m - z_{m-1} + dz_m) \mid Q_1(z_{m-1}) = \ell_{m-1}, Q_3(z_{m-1}) = k_{m-1}, \ldots,$$

$$Q_1(y) = j_1, Q_3(y) = j_2, Z_{m-1} = z_{m-1}, \ldots, Z_1 = z_1, Q_2(y) = m, S_2 = u, S_1 = y, R = r_1\}$$

$$= \begin{cases} 1 & \text{if } z_m = u \\ 0 & \text{otherwise;} \end{cases} \tag{16}$$

and for $h = 1, \ldots, m-1$

$$\Pr\{Z_{m-h} - Z_{m-h-1} \in (z_{m-h} - z_{m-h-1}, z_{m-h} - z_{m-h-1} + dz_{m-h}) \mid Q_1(z_{m-h-1}) = \ell_{m-h-1}$$

$$Q_3(z_{m-h-1}) = k_{m-h-1}, \ldots, Q_1(y) = j_1, Q_3(y) = j_2, Z_{m-h-1} = z_{m-h-1}, \ldots, Z_1 = z_1,$$

$$Q_2(y) = m, S_2 = u, S_1 = y, R = r_1\}$$

$$= \frac{\cdot\, h}{u - z_{m-h}} \left(\frac{u - z_{m-h}}{u - z_{m-h-1}}\right)^{h-1} \text{ on } 0 < z_{m-h-1} < z_{m-h-1} + (z_{m-h-1}\ z_{m-h-1}) < u. \tag{17}$$

Proof. The proof of (16) is trivial since C departs queue 2 at $y+u$ with probability 1 on $S_2 = u, S_1 = y$. The remainder of the proof deals with showing (17).

It is clear that for

$$y < y + z_{m-h-1} < y + z_{m-h} < y+u$$

given $S_2 = u, Q_2(y) = m, S_1 = y, R = r_1$, and $Z_{m-h-1}, \ldots, Z_1, Z_{m-h-1} - Z_{m-h-1}$ is independent of $Q_1(z_{m-h-1}), Q_3(z_{m-h-1}), \ldots, Q_1(y), Q_3(y)$. Hence (17) equals

$$\Pr\{Z_{m-h} - Z_{m-h-1} \in (z_{m-h} - z_{m-h-1}, z_{m-h} - z_{m-h-1} + dz_{m-h}) \mid$$

$$Z_{m-h-1} = z_{m-h-1}, \ldots, Z_1 = z_1, S_2 = u, S_1 = y, R = r_1, Q_2(y) = m\}$$

$$= \frac{\Pr\{Z_{m-h}-Z_{m-h-1} \in (z_{m-h}-z_{m-h-1}, z_{m-h}-z_{m-h-1}+dz_{m-h}), Q_2(y)=m \mid Z_{m-h-1}=z_{m-h-1},\ldots,Z_1=z_1,S_2=u,S_1=y,R=r_1\}}{\Pr\{Q_2(y)=m \mid Z_{m-h-1}=z_{m-h-1},\ldots,Z_1=z_1,S_2=u,S_1=y,R=r_1\}} . \tag{18}$$

Let $D(a,b)$ be the number of departures from queue 2 in the interval (a,b). Then on $S_2=u, S_1=y, R=r_1$,

$$D(y,y+u) = Q_2(y) - 1$$

since C sees exactly $Q_2(y) - 1$ customers ahead of him at y, the time at which C enters queue 2, and since the queueing discipline is FIFO. These $Q_2(y) - 1$ customers must complete service at queue 2 in $(y,y+u)$ and are the only customers to complete service at queue 2 during this interval. Hence (18) becomes

$$\frac{\Pr\{Z_{m-h}-Z_{m-h-1} \in (z_{m-h}-z_{m-h-1}, z_{m-h}-z_{m-h-1}+dz_{m-h}), D(y+z_{m-h-1},y+u)=h-1 \mid Z_{m-h-1}=z_{m-h-1},\ldots,Z_1=z_1,S_2=u,S_1=y,R=r_1\}}{\Pr\ D(y+z_{m-h-1},y+u)=h \mid Z_{m-h-1}=z_{m-h-1},\ldots,Z_1=z_1,S_2=u,S_1=y,R=r_1\}} . \tag{19}$$

Since on $S_2=u,S_1=y,R=r_1$ the departure process from queue 2 on the interval $(y,y+u)$ is a Poisson process (cf. Simon and Foley (1979)) we get that $Z_{m-h}-Z_{m-h-1}, D(y+z_{m-h},y+u)$ and $D(y+z_{m-h-1},y+u)$ are independent of $Z_{m-h-2},\ldots,Z_1$ given $Z_{m-h-1},S_2=u,S_1=y,R=r_1$ and depend on Z_{m-h-1} in that we need $Z_{m-h-1}+(Z_{m-h}-Z_{m-h-1}) < u$. So for $0 < y+z_{m-h-1} < y+z_{m-h-1} + (z_{m-h}-z_{m-h-1}) = y+z_{m-h} < u$ (19) equals

$$\frac{\Pr\{Z_{m-h}-Z_{m-h-1} \in (z_{m-h}-z_{m-h-1}, z_{m-h}-z_{m-h-1}+dz_{m-h}) \mid S_2=u, S_1=y R=r_1\} \cdot \Pr\{D(y+z_{m-h}, y+u)=h-1 \mid S_2=u, S_1=y, R=r_1\}}{\Pr\{D(y+z_{m-h-1}, y+u)=h \mid S_2=u, S_1=y, R=r_1\}}$$

$$= \frac{p\lambda e^{-p\lambda(z_{m-h}-z_{m-h-1})} \frac{(p\lambda(u-z_{m-h}))^{h-1}}{(h-1)!} e^{-p\lambda(u-z_{m-h})}}{\frac{(p\lambda(u-z_{m-h-1}))^h}{h!} e^{p\lambda(u-z_{m-h-1})}}$$

$$= \frac{h}{u-z_{m-h-1}} \left(\frac{u-z_{m-h}}{u-z_{m-h-1}}\right)^{h-1} \quad \text{as desired.} \quad \square$$

5. <u>Conclusions</u>. Equation (15) still must be calculated. To calculate (15) we need the time dependent departure process from the M/M/1 queue. Rosenshine and Pegden (1981) have determined this for the special case where the queue starts empty. However, we need the distribution of this process for an arbitrary initial queue length. Even if (15) can be calculated, we still need to put all the pieces together. That is, (15) into (14), (14) into (13), (13) into (12), (12) into (11), (10) and (11) into (9) and (5), (6), (7), and (9) into (2). These will be subjects of later papers.

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