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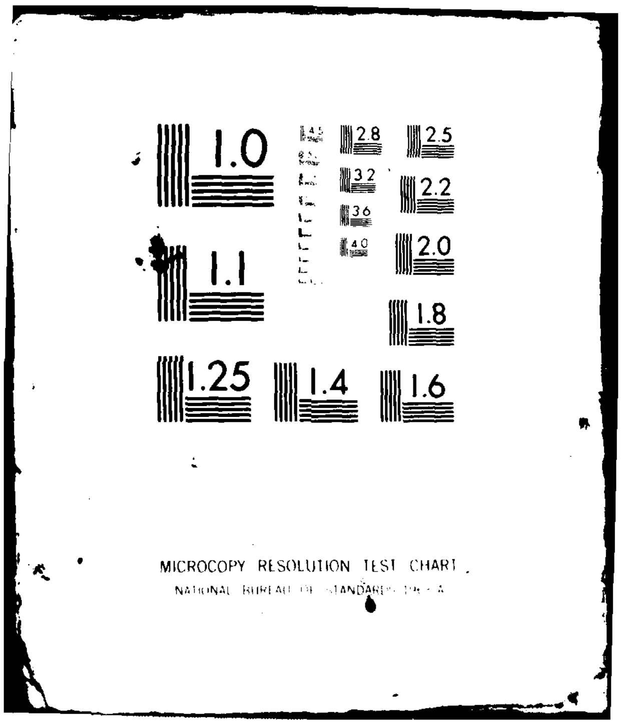
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MODELING AND SOLUTION TECHNIQUES FOR

ADVANCED ECONOMIC ANALYSIS

by

John H. Estes IV

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

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ABSTRACT

Both the cash flow and the timing of a cash flow are often found to be random variables in economic analysis. Very few analytical tools exist to handle such problems. This research develops analytical tools to aid in the analysis of such problems. A total of sixteen basic models are developed for impulsive, single cash flows and series cash flows involving capital investment problems with mutually exclusive alternatives. Each model is developed under the assumption of deterministic, constant interest rates. In addition to the development of models where cash flow and timing of cash flow are independent, dependent models are also presented. The use of these model elements by an analyst will allow the utilization of available data in a more realistic and accurate mode. The models are also of significant worth in verifying simulation output.

Other areas covered in this research are a spanning set of models found in the literature, the development of a taxonomic structure for capital investment problems, a proposed network logic for solving economic problems, and applications of this research.

MODELING AND SOLUTION TECHNIQUES FOR

ADVANCED ECONOMIC ANALYSIS

by

John H. Estes IV

has been approved

November 1981

APPROVED:

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Supervisory Committee

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I express sincere thanks to my wife, Leanne, who always said "you can" when I had my doubts, and to my children, Tracy and Johnny, for their many days of patience while their father worked on this project. My parents deserve a special thanks for their confidence during my educational and career endeavors.

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Chapter 1

INTRODUCTION

This research is concerned with modeling and solving specific classes of long-term capital investment problems in engineering economics involving mutually exclusive alternatives with deterministic and/or stochastic parameters. The analysis is based on the net present worth for each alternative.

Investment decisions are among the most difficult and important decisions with which management must cope. There are several reasons as to why a detailed analysis of possible alternatives has such a significant impact. First, the decisions involve large capital expenditures. Second, the decisions normally have long lasting impact. The concept of irreversibility in the decision can cause a poor decision to literally destroy the firm. Many capital acquisitions costing several thousands of dollars may have only scrap value if they fail to achieve the objectives of the organization. In addition, the decision may very likely commit the firm to a plan that will last several years prior to reaching a point where a new course of action can be determined. Third, and the true justification for this research effort, the alternative actions invariably involve a high degree of uncertainty. They involve estimates concerning many different variables. When the variance and probabilistic characteristics of these variables are not considered in the decision process, the decisions are based upon incomplete information, and in some cases, can be incorrect.

Statement of the Problem

The requirement to which this research is addressed is the development of solution techniques for solving complex alternative selection problems in capital investment. Such techniques must be able to handle a wide variety of complex discrete and stochastic parameters. The general orientation of the research is to define a taxonomy for capital investment model elements, to formulate and analyze each type of model element, and to establish solution procedures to solve complex problems in the area of study.

Approach to the Problem

The approach to the problem of alternative selection among capital investments will involve three basic phases. The first phase will be to develop a detailed classification for capital investment models. The second phase will be to discuss analytical tools for solutions to capital investment problems and the third phase will be the actual solution procedures to follow in solving a complex problem.

During phase one of the research, representative models found in the field of economic analysis will be reviewed. The review is not meant to provide an exhaustive enumeration of models. It will introduce the reader to the current state-of-the-art in solving capital investment problems.

Fleischer and Ward (1977) have developed a multiple-classification taxonomy for economic analysis which they claim to embrace all models in the field. The taxonomy is based upon separate descriptive classifications for cash flow, interest rate and planning horizon. Using this taxonomy, a more detailed classification structure will be proposed. This classification structure will act as a guide to illuminate those areas where additional research is required.

Specific models will be developed and selected for further study during phase one of the research. These models will be limited to those alternative selection problems involving deterministic, constant interest rates. The cash flow and time parameters will be allowed to take on discrete or continuous and deterministic or stochastic characteristics. In addition, the area of dependence between cash flow and time will be discussed for those models using ramp, decay, and growth functions.

The second phase of the research will be to develop analytic tools to solve capital investment problems. A network illustration of a problem will be discussed as an aid in illustrating the interrelationships between cash flow, time, and interest rate. The solution procedures will assume independent parameters.

Phase three of the research will be to solve a complex alternative selection problem using a general model and to show how network analysis can be used as a tool in the solution process.

Organization of the Dissertation

The remainder of Chapter 1 will be devoted to the background of engineering economics. Chapter 2 contains a spanning set of models used to solve capital investment problems.

Due to the wide variety of problems that can be viewed as capital investment problems, Chapter 3 will present a taxonomy of such problems and a general model development. In addition, Chapter 3 contains a discussion of the problems addressed in this research.

Analytical tools for solving capital investment selection problems will be discussed in Chapter 4. A given alternative will be evaluated by calculating a present worth and variance of present worth. The remainder of Chapter 4 will be devoted to the development of a detailed model for solving complex capital investment problems. Chapter 5 will be a discussion of results.

The development and justification of a network representation of a capital investment problem will be given in Chapter 6.

An analysis of a specific group of alternatives is presented in Chapter 7. This analysis is presented on the basis of expected present worth and variance of present worth for each alternative.

Conclusions and recommendations are summarized in Chapter 8. Selected mathematical proofs and derivations are given in the Appendices.

Background for Engineering Economics

Representative textbooks by Fabrycky and Thuesen (1964), Morris (1960), Terborgh (1949), and Taylor (1964) give general credit to Wellington, Fish, Goldman, Grant, and Dean for the development of the three basic methods of engineering economic analysis (Rate-of-return, Present value, and Annual cost).

Buck (1975:81) attributes the roots of engineering economics to the economics of railroad building at the end of the nineteenth century. The initial work in this area was accomplished by Wellington (1908), a civil engineer, who was interested in the structural selection aspects of railroad building. He assumed that railroads, due to their monopolistic nature, could never be destroyed. Also, he set forth the principles of compounding in assuming that railroads were built for future expansion as well as present requirements. Future traffic was postulated to increase on a compound interest basis. His book was addressed to engineers designing railroad systems. The focal point of his work centered on the selection of structures based upon capitalized cost.

Another civil engineer, J. C. L. Fish (1923), published one of the first books on engineering economics. He believed that the central concern of engineering economy should be in choice of investment, rather than structure.

Goldman (1923) introduced a comparative value concept using compound interest calculations similar to those used in textbooks today (Lesser, 1969). His work leaned heavily on the use of compound interest calculations to compare different alternatives.

In 1930, Grant published the first of a series of books on engineering economy (Grant, 1930). For the first time, short-term investments were studied in depth. Grant (1938: 362-363) pointed out several obstacles to engineering economy including problems with data leading to "hunches" and oversimplification. Other problems of bias on the part of the estimator, impractical alternatives and legal obstacles were also pointed out. Despite these warnings, engineering economy is still largely an exercise in financial arithmetic with serious omission of the real problems in analysis. Grant's comments (1938:81) concerning safety are noteworthy in that they also apply to any risk in the decision process.

> There is no correct way to introduce into an economy study this requirement for a margin of safety before undertaking a proposed investment in plant.

His suggestions to include safety in analysis are by: using conservative data; increasing the required interest rate; using a mandatory payback period; or recognizing the need of safety as an irreducible element in the final choice between alternatives. These suggestions are based upon deterministic changes in the data. As stated by Arthur Lesser (1969:111), "Eugene L. Grant can truthfully be called the father of engineering economy".

Rate-of-return analysis was introduced by Joel Dean (1951). A great deal of argument has been noted in the literature over common errors and solution problems with this method; however, Taylor (1964:125) notes that it is preferred by top management.

The early authors such as Wellington, Fish, Goldman, Grant, and Dean used data from mortality tables or curves. Rather than using the probabalistic characteristics of these data sources, deterministic data were used by calculating means and using these figures as if certainty was assumed. The models also required that the data be put into a specified format by whatever "appropriate" methods and the "appropriate" algebraic manipulations be used as stated. There is very little discussion as to how to handle risk and uncertainty other than that these elements must be considered in the decision process.

Modern textbooks have developed some of the finer details of analysis by injecting risk and uncertainty, utility theory, and mathematical and computer models. The list of references for this information is voluminous; however, specific references to some of the more sophisticated works is required. DeGarmo and Canada (1973), Morris (1977), Raiffa (1970), Bussey (1978), and Reisman and Rao (1972) are some of the works which provide powerful tools for analysis.

The work of Reisman and Rao (1972) is perhaps the most detailed effort to allow for stochastic extension of model elements. Interest rates and inflation are considered in both discrete and continuous modes and can take on random patterns (Fleischer, 1975:79). The equations for specific cases are derived and stated in a set of tables.

Summary

A review of literature points to the concern in society for economic analysis. The models developed to accomplish the analysis use a large number of parameters. Since one of the objectives of this research is to develop a general model for economic analysis, the next chapter will review a representative set of models and model parameters.

Chapter 2

BACKGROUND FOR ANALYSIS MODELS

There are several well-developed models used in capital investment decisions. Since one of the objectives of this research is to develop a general model to evaluate capital investment problems, a review of representative models is essential. The selected models are not an exhaustive set, but research indicates that they are a spanning set of models used by industry and represent the "current state-of-the-art". The models span analysis based on parameters which are deterministic to parameters which are stochastic, from models which have only a few parameters of interest (less than or equal to five) to models having as many as thirty-one parameters, and from models which are concerned with single item analysis to models which consider a string of replacements for the equipment under analysis.

A review of these models will point out the parameters which were considered for inclusion in the taxonomy presented in Chapter 3. The purpose of the review is not to critique these models, but should orient the reader to the fact that many gaps are left in the analysis of a problem due to inapplicability of a known model or the requirement to make simplifying assumptions to the data used in the analysis. The review will also be used for the general model to be introduced at the end of Chapter 3.

The models selected for review are the Machinery and Allied Products Institute (MAPI) Method, Morris Model, Bowman and Fetter Models, Alchian Model, Reisman Models, Bernhard Model, Hillier Models, and Canada/Wadsworth Model. Each model is presented with a brief discussion of the history of the model along with its mathematical formulation. The applicability of each model and assumptions made from the parameters are also covered.

MAPI Method

The MAPI Method was first published in 1958 by George Terborgh (1958). The MAPI procedures produce an "urgency rating" which is based upon an after-tax rate-of return analysis of the net project investment (Terborgh, 1958:153). The basic method is to:

1. Select an ownership period which may or may not be equivalent to the economic life of the proposed asset.

2. The existing asset takes on a one year life. The analysis is a one-more-year rate-of-return analysis to

compare cost between keeping the existing asset an additional year and replacing the existing asset with the proposed asset.

3. The formulas capitalize future sums beyond the first year at 8.25%. There are four formulas used which vary only by the pattern of equipment "inferiority" used. By pattern of equipment "inferiority", the MAPI method is evaluating the rate of accumulation of depreciation and obsolescence over the service life. This pattern is assumed to be straight-line, double-rate declining-balance, sum-of-digits, or expensing.

4. An urgency rating is calculated as a percentage on the extra investment after recovery of all costs of buying the proposed asset now rather than keeping the existing asset one additional year.

One of the four formulas is:

$$C = \frac{n(Q^n-w^n)(Q-1)^2 - (1-b)\,P\left[(Q^n-1) - n(Q-1)\right]}{nQ^n(Q-1) - (Q^n-1)} - (Q-1),$$

where

$$P = w^n\left[1 - w + py + \frac{(1-p)z}{1-b}\right],$$

$$Q = 1 + i - bpy,$$

$$n = \text{service life},$$

p = fixed debt ratio,

y = fixed interest rate on debt capital,

z = fixed rate-of-return on equity after tax,

b = fixed income tax rate,

i = fixed capitalization rate,

w^n = salvage as a decimal of original cost,

w = ratio of salvage in year n+1 to salvage in year n,

C = next-year-capital consumption expressed as a ratio of investment P.

This formula is for an accumulated depreciation following the sum-of-the-year's digits tax procedures (Taylor, 1964:374). The actual solution to the formula is graphically solved and a set of worksheets is required. The model is totally deterministic and assumes discrete parameters.

<u>Morris Model</u>

William T. Morris (1964:220-221) presents a generalized equipment replacement model:

$$TC(N_1, N_2, \ldots, N_k, \ldots) = I_0 + \sum_{j=1}^{N_1} \frac{C_{0j}}{(1+i)^j}$$

$$- \frac{S_{0N_1}}{(1+i)^{N_1}} + \frac{I_{N_1}}{(1+i)^{N_1}} + \sum_{j=1}^{N_2} \frac{C_{Nj}}{(1+i)^{N_1+j}}$$

$$- \frac{S_{N_1N_2}}{(1+i)^{N_1+N_2}} + \frac{I_{N_2}}{(1+i)^{N_1+N_2}} + \sum_{j=1}^{N_2} \frac{C_{N_2j}}{(1+i)^{N_1+N_2+j}} - \ldots,$$

where

I_t = initial investment in a machine purchased at the end of period t,

C_{tj} = operation and maintenance costs for the jth year of a machine purchased at the end of period t,

S_{tj} = salvage value at the end of the jth year of use for a machine purchased at the end of period t,

N_k = life of the kth machine in the sequence of replacements.

The model is deterministic and involves only discrete parameters with single yearly costs.

Bowman and Fetter Models

Bowman and Fetter (1957:367-375) present two models. The first model is for a chain of machines or replacements using continuous cost and revenue functions with continuous interest:

$$V = \left[\int_0^T \left[R(t) - E(t)\right] e^{-it}dt - B + S(T)e^{-iT}\right] \left[(1+e^{-iT}+e^{-2iT}+\ldots)\right],$$

$$V = \left[\int_0^T R(t)e^{-it}dt - B + S(t)e^{-iT}\right] \frac{1}{1-e^{-iT}} ,$$

where

V = present worth of the series of investments,

B = constant initial cost for each investment,

T = life of a piece of equipment,

R(t) = revenue function,

E(t) = expense function,

S(T) = salvage value at T,

i = annual interest rate.

The second model uses the same basic parameters in a discrete mode.

$$V = \left[\sum_{t=1}^{T} \frac{R_t - E_t}{(1+i)^t} - B + \frac{S(T)}{(1+i)^T} \right] \left[1 + \frac{1}{(1+i)^T} + \frac{1}{(1+i)^{2T}} + \cdots \right],$$

$$V = \left[\sum_{t=1}^{T} \frac{R_t - E_t}{(1+i)^t} - B + \frac{S(T)}{(1+i)^T} \right] \left[\frac{(1+i)^T}{(1+i)^T - 1} \right],$$

where

R_t = revenue for a given period,

E_t = expense for a given period.

The two models require that all investments be identical with respect to all parameters.

Alchian Model

The Alchian Model is composed of an expression for expenses and an expression for revenues (Alchian, 1952).

$$E = \int_0^n \left[A_1 + B_1(1-e^{-W_1, t}) \right] e^{-rt} dt + (C - C_0 e^{-d_1 n}) e^{-rn}$$

$$+ C(1-ke^{-dL}) \sum_{j=1}^{\infty} e^{-r(n+Lj)}$$

$$+ \sum_{j=0}^{\infty} \int_0^L \left[Ae^{-z(n+jL)} + B(1-e^{-wt}) u^{Lj} \right] e^{-r(n+jL+T)} dT + C_0,$$

$$R = \int_0^n (P_1 e^{-x, t}) e^{-rt} dt$$

$$+ \sum_{j=0}^{\infty} \int_0^L \left[P + Q(1-e^{-g(n+Lj)}) \right] e^{-sT-r(n+Lj+T)} dT,$$

where

A = initial annual rate of operating and maintenance costs of new item now available,

A_1 = current annual rate of operating and maintenance costs of incumbent item,

$A + B$ = limiting annual rate of operating and maintenance costs (deterioration) of new item,

A_1+B_1 = limiting rate for operating and maintenance costs of incumbent item,

C = purchase price of new items,

C_0 = salvage value of currently used item,

d = coefficient related to rate of decline of salvage value of new item,

d_1 = coefficient related to rate of decline of salvage value of current item,

g = coefficient for rate at which $P + Q$ is approached,

k = percentage of C remaining as salvage value immediately after purchase ($1-k$ = loss due to acquisition and installation),

L = projected replacement period of new equipment,

n = economic life of currently used equipment,

P = initial annual rate of revenue of new item,

P_1 = current annual rate of revenue of current item,

$P + Q$ = limit approached by initial rate of revenue (prices of services and technological changes in new machines) as time passes,

r = rate of interest,

s = coefficient for rate at which annual revenue of a new machine changes with age of a machine,

s_1 = coefficient related to rate of change of annual revenue of current item,

u = coefficient related to rate at which B changes because of new technology and prices,

w = coefficient of rate at which operating and maintenance costs of any new item approach the limiting rate (because of deterioration),

w_1 = coefficient of rate at which $A_1 + B_1$ is approached (because of deterioration),

z = coefficient related to the rate at which the initial operating and maintenance costs, A, fall as new items are developed,

$C_0(t)$ = turn-in value of present machine t years from now,

$A_1(t)$ = rate of operating and maintenance costs of present machine in year t,

$D(T)$ = turn-in value of successor machines at age T as a fraction of C,

$A(t,T)$ = rate of operating and maintenance costs, per year, of a machine purchased at time t for the Tth year of its age,

e^{-rt} = present value of one dollar t years hence at rate of interest r (continuously compounded),

t = time in year-units,

$j + 1$ = number of items in series of machines; $j = 0$ for the current or first machine,

$V_1(t)$ = value of annual rate of services in the year t for an existing machine,

$V(t,T)$ = value of annual services in the Tth year of life of operation of a machine installed $n + L(j-1)$ years from now.

The goal of this model is to select the combination of parameters which yields the maximum difference R - E. This goal is to optimize the present worth of the appropriate replacement model by use of computer generated tables or dynamic programming.

Reisman Models

Arnold Reisman has published detailed models for capital equipment investments. There are three basic approaches used. The first approach is to consider the initial purchase plus installation cost and the salvage value for each replacement as a discrete event. The expense and revenue parameters are considered to be continuous and all discounting is continuous. The model by Reisman (1971:73) is:

$$P = \sum_{j=0}^{n-1} \left[B_j e^{-r \sum_{h=0}^{j} T_h} \right] - \sum_{j=0}^{n-1} \left[S_j(T_{j+1}) e^{-r \sum_{h=0}^{j} (T_{h+1})} \right]$$
$$+ \sum_{j=0}^{n-1} \left[e^{-r \sum_{h=0}^{j} T_h} \int_0^{T_{j+1}} E_j(t) e^{-rt} dt \right]$$
$$- \sum_{j=0}^{n-1} \left[e^{-r \sum_{h=0}^{j} T_h} \int_0^{T_{j+1}} R_j(t) e^{-rt} dt \right],$$

where

T_h = time at which the hth replacement item is installed,

B_j = cost of the jth replacement item which is considered to be the total cost of purchase plus installation,

S_j = salvage value of the jth replacement,

$E_j(t)$ = expense function for the jth replacement which is assumed to be a continuous function of time,

$R_j(t)$ = revenue function for the jth replacement which is assumed to be a continuous function of time,

n-1 = total replacements for the study period.

The next model developed by Reisman (1971:76) considers all cash flows to be discrete, but the compounding remains continuous. He modifies his basic model further by noting that the jth replacement can have multiple cash flows. Both revenue and expense cash flows are allowed to have k cash flows within the economic life of the jth replacement. The model is:

$$P = \left[\sum_{j=0}^{n-1} B_j e^{-r\sum_{h=0}^{j} T_h}\right] - \left[\sum_{j=0}^{n-1} S_j(T_{j+1}) e^{-r\sum_{h=0}^{j}(T_{h+1})}\right]$$

$$+\left[\sum_{j=0}^{n-1} e^{-r\sum_{h=0}^{j} T_h}\left(\sum_{p=0}^{k} E_{jp} e^{-r\sum_{q=0}^{p} T_{(j+1),q}}\right)\right]$$

$$-\left[\sum_{j=0}^{n-1} e^{-r\sum_{h=0}^{j} T_h}\left(\sum_{p=0}^{k} R_{jp} e^{-r\sum_{q=0}^{p} T_{(j+1),q}}\right)\right] .$$

The following model treats all cash flows and discounting as discrete.

$$P = \left[\sum_{j=0}^{n-1} B_j (1+i)^{\left[-\sum_{h=0}^{j} T_h\right]}\right] - \left[\sum_{j=0}^{n} S_j(T_{j+1})(1+i)^{\left[-\sum_{h=0}^{j} T_{h+1}\right]}\right]$$

$$+\left[\sum_{j=0}^{n-1} (1+i)^{\left[-\sum_{h=0}^{j} T_h\right]}\right] X \left[\sum_{p=0}^{k} E_{jp}(1+i)^{\left[-\sum_{q=0}^{p} T_{(j+1),q}\right]}\right]$$

$$-\left[\sum_{j=0}^{n-1} (1+i)^{\left[-\sum_{h=0}^{j} T_h\right]}\right] X \left[\sum_{p=0}^{k} R_{jp}(1+i)^{\left[-\sum_{q=0}^{p} T_{(j+1),q}\right]}\right] .$$

It should be noted that k cash flows are allowed for both revenue and expense functions for each replacement.

The next set of important work accomplished by Reisman was published jointly with Arza K. Rao (1973). The monograph is by far too extensive to review in this paper. The research sets up seven deterministic models, similar in structure to the before mentioned Reisman models, which are used frequently in engineering economics: compound amount of a single payment, present worth of a single payment, amount of an annuity, periodic deposits to accumulate a future amount, present worth of an annuity, capital recovery, and present worth of a deferred annuity. Each of the seven models is then evaluated for discrete and continuous compounding. The models are further extended by allowing the cash flows, rates of discounting or compounding, and rate of inflation to be random variables with specific, known distributions. In addition, the interest rate and inflation rate are allowed to be time dependent. The timing between cash flows are assumed to be deterministic and discrete.

Bernhard Model

Richard H. Bernhard (1962:19) presents a basic model for a "proposed productive investment".

$$P = Q_0 + \frac{Q_1}{(1+i_1)} + \frac{Q_2}{(1+i_1)(1+i_2)} + \cdots \frac{Q_s}{(1+i_1)(1+i_2)\ldots(1+i_s)}$$

$$+ \cdots + \frac{Q_n}{(1+i_1)(1+i_2)\ldots(1+i_n)},$$

where

P = present worth,

Q_s = net incremental return to be gained at the end of period s. $(s = 0,1,2,\ldots,n)$,

i_s = rate of interest on borrowing or lending in any quantity during period s.

The basic model considers all cash flows as lump sum end-of-period. The deviation from traditional models for discrete cash flows is in recognizing that the interest may vary between periods.

Hillier Models

The Hillier Models (1963:449) consider cash flows as random variables. The specific relationship between the random variables is assumed to be independent or perfectly correlated. Fleisher (1975:77) points out that Hillier's work was the most important contribution to capital budgeting in the 1960's. Although specifically designed for capital budgeting, the basic models handle any random variable cash flow with known means and standard deviations. The means and variances for the present worth of the cash flow are then handled analytically. The present worth for an investment is:

$$P = \sum_{j=0}^{n} \frac{X_j}{(1+i)^j},$$

where

X_j = random variable cash flow during period j having mean μ_j and standard deviation σ_j,

i = interest rate.

Assuming that the cash flows are all independent, the mean for the present worth of the entire cash flow is:

$$\mu_p = \sum_{j=0}^{n} \frac{X_j}{(1+i)^j}.$$

The corresponding variance is:

$$\sigma^2_p = \sum_{j=0}^{n} \frac{\sigma_j^2}{(1+i)^{2j}}.$$

These observations are fairly straightforward; however, Hillier next develops the random variable X_j as:

$$X_j = Y_j + Z_j^{(1)} + Z_j^{(2)} + \ldots Z_j^{(m)},$$

where

Y_j = independent cash flow in period j,

$Z_j^{(k)}$ = kth distinct cash flow which is perfectly correlated with the corresponding cash flow in other periods.

Since Hillier assumes that the random variables are close to normal, the present worth should be close to normal with

$$\mu_p = \sum_{j=0}^{n} \left(\frac{\mu_j}{(1+i)^i} \right) = \sum_{j=0}^{n} \frac{E(Y_j) + \sum_{k=1}^{m} E(Z_j^{(k)})}{(1+i)^j},$$

and

$$\sigma_p^2 = \sum_{j=0}^{n} \left(\frac{\text{Var}\ (Y_j)}{(1+i)^{2j}} \right) + \sum_{k=0}^{m} \left(\sum_{j=0}^{n} \left[\frac{\sqrt{\text{Var}\ (Z_j^{(k)})}}{(1+i)^i} \right] \right)^2.$$

There are two deficiencies in the models. First, the assumption of normality may be questionable in many cash flow structures. If it is assumed that the number of cash flows is large, the resulting present worth will still have an approximate normal distribution; however, if the number of cash flows is not large, then only limited statements can be made using the Tchebycheff inequality.

The second questionable assumption is in considering that the correlated cash flows are perfectly correlated. The basic concept of cash flows being completely independent or perfectly correlated is at best questionable (Hillier, 1963:449). To consider cash flows which fall somewhere between mutual independence and complete correlation requires a covariance matrix with proper weighting factors for each term. From an analytical point of view, the development might not be too difficult; however, from a practical standpoint, it would appear to be difficult enough to obtain good values for means and standard deviations and not very realistic to obtain covariance type data from investment analysts.

Canada and Wadsworth Model

The model proposed by Canada and Wadsworth (1968) is

$$PV(\$) = P + D(1-e^{-rT})/r + Se^{-rT},$$

where

$PV(\$)$ = present worth of the cash flow,

P = initial investment,

S = salvage value,

r = nominal continuous interest.

Since D is a constant receipt or disbursement for each year, the basic model differs from classical models only in using a uniform series present worth calculation. The importance of the Canada and Wadsworth work is that they extended the traditional model for conditions where two variables such as salvage value and time could be dependent. This was accomplished by allowing the variables S, T, D, and P to be dependent. If any two of the variables, say S and T, are dependent while the remaining variables are independent, then the variance of PV($) is approximately equal to

$$\left(\frac{\delta PV(\$)}{P}\right)^2 V(P) + \left(\frac{\delta PV(\$)}{D}\right)^2 V(D) + \left(\frac{\delta PV(\$)}{T}\right)^2 V(T)$$

$$+ \left(\frac{\delta PV(\$)}{S}\right)^2 V(S) + 2\left(\frac{\delta PV(\$)}{T}\right)\left(\frac{\delta PV(\$)}{S}\right) \text{Cov}(T,S),$$

where

V = variance,

δ = partial derivative operator.

An additional term is added for each set of dependent variables.

Summary

The models presented in this chapter illuminate the large number of interrelated parameters which are considered to be of importance in economic analysis. Authors have a rather significant difference of opinion as to which set of parameters should be used. Although each model is correct when viewed from the specific assumptions made in the modeling environment, departures from these assumptions invalidate the model.

The next chapter will be devoted to examining the parameters used in economic analysis to develop a taxonomic structure to classify different problems or models. This effort will be used to approach the modeling problem by examining various model components or elements. The taxonomic structure will represent an orderly method of presenting models and will also illuminate those areas where additional research is needed.

Chapter 3

TAXONOMY FOR CAPITAL EQUIPMENT EXPENDITURES

The three basic parameters for all capital equipment expenditures are cash flow, time, and interest rate. Each of these parameters can take on a multitude of various characteristics such as being discrete or continuous, deterministic or stochastic, independent or dependent, etc. A review of the literature indicates that the majority of these parameters, along with their associated characteristics, have not been researched to the point of model development. This chapter will be devoted to the design of a taxonomic structure for capital equipment expenditure problems.

Two major benefits are derived from the taxonomic structure. First, it is imperative that a classification scheme be used to point out areas where research is limited or non-existent. Second, the taxonomic structure presented in this chapter is used to illustrate various models in a logical or orderly format.

Cash Flow Classification

The cash flows in a problem are not limited by the classical approaches to solving engineering economic problems. In the classical approach, one considers all cash flows to be of the same class (e.g. deterministic, continuous, etc.). However, in the real-world problem, one might consider some costs such as scheduled maintenance to be deterministic, lump sum, and independent; whereas, another cost such as electrical consumption might be deterministic, continuous, and dependent upon operating time, while a revenue of salvage value might be stochastic, lump sum, and time dependent. With this in mind, it appears that one must first consider a cash flow model as being composed of different cash flow model elements. Cash flow model elements can be dependent upon each other, as well as dependent upon time. When one is working with dependent cash flows, a knowledge of the covariance matrix for the cash flows is required to calculate the variance of the present worth. If one can assume that the cash flows are independent, the computation of an expected present worth and variance of the present worth is significantly simplified.

It is also important to distinguish between expenses and revenues. A simple sign convention, as normally used, takes care of this problem.

For simplicity, one might consider cash flows as being either continuous or discrete. Figure 3.1 is a representation of possible discrete cash flows and Figure 3.2 represents possible continuous cash flows. A continuous cash flow as

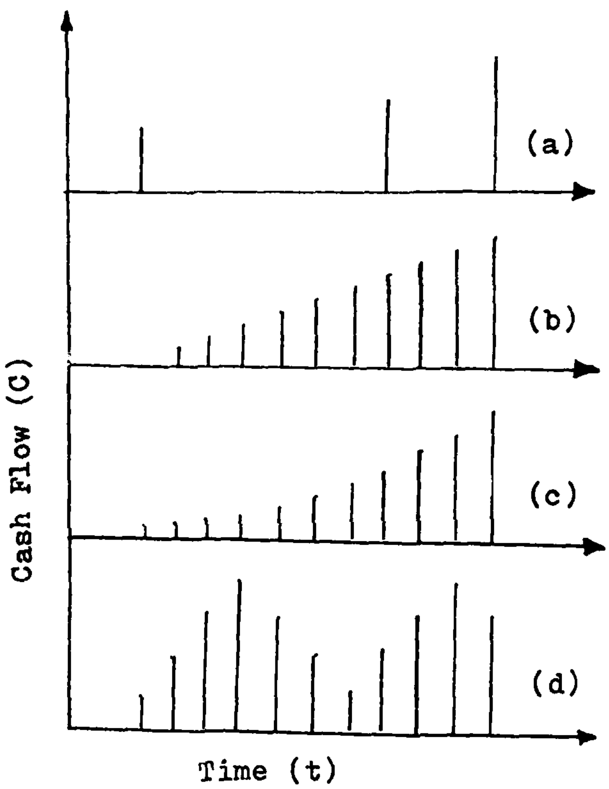

Figure 3.1

Examples of Discrete Cash Flows

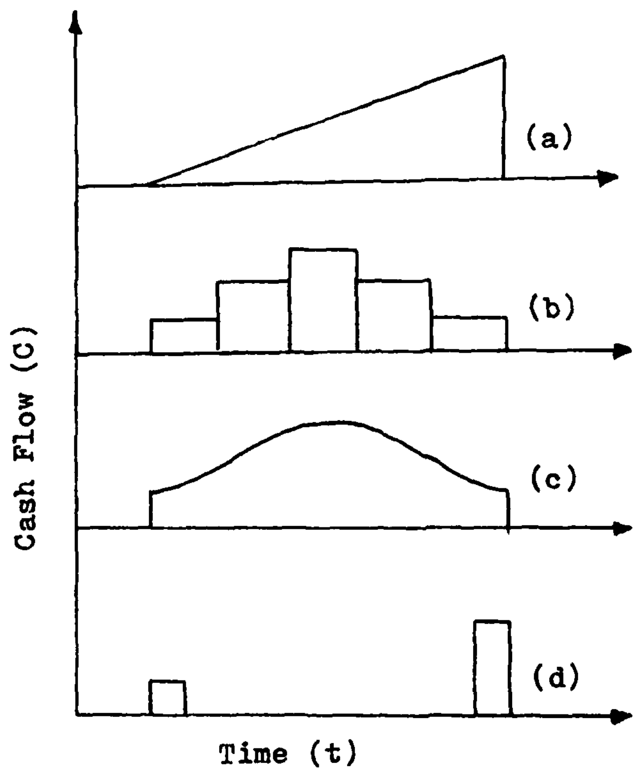

Figure 3.2

Examples of Continuous Cash Flows

used in this research, includes piecewise continuous cash flow functions. By piecewise continuous, it is meant that within a time interval there are a finite number of discontinuities and that the right and left limits for each cash flow within a subinterval exists. For example, Figures 3.2(b) and 3.2(d) represent piecewise continuous cash flows.

When discussing discrete cash flows, it seems relevant from a mathematical standpoint to distinguish an impulsive cash flow from a series cash flow. An impulsive cash flow is essentially a cash flow which is not related to any other cash flow (e.g. independent). Examples of impulsive cash flows might be the initial purchase cost or the salvage value of a piece of equipment. Obviously, any series type cash flow composed of n elements could be handled as n impulse functions. However, the ease of mathematically handling series cash flows justifies the additional effort in developing both models.

Impulsive Cash Flows

Several authors such as Fleischer and Ward (1977:14) and Reisman (1971) argue that continuous cash flows and continuous discounting should be used for analysis. The concept of continuous cash flow and continuous discounting arises from the argument that expenses and earnings are created every second, minute, hour and day of plant operations.

Using continuous cash flows and continuous discounting requires that discrete cash flows be treated as impulse functions. As developed by Spiegel (1967:255-258), the unit impulse function $(t-t_0)$ is defined by its integral property;

$$\int_{-\infty}^{\infty} F(t)\ \delta(t-a)dt = F(a), \tag{3.1}$$

where $F(t)$ is any function that is continuous at $t = a$. If $F(t)$ is set equal to 1, then

$$\int_{-\infty}^{\infty} \delta(t-a)dt = H(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a, \end{cases} \tag{3.2}$$

where $H(t-a)$ is the step at $t = a$ as illustrated in Figure 3.3 and is referred to as the Heaviside unit step function.

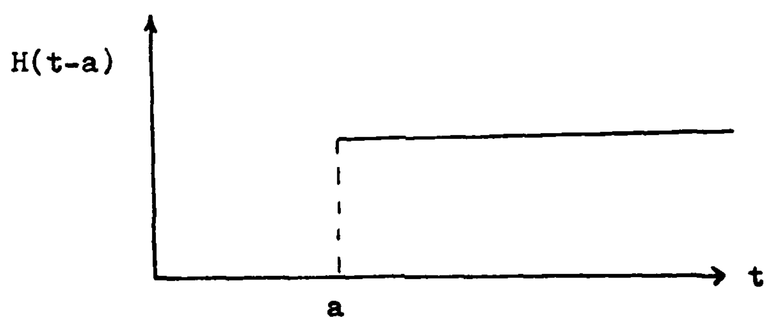

Figure 3.3

Heaviside Unit Step Function

One can now look at the Laplace transform of $\delta(t-a)$ as being equal to

$$\int_0^\infty e^{-rt}\,\delta(t-a)dt = e^{-ra}. \qquad (3.3)$$

By Equation (3.1), one can formally derive an expression using a cash flow (C):

$$\int_0^\infty Ce^{-rt}\,\delta(t-a)dt = Ce^{-ra}. \qquad (3.4)$$

Equation (3.4) is the familiar form for the present worth of a single future cash flow occurring at time = a and with continuous interest rate r.

Table 3.1 shows a variety of selected functions and their associated present worth formulas. These functions have been used by Hill and Buck (1974:121) and other authors due to their common occurrence in cash flow problems. The step function is found commonly in operating costs, the ramp function is typically used with maintenance and deterioration, the decay function can be associated with startup and learning costs, and the growth function is commonly found in wear-in maintenance costs. These formulas are developed under the assumption of continuous compounding. They may be derived for discrete compounding by denoting i to be the effective interest rate and deriving:

$$i = e^r - 1$$

$$\frac{1}{(i+1)^t} = e^{-rt}\;. \qquad (3.5)$$

Table 3.1

Selected Single Cash Flows and Transforms for Continuous Compounding

Type	F(t)	Present Worth (Laplace Transform)
Step	C	Ce^{-rt}
Ramp	Ct	Cte^{-rt}
Decay	Ce^{-at}	$Ce^{-(a+r)t}$
Growth	$C(1-e^{-at})$	$Ce^{-rt}(1-e^{-at})$

Legend:

C is the scale factor

t is the time interval

r is the continuous interest rate

a is a constant

A simple substitution of Equation 3.5 allows the development of the present worth with discrete interest as presented in Table 3.2.

The procedure for handling impulse functions is rather simple. Select the appropriate type of cash flow which is representative of the expense or revenue being considered and multiply by e^{-rt} or $\frac{1}{(1+i)^t}$. The justification for this effort will be further developed under continuous cash flows.

Series Cash Flows

As pointed out by Taylor (1964:31), disbursements for some equipment increase with the life of that equipment. By similar logic, other disbursements and/or revenues may increase or decrease during the life of the equipment. Typically, the cash flows are placed into partitions with signed components and multiplied by the appropriate series factor. These series factors require single calculations for each time series rather than by treating each point in time as a separate calculation. This analytic advantage increases with the number of points considered. Traditional approaches consider cash flows which are uniform, arithmetic, or geometric series starting at time equal to zero.

Table 3.2

Selected Single Cash Flows and Transforms for Discrete Compounding

Type	F(t)	Present Worth
Step	C	$\frac{C}{(1+i)^t}$
Ramp	Ct	$\frac{Ct}{(1+i)^t}$
Decay	Ce^{-at}	$\frac{Ce^{-at}}{(1+i)^t}$
Growth	$C(1-e^{-at})$	$\frac{C(1-e^{-at})}{(1+i)^t}$

Hill and Buck (1974:120-125) have expanded the traditional analytic tools by using Zeta transforms which serve engineering economic analysis of discrete time series much in the same manner as the Laplace transforms do those with continuous cash flow functions.

Consider a function F(n) which describes a cash flow at time n. The present worth of this cash flow at an effective interest rate i is:

$$F(n)(1+i)^{-n}.$$

This assumes that i has been adjusted or that n is a multiple of the interest compounding period. Given that F(n) is a series of end-of-period cash flows, the present worth would be:

$$\sum_{n=0}^{N} F(n)(1+i)^{-n},$$

where F(N) is the last cash flow. Hill and Buck (1974:83) note that this looks like the Zeta transform

$$\sum_{n=0}^{\infty} F(nT)(1+zT)^{-n}$$

after one replaces i with z, considers T=1, and assumes an infinite series of cash flows. One notes that Zeta transforms can be used directly to obtain present worth calculations for any series having a Zeta transform. Since

Zeta transforms are not commonly found in most handbooks, the Z transform tables can be used by noting that

$$Z = 1+zT.$$

Table 3.3 is generated by using Z transforms and then converting to the appropriate Zeta transformation (letting i=z). To illustrate this procedure the Z transform for $F(nT) = CnT$ is

$$\frac{CTZ}{(Z-1)^2},$$

which is equal to the Zeta transform

$$\frac{CT(1+iT)}{(1+iT-1)^2} = \frac{CT(1+iT)}{(iT)^2}.$$

Authors, such as Taylor (1964:26), develop series cash flows which start at time equal to one rather than zero. To convert Table 3.3 to formulas for cash flows starting at time equal one, the cash flow function need be only evaluated at $n=0$ and subtracted from the appropriate Zeta transform entry.

For discrete, series cash flows which do not start at time equal to zero and do not continue infinitely, adjustments have been derived by Hill and Buck (1974:123). Table 3.4 represents series cash flows which are translated forward b time units, start at time h (where $h \geq b$), and stop at time k-1 (where $k > h$).

Table 3.3

Zeta Transforms for Discrete Series
Cash Flows at Interest Rate i

Type of Series	F(nt)	Zeta Transform
Step	C	$\frac{C(1+iT)}{iT}$
Ramp	CnT	$\frac{C(1+iT)T}{(iT)^2}$
Decay	Ce^{-arT}	$\frac{C(1+iT)}{iT+1-e^{-aT}}$
Growth	$C(1-e^{-arT})$	$C(1+iT)\left[\frac{1}{iT} - \frac{1}{iT+1-e^{-aT}}\right]$

Source:

Thomas W. Hill and James R. Buck. "Zeta Transforms, Present Value, and Economic Analysis." AIIE Transactions, 6, No. 2 (1974), 121.

Table 3.4

Present Value of Shifted/Translated Series Cash Flows with Discrete Interest i

Type of Series	F(nT)	Zeta Transform
Step	C	$\frac{C}{i}\left[(1+i)^{1-h} - (1+i)^{1-k}\right]$
Ramp	Cn	$\frac{C}{i^2}\left[(1+i)^{1-h} - (1+i)^{1-k}\right] + \frac{C}{i}\left[(h-b)(1+i)^{1-h}\right] - \frac{C}{i}\left[(k-b)(1+i)^{1-k}\right]$
Decay	Ce^{-an}	$\frac{C(1+i)^{1-b}}{1+i-e^{-a}}\left[\frac{e^{-ha}}{(1+i)^h} - \frac{e^{-ka}}{(1+i)^k}\right]$
Growth	$C(1-e^{-an})$	$\frac{C(1+i)^{1-b}}{1+i-e^{-a}}\left[\frac{1+i-e^{-a}-ie^{-hi}}{i(1+i)^h} - \frac{1+i-e^{-a}+ie^{-ka}}{i(1+i)^k}\right]$

<u>Source</u>:

Thomas W. Hill and James R. Buck. "Zeta Transforms, Present Value, and Economic Analysis." <u>AIIE Transactions</u>, 6, No. 2 (1974), 123.

The logical extension of the material in Table 3.4 is to develop the same discrete series cash flows under conditions of continuous compounding. By observing the relationship between i and r in Equation 3.5, the results displayed in Table 3.5 can be easily derived.

Continuous Cash Flows

The derivations for continuous cash flows can be theoretically derived with both continuous and discrete compounding of interest. However, when discrete interest is used, the model is really a subset of discrete cash flow with discrete interest since the integration of the cash flow function can be treated as a single lump sum element. For this reason, discrete compounding is part of the taxonomy only for the need of completeness. In application, one would simply integrate the cash flow function over time and multiply by

$$\frac{1}{(1+i)^t}.$$

A continuous cash flow starting at time h and continuing to time k is a function of time, C(t). This function is then multiplied by e^{-rt} to discount the cash flow to the present and integrated over the time period to develop an equivalent present value as:

$$\int_n^k C(t)\, e^{-rt}\, dt. \qquad (3.6)$$

Table 3.5

Present Value of Shifted/Translated Series Cash Flows with Continuous Interest r

Type of Series	F(nT)	Zeta Transform
Step	C	$\frac{C}{e^{r}-1}\left[e^{r(1-h)} - e^{r(1-k)}\right]$
Ramp	Cn	$\frac{C}{(e^{r}-1)^{2}}\left[e^{r(1-h)} - e^{r(1-k)}\right]$
		$+ \frac{C}{e^{r}-1}\left[(h-b)(e^{r(1-h)})\right]$
		$+ \frac{C}{e^{r}-1}\left[(k-b)e^{r(1-k)}\right]$
Decay	Ce^{-an}	$\frac{Ce^{r(1-b)}}{e^{r}-e^{-a}}\left[e^{-h(a-r)} - e^{-k(a-r)}\right]$
Growth	$C(1-e^{-an})$	$\frac{Ce^{r(1-b)}}{e^{r}-e^{-a}}\left[\frac{e^{r}-e^{-a}-(e^{r}-1)e^{-ha}}{(e^{r}-1)e^{rh}}\right.$
		$\left.- \frac{-e^{r}-e^{-a}+(e^{r}-1)e^{-ka}}{(e^{r}-1)e^{rk}}\right]$

Buck (1975:81) and other authors give credit to Grubbstrom (1967) for recognizing Equation 3.6 as a Laplace transform. This is a significant realization since a table of Laplace transforms found in many mathematical handbooks such as Selby (1975:506-515) can be used to convert many continuous cash flow functions to a present value equivalent. Table 3.6 is an extension of Table 3.5 for continuous cash flow functions with continuous compounding.

The use of Laplace, Z, and Zeta transforms in analysis have two important properties when available for use in modeling engineering economic problems. One of the two properties is the linearity under addition and multiplication by a constant which allows one to take transforms of component cash flows, scale each, and then sum them. Another useful property is the ability to shift the time pattern forward. A number of transforms have been derived and are available in several references. When these transforms do not exist, they must be either derived or more involved calculations must be made.

Cash Flow Parameters

In addition to classifying each flow as impulsive, series, or continuous one must also look at the parameters

Table 3.6

Present Value of Shifted/Translated Continuous Cash Flow with Continuous Interest r

Type of Cash Flow	C(t)	Present Value (Laplace) Transform
Step	C	$\frac{C(e^{-rh}-e^{-rk})}{r}$
Ramp	Ct	$\frac{C(e^{-rh}-e^{-rk})}{r^2}$
		$+\frac{C}{r}\left[(h-b)e^{-rh}-(k-b)e^{-rk}\right]$
Decay	Ce^{-at}	$\frac{Ce^{-br}}{a+j}\left[e^{-(a+r)h}-e^{-(a+r)k}\right]$
Growth	$C(1-e^{-at})$	$\frac{C}{r}(e^{-rh}-e^{-rk})$
		$-\frac{Ce^{-rb}}{a+r}\left[e^{-(a+r)h}-e^{-(a+r)k}\right]$

associated with the actual cash flow. In the model elements presented prior to this point, deterministic cash flows have been assumed. This assumption is usually made in traditional analysis; however, the actual cash flows can also be random processes. The assumption that cash flows can be random variables has been incorporated into the cash flow taxonomic structure illustrated in Figure 3.4.

When a specific cash flow is studied, the timing of that cash flow must also be studied. The timing of a cash flow is the next subject of consideration.

Time Classification

Once a cash flow has occurred, the duration or time of occurrence is known. However, prior to occurrence, the timing of the cash flow may be either deterministic or stochastic. Once again, discrete timing is divided into impulsive timing for a single time element or series timing for a number of time periods. Figure 3.5 represents the time classification.

After the cash flow and the timing of that cash flow have been determined, the decision must be made for an appropriate discount rate. This is the third element to be classified.

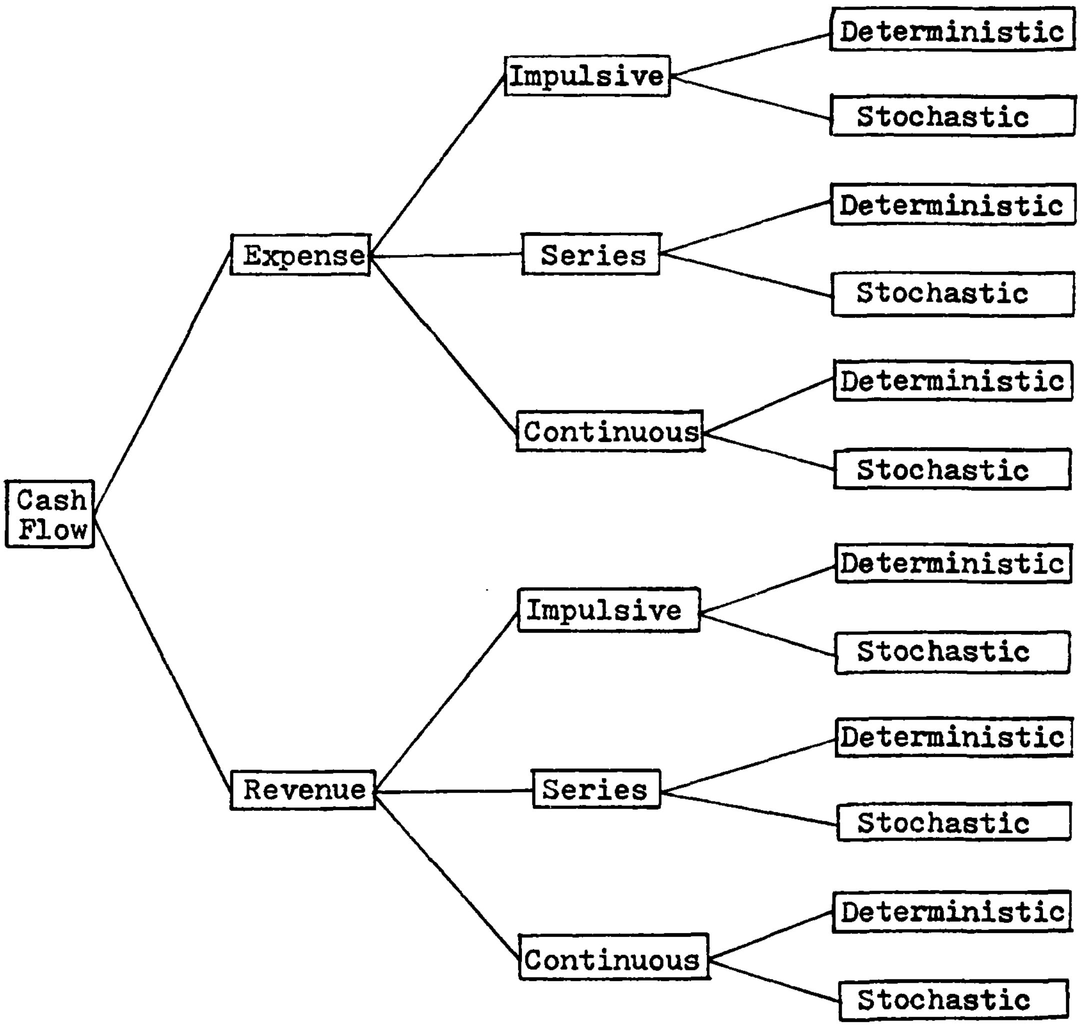

Figure 3.4

Cash Flow Taxonomic Structure

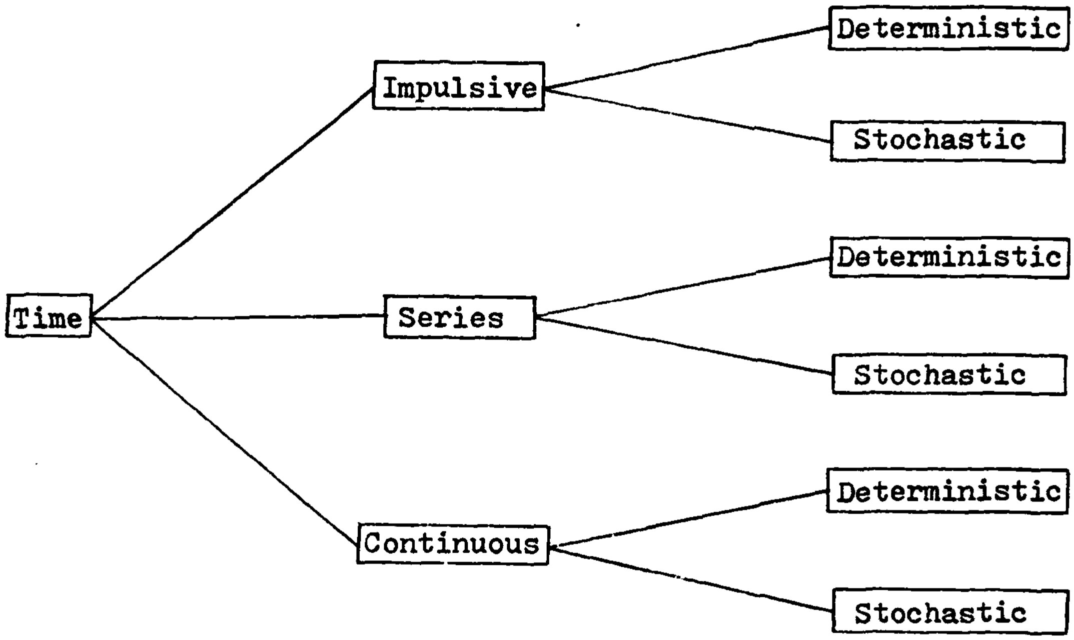

Figure 3.5

Timing of Cash Flow Taxonomic Structure

Interest Rate Classification

Three basic interest rate classes are considered by Fleisher and Ward (1977:21). These are constant, stepwise constant, and variable constant interest rates. Three examples of possible interest rates are illustrated in Figures 3.6, 3.7, and 3.8.

The interest rate is defined over the entire time interval of a specific cash flow. The actual discounting takes place at the end of a unit of time or continuously throughout the period. In addition, the interest rate can be either deterministic or stochastic. In the deterministic constant case, the interest rate (i) is fixed over the entire discounting period. In the stochastic constant case, the interest rate (i) is a random variable that, once determined, is constant over the entire discounting period. In the deterministic stepwise constant case, the rate is constant between each of a finite number of discontinuities. The end points may not agree with the discounting time period which will require an adjustment of the discount rate. The stochastic stepwise constant case has a constant mean for each time interval and each random variable i_1, i_2, ... i_n may be independent or correlated with one another. In the deterministic variable case,

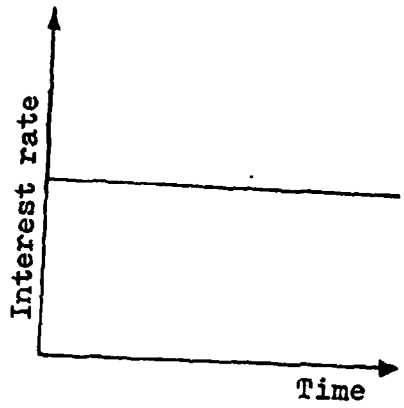

Figure 3.6

Constant Interest Rate

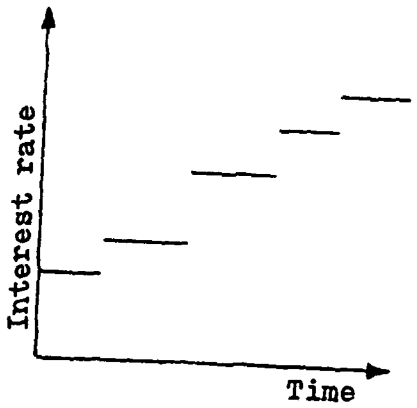

Figure 3.7

Stepwise Constant Interest Rate

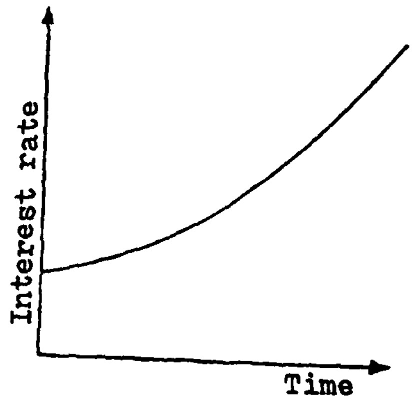

Figure 3.8

Variable Interest Rate

one discounts the cash flow by multiplying by

$$e^{-\int_0^t r(t)dt}.$$

This is a continuous time process. A similar expression is used for the stochastic variable case. Figure 3.9 illustrates the taxonomic structure for the interest rate.

Inflation has been considered by several authors as a factor which must be considered within an analysis. The work of Reisman and Rao (1973) develops both the justification and procedures for handling inflation. For the purpose of the taxonomic structure, it is assumed that the discount rate must be appropriately modified to reflect inflationary factors and does not have inflation as a separate classification.

The taxonomies in Figures 3.4, 3.5, and 3.9 must be combined to visually depict the taxonomy for capital equipment expenditures. This has been accomplished in Figure 3.10. A particular calculation for present worth would be formulated by taking three paths through the network. As an example, a traditional engineering economic problem is:

Cash Flow - expense, impulsive, deterministic,
Time - impulsive, deterministic,
Interest Rate - constant, deterministic.

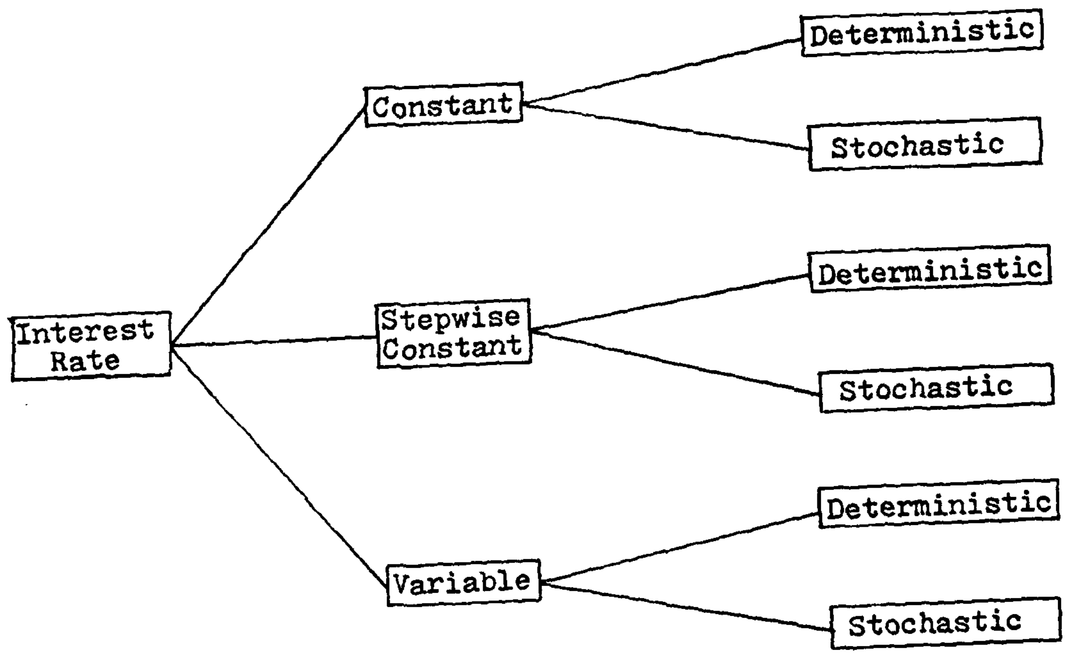

Figure 3.9

Taxonomic Structure for Interest Rate

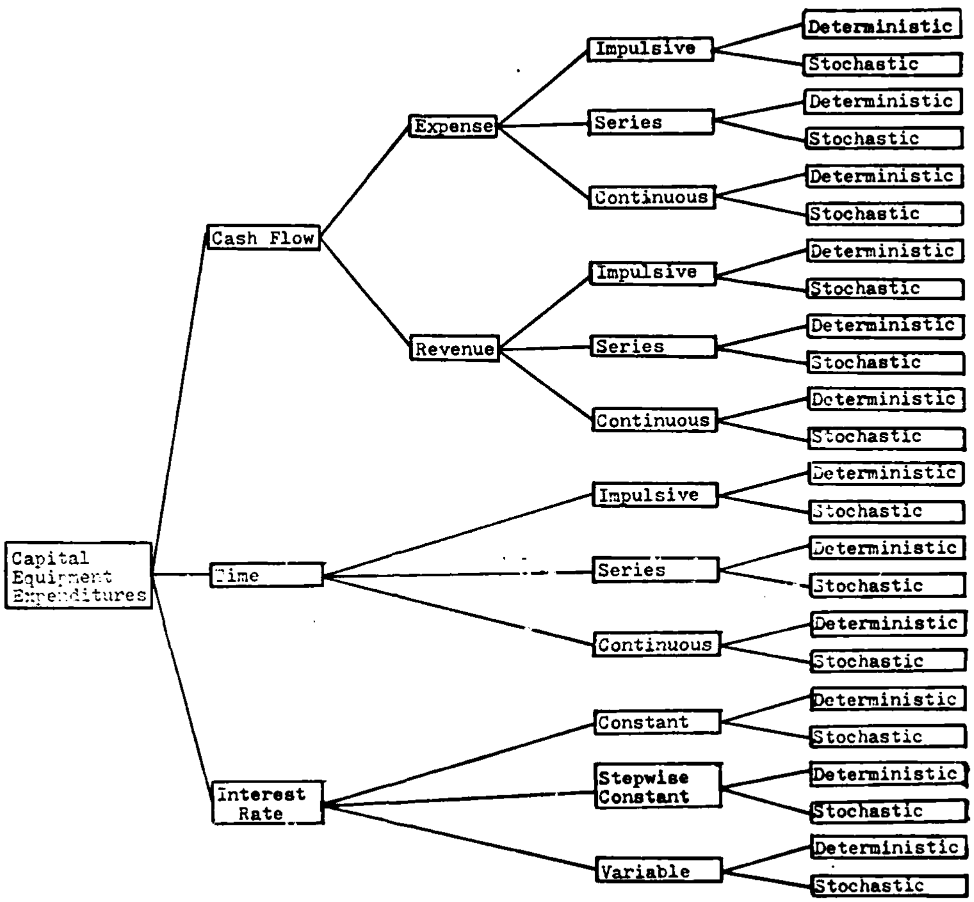

Figure 3.10

Taxonomic Structure for Capital Equipment Expenditures

Note that a classification distinction has not been made for continuous versus discrete interest since it has been shown that this involves a simple mathematical substitution. In addition, the subject of dependence among the elements of cash flow, time, and/or interest rate has not been visually depicted. Rather than redrawing Figure 3.10, note that a heavy dashed line could be placed between any of the three basic parameters to signify dependence. The path through the impulsive/series classification must agree for both cash flow and time.

State-of-the-Art

The current state-of-the-art for economic analysis problems is reviewed by Fleischer and Ward (1977:24, exhibit 3). Their work is reproduced in Table 3.7. An extensive review of the literature since 1977 does not indicate any additional entries. As can be noted from Table 3.7, a great deal of research is still open in the field of engineering economics. Expected values for some decision criterion such as present worth (Young and Contreras, 1975) have been developed for some of the missing entries in Table 3.7; however, the corresponding variances have not been developed. In addition, only the works of Hillier (1963, 1966) develop dependent models.

Table 3.7

Taxonomic Structure and Current
State-of-the-Art

Cash Flow				Interest Rate					
Amount		Timing		Constant		Stepwise Constant		Variable	
How	Cert	When	Cert	Det	Sto	Det	Sto	Det	Sto
Impulsive	Det	Disc	Det	Cla	1	Cla	1	Cla	
			Sto						
		Cont	Det	Cla		Cla		Cla	
			Sto	2					
	Sto	Disc	Det	4,3,2	1		1		
			Sto						
		Cont	Det						
			Sto						
Continuous	Det	Disc	Det	Cla		Cla		Cla	
			Sto						
		Cont	Det	Cla		Cla		Cla	
			Sto						
	Sto	Disc	Det	2					
			Sto						
		Cont	Det	2					
			Sto						

1-Reisman and Rao (1973)
2-Ward (1975)
3- Hillier (1969)
4-Hillier (1963)

Cert-Certainty
Det-Deterministic
Sto-Stochastic
Cla-Classical
Cont-Continuous
Disc-Discrete

Source:

G. A. Fleischer and T. L. Ward. "Classification of Compound Interest Models in Economic Analysis." Engineering Economist, (Fall, 1977), 24.

Any one of the missing entries in Table 3.7 is worthy of research. The wide use of minimum attractive rate of return (MARR) in economic analysis indicates that the area for initial research should be restricted to the missing entries in Table 3.7 under constant, deterministic interest. Grant, Ireson, and Leavenworth (1976:97-100,189-195) pointed out the value in using MARR in that the purpose of an economic study is to determine if an investment should be made. They suggested that this be done on the basis of determining whether the investment can be recovered with at least a stipulated MARR. For these reasons, the models selected for research in this paper were restricted to the missing entries in the impulsive cash flow section of Table 3.7 under constant, deterministic interest. Normally, an impulsive cash flow is considered to be a single payment or expense such as salvage value or initial purchase cost. This research expands impulsive cash flow research to include series (multiple impulsive) cash flows.

In view of the developed taxonomy, the models for Table 3.7 are illustrated in Figure 3.11 by taking one branch through cash flow and one through time. The path through the impulsive/series level must agree for both cash flow and time. The dashed line indicates dependence between time and cash flow. The lack of a dashed line

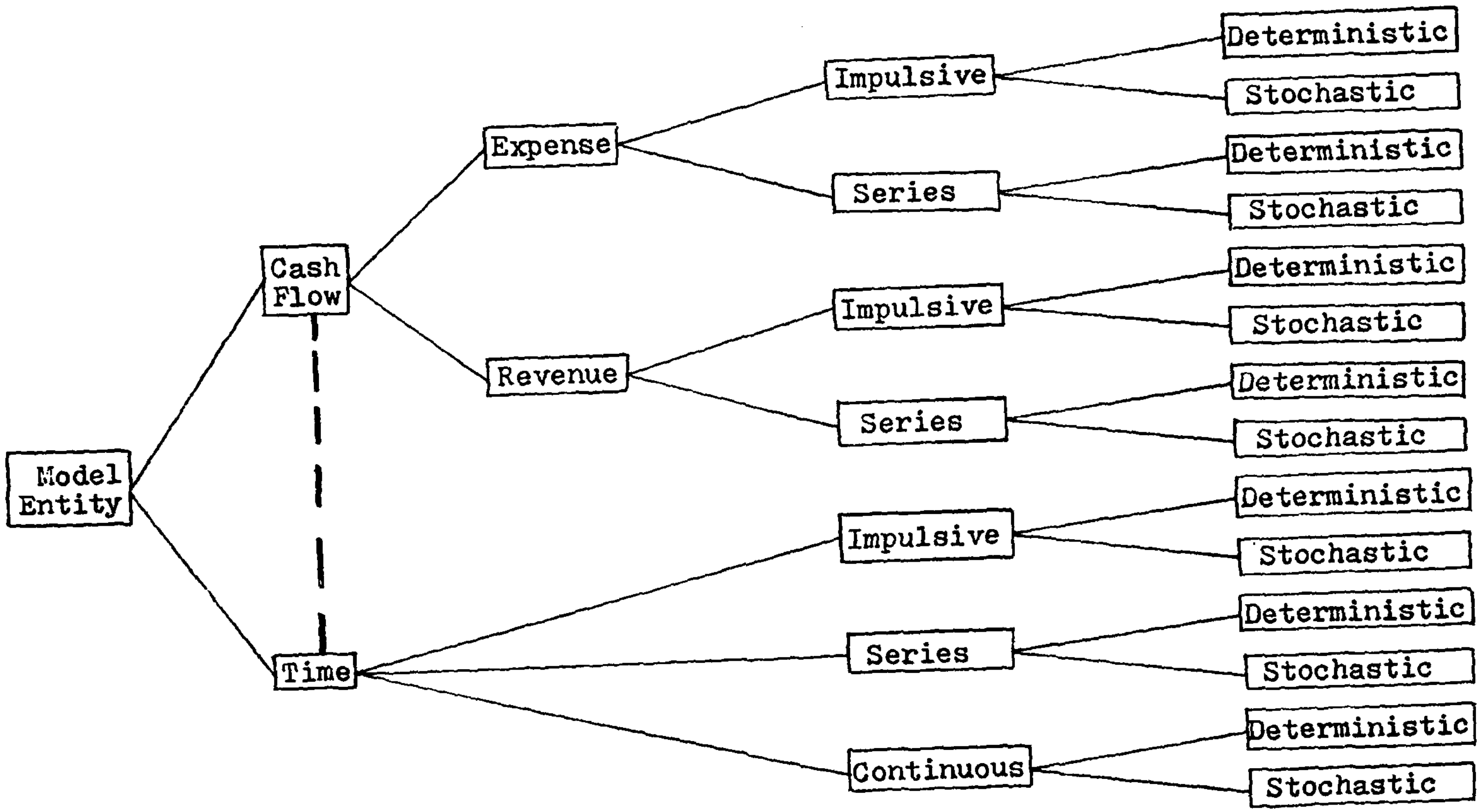

Figure 3.11

Taxonomic Structure for Research Models

would indicate independence. If we assume that revenue and expense can be treated with an appropriate sign convention, the specific models can be enumerated with a short form classification.

Short Form Classification

The concepts for the classification scheme have been addressed in prior sections. A short form classification scheme is developed in this section to enumerate those models studied in the research. The proposed form is:

$$C_{XY}T_{ZY},$$

where

C = Cash Flow

T = Time

$$X = \begin{cases} i & \text{impulsive cash flow} \\ s & \text{series cash flow} \end{cases}$$

$$Y = \begin{cases} d & \text{deterministic} \\ s & \text{stochastic} \end{cases}$$

$$Z = \begin{cases} i & \text{impulsive time} \\ s & \text{series time} \\ c & \text{continuous time} \end{cases}$$

A bar placed over C and T indicates dependence between cash flow and time. Notice that interest rate is not included since each model assumes deterministic, constant interest. Using this classification, the specific models to be researched are presented in Table 3.8 for independent cases. Since these models are repeated for dependent cases, a total of 16 models are developed. It is important to note that each model is a cash flow entity where a model for the actual present worth of a particular alternative could be composed of a large number of independent models for each cash flow entity. Each of these models are developed in the next chapter.

General Model

The general model for a present worth analysis is stated as:

$$E(PV) = E(P) + E(E) - E(R) - E(S),$$

where

E = Expected value operator

P = Purchase price

E = Expenses

R = Revenues

S = Salvage value.

Any one of the elements (P, E, R, S) may or may not contain multiple cash flows. This general model is expanded in the next chapter to offer more detail to the analyst.

Table 3.8

Independent Parameter Research Problems

Model	Cash Flow	Time
$C_{id}T_{is}$	impulsive deterministic	impulsive stochastic
$C_{sd}T_{ss}$	series deterministic	series stochastic
$C_{is}T_{is}$	impulsive stochastic	impulsive stochastic
$C_{ss}T_{ss}$	series stochastic	series stochastic
$C_{is}T_{cd}$	impulsive stochastic	continuous deterministic
$C_{ss}T_{cd}$	series stochastic	continuous deterministic
$C_{is}T_{cs}$	impulsive stochastic	continuous stochastic
$C_{ss}T_{cs}$	series stochastic	continuous stochastic

Summary

The taxonomic structure for capital equipment expenditures illustrated in Figure 3.10 represents an orderly and logical format to present models in the following chapter. Several characteristics of the parameters have been left out. A more detailed structure might include the actual functions being used for each parameter and considerations for inflation. As pointed out by Fleischer and Ward (1977:28), "A truly comprehensive classification scheme would be so extraordinarily complex that it would be virtually without utility."

Chapter 4 presents models for the selected taxons in Table 3.8 and concludes with a more detailed model for analysis of capital equipment expenditure problems.

Chapter 4

ANALYTIC TOOLS

The models proposed for study during this research assume lump sum cash flow elements which are either impulsive (single cash flow) or series. Analytic tools for independent and dependent models are presented in this chapter with a concluding section on a general model development for a capital investment alternative selection problem.

A complete model for a specific investment alternative may contain multiple impulsive or series cash flows, however, each cash flow element is assumed to be independent of all other cash flow elements. This assumption is required if one is to consider time as a random variable. The interest rate is assumed to be independent of both cash flow and time. In addition, the interest rate is assumed to be constant over the time horizon and deterministic in nature for all model elements. Although models are developed where cash flow is dependent on time, the nature of this dependency is restricted to ramp, decay, and growth functions.

Expected value of present worth and variance of present worth are developed for all of the independent model elements, and expected value of present worth and variance of present worth are developed for all impulsive, dependent model elements. The variance for the $C_{sd}T_{ss}$ dependent, ramp model element is also derived. The complexity of the expression degrades the use of it as an analytical tool. For this reason, the remaining series, dependent models do not have a variance expression; however, the expected present worth is derived for all of the series, dependent model elements.

Due to the large number of discrete and continuous distributions, a complete listing of analytical tools for all distributions would be too extensive. The approach used in this research is to develop general formulas and then show applications with commonly used distributions.

Nomenclature

The short form classification system developed in Chapter 3 is used to introduce each model. The symbol T is used for time, and t is used for a specific numerical value of time. Similarly, C is used for a cash flow designation, and c as a specific value. The symbols μ and σ^2 are used for mean and variance respectively and are subscripted with c or t for time or cash flow. The symbols

E and V are used for the expected value operator and variance operator respectively. Present worth is denoted by PW. Due to the extensive use of moment generating functions, M_t and M_c are used for the moment generating functions for the time and cash flow distributions.

The independent models are presented by first deriving the impulsive model elements and then extending that model for series model elements. This format was selected as a logical development for the independent models. Since variance expressions are not developed for the series, dependent model elements, all of the impulsive, dependent model elements are presented prior to the series, dependent model elements.

The last section in this chapter contains a more detailed general model which extends the general model presented in Chapter 3 to include the model elements presented in this chapter.

Independent Model Elements

Cash flow elements in this section assume all cash flows are independent of time. As is normally done, the present is considered to be at time equal to zero.

$C_{id}T_{is}$ - Independent Model Element

The first model discussed is when the cash flow is impulsive and deterministic, such as with an overhaul, and time is impulsive and stochastic. If one designates C as some future cash flow (impulsive and deterministic) then the present worth of C is

$$PW = C \cdot e^{-rT},$$

and the expected present worth, due to independence of cash flow and time, is

$$E(PW) = C \cdot E(e^{-rT}). \tag{4.1}$$

Since T is a random variable with a given probability law, the solution to Equation 4.1 is easily found by

$$C \cdot E(e^{-rT}) = C \cdot \sum_{t=0}^{\infty} e^{-rt} P(t) \tag{4.2}$$

where

$$t = (0, 1, 2, 3, \ldots),$$

$$P(t) = \text{probability of } T \text{ taking on a specific value } t.$$

Since

$$\sum_{t=0}^{\infty} e^{-rt} P(t)$$

is the moment generating function for P(t), Equation 4.2 can be written as

$$E(PW) = CM_t(-r), \tag{4.3}$$

where $M_t(-r)$ designates the moment generating function of P(t). The derivation of the expected present worth for this model was derived by Young and Contreras (1975).

Table 4.1 represents some common discrete distributions and their respective moment generating functions. To find the expected present worth for a $C_{id}T_{is}$ model element, one merely multiplies the cash flow, C, by the appropriate moment generating function for the time distribution.

Before proceeding to the development for the variance of the present worth for this model, an important property for present worth analysis is required:

$$E(e^{-rT}) \geq e^{-rE(T)} \qquad (4.4)$$

The vast majority of engineering economic analysis assumes that Equation 4.4 holds as a strict equality, as an approximate equality, or simply ignores the effects of random variables. Obviously, in those cases where the data is completely deterministic, Equation 4.4 does hold as a strict equality.

The proof of Equation 4.4 is derived by a Taylor series for e^{-rT} expanded about the point $T = \mu_t$ and is found in Appendix I (1). Ignoring the subject of variance, Equation 4.4 points out where a present worth analysis can be in error. When one of the following conditions exist, the present worth analysis of a group of mutually exclusive alternatives could lead to an incorrect selection.

Table 4.1

Discrete Probability Laws and Their Moment Generating Functions

Distribution	Probability Law	$M_t(-r)$
Binomial	$P(t) = \binom{n}{t} p^t q^{n-t}$, $t=0,1,\ldots,n$	$(pe^{-r}+q)^n$
Geometric	$P(t) = pq^{t-1}$, $t=1,2,3,\ldots$ 0, otherwise	$\dfrac{pe^{-r}}{1-qe^{-r}}$
Poisson	$P(t) = \dfrac{e^{-\lambda}\lambda^t}{t!}$, $t=0,1,2,3,\ldots$ 0, otherwise	$e^{\lambda(e^{-r}-1)}$

1. One or more of the alternatives involve random variables. Since the present worth is underestimated for an alternative involving time as a random variable, an incorrect selection is possible.

2. A budget limit for the selection of alternatives is required. Ignoring the idea that time is a random variable may place the least cost item under the budget limit when it is in fact over the budget limit.

The significance of this error is important and must be evaluated. As an example, let us assume that the time between major breakdowns for a capital investment follows the Poisson distribution with mean = λ = 1 year. Furthermore, let us assume that the repair cost is independent of the time before a breakdown and is equal to \$10,000. The nominal interest rate is 10%. The expected present worth by Equation 4.3 is:

$$E(PW) = C \cdot M_t(-r)$$
$$= \$10{,}000\ e^{(e^{-.10}-1)} = \$9{,}092.25.$$

Using $E(PW) = C \cdot e^{-r\mu_t}$, leads to a solution of \$9048.37. This is a .483% error. Obviously, this is a very small error and should not significantly effect the selection process if this alternative was among other similar capital investment alternatives. Now let us assume that the nominal interest is allowed to increase. The percent error can be obtained from:

$$\frac{CM_t(-r) - Ce^{-r\mu t}}{CM_t(-r)} \cdot 100.$$

For the Poisson distribution, the error is

$$\frac{e^{\lambda e^{-r}} - e^{\lambda(1-r)}}{e^{\lambda e^{-r}}} \cdot 100.$$

Table 4.2 illustrates the error as the nominal interest rate increases. The question of what is a significant error is rather academic; however, the error does increase rapidly as one uses nominal interest rates over 10%. The current use of interest rates well in excess of 10% would support the requirement that random variables should be taken into consideration for a present worth analysis.

The variance for the present worth of the $C_{id}T_{is}$ independent model can be derived from

$$V(PW) = E(PW^2) - \left[E(PW)\right]^2$$

$$= C^2\left[E(e^{-rT})^2 - \left[E(e^{-rT})\right]^2\right].$$

The derivation is given in Appendix I (2) and is shown to reduce to:

$$V(PW) = C^2\left[M_t(-2r) - \left[M_t(-r)\right]^2\right].$$

Table 4.2

Error for $C_{id}T_{is}$ Independent Model
(Time Poisson Distributed
With $\lambda = 1$ year)

r	e^{-r}	$e^{(e^{-r}-1)}$	% Error
0	1	1	0
.10	.9048	.9092	.483
.20	.8187	.8342	1.856
.30	.7408	.7717	4.000
.40	.6703	.7192	6.790
.50	.6065	.6747	10.105
.60	.5488	.6369	13.823

In the example problem, the variance for the present worth would be:

$$V(PW) = C^2\left[M_t(-2r) - \left[M_t(-r)\right]^2\right]$$

$$= (\$10,000)^2\left[e^{(e^{-2(.10)}-1)} - \left[e^{(e^{-.10}-1)}\right]^2\right]$$

$$= \$752,043.94.$$

The standard deviation is:

$$\sigma = \$867.20.$$

Without the use of moment generating functions, Poisson tables (PARZEN, 1960: 444) must be used. A series of 17 calculations are required to obtain the expected present worth and a series of 9 calculations are required to obtain the variance.

As stated earlier, one must allow for more than one cash flow (all independent) for a given alternative. For alternatives which involve $C_{id}T_{is}$ model elements exclusively, the expected present worth for the k^{th} alternative is:

$$\sum_{j=1}^{n_k} C_{jk}\, M_{t_{jk}}(-r),$$

where

n_k = number of impulsive, deterministic cash flows for the k^{th} alternative,

C_{jk} = the j^{th} impulsive, deterministic cash flow for the k^{th} alternative,

$M_{t_{jk}}(-r)$ = the moment generating function for the time parameter associated with the j^{th} cash flow of the k^{th} alternative.

The selection model (based on mean present worth) is:

$$\operatorname*{Min}_{k} \sum_{j=1}^{n_k} C_{jk} M_{t_{jk}}(-r).$$

Alderfer and Bierman (1970:341) suggest the logical argument that selection among alternatives with tied mean present worths should be based on which of the tied alternatives has a smaller variance and is used for the tie-breaker in this research. Variance is also discussed in Chapter 7 for making probability statements concerning the present worth of an alternative.

The variance for the present worth of the k^{th} alternative is:

$$V(PW_k) = \sum_{j=1}^{n_k} C_{jk}^{2} \left[M_{t_{jk}}(-2r) - \left[M_{t_{jk}}(-r) \right]^2 \right].$$

In the event that the moment generating function for a distribution does not exist, calculations using Equation 4.2 must be used to find the expected present worth. The variance of the present worth must be found by:

$$V(PW) = C^2 \left(\sum_{t=0}^{\infty} e^{-2rt} P(t) - \left(\sum_{t=0}^{\infty} e^{-rt} P(t) \right)^2 \right).$$

However, this would be a rather unusual calculation in that the moment generating functions for most distributions do exist and are found in a multitude of handbooks and textbooks. The appropriate interrelationships between the Z, Zeta, Laplace transforms, and moment generating functions demonstrated in Chapter 3 allow for a large number of distributions to be handled without lengthy calculations. The remainder of this research assumes that the moment generating function, or an appropriate transform, does exist.

Alternative selection models for the remainder of the model elements are discussed in the final section of this chapter. The introduction of the alternative selection model for $C_{id}T_{is}$ model elements was presented at this point to orient the reader with respect to a decision choice methodology of minimum expected present worth among alternatives.

The next model developed is an extension of the $C_{id}T_{is}$ model for cash flow elements which are impulsive and series in nature.

$C_{sd}T_{ss}$ - Independent Model Element

If one now considers a series type cash flow which is deterministic in nature and has an impulsive, stochastic time of occurrence for each cash flow, the present worth is calculated by:

$$PW = c_1 e^{-rt_1} + c_2 e^{-r(t_1+t_2)} + c_3 e^{-r(t_1+t_2+t_3)} + \ldots + c_n e^{-r(t_1+t_2+t_3+\ldots t_n)} + \ldots \quad .$$

If the c_i's are different in magnitude, then one must solve for the present worth by considering each cash flow as a $C_{id}T_{is}$ model element. However, if the c_i's are a constant function(denoted as C), then the expected present worth can be shown to be equal to:

$$E(PW) = C \frac{M_t(-r)}{1-M_t(-r)} \, , \qquad (4.5)$$

and the corresponding variance can be shown to be equal to:

$$V(PW) = C^2 \frac{\left[M_t(-2r) - \left[M_t(-r)\right]^2 \right]}{\left[1-M_t(-2r)\right] \left[1-M_t(-r)\right]^2} \, . \qquad (4.6)$$

The derivations for Equations 4.5 and 4.6 are quite lengthy and are presented in Appendix I (3).

As an example of the use of Equations 4.5 and 4.6, let one assume that a preventive maintenance program costs \$100. each time maintenance is done. The time between maintenance is assumed to be a function of running time, which in turn is a function of demand, the time between maintenance requirements is assumed to be Poisson distributed with a mean $= \lambda =$ 1 month. The nominal interest rate is 24% (2% per month).

By Equation 4.5, the expected present worth is:

$$E(PW) = \frac{100\ (e^{(e^{-.02}-1)}}{(e^{(e^{-.02}-1)}} = \$5{,}000.33,$$

$$V(PW) = \frac{(100)^2 \left[e^{(e^{-.04}-1)} - \left| e^{(e^{-.02}-1)} \right|^2 \right]}{\left[1 - e^{(e^{-.04}-1)} \right] \left[1 - e^{(e^{-.02}-1)} \right]^2}$$

$$= \$255{,}008.66\ ,$$

and the standard deviation is

$$\sigma = 504.98.$$

Using the traditional formulas (Taylor, 1964:26) with the timing assumed to be deterministic (one month intervals) result in a present worth of $5,000. The insignificant error of 33 cents is expected with the low monthly interest rates; however, having the variance gives one more information for a decision. This is illustrated in Chapter 7.

$C_{is}T_{is}$ - Independent Model Element

When both cash flow and time are impulsive and stochastic, one can calculate the expected present worth (assuming independence) from:

$$PW = C\ e^{-rt}\ ,$$

$$E(PW) = E(C) \cdot E(e^{-rt}). \qquad (4.7)$$

Since C is not limited to being discrete, one must develop formulas for continuous cash flow. Table 4.1

suffices for discrete cash flows by merely substituting $M_c(-r)$ for $M_t(-r)$. Table 4.3 represents the moment generating functions for a few common continuous distributions.

The mean and variance, as developed by Whitehouse (1973:24), for continuous distributions are:

$$E(C) = \mu_c = \left.\frac{dM_c(-r)}{d(-r)}\right|_{-r=0}$$

$$E(C^2) = \left.\frac{d^2\, M_c(-r)}{d(-r)^2}\right|_{-r=0}$$

$$\sigma_c^{\,2} = E\,(C-\mu_c)^2 = E(C^2) - \left[E(C)\right]^2$$

$$= \left.\frac{d^2M_c(-r)}{d(-r)^2} - \left(\frac{dM_c(-r)}{d(-r)}\right)^2\right|_{-r=0}$$

In terms of moment generating functions, Equation 4.7 can be expressed as:

$$E(PW) = \left[\left.\frac{dM_c(-r)}{d(-r)}\right|_{-r=0}\right] M_t(-r)\ . \tag{4.8}$$

Equation 4.8 can be written simply as

$$E(PW) = \mu_c\ M_t(-r)\ . \tag{4.9}$$

The variance for the present worth follows directly from the classical variance of the product of two independent random variables (Duncan, 1965:94):

$$V(PW) = \mu_c^{\,2}\sigma_{e\text{-}rt}^{\,2} + (\mu_{e\text{-}rt})^2\sigma_c^{\,2} + (\sigma_{e\text{-}rt})^2\sigma_c^{\,2}. \tag{4.10}$$

Table 4.3

Continuous Probability Laws and Their Moment Generating Functions

Distribution	Probability Law	$M_c(-r)$
Exponential	$f(t) = ae^{-at}, \; t \geq 0$ $0, \; t < 0$	$\frac{a}{a+r}$
Gamma	$f(t) = \frac{a}{\Gamma(b)}(at)^{b-1}e^{at}, \; t \geq 0$ $0, \; t < 0$	$\left(\frac{a}{a+r}\right)^b$
Normal	$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2},$ $-\infty < t < \infty$	$e^{-r\mu+\frac{1}{2}r^2\sigma_c^2}$
Uniform	$f(t) = \frac{1}{b-a}, \; a < t < b$ 0, otherwise	$\frac{e^{-rb} - e^{-ra}}{-r(b-a)}$

μ_c = Mean cost or revenue

$\mu_{e^{-rt}}$ = Mean of (e^{-rt})

$\sigma_{e^{-rt}}^2$ = Variance of (e^{-rt})

σ_c^2 = Variance of the cost or revenue distribution.

As previously noted,

$$\mu_{e^{-rt}} = E(e^{-rt}) = M_t(-r),$$

and

$$\sigma_{e^{-rt}}^2 = M_t(-2r) - \left[M_t(-r)\right]^2 .$$

Equation 4.10 can now be written as:

$$V(PW) = \mu_c^2 \left[M_t(-2r) - \left[M_t(-r)\right]^2 \right] + (M_t(-r))^2 \sigma_c^2 + \left[M_t(-2r) - \left[M_t(-r)\right]^2\right] \sigma_c^2 . \quad (4.11)$$

Equation 4.11 reduces to:

$$V(PW) = \mu_c^2 \left[M_t(-2r) - (M_t(-r))^2\right] + \left[M_t(-2r)\right] \sigma_c^2 \quad (4.12)$$

Equation 4.9 and 4.12 are used to calculate the mean present worth and variance of present worth for the $C_{is}T_{is}$ model elements.

As an example, let one assume that the cost of a major breakdown is normally distributed with a mean of $10,000 and a standard deviation of $100. Furthermore, the time between major breakdowns is poisson distributed with a mean of 1 year. The nominal interest rate is 10%.

The expected present worth is

$$E(PW) = \mu_c \cdot M_t(-r)$$
$$= 10{,}000 \cdot e^{1(e^{-.10}-1)}$$
$$= \$9092.25 \ .$$

The variance of the present worth is calculated from Equation 4.10 as

$$V(PW) = (\$10{,}000)^2 \left[e^{(e^{-.20}-1)} - \left(e^{(e^{-.10}-1)}\right)^2\right] + \left[e^{(e^{-.20}-1)}\right](100)^2$$
$$= \$760{,}386.05 \ ,$$

and

$$\sigma_{pw} = \$872.00 \ .$$

The expected present worth has not changed from the $C_{id}T_{is}$ independent model; however, the variance has increased due to the cash flow being a random variable.

$C_{ss}T_{ss}$ - Independent Model Element

When the independent parameters of cash flow and time are both series and stochastic, the expected value of the present worth is derived from:

$$PW = C_1 e^{-rt_1} + C_2 e^{-r(t_1+t_2)} + C_3 e^{-r(t_1+t_2+t_3)} + \ldots,$$
$$E(PW) = E(C) \cdot \sum_{j=1}^{\infty} \left[E(e^{-rt})\right]^j .$$

In terms of moment generating functions, the expected present worth can be written as

$$E(PW) = \left(\frac{d\, M_c(s)}{d\, s}\bigg|_{s=0}\right) \cdot \left(\frac{M_t(-r)}{1-M_t(-r)}\right) . \qquad (4.13)$$

Since the mean cash flow element would normally be given rather than its corresponding moment generating function,

$$E(PW) = \mu_c \left[\frac{M_t(-r)}{1-M_t(-r)}\right] \quad . \tag{4.14}$$

The derivation of the variance for the present worth is included in Appendix I(4) and is:

$$V(PW) = \sigma_c^{\,2}\ \frac{M_t(-2r)}{1-M_t(-2r)} + \mu_c^{\,2}\left[\frac{M_t(-2r)-\left[M_t(-r)\right]^2}{\left[1-M_t(-2r)\right]\left[1-M_t(-r)\right]^2}\right]. \tag{4.15}$$

Equation 4.14 and 4.15 are used to calculate the expected present worth and variance of present worth for the $C_{ss}T_{ss}$ model elements.

As an example problem, let us assume that the damage to a bridge during a flood season is exponentially distributed with a mean of $2,800,000 and variance of $200,000. The time between major floods, which cause this damage, is poisson distributed with a mean of 20 years and variance of 10 years. The nominal interest rate is 25%.

By Equation 4.14 the expected present worth is calculated as:

$$E(PW) = \mu_c \ \frac{M_t-(r)}{1-M_t(-r)}$$

$$= 2{,}800{,}000 \ \frac{e^{20(e^{-.25}-1)}}{1-e^{20(e^{-.25}-1)}}$$

$$= \$33{,}969.03 \ .$$

and the variance is calculated using Equation 4.15 as:

$$V(PW) = \$1,932,278,896.$$

The standard deviation is \$43,957.71. Using totally deterministic data, the expected present worth could be calculated from:

$$PW = \mu_c \sum_{n=1}^{\infty} \left[e^{-r(20)}\right]^n = \mu_c \frac{e^{-20r}}{1-e^{-20r}}$$

$$= 2,800,000 \quad \frac{e^{-20(.25)}}{1-e^{-20(.25)}}$$

$$= 2,800,000 \quad (.00678)$$

$$= \$18,994.23 .$$

The error in the expected present worth is:

$$\frac{33,969.03 - 18,994.23}{33,969.03} \times 100 = 44.1\% .$$

$C_{is}T_{cd}$ - Independent Model Element

The cash flow model elements which are impulsive and stochastic, with time being continuous and deterministic, can be found in examples of monthly receipts or revenues where the cash flow per month is expressed as a random variable. The expected present worth is developed by once again looking at:

$$PW = Ce^{-rt}$$

Since t, in this case, is known; then

$$E(PW) = E(C)\ e^{-rt} = \mu_c e^{-rt} \tag{4.16}$$

One should note that this is the first model developed where common techniques of using the mean of a random variable in a deterministic manner yields a correct solution.

Since t is a constant, the variance for the present worth is expressed as:

$$V(PW) = \sigma_c{}^2\ e^{-2rt}\ .$$

$C_{ss}T_{cd}$ - Independent Model Element

When the cash flow is series, stochastic and the time is continuous, deterministic, then the present worth is expressed as:

$$PW = C_1 e^{-rt_1} + C_2 e^{-r(t_1+t_2)} + C_3 e^{-r(t_1+t_2+t_3)}$$
$$+ C_4 e^{-r(t_1+t_2+t_3+t_4)} + \ldots\ldots \tag{4.17}$$

If one considers that each time interval is equal, then Equation 4.17 reduces to

$$PW = \sum_{j=1}^{\infty} C_j\ (e^{-rt})^j\ . \tag{4.18}$$

When one takes the expected value of both sides of Equation 4.18, while realizing that $E(C_j) = E(C_k) = E(C)$ for all j and k, then one has:

$$E(PW) = \sum_{j=1}^{\infty} E(C)\ (e^{-rt})^j \ ,$$

where e^{-rt} is a constant, the expected present worth can then be expressed as:

$$E(PW) = \mu_c \frac{e^{-rt}}{1-e^{-rt}} \ . \qquad (4.19)$$

Letting t = 1, then

$$E(PW) = \mu_c \frac{e^{-r}}{1-e^{-r}} \ ,$$

or

$$E(PW) = \frac{\mu_c}{i} \ ,$$

which is the traditional expression for the present worth of an infinite stream of equal end-of-period cash flows at effective interest i.

If the time periods are not equal, then

$$E(PW) = \mu_c \sum_{j=1}^{\infty} e^{-r\sum_{i=1}^{j} t_i} \ ,$$

and this calculation can be accomplished by finding the mean and variance of the distribution of times and using equations for the $C_{ss}T_{ss}$ model elements. Since the model would then be a $C_{ss}T_{ss}$ model, the $C_{ss}T_{cd}$ model must assume constant, equal time periods.

The variance for the present worth of a $C_{ss}T_{cd}$ model element can be derived in a lengthy manner as was accomplished for the $C_{ss}T_{ss}$ variance, or one can simply note that the moment generating function for a constant is

$$M_t(-r) = e^{-rt}$$

Using the expression for variance of present worth in the $C_{ss}T_{ss}$ model (Equation 4.15), and substituting

$$M_t(-2r) = e^{-2rt} \text{ and } M_t(-r) = e^{-rt} ,$$

leads to the expression for the variance of the present worth for the $C_{ss}T_{cd}$ model element:

$$V(PW) = \sigma_c^2 \frac{e^{-2rt}}{1-e^{-2rt}} + \frac{\mu_c^2 \left[e^{-2rt} - (e^{-rt})^2\right]}{(1-e^{-2rt})(1-e^{-rt})^2} . \qquad (4.20)$$

Equation 4.20 reduces to:

$$V(PW) = \sigma_c^2 \frac{e^{-2rt}}{1-e^{-2rt}} . \qquad (4.21)$$

An example of the $C_{ss}T_{cd}$ model element is dispersments for operating expenses which are received at random points during a month and paid at the end of each month.

$C_{is}T_{cs}$ - Independent Model Element

When the cash flow elements are impulsive and series in nature with the time between occurrences being continuous and stochastic, one should immediately note the parallel with the $C_{is}T_{is}$ model. The only difference is time being treated as a continuous element. When dealing with moment generating functions, the distinction between discrete and continuous time is not required in the model. The expected present worth and variance of present worth are (from Equations 4.9 and 4.12):

$$E(PW) = \mu_c M_t(-r)$$

and

$$V(PW) = \mu_c^{\,2}\left[M_t(-2r) - \left(M_t(-r)\right)^2\right] + M_t(-2r)\cdot\sigma_c^{\,2}$$

respectively. Table 4.3 can be used for the moment generating function for the time variables by simply substituting $M_t(-r)$ for $M_c(-r)$ in the table. When the moment generating functions for a specific time distribution are not available, then one must calculate the expected present worth as:

$$E(PW) = \mu_c \int e^{-rt} f(t)dt \;.$$

The summation sign in the $C_{is}T_{is}$ model has been replaced with an integral sign since one now has a continuous distribution to deal with. A similar substitution is required to calculate variance. Once again, it should be pointed out that this research assumes the distributions of interest (both continuous and discrete) have moment generating functions.

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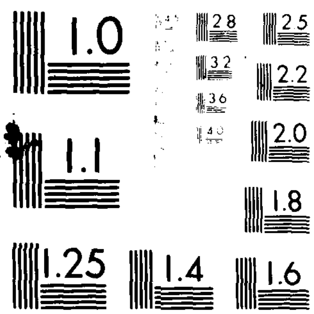

MICROCOPY RESOLUTION TEST CHART

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$C_{ss}T_{cs}$ - Independent Model Element

When cash flow is series and stochastic and time is continuous and stochastic, the model is the same as the one developed for $C_{ss}T_{ss}$ and from Equations 4.14 and 4.15:

$$E(PW) = \mu_c \frac{M_t(-r)}{1-M_t(-r)} ,$$

$$V(PW) = \sigma_c^2 \frac{M_t(-2r)}{1-M_t(-2r)} + \mu_c^2 \left[\frac{M_t(-2r)-\left(M_t(-r)\right)^2}{\left(1-M_t(-2r)\right)\left(1-(M_t-r)\right)^2} \right] .$$

Once again, it is assumed that the moment generating functions for the continuous time distributions exist.

Summary of Independent Model Elements

Table 4.4 is a summary of the independent model elements. Although each model has been derived separately with infinite cash flow streams, one should note that all models are special cases of one of two general models. The $C_{id}T_{is}$, $C_{is}T_{cd}$, and $C_{is}T_{cs}$ are all special cases of the $C_{is}T_{is}$ model. Also $C_{sd}T_{ss}$, $C_{ss}T_{cd}$, and $C_{ss}T_{cs}$ are special cases of the $C_{ss}T_{ss}$ model. The $C_{ss}T_{ss}$ model is extended in the next section for cases where cash flow streams are finite.

Table 4.4

Summary of Independent Models

Model Element	E(PW)	V(PW)
$C_{id}T_{is}$	$C\, M_t(-r)$	$C^2\left[M_t(-2r) - \left[M_t(-r)\right]^2\right]$
$C_{sd}T_{ss}$	$C\, \frac{M_t(-r)}{1-M_t(-r)}$	$C^2\, \frac{M_t(-2r) - \left[M_t(-r)\right]^2}{\left[1-M_t(-2r)\right]\left[1-M_t(-r)\right]^2}$
$C_{is}T_{is}$	$\mu_c\, M_t(-r)$	$\mu_c^2\left[M_t(-2r)-\left[M_t(-r)\right]^2\right] + \sigma_c^2\, M_t(-2r)$
$C_{ss}T_{ss}$	$\mu_c\, \frac{M_t(-r)}{1-M_t(-r)}$	$\sigma_c^2\, \frac{M_t(-2r)}{1-M_t(-2r)} + \mu_c^2\left[\frac{M_t(-2r)-\left[M_t(-r)\right]^2}{\left[1-M_t(-2r)\right]\left[1-M_t(-r)\right]^2}\right]$
$C_{is}T_{cd}$	$\mu_c\, e^{-rt}$	$\sigma_c^2\, e^{-2rt}$
$C_{ss}T_{cd}$	$\mu_c\, \frac{e^{-rt}}{1-e^{-rt}}$	$\sigma_c^2\, \frac{e^{-2rt}}{1-e^{-2rt}}$
$C_{is}T_{cs}$	$\mu_c\, M_t(-r)$	$\mu_c^2\left[M_t(-2r)-\left[M_t(-r)\right]^2\right] + \sigma_c^2\, M_t(-2r)$
$C_{ss}T_{cs}$	$\mu_c\, \frac{M_t(-r)}{1-M_t(-r)}$	$\sigma_c^2\, \frac{M_t(-2r)}{1-M_t(-2r)} + \mu_c^2\left[\frac{M_t(-2r)-\left[M_t(-r)\right]^2}{\left[1-M_t(-2r)\right]\left[1-M_t(-r)\right]^2}\right]$

Extension for Finite Cash Flow Streams

The series, independent model elements assume an infinite number of cash flows. Although these formulas may be fairly accurate when performing analysis on long-life items and/or analysis with high interest rates, these formulas would not be valid for analysis on shorter-life items and/or analysis with low interest rates.

Rather than developing four separate models for finite cash flow streams for the $C_{sd}T_{ss}$, $C_{ss}T_{cd}$, $C_{ss}T_{cs}$, and $C_{ss}T_{ss}$ model elements, only the $C_{ss}T_{ss}$ model element for finite cash flows is developed. This is due to the fact that the remaining three model elements are special cases of this model element.

The present worth for the $C_{ss}T_{ss}$ finite cash flow model elements can be expressed as:

$$
\begin{aligned}
PW &= C_1\, e^{-rt_1} + C_2\, e^{-r(t_1+t_2)} \\
&\quad + C_3\, e^{-r(t_1+t_2+t_3)} + \ldots + C_n\, e^{-r(t_1+t_2+t_3+\ldots t_n)} \\
&= \sum_{j=1}^{n} C_j\, e^{\sum_{i=1}^{j} -rt_i} .
\end{aligned}
$$

Since both the C_j's and t_i's are each independent, identically distributed random variables,

$$
E(PW) = E(C) \cdot \sum_{k=1}^{n} \left[E(e^{-rt})\right]^k
$$

Letting $X = E(e^{-rt})$, the summation term can be expressed as:

$$S_n = \sum_{k=1}^{n} X^k = X + X^2 + X^3 + \ldots X^n \; .$$

Multiply S_n by X and then subtracting from S_n leads to

$$S_n(1-X) = X - X^{n+1} \; .$$

Hence,

$$S_n = \frac{X - X^{n+1}}{1-X} \; .$$

Substituting $X = E(e^{-rt})$ leads to

$$E(PW) = E(C) \cdot \frac{E(e^{-rt}) - \left[E(e^{-rt})\right]^{n+1}}{1-E(e^{-rt})} \; .$$

Using $E(C) = \mu_c$

and

$$E(e^{-rt}) = M_t(-r)$$

leads to the expected present worth for the $C_{ss}T_{ss}$ finite cash flow stream model element:

$$E(PW) = \mu_c \left[\frac{M_t(-r) - \left[M_t(-r)\right]^{n+1}}{1-M_t(-r)} \right] \tag{4.22}$$

To develop the variance of present worth for the model, one must first find an expression for $E(PW^2)$.

$$(PW^2) = C_1{}^2\, e^{-2rt_1} + C_2{}^2\, e^{-2r(t_1+t_2)}$$

$$+\, C_3{}^2\, e^{-2r(t_1+t_2+t_3)} + \ldots + C_n{}^2\, e^{-2r(t_1+t_2+t_3+\ldots t_n)}$$

$$+\, 2C_1C_2\, e^{-2rt_1-rt_2} + 2C_1C_3\, e^{-2rt_1-rt_2-rt_3} + \ldots$$

$$+\, 2C_2C_3\, e^{-2rt_1-2rt_2-rt_3} + \ldots$$

$$+\, 2C_{n-1}C_n\, e^{-2rt_1-2rt_2-2rt_3-\ldots-2rt_{n-1}-rt_n} \quad .$$

Taking the expected value of (PW^2) and using:

$$E(C_i \cdot C_j) = E(C)^2 \text{ for all i and j,}$$

$$E(e^{-rt_i}) = E(e^{-rt_j}) = E(e^{-rt}) = Y,$$

and

$$E(e^{-2rt_i}) = E(e^{-2rt_j}) = E(e^{-2rt}) = X,$$

leads to

$$E(PW^2) = E(C_j{}^2) \sum_{j=1}^{n} X^j + 2\left[E(C)\right]^2 \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} X^k Y^j .$$

Using $V(C) = E(C^2) - \left[E(C)\right]^2,$

leads to

$$E(PW^2) = \left[V(C) + \left|E(C)\right|^2 \right] \sum_{j=1}^{n} X^j + 2\left|E(C)\right|^2 \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} X^k Y^j \ .$$

Now using

$$S_n = \sum_{j=1}^{n} X^j = X + X^2 + X^3 + \ldots + X^n$$

$$S_n(1-X) = X - X^{n+1}$$

$$S_n = \frac{X - X^{n+1}}{1 - X} \text{ ,}$$

and

$$\begin{aligned} G_n = \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} X^k Y^j &= X^1Y^1 + X^1Y^2 + X^1Y^3 + \ldots + X^1Y^{n-1} \\ &+ X^2Y^1 + X^2Y^2 + X^2Y^3 + \ldots + X^2Y^{n-2} \\ &+ X^3Y^1 + X^3Y^2 + X^3Y^3 + \ldots + X^3Y^{n-3} \\ &+ \\ &\vdots \\ &+ X^{n-2}Y + X^{n-2}Y^2 \\ &+ X^{n-1}Y \end{aligned}$$

$$\begin{aligned} \frac{X}{Y} G_n &= X^2 + X^2Y + X^2Y^2 + \ldots + X^2Y^{n-2} \\ &+ X^3 + X^3Y + X^3Y^2 + \ldots + X^3Y^{n-3} \\ &+ X^4 + X^4Y + X^4Y^2 + \ldots + X^4Y^{n-4} \\ &+ \\ &\vdots \\ &+ X^{n-2} + X^{n-2}Y + X^{n-2}Y^2 \\ &+ X^{n-1} + X^{n-1}Y \\ &+ X^n \end{aligned}$$

$$G_n\left(\frac{X}{Y} - 1\right) = X^2 + X^3 + X^4 + \ldots + X^n$$

$$- \left[X^1Y^1 + X^1Y^2 + X^1Y^3 + \ldots + X^1Y^{n-1}\right]$$

$$= \sum_{j=2}^{n} X^j - X \sum_{j=1}^{n-1} Y^j$$

$$G_n = \frac{\sum_{j=2}^{n} X^j - X \sum_{j=1}^{n-1} Y^j}{\left(\frac{X}{Y} - 1\right)} ,$$

leads to

$$E(PW^2) = (V(c) + \left[E(C)\right]^2)\cdot\left(\frac{X - X^{n+1}}{1 - X}\right)$$

$$+ \frac{2\left[E(C)\right]^2 \left[\sum_{j=2}^{n} X^j - X\sum_{j=1}^{n-1} Y^j\right]}{\left(\frac{X}{Y} - 1\right)} .$$

Now,

$$\sum_{j=2}^{n} X^j = \frac{\left[M_t(-2r)\right]^2 - \left[M_t(-2r)\right]^{n+1}}{1 - M_t(-2r)} ,$$

and

$$X \sum_{j=1}^{n-1} Y^j = \frac{M_t(-2r)\ M_t(-r)\left[1 - M_t(-r)\right]^{n-1}}{1 - M_t(-r)}$$

Hence:

$$E(PW^2) = \left[V(c)+\left[E(C)\right]^2\right]\left[\frac{M_t(-2r)-(M_t(-2r))^{n+1}}{1 - M_t(-2r)}\right]$$

$$+\ 2\left[E(C)\right]^2\left[\frac{\dfrac{\left[M_t(-2r)\right]^2 - (M_t(-2r))^{n+1}}{1 - M_t(-2r)}}{\dfrac{M_t(-2r)}{M_t(-r)} - 1}\right]$$

$$-\ 2\left[E(C)\right]^2\left[\frac{\dfrac{M_t(-2r)\ M_t(-r)(1-(M_t(-r))^{n-1})}{1 - M_t(-r)}}{\dfrac{M_t(-2r)}{M_t(-r)} - 1}\right]$$

V(PW) is found by subtracting $\left[E(PW)\right]^2$ and simplifying:

$$V(PW) = (\sigma_c^2+\mu_c^2)\ \frac{M_t(-2r)-\left[M_t(-2r)\right]^{n+1}}{1 - M_t(-2r)}$$

$$+\ 2\mu_c^2 \cdot \frac{\left[M_t(-2r)\right]^2-\left[M_t(-2r)\right]^{n+1}}{1 - M_t(-2r)} \cdot \frac{M_t(-r)}{M_t(-2r)-M_t(-r)}$$

$$-\ 2\mu_c^2\ \frac{M_t(-2r)\left[M_t(-r)\right]^2\left[1-\left[M_t(-r)\right]^{n-1}\right]}{\left[1-M_t(-r)\right]\left[M_t(-2r)-M_t(-r)\right]}$$

$$-\ \mu_c^2\ \frac{\left[M_t(-r)-\left[M_t(-r)\right]^{n+1}\right]^2}{\left[1 - M_t(-r)\right]^2} \qquad (4.23)$$

Expressions for finite cash flow streams for the ramp, decay, and growth are developed later in this chapter.

Dependent Model Elements

Cash flow elements in this section assume dependence with time. The common ramp, decay, and growth functions are used as dependence models. These functions are widely used in economic analysis (ramp functions for maintenance and deterioration, decay functions for startup and learning costs, and growth functions for wearin maintenance costs). Figure 4.1, 4.2, and 4.3 illustrate sample ramp, decay, and growth functions.

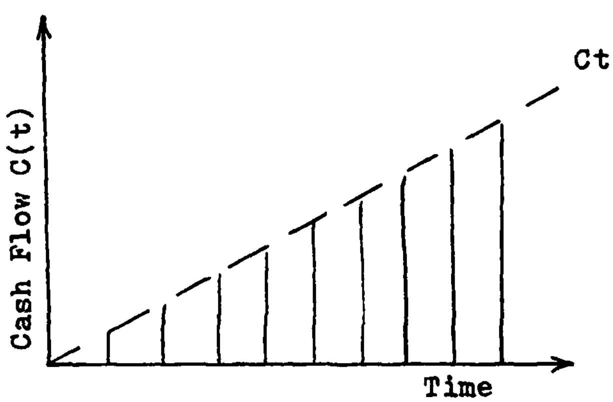


Figure 4.1

Ramp Function

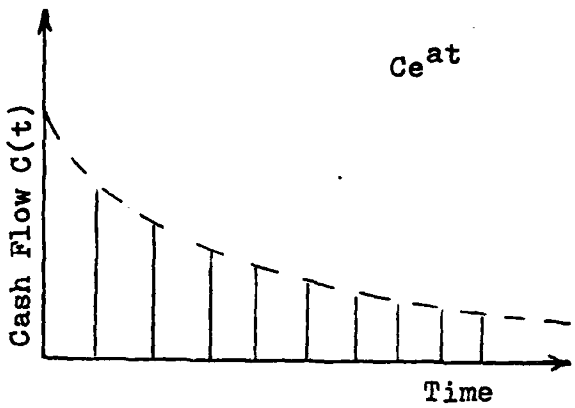


Figure 4.2

Decay Function

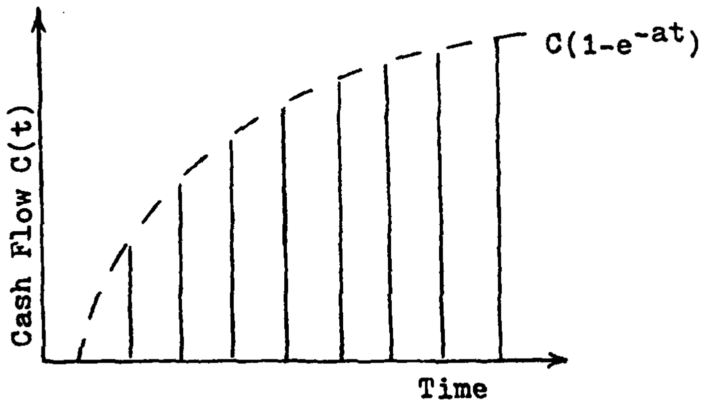


Figure 4.3

Growth Function

$C_{id}T_{is}$ - Dependent Model Elements

The models for dependence between cash flow and time where cash flow is impulsive, deterministic and time is impulsive, stochastic are presented for ramp, decay, and growth functions.

$C_{id}T_{is}$ - Dependent, Ramp Model Element. The present worth for this model is:

$$PW = Ct\ e^{-rt} \tag{4.24}$$

The expected present worth can be stated as:

$$E(PW) = E(C) \cdot E(t\ e^{-rt}). \tag{4.25}$$

In this case, the cash flow (C) is deterministic, and $E(C) = C$. Also,

$$E(t\ e^{-rt}) = E\left[-\frac{d}{dr}\ e^{-rt}\right] = -M_t'(-r).$$

Hence, Equation 4.25 can be reduced to:

$$E(PW) = -CM_t'(-r)$$

for the expected present worth.

To derive the expression for the variance of the $C_{id}T_{is}$ dependent, ramp model element one must first derive an expression for $E(PW^2)$:

$$E(PW^2) = E\ (Cte^{-rt})^2 = E(C^2t^2e^{-2rt})\ . \tag{4.26}$$

However, one notes:

$$M_t(-2r) = E(e^{-2rt}),$$

$$M_t'(-2r) = E(-2te^{-2rt}),$$

and

$$M_t''(-2r) = E(4t^2e^{-2rt}).$$

As a result, Equation 4.26 can be expressed as

$$E(PW^2) = \frac{C^2M_t''(-2r)}{4} \quad ,$$

and the variance for the model can then be expressed as

$$V(PW) = C^2\left[\frac{M_t''(-2r)}{4} - \left|M_t'(-r)\right|^2\right] \quad . \qquad (4.27)$$

$C_{id}T_{is}$ - Dependent, Decay Model Element. The present worth for the $C_{id}T_{is}$ dependent decay model element is:

$$PW = C\ e^{-at}\ e^{-rt}$$

$$= C\ e^{-(a+r)t} \qquad (4.28)$$

Recognizing that

$$E(e^{-(a+r)t}) = M_t\left|-(a+r)\right| \quad ,$$

and taking the expected value of both sides of Equation 4.26 leads to the expected present worth for the model element:

$$E(PW) = C\ M_t\left|-(a+r)\right| \ .$$

The variance for the model element can be derived by noting that

$$PW^2 = \left[C(e^{-(a+r)t})\right]^2$$

$$E(PW^2) = E\left|C^2 e^{-2(a+r)t}\right|$$
$$= C^2 M_t \left|-2(a+r)\right| \quad .$$

Hence,

$$V(PW) = E(PW^2) - \left[E(PW)\right]^2$$
$$= C^2 \left[M_t \left|-2(a+r)\right| - \left[M_t \left|-(a+r)\right|\right]^2 \right]$$

<u>$C_{id}T_{is}$ - Dependent, Growth Model Element.</u> The present worth for the $C_{id}T_{is}$ dependent growth model element is composed of two previously mentioned model elements (step and decay):

$$PW = C\ e^{-rt} - C\ e^{-at}\ e^{-rt} \ .$$

The term $C\ e^{-rt}$ is the step function and $C\ e^{-at}\ e^{-rt}$ is the decay function. When these two terms are subtracted, they become the growth function:

$$PW = C(1-e^{-at})\ e^{-rt} \ . \qquad (4.29)$$

The expected present worth is found by taking the expected value of both sides of Equation 4.29 and solving in terms of moment generating functions:

$$E(PW) = C\left[M_t(-r) - M_t\left[-(a+r)\right]\right]$$

The variance is calculated as previously done by first calculating $E(PW^2)$:

$$PW = C\left[e^{-rt} - e^{-(a+r)t}\right]$$

$$PW^2 = C^2\left[e^{-2rt} - 2e^{-(a+2r)t} + e^{-2(a+r)t}\right]$$

$$E(PW^2) = C^2\left[M_t(-2r) - 2M_t\left[-(a+2r)\right] + M_t\left[-2(a+r)\right]\right].$$

$$V(PW) = E(PW^2) - \left[E(PW)\right]^2$$

$$= C^2\left[M_t(-2r) - 2M_t\left[-(a+2r)\right] + M_t(-2(a+r)) - \left(M_t(-r) - M_t\left[-(a+r)\right]\right)^2\right]$$

Table 4.5 is a summary of the $C_{id}T_{is}$ dependent model elements.

$C_{is}T_{is}$ - Dependent Model Elements

The model for the present worth of the $C_{is}T_{is}$ dependent model elements assumes cash flow as an impulsive, stochastic element (expressed as a function of time). The time parameter for the model elements is assumed to be impulsive, stochastic. The means and variances for the cash flow and time parameters are considered as known and from known distributions.

Table 4.5

$C_{id}T_{is}$ Dependent Model Elements

Model	E(PW)	V(PW)
Ramp	$-C\ M_t'(-r)$	$C^2\left[\frac{M_t''(-2r)}{4} - \left(M_t(-r)\right)^2\right]$
Decay	$C\ M_t\ \left(-(a+r)\right)$	$C^2\left[M_t\ \left(-2(a+r)\right) - \left[M_t\left(-(a+r)\right)\right]^2\right]$
Growth	$C\left[M_t(-r) - M_t(-(a+r))\right]$	$C^2\left[M_t(-2r) - 2M_t(-(a+2r)) + M_t(-2(a+r)) - \left(M_t(-r) - M_t\left(-(a+r)\right)\right)^2\right]$

<u>$C_{is}T_{is}$ - Dependent, Ramp Model Element.</u> The $C_{is}T_{is}$ dependent, ramp model element has a present worth which is expressed as

$$PW = Ct\ e^{-rt},$$

where C and t are both random variables.
The expected present worth is then:

$$E(PW) = E(Ct\ e^{-rt})\ .$$

By the mathematical statement in Equation 4.24 for the $C_{id}T_{is}$ dependent, ramp model element,

$$\begin{aligned} E(PW) &= E(C) \cdot E(t\ e^{-rt}) \\ &= \mu_c \cdot (-M_t'(-r)) \\ &= -\mu_c\ M_t'(-r)\ . \end{aligned}$$

To derive the $E(PW^2)$ term in the variance expression for the $C_{is}T_{is}$ dependent, ramp model element:

$$E(PW^2) = E\left[(Ct\ e^{-rt})^2\right], \tag{4.30}$$

one must first rewrite Equation 4.30 as:

$$E(PW^2) = E(C^2) \cdot E(t^2\ e^{-2rt})\ .$$

By definition of variance,

$$\sigma_c^2 = E(C^2) - \left[E(C)\right]^2 .$$

Also,

$$E(t^2\ e^{-2rt}) = \frac{M_t''(-2r)}{4}\ .$$

Hence

$$E(PW^2) = (\sigma_c^2 + \mu_c^2)\ \frac{M_t''(-2r)}{4} .$$

As a result,

$$V(PW) = E(PW^2) - \left[E(PW)\right]^2$$

$$= (\sigma_c^2 + \mu_c^2)\ \frac{M_t''(-2r)}{4} - \left[\mu_c\ M_t'(-r)\right]^2$$

$$= \mu_c^2 \left[\frac{M_t''(-2r)}{4} - \left|M_t'(-r)\right|^2\right] + \sigma_c^2\ \frac{M_t'(-2r)}{4} .$$

<u>$C_{is}T_{is}$ - Dependent, Decay Model Element.</u> The present value for the $C_{is}T_{is}$ dependent, decay model element is expressed as:

$$PW = C\ e^{-at}\ e^{-rt} ,$$

and the expected present worth is expressed as:

$$E(PW) = E(C) \cdot E\left|e^{-(a+r)t}\right|$$

$$= \mu_c \cdot M_t\left|-(a+r)\right| .$$

The expression for $E(PW^2)$ in the variance for the model element is

$$E(PW^2) = E\left|C^2\ e^{-2(a+r)t}\right|$$

$$= E(C^2) \cdot E\left|e^{-2(a+r)t}\right|$$

$$= (\sigma_c^2 + \mu_c^2) \cdot M_t\left|-2(a+r)\right| .$$

Hence, the variance for the model is

$$V(PW) = (\sigma_c^2 + \mu_c^2) \cdot M_t\left|-2(a+r)\right| - \mu_c^2\left|M_t(-(a+r))\right|^2$$

$$= \mu_c^2\left[M_t\left|-2(a+r)\right| - \left|M_t\left|-(a+r)\right|\right|^2\right] + \sigma_c^2 M_t\left|-2(a+r)\right| .$$

$C_{is}T_{is}$ - Dependent, Growth Model Element. The present worth for the $C_{is}T_{is}$ dependent, growth model element (impulsive, stochastic cash flow and impulsive, stochastic timing) can be represented as:

$$PW = C(1-e^{-at})\, e^{-rt} \; . \tag{4.31}$$

The expected present worth can be calculated by reformulating Equation 4.31:

$$PW = C(e^{-rt} - e^{-(a+r)t}) \; ,$$

and then taking expected values to yield:

$$\begin{aligned} E(PW) &= E(C)\left[E\left(e^{-rt} - e^{-(a+r)t}\right)\right] \\ &= E(C)\left[\,E(e^{-rt}) - E\left(e^{-(a+r)t}\right)\right] \\ &= \mu_c\left[M_t(-r) - M_t\left(-(a+r)\right)\right] \; . \end{aligned} \tag{4.32}$$

Using the standard definition of variance, one must now obtain a term for PW^2,

$$PW^2 = C^2\left[e^{-2rt} - 2e^{-(a+2r)t} + e^{-2(a+r)t}\right] .$$

The expected present worth is then,

$$E(PW^2) = E(C^2)\left[E(e^{-2rt}) - 2E(e^{-(a+2r)t}) + E(e^{-2(a+r)t})\right] .$$

This expression can be transformed into

$$E(PW^2) = (\sigma^2+\mu_c^{\,2})\left[M_t(-2r) - 2M_t(-(a+2r)) + M_t(-2(a+r))\right] .$$

The variance follows as:

$$\begin{aligned} V(PW) = {} & (\sigma^2+\mu_c^{\,2})\left[M_t(-2r) - 2M_t(-(a+2r)) + M_t(-2(a+r))\right] \\ & - \mu_c^{\,2}\left[M_t(-r) - M_t(-(a+r))\right]^2 \; . \end{aligned}$$

Table 4.6 is a summary of the expected present worth and variance of present worth for the $C_{is}T_{is}$ dependent model elements.

Table 4.6

$C_{is}T_{is}$ Dependent Model Elements

Model	E(PW)	V(PW)
Ramp	$-\mu_c M_t'(-r)$	$\mu_c^2\left[\frac{M_t''(-2r)}{4} - [M_t'(-r)]^2\right] + \sigma_c^2 \frac{M_t''(-2r)}{4}$
Decay	$\mu_c M_t(-(a+r))$	$\mu_c^2\left[M_t(-2(a+r)) - [M_t(-(a+r))]^2\right] + \sigma_c^2 M_t(-2(a+r))$
Growth	$\mu_c\left[M_t(-r) - M_t(-(a+r))\right]$	$(\sigma_c^2+\mu_c^2)\left[M_t(-2r)-2M_t(-(a+2r))+M_t(-2(a+r))\right] - \mu_c^2\left[M_t(-r) - M_t(-(a+r))\right]^2$

$C_{is}T_{cd}$ - Dependent Model Elements

The presentation for the $C_{is}T_{cd}$ dependent model elements will follow the same format as presented in the $C_{id}T_{is}$ dependent model elements. In the $C_{is}T_{cd}$ model, the time is continuous, deterministic. Since the cash flow remains impulsive, stochastic; one must only determine the differences between the two models where the time parameter is deterministic.

$C_{is}T_{cd}$ - Dependent, Ramp Model Element. The present worth for the $C_{is}T_{cd}$ dependent, ramp model element is:

$$PW = Ct\ e^{-rt}\ ,$$

where $t\ e^{-rt}$ is a constant.

From Equation 4.25 of the $C_{id}T_{is}$ dependent, ramp model element, the expected present worth of the $C_{is}T_{cd}$ dependent, ramp model element is

$$\begin{aligned} E(PW) &= E(C)\cdot(t\ e^{-rt}) \\ &= \mu_c\ t\ e^{-rt}\ . \end{aligned}$$

Since $t\ e^{-rt}$ is a constant, the variance for the $C_{id}T_{cd}$ dependent, ramp model element is

$$\begin{aligned} V(PW) &= \sigma_c{}^2\ (t\ e^{-rt})^2 \\ &= \sigma_c{}^2\ t^2\ e^{-2rt}\ . \end{aligned}$$

$C_{is}T_{cd}$ - Dependent, Decay Model Element. The present worth for the $C_{is}T_{cd}$ dependent, decay model element is

$$PW = C\ e^{-at}\ e^{-rt}$$
$$= C\ e^{-(a+r)t}\ .$$

The expected present worth is then

$$E(PW) = \mu_c \cdot E\ e^{-(a+r)t}$$
$$= \mu_c \cdot e^{-(a+r)t}\ .$$

Since $e^{-(a+r)t}$ is a constant,

$$V(PW) = \sigma_c{}^2\ e^{-2(a+r)t}\ .$$

$C_{is}T_{cd}$ - Dependent, Growth Model Element. The present worth for the $C_{is}T_{cd}$ dependent, growth model element is

$$PW = C(1-e^{-at})(e^{-rt})$$
$$= C(e^{-rt} - e^{-(a+r)t})\ ,$$

where r, t, and a are constants.

Since C is the only random variable, the expected present worth follows as:

$$E(PW) = E(C) \cdot E(e^{-rt} - e^{-(a+r)t})$$
$$= \mu_c \cdot \left[e^{-rt} - e^{-(a+r)t}\right]\ .$$

The variance for the model is

$$V(PW) = \sigma_c{}^2 \left[e^{-rt} - e^{-(a+r)t}\right]^2$$
$$= \sigma_c{}^2 \left[e^{-2rt} - 2e^{-(a+2r)t} + e^{-2(a+r)t}\right]\ .$$

Table 4.7 is a summary of the expected present worth and variance of present worth for the $C_{is}T_{cd}$ dependent model element.

Table 4.7

$C_{is}T_{cd}$ Dependent Model Elements

Model	E(PW)	V(PW)
Ramp	$\mu_c\ t\ e^{-rt}$	$\sigma_c^{\ 2}\ t^2\ e^{-2rt}$
Decay	$\mu_c\ e^{-(a+r)t}$	$\sigma_c^{\ 2}\ e^{-2(a+r)t}$
Growth	$\mu_c \cdot \left[e^{-rt} - e^{-(a+r)t}\right]$	$\sigma_c^{\ 2}\left[e^{-2rt} - 2e^{-(a+2r)t} + e^{-2(a+r)t}\right]$

$C_{is}T_{cs}$ - Dependent Model Elements

As previously noted in this research, treating time as continuous or discrete does not alter the mathematical development. This was due to the use of moment generating functions which are assumed to be known for both discrete and continuous distributions. The expected present worth and variance of present worth for each of the $C_{is}T_{cs}$ dependent model elements will be the same as for the $C_{is}T_{is}$ dependent model elements.

$C_{is}T_{cs}$ - Dependent, Ramp Model Element. The expected present worth is

$$E(PW) = -\mu_c \, M_t'(-r) \, ,$$

and the variance of present worth is

$$V(PW) = \mu_c^2 \left[\frac{M_t''(-2r)}{4} - \left[M_t'(-r) \right]^2 \right] + \sigma_c^2 \, \frac{M_t''(-2r)}{4} \, .$$

$C_{is}T_{cs}$ - Dependent, Decay Model Element. The expected present worth is

$$E(PW) = \mu_c \cdot M_t(-(a+r)) \, ,$$

and the variance of present worth is

$$V(PW) = \mu_c^2 \left[M_t(-2(a+r)) - \left[M_t(-(a+r)) \right]^2 \right] + \sigma_c^2 \, M_t(-2(a+r)) \, .$$

$C_{is}T_{cs}$ - Dependent, Growth Model Element. The expected present worth is

$$E(PW) = \mu_c \left[M_t(-r) - M_t(-(a+r)) \right],$$

and the variance of present worth is

$$V(PW) = (\sigma_c^2 + \mu_c^2)\left[M_t(-2r) - 2M_t(-(a+r)) + M_t(-2(a+r)) \right]$$
$$- \mu_c^2 \left[M_t(-r) - M_t(-(a+r)) \right]^2 .$$

For consistency in the presentation of models, Table 4.8 contains the $C_{is}T_{cs}$ dependent model elements.

Summary of Impulsive, Dependent Cash Flow Elements

The $C_{id}T_{is}$, $C_{is}T_{cd}$, and $C_{is}T_{cs}$ models presented in this research under the dependent model elements are all special cases of the $C_{is}T_{is}$ dependent model elements. This conclusion parallels the findings for the independent model elements.

The expected present worth will be derived for all of the dependent, series models in the next section. The variance will be derived for the $C_{sd}T_{ss}$ dependent, ramp function.

Table 4.8

$C_{is}T_{cs}$ Dependent Model Elements

Model	E(PW)	V(PW)
Ramp	$-\mu_c M_t'(-r)$	$\mu_c^2\left[\frac{M_t''(-2r)}{4} - \left[M_t'(-r)\right]^2\right] + \sigma_c^2 \cdot \frac{M_t''(-2r)}{4}$
Decay	$\mu_c M_t -(a+r)$	$\mu_c^2\left[M_t(-2(a+r)) - \left[M_t(-(a+r))\right]^2\right] + \sigma_c^2 M_t(-2(a+r))$
Growth	$\mu_c\left[M_t(-r) - M_t(-(a+r))\right]$	$(\sigma_c^2+\mu_c^2)\left[M_t(-2r)-2M_t(-(a+r)) +M_t(-2(a+r))\right] -\mu_c^2\left[M_t(-r)-M_t(-(a+r))\right]^2$

$C_{sd}T_{ss}$ - Dependent Model Elements

The models for dependence between cash flow and time when the cash flow is series, deterministic and the time is series, stochastic are presented in this section for ramp, decay, and growth functions.

$C_{sd}T_{ss}$ - Dependent, Ramp Model Element. The present worth for this model is

$$PW = Ct_1\ e^{-rt_1} + C(t_1+t_2)\ e^{-r(t_1+t_2)}$$
$$+\dots+ C(t_1+t_2+\dots+t_n)\ e^{-r(t_1+t_2+\dots+t_n)}$$
$$+\dots ,$$

where

t_i is a random variable,

C is a constant.

The present worth is reformulated as:

$$PW = Ct_1\ e^{-rt_1}\left[1 + e^{-rt_2} + e^{-r(t_2+t_3)} + e^{-r(t_2+t_3+t_4)} +\dots\right]$$
$$+ Ct_2\ e^{-rt_2}\ e^{-rt_1}\left[1+e^{-rt_3}+e^{-r(t_3+t_4)}+e^{-r(t_2+t_3+t_4)}+\dots\right]$$
$$+ Ct_3e^{-rt_3}e^{-rt_2}e^{-rt_1}\ \left[1+e^{-rt_4}+e^{-r(t_4+t_5)}+e^{-r(t_4+t_5+t_6)}+\dots\right]$$
$$+\dots\ .$$

The expected present worth can now be written as

$$E(PW) = C\ E(te^{-rt}) \sum_{n=0}^{\infty}\left[E(e^{-rt})\right]^n \cdot \sum_{n=0}^{\infty}\left[E(e^{-rt})\right]^n$$

$$= \frac{-CM_t'(-r)}{\left[1-M_t(-r)\right]^2} \tag{4.33}$$

This is proven by noting

$$E(e^{-rt_i}) = E(e^{-rt_j}) \text{ for all } i,j$$

and

$$E(t_i e^{-rt_i}) = E(t_j e^{-rt_j}) \text{ for all } i,j.$$

Analytically, Equation 4.33 is not too difficult to use; however, the expression for the variance of present worth for the $C_{sd}T_{ss}$ dependent, ramp model element is a great deal more detailed. As before, to calculate the variance, one must first generate PW^2 and then calculate $E(PW^2)$. This calculation is contained in Appendix I (5) and the resulting variance for the model is:

$$V(PW) = C^2\left[\left[\frac{M_t''(-2r)}{4\left[1-M_t(-2r)\right]^2} + \frac{M_t'(-2r)^2}{\left[1-M_t(-2r)\right]^3}\right]\left[1+\frac{2M_t(-r)}{1-M_t(-r)}\right] + \frac{M_t'(-2r)M_t(-r)}{\left[1-M_t(-2r)\right]\left[1-M_t(-r)\right]^3} - \frac{\left[M_t'(-r)\right]^2}{\left[1-M_t(-r)\right]^4}\right]. \quad (4.34)$$

Although this variance expression can be further simplified, it is complete enough at this point to draw a conclusion as to the usefulness of such an expression. The original intent of the analytical solution approach was to develop models which were mathematically correct and realistic to use. Without the use of simulation, it would appear that calculating variance for dependent, series models is not of significant value. The remainder of the $C_{sd}T_{ss}$ dependent models and the remainder of the series, dependent models will be limited to the development of expected present worths.

$C_{sd}T_{ss}$ - Dependent, Decay Model Element. The present worth for the $C_{sd}T_{ss}$ dependent, decay model element can be represented as:

$$PW = Ce^{-(a+r)t_1} + Ce^{-(a+r)(t_1+t_2)} + Ce^{-(a+r)(t_1+t_2+t_3)} + \ldots$$

Noting that

$$E(e^{-(a+r)t_i}) = E(e^{-(a+r)t_j}) \text{ for all i and j,}$$

and

$e^{-(a+r)t_i}$ is independent of $e^{-(a+r)t_j}$ for all i and j,

one can express

$$E\left[e^{-(a+r)(t_1+t_2+\ldots t_n)}\right]$$

as

$$E\left[e^{-(a+r)t}\right]^n.$$

Hence:

$$E(PW) = \sum_{n=1}^{\infty} C\left[E(e^{-(a+r)t})\right]^n$$

$$= C\,\frac{E(e^{-(a+r)t})}{1-E(e^{-(a+r)t})}$$

$$= C\,\frac{M_t(-(a+r))}{1-M_t(-(a+r))} \quad .$$

$C_{sd}T_{ss}$ - Dependent, Growth Model Element. The present worth for the $C_{sd}T_{ss}$ dependent, growth model element can be represented as:

$$PW = C\left[e^{-rt} - e^{-(a+r)t}\right] + C\left[e^{-r(t_1+t_2)} - e^{-(a+r)(t_1+t_2)}\right]$$
$$+ C\left[e^{-r(t_1+t_2+t_3)} - e^{-(a+r)(t_1+t_2+t_3)}\right] + \ldots .$$

As pointed out earlier in the research, the growth function is the sum of a step function and a decay function. As a result, the expected present worth can be expressed as:

$$E(PW) = C\sum_{n=1}^{\infty}\left[E(e^{-rt})\right]^n - C\sum_{n=1}^{\infty}\left[E(e^{-(a+r)})\right]^n$$
$$= C\left[\frac{M_t(-r)}{1-M_t(-r)} - \frac{M_t(-(a+r))}{1-M_t(-(a+r))}\right].$$

The expected present worth for the $C_{sd}T_{ss}$, dependent model elements are presented in Table 4.9.

$C_{ss}T_{ss}$ - Dependent Model Elements

The models for dependence between cash flow and time when the cash flow is series, stochastic and the timing of that cash flow is series, stochastic are presented in this section. Only the expected present worth for the ramp, decay, and growth models are derived.

Table 4.9

$C_{sd}T_{ss}$ Dependent Model Elements

Model	E(PW)
Ramp	$\dfrac{-\,C\,M_t'(-r)}{(1-M_t(-r))^2}$
Decay	$\dfrac{C\,M_t(-(a+r)}{1-M_t(-(a+r))}$
Growth	$C\left[\dfrac{M_t(-r)}{1-M_t(-r)} - \dfrac{M_t(-(a+r))}{1-M_t(-(a+r))}\right]$

$C_{ss}T_{ss}$ - Dependent, Ramp Model Element. The present worth for the $C_{ss}T_{ss}$ dependent, ramp model element can be represented as

$$PW = C_1(t_1 e^{-rt_1}) + C_2(t_1+t_2)e^{-r(t_1+t_2)} + C_3(t_1+t_2+t_3)e^{-r(t_1+t_2+t_3)} + \ldots \quad . \qquad (4.35)$$

The expected present worth is derived in Appendix I (6) and is shown to be:

$$E(PW) = \mu_c \left[\frac{-M_t'(-r)}{\left[1-M_t(-r)\right]^2} \right].$$

$C_{ss}T_{ss}$ - Dependent, Decay Model Element. The present worth for the $C_{ss}T_{ss}$ dependent, decay model element is

$$PW = C_1\ e^{-(a+r)t_1} + C_2\ e^{-(a+r)(t_1+t_2)} + \ldots + C_n\ e^{-(a+r)(t_1+t_2+\ldots+t_n)} + \ldots \quad .$$

Since the t_i's and C_i's are independent and identically distributed random variables,

$$E(PW) = \mu_c \sum_{n=1}^{\infty} \left[E(e^{-(a+r)t})\right]^n = \mu_c \frac{M_t(-(a+r))}{1-M_t(-(a+r))} \quad .$$

$C_{ss}T_{ss}$ - Dependent, Growth Model Element. The present worth for the $C_{ss}T_{ss}$ dependent, growth model element is:

$$PW = C_1(1-e^{-at_1})(e^{-rt_1}) + C_2(1-e^{-a(t_1+t_2)})e^{-r(t_1+t_2)} + \ldots$$

$$+ C_n(1-e^{-a(t_1+t_2+\ldots+t_n)})e^{-r(t_1+t_2+\ldots+t_n)} + \ldots$$

$$= C_1(e^{-rt_1}-e^{-(a+r)t_1}) + C_2(e^{-r(t_1+t_2)}-e^{-(a+r)(t_1+t_2)})$$

$$+ \ldots + C_n(e^{-r(t_1+t_2+\ldots+t_n)}-e^{-(a+r)(t_1+t_2+\ldots+t_n)}) + \ldots .$$

Taking expected values, while noting that

$$E(C_i) = E(C_j) = \mu_c$$

and

$$E(e^{-rt_i}) = E(e^{-rt_j}) = M_t(-r) \text{ for all } i \text{ and } j ,$$

leads to

$$E(PW) = \mu_c \left[\sum_{n=1}^{\infty} (M_t(-r))^n - \sum_{n=1}^{\infty} \left[(M_t(-(a+r))\right]^n\right]$$

$$= \mu_c \left[\frac{M_t(-r)}{1-M_t(-r)} - \frac{M_t(-(a+r))}{1-M_t(-(a+r))}\right] .$$

The expected present worth for the $C_{ss}T_{ss}$, dependent model elements are presented in Table 4.10.

$C_{ss}T_{cd}$ - Dependent Model Elements

The models for dependence between cash flow and time when the cash flow is series, stochastic and the time is continuous, deterministic are presented in this section

Table 4.10

$C_{ss}T_{ss}$ Dependent Model Elements

Model	E(PW)
Ramp	$\mu_c \left[\dfrac{-M_t'(-r)}{\left[1-M_t(-r)\right]^2} \right]$
Decay	$\mu_c \left[\dfrac{M_t(-(a+r))}{1-M_t(-(a+r))} \right]$
Growth	$\mu_c \left[\dfrac{M_t(-r)}{1-M_t(-r)} - \dfrac{M_t(-(a+r))}{1-M_t(-(a+r))} \right]$

for ramp, decay, and growth functions. Working from the previous $C_{ss}T_{ss}$, dependent model elements, one need not derive these models directly.

<u>$C_{ss}T_{cd}$ - Dependent, Ramp Model Element.</u> The $C_{ss}T_{cd}$ dependent, ramp model element is indirectly derived from the $C_{ss}T_{ss}$ dependent, ramp model element. Recalling that the expected present worth for the $C_{ss}T_{ss}$ dependent, ramp model element is

$$E(PW) = \mu_c \left[\frac{-M_t'(-r)}{\left[1-M_t(-r)\right]^2} \right] ,$$

one need only recognize that constants do have moment generating functions (e.g. for time as a constant):

$$M_t(-r) = e^{-rt} ,$$

$$-M_t'(-r) = t\ e^{-rt} .$$

Hence, the expected present worth for the $C_{ss}T_{cd}$ dependent, ramp model element is

$$E(PW) = \frac{\mu_c (t\ e^{-rt})}{(1-e^{-rt})^2} .$$

<u>$C_{ss}T_{cd}$ - Dependent, Decay Model Element.</u> The $C_{ss}T_{cd}$ dependent, decay model element is also derived from the $C_{ss}T_{ss}$ dependent, decay model element.

Using time (t) as a constant,

$$M_t(-(a+r)) = e^{-(a+r)t} .$$

Hence, the expected present worth for the $C_{ss}T_{cd}$ dependent, decay model element is

$$E(PW) = \frac{\mu_c \; M_t(-(a+r))}{1-M_t(-(a+r))}$$

$$= \mu_c \frac{e^{-(a+r)t}}{1-e^{-(a+r)t}} .$$

<u>$C_{ss}T_{cd}$ - Dependent, Growth Model Element.</u> The $C_{ss}T_{cd}$ dependent, growth model element is derived from the $C_{ss}T_{ss}$ dependent, growth model element as follows:

$$E(PW) = \mu_c \left[\frac{M_t(-r)}{1-M_t(-r)} - \frac{M_t(-(a+r))}{1-M_t(-(a+r))}\right]$$

$$= \mu_c \left[\frac{e^{-rt}}{1-e^{-rt}} - \frac{e^{-(a+r)t}}{1-e^{-(a+r)t}}\right]$$

The expected present worth for the $C_{ss}T_{cd}$ dependent model elements are presented in Table 4.11.

<u>$C_{ss}T_{cs}$ - Dependent Model Elements</u>

When cash flow is series and stochastic and time is continuous and stochastic, the models are the same as those presented under the $C_{ss}T_{ss}$ dependent model elements (see Table 4.10).

Table 4.11

$C_{ss}T_{cd}$ Dependent Model Elements

Model	E(PW)
Ramp	$\dfrac{\mu_c\ t\ e^{-rt}}{(1-t\ e^{-rt})^2}$
Decay	$\mu_c\ \dfrac{e^{-(a+r)t}}{1-e^{-(a+r)t}}$
Growth	$\mu_c \left[\dfrac{e^{-rt}}{1-e^{-rt}} - \dfrac{e^{-(a+r)t}}{1-e^{-(a+r)t}}\right]$

Summary of Series Dependent Model Elements

The series, dependent model elements are all special cases of the $C_{ss}T_{ss}$ dependent model elements presented in Table 4.10. The $C_{ss}T_{ss}$ dependent model elements are extended in the next section for cases where cash flow streams are finite.

Extension of $C_{ss}T_{ss}$ Dependent Model Elements for Finite Cash Flow Streams

The $C_{ss}T_{ss}$ dependent model element for finite cash flow streams are presented in Table 4.12. The derivation of the models are found in Appendix I (7), (8), and (9).

Expansion of General Model

The general model for a present value analysis was presented in Chapter 3 as:

$$E(PV) = E(P) + E(E) - E(R) - E(S) \,, \tag{4.36}$$

where

E = Expected value operator

P = Purchase price

E = Expense

R = Revenue

S = Salvage.

Table 4.12

$C_{ss}T_{ss}$ Dependent, Finite n Cash Flow Model Elements

Model	Expected Present Worth
Ramp	$-\mu_c M_t'(-r)\left[\dfrac{1+\left[M_t(-r)\right]^n\left[M_t(-r)n-n-1\right]}{\left[1-M_t(-r)\right]^2}\right]$
Decay	$\mu_c\left[\dfrac{M_t(-(a+r)) - \left[M_t(-(a+r))\right]^{n+1}}{1-M_t(-(a+r))}\right]$
Growth	$\mu_c\left[\dfrac{M_t(-r)-\left[M_t(-r)\right]^{n+1}}{1-M_t(-r)} - \dfrac{M_t(-(a+r))-\left[M_t(-(a+r))\right]^{n+1}}{1-M_t(-(a+r))}\right]$

Let us now assume that there are i independent alternatives and each of the alternatives can have up to k model elements associated with P, E, R, and S. The expected present worth of the i^{th} alternative is then:

$$E(PV_i) = \sum_{j=1}^{k} E(P_k) + \sum_{j=1}^{k} E(E_k)$$
$$- \sum_{j=1}^{k} E(R_k) - \sum_{j=1}^{k} E(S_k) \quad .$$

The analyst must now identify the type of cash flow model element and compute the expected present worth for that model element using Tables 4.9 through 4.12. Once this has been accomplished, the expected present worth would be added to the appropriate category of purchase price, expense, revenue, or salvage value. Equation 4.36 can then be used to calculate the expected present value for that alternative.

The variance can be calculated in a similiar manner and is equal to

$$V(PW_i) = \sum_{j=1}^{k} V(P_i) + \sum_{j=1}^{k} V(E_k) + \sum_{j=1}^{k} V(R_k) + \sum_{j=1}^{k} V(S_k) \; .$$

Many additional model element types, not discussed in this research, can also be included in the general model for the calculation of expected present value. The general model for the variance of present value would also be valid as long as the cash flows were considered to be independent. Dependent cash flow elements could be handled with an expanded model as long as time is not a random variable. If time is a random variable, it is not worthwhile to develop a covariance matrix for the cash flows because of the difficulty involved.

The actual selection among alternatives would normally be based upon minimum expected present value. There are two possible cases where variance of present value would enter the selection procedure. First, the expected present values of two alternatives could be either tied or too close for the analyst to be positive which was the better choice. In this case, the alternative with the smaller variance would be selected. The second case is where the variance of the cheapest alternative is greater than the variance of an alternative close to the expected present value of the cheapest alternative. Although more research is required to justify this viewpoint, there is some intuitive appeal to minimize uncertainty even if a slightly higher cost is involved.

The next chapter contains an indepth discussion of the contributions of this research.

Chapter 5

DISCUSSION OF MODELS

This research has extended the present knowledge of capital investment problems in several ways. A detailed taxonomic structure has been developed which is utilized in this research as an outline for model development. The taxonomic structure also serves as a set of designations for new models which should aid in organization, storage, retrieval, and research of these models. Many specific models have also been developed. These models should be useful to the practitioner and of motivational value to structuring realistic classroom presentations. Each of these areas are addressed in this chapter.

Taxonomic Structure

There is a definite requirement for a taxonomic structure for capital investment problems and economic analysis problems. Fleischer and Ward (1977:24) developed such a taxonomic structure and their work is presented in Table 3.7. The intent of their work was three-fold. First, the taxonomic structure serves as a "road map" for locating and selecting appropriate models. Second, the taxonomic structure serves as a method of organization, storage, and retrieval of models. Third, the taxonomy permits enumeration of simpler models. As part of their research effort, a literature search was accomplished to designate which areas of economic analysis had been researched and to determine where additional research was required. Table 3.7 is reproduced in Table 5.1 with additional entries for the contributions of this research. The entries 5a, 5b, 5c, and 5d in the table highlight where the research was accomplished.

The taxonomic structure developed by Fleischer and Ward (1977) was not detailed enough to permit the type of enumeration of models described in this research. Additional detail was added to facilitate the research. A review of Table 3.7 and Figure 3.11 highlights the additions to the taxonomic structure. The cash flow has been subdivided into expenses and revenues. An additional category has been made for independent and dependent models. The impulsive cash flow has been subdivided into those types of cash flows which only occur once (designated as impulsive) and those cash flows which are repeated with some type of frequency (designated as series). Additional models have been developed for series cash flows, both independent and dependent, where the cash flow streams are finite.

Independent Models

Independent models are developed in Chapter 4 with a summary in Table 4.4. The $C_{id}T_{is}$, $C_{is}T_{cd}$, and $C_{is}T_{is}$ models

Table 5.1

Taxonomic Structure and Current State-of-the-Art

Cash Flow				Interest Rate					
Amount		Timing		Constant		Stepwise Constant		Variable	
How	Cert	When	Cert	Det	Sto	Det	Sto	Det	Sto
Impulsive	Det	Disc	Det	Cla	1	Cla	1	Cla	
			Sto	5a					
		Cont	Det	Cla		Cla		Cla	
			Sto	2					
	Sto	Disc	Det	4,3,2	1		1		
			Sto	5b					
		Cont	Det	5c					
			Sto	5d					
Continuous	Det	Disc	Det	Cla		Cla		Cla	
			Sto						
		Cont	Det	Cla		Cla		Cla	
			Sto						
	Sto	Disc	Det	2					
			Sto						
		Cont	Det	2					
			Sto						

1-Reisman and Rao (1973)
2-Ward (1975)
3- Hillier (1969)
4-Hillier (1963)
5-Estes (current research)

Cert-Certainty
Det-Deterministic
Sto-Stochastic
Cla-Classical
Cont-Continuous
Disc-Discrete

Source:

G. A. Fleischer and T. L. Ward. "Classification of Compound Interest Models in Economic Analysis." Engineering Economist, (Fall, 1977), 24.

are shown to be special cases of the $C_{is}T_{is}$ model which assumes that cash flow and timing of cash flow are impulsive and stochastic in nature. The $C_{sd}T_{ss}$, $C_{ss}T_{cd}$, and $C_{ss}T_{cs}$ models are special cases of the $C_{ss}T_{ss}$ model which assumes that the cash flows, and associated timing of the cash flows, are series in nature with stochastic parameters. Expected present worth and variance of present worth are derived for each model. An extension for the $C_{ss}T_{ss}$ model is presented next for models which have a finite cash flow stream. The expected present worth and variance of present worth for the finite cash flow $C_{ss}T_{ss}$ model are presented in Equations 4.22 and 4.23 respectively.

These models are of interest in that they handle data for capital investment where such variables as initial cost, cost and timing of breakdowns, and salvage values must be considered as random variables.

Dependent Models

Dependent models are also developed in Chapter 4. The assumption made in these models is dependence between cash flow and time. The ramp, decay, and growth functions are developed for each of the eight basic model elements previously derived for the independent models. The $C_{is}T_{is}$

models are presented in Table 4.6. The remaining impulsive models are all special cases of this model.

The series, dependent models ($C_{sd}T_{ss}$, $C_{ss}T_{cd}$, $C_{ss}T_{cs}$) are special cases of the $C_{ss}T_{ss}$ dependent models presented in Table 4.10. Variance of present worth is developed for the $C_{sd}T_{ss}$ dependent, ramp model. As can be seen in Equation 4.34, the complexity of the variance expression makes it of little value as an analytical tool. In addition, attempts to derive a general form for the $C_{ss}T_{ss}$ variance were not successful. Expressions for the expected present worth for the $C_{ss}T_{ss}$ finite cash flow dependent model elements are presented in Table 4.12. The development of dependent series models is considered to be of relevance due to the use of ramp functions with maintenance and deterioration costs, decay functions with start up and learning costs, and growth functions with wearin maintenance costs.

Summary

A total of nine independent models are developed with expected present worth and variance of present worth. In addition, twelve dependent models are presented with expected present worth and variance of present worth for impulsive

Table 5.2

Mapping of Current Research to Table 5.1 (5a)

Fleischer and Ward Taxon		Current Research Models Developed	Location in Research
Cash Flow			
Amount-impulsive, deterministic Timing-discrete, stochastic	$C_{id}T_{is}$	Independent Model E(PW), V(PW)	Table 4.4
Interest rate-constant	$C_{sd}T_{ss}$	Independent Model E(PW), V(PW)	Table 4.4
	$C_{id}T_{is}$	Dependent Ramp Model E(PW), V(PW)	Table 4.5
	$C_{id}T_{is}$	Dependent Decay Model E(PW), V(PW)	Table 4.5
	$C_{id}T_{is}$	Dependent Growth Model E(PW), V(PW)	Table 4.5
	$C_{sd}T_{ss}$	Dependent Ramp Model E(PW), V(PW)	Table 4.9, Eqn 4.34
	$C_{sd}T_{ss}$	Dependent Decay Model E(PW)	Table 4.9
	$C_{sd}T_{ss}$	Dependent Growth Model E(PW)	Table 4.9

Table 5.3

Mapping of Current Research to Table 5.1 (5b)

Fleischer and Ward Taxon	Current Research Models Developed	Location in Research
Cash Flow Amount-impulsive, stochastic Timing-discrete, stochastic	$C_{is}T_{is}$ Independent Model E(PW), V(PW)	Table 4.4
Interest rate-constant	$C_{ss}T_{ss}$ Independent Model E(PW), V(PW)	Table 4.4
	$C_{ss}T_{ss}$ Independent, Finite Cash Flow Model E(PW), V(PW)	Equations 4.22, 4.23
	$C_{is}T_{is}$ Dependent Ramp Model E(PW), V(PW)	Table 4.6
	$C_{is}T_{is}$ Dependent Decay Model E(PW), V(PW)	Table 4.6
	$C_{is}T_{is}$ Dependent Growth Model E(PW), V(PW)	Table 4.6

Table 5.3 (continued)

Fleischer and Ward Taxon	Current Research Models Developed	Location in Research
	$C_{ss}T_{ss}$ Dependent Ramp Model E(PW)	Table 4.10
	$C_{ss}T_{ss}$ Dependent Decay Model E(PW)	Table 4.10
	$C_{ss}T_{ss}$ Dependent Growth Model E(PW)	Table 4.10
	$C_{ss}T_{ss}$ Dependent, Finite Cash Flow, Ramp Model E(PW)	Table 4.12
	$C_{ss}T_{ss}$ Dependent, Finite Cash Flow, Decay Model E(PW)	Table 4.12
	$C_{ss}T_{ss}$ Dependent, Finite Cash Flow, Growth Model E(PW)	Table 4.12

Table 5.4

Mapping of Current Research to Table 5.1 (5c)

Fleischer and Ward Taxon	Current Research Models Developed	Location in Research
Cash Flow Amount-impulsive, stochastic Timing-continuous, deterministic	$C_{is}T_{cd}$ Independent Model E(PW), V(PW)	Table 4.4
Interest rate-constant	$C_{ss}T_{cd}$ Independent Model E(PW), V(PW)	Table 4.4
	$C_{is}T_{cd}$ Dependent Ramp Model E(PW), V(PW)	Table 4.7
	$C_{is}T_{cd}$ Dependent Decay Model E(PW), V(PW)	Table 4.7
	$C_{is}T_{cd}$ Dependent Growth Model E(PW), V(PW)	Table 4.7
	$C_{ss}T_{cd}$ Dependent Ramp Model E(PW)	Table 4.11
	$C_{ss}T_{cd}$ Dependent Decay Model E(PW)	Table 4.11
	$C_{ss}T_{cd}$ Dependent Growth Model E(PW)	Table 4.11

Table 5.5

Mapping of Current Research
to Table 5.1 (5d)

Fleischer and Ward Taxon	Current Research Models Developed	Location in Research
Cash Flow Amount-impulsive, stochastic Timing-continuous, stochastic	$C_{is}T_{cs}$ Independent Model E(PW), V(PW)	Table 4.4
Interest rate-constant	$C_{ss}T_{cs}$ Independent Model E(PW), V(PW)	Table 4.4
	$C_{is}T_{cs}$ Dependent Ramp Model E(PW), V(PW)	Table 4.8
	$C_{is}T_{cs}$ Dependent Decay Model E(PW), V(PW)	Table 4.8
	$C_{is}T_{cs}$ Dependent Growth Model E(PW), V(PW)	Table 4.8
	$C_{ss}T_{cs}$ Dependent Ramp Model E(PW)	Table 4.10
	$C_{ss}T_{cs}$ Dependent Decay Model E(PW)	Table 4.10
	$C_{ss}T_{cs}$ Dependent Growth Model E(PW)	Table 4.10

dependent models. For the series, dependent models, only expected present worth is presented for eleven models and the finite cash flow stream model. A total of 36 models have been presented in Chapter 4 with the majority having both expected present worth and the variance of present worth developed.

In order to clarify the specific areas of comparison between the taxonomic structure accomplished by Fleischer and Ward (1977:24) with this research, Table 5.1 should be compared to Tables 5.2 through 5.5.

The next chapter contains a development and justification for a network logic to solving problems which utilize the analytical tools developed in this research.

Chapter 6

NETWORK REPRESENTATION OF A CAPITAL INVESTMENT ALTERNATIVE

All textbooks which discuss alternative selection models for engineering economics develop rate-of-return formulas. These formulas are developed by the use of both mathematical equations and visual depictions of the process. As stated by A. Alan B. Pritsker (1979:1),

> The modeling of a system is made easier if: 1) physical laws are available that pertain to the system; 2) a pictorial or graphical representation can be made of the system; and 3) the variability of system inputs, elements and outputs is manageable.

This research has attempted to improve upon models for capital investment problems by providing useful tools to accomplish the modeling of an economic process. The research is expanded in this chapter to offer suggestions as to how a pictorial representation can be made of the process.

Present Worth Development

Given a future sum E, a continuous interest rate r, and time of occurrence t, the present worth PW is $E\ e^{-rt}$. Figure 6.1 represents a network configuration of the problem. The nodes of the network can be used to generate and collect dollar values. The branches are used to designate time between nodes. The symbol E was chosen to designate an expense.

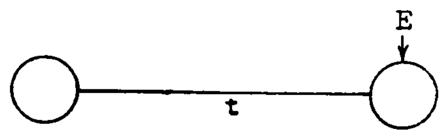


Figure 6.1

Present Worth of a Single Cash Flow

Let us now extend this pictorial network by allowing an initial expense P occurring at t=0 and a series of n expenses (E_1, E_2, E_3,...E_n). Figure 6.2 illustrates this cash flow.

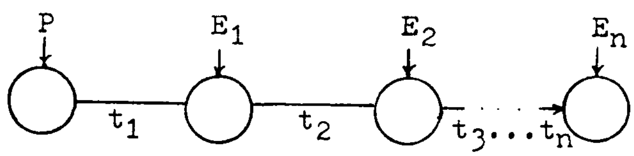


Figure 6.2

Present Worth of a Multiple Cash Flow

The present worth is

$$PW = P + \sum_{i=1}^{n} E_i \, e^{-rt_i} .$$

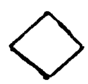
If one were to assume that all of the expenses and the time between expenses are random variables, Figure 6.2 is still valid. Only the method used to calculate the present worth has changed. Obviously for simple cash flows, such as those discussed to this point, a picture is probably not required to aid the analyst in solving the problem at hand. Now let us expand upon the nature of the expenses involved.

The following example problem is presented as a more realistic real-world problem: A manufacturing firm has decided to expand their plant to produce a new product. A lathe is required in one area of the production line. Several lathes which have passed an initial screening to insure that they can meet or exceed minimum production standards have been suggested and are now ready for economic evaluation. The data for the first lathe to be considered has been accumulated from the producer of the lathe and in-house estimates. The initial cost of the lathe is $50,000. The installation cost is dependent upon suspected electrical problems and is estimated to be from a triangular distribution with a mean of $10,000 and the endpoints of $2,000 and $15,000. The salvage value for the lathe is normally distributed with a mean of $5,000 and a variance of $750. This particular lathe has been extensively used in industrial applications and Table 6.1 is a summary of the repair data provided by the manufacturer. Time to failure is normally distributed with a mean of 60 days and a variance of 10 days. The cost of down-time is a constant $500 per day. Monthly operational costs for preventive maintenance and electrical consumption is $600. The compounding period is daily and the rate of return is 20%. Service life is 5 years.

Figure 6.3 represents a possible network type configuration for the problem. The nodes, designated as

k

(where k stands for the node number) is where any expenses generated at that time are introduced. Transactions passing thru the network would collect present worth for all expenses up to that point in time. The symbol



is used as a decision node to designate that some parameters must be tested prior to determination of branching. The use of a triangle on the output portion of a node is used to designate probabilistic branching. The absence of a time specification on a branch assumes that zero time is associated with that branch. The actual present worth for

Table 6.1

Repair Data for Lathe 1

Type of Failure	Probability	Cost ($)	Probability of Repair	Down-time (days)
Major	.1	N(1K, 100)	.90	4
Normal	.5	U(100, 500)	.75	2
Minor	.4	E(100)	.50	1

Legend:

N is normal distribution

U is uniform distribution

E is exponential distribution

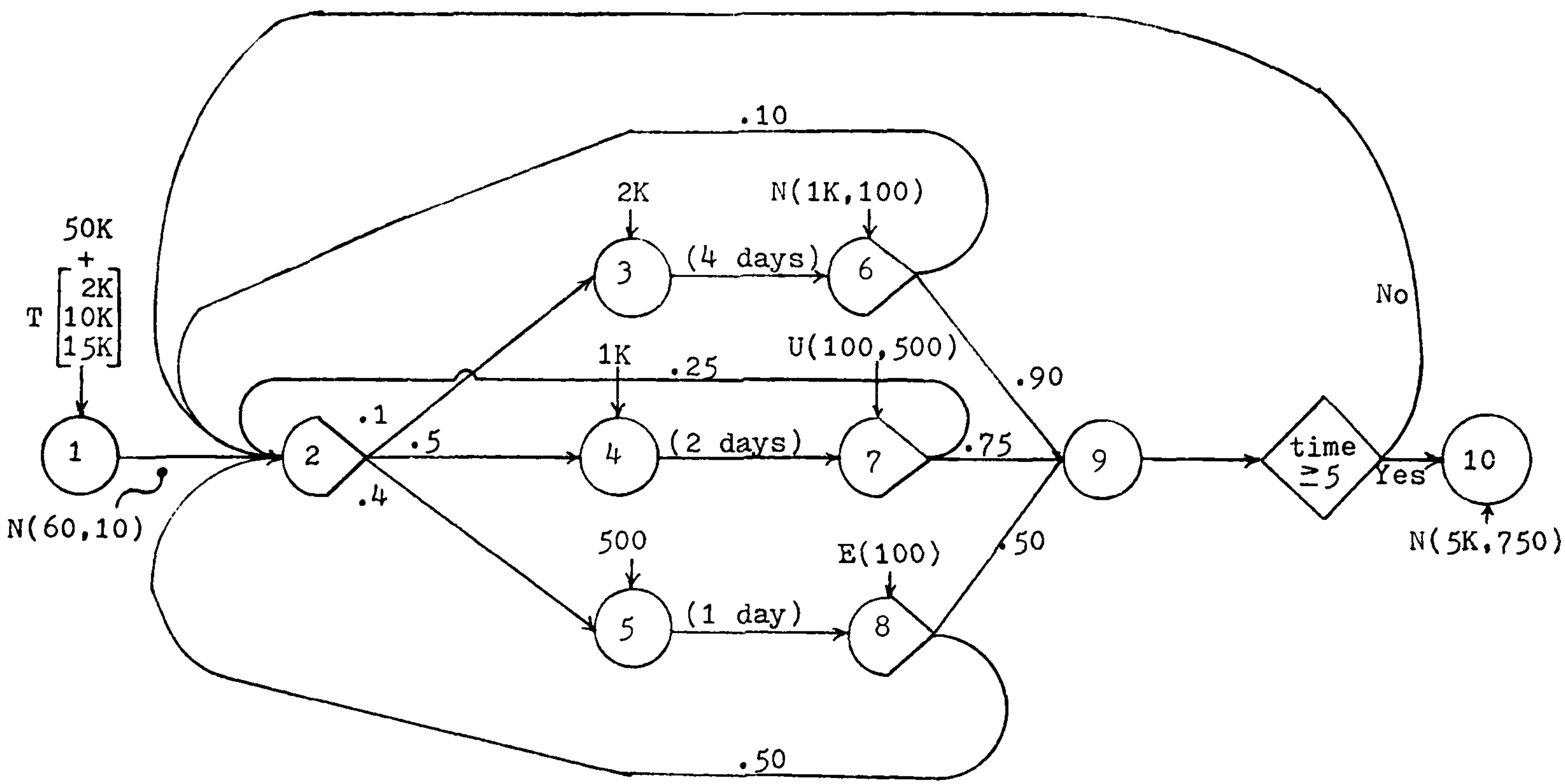


Figure 6.3

Network for Repair Evaluation

one realization of the network could be carried along with cash transaction by an attribute.

The approach suggested to solve this problem is by using simulation; however, many analytical tools (including those introduced in this research) could be used as actual nodes in the network.

The example problem could be extended further to approach the real-world more closely by adding taxes, revenues, depreciation, etc. As the complexity of the problem increases, the need for a pictorial representation is justified.

Preliminary research indicates that QGERTS, developed by A. Alan B. Pritsker is adaptable to performing the analysis. It would be a major undertaking, requiring extensive re-writing and additions to the current program; however, it is suggested in Chapter 8 of this work that this task be undertaken. Other possible simulation programs which appear to have potential are SLAM and GPSS.

Summary

The intent of this chapter is not to design a simulation language to be used in solving capital investment problems. The chapter does layout a network procedure to use in solving problems. The next chapter introduces some applications using the results of this research and the network logic will be used as an illustrative tool.

Chapter 7

APPLICATIONS FOR RESEARCH MODELS

There are three basic areas where this research is useful. The first area is in adding more realistic analytical tools to those already used in traditional or classical engineering economic problems, the second area is in the validation and verification of simulation models, and the third area is in the direct use of this research in the development of a simulation language for engineering economics. Each of these areas are addressed in this chapter.

Extensions to Traditional Engineering Economic Problems

This section of applications is concerned with adding this research to the set of analytical tools used to solve traditional engineering economic problems. Areas covered are capitalized cost comparisons, replacement economy, and lease or buy decisions. The research is not limited in application to these areas; however, these areas were selected to show the value of the research in a traditional type problem.

Capitalized-Cost Comparisons

Capitalized-cost comparisons are present worth analysis where the comparison period is assumed to be infinitely long. Taylor (1964: 95) points out that capitalized cost predominated in the early days of railroad construction and expansion due to the viewpoint that roadbeds, tracks, and bridges were considered to have the "mathematical equivalent of perpetual lives". This is due to the very small error introduced in using discounting factors with infinite lives versus long life. This error is decreased even more with higher interest rates. Capitalized costs comparisons are valid for many analysis problems such as tunnels, dams, aqueducts, bridges, and interstate highways. A few typical analysis problems are illustrated in the following examples.

Example 7.1 The following example is modified from Taylor (1964: 95, ex. 7.8).

A dam costing $100,000 to construct will cost $15,500 a year to operate and maintain. Another design costing $150,000 to build will cost $10,000 a year to operate and maintain. Both installations are felt to be permanent. The annual dispersments are assumed to be lump sum, year end payments and the interest is continuous at 25%.

Taylor's solution would be to calculate the present worth of the first design as:

$$\$100,000 + \$15,500 \cdot \frac{1}{e^{.25}-1} = \$154,572.58 \ ,$$

and the second design as:

$$\$120,000 + \$10,000 \cdot \frac{1}{e^{.25}-1} = \$155,208.12$$

The first design is, in terms of present dollars, \$635.54 better. Now, let one assume that the annual dispersments are dependent upon funding by the government and that this funding is normally distributed with a mean of one year and a variance of 6 months. The first cost of both designs is assumed to be, as before, a constant.

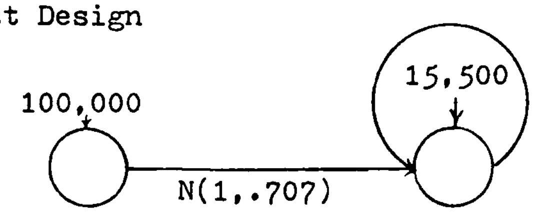
Figure 7.1 illustrates a network for the decision process. The $C_{sd}T_{ss}$ model element is used to calculate the present worth of the first design as:

$$\$100,000 + \$15,500 \left[\frac{M_t(-.25)}{1-M_t(-.25)}\right] .$$

$M_t(-.25)$ for the normal distribution with mean 1 and variance of .5 is $e^{(-.25)+\frac{1}{2}(-.25)^2\,(.5)}$. Hence, the first design will have a present worth of

$$\$100,000 + \$15,500 \ \frac{e^{-.25+\frac{1}{2}(-.25)^2(.5)}}{1-e^{-.25+\frac{1}{2}(-.25)^2(.5)}} = \$158,685.79$$

First Design



Second Design

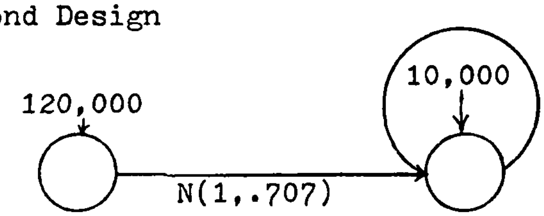


Figure 7.1

Dam Construction

and the second design will have a present worth of

$$\$120{,}000 + \$10{,}000 \frac{e^{-.25+\frac{1}{2}(-.25)^2(.5)}}{1-e^{-.25+\frac{1}{2}(-.25)^2(.5)}} = \$157{,}861.80 \ .$$

Considering the funding as taking place at random points, the advantage has changed to the second design. The wrong alternative has been selected by using deterministic data. In addition, if one assumes that the expected true cost of the first design is \$158,685.79, using the estimated cost of \$154,572.58 yields an error of \$4113.21. Obviously, this data was selected to make the point that both incorrect selections and incorrect estimates can be made when data is used in a deterministic manner even though the data is known to be random variables.

The variance for the present worth is found by using only the variances of the $C_{sd}T_{ss}$ model element.

$$V(PW) = C^2 \frac{\left[M_t(-2r)-(M_t(-r))^2\right]}{\left[1-M_t(-2r)\right]\left[1-M_t(-r)\right]^2}$$

where C = \$10,000

$$M_t(-2r) = e^{-2(.25)+2(-.25)^2(.5)}$$

$$= .6456$$

$$M_t(-r) = e^{-(.25)+\frac{1}{2}(-.25)^2(.5)}$$

$$= .7911$$

$$V(PW) = \$128{,}416{,}795.60$$

$$\sigma_{pw} = \$11{,}332.11$$

It should be noted that in finding the expected value and variance of the cash flow a specific distribution of cash flow has been assumed. As a result, the distribution of the present value can be assumed to be normally distributed. Since the variance of the present value is known, more information is known than by the first method.

To illustrate this, let us assume that the analyst used the data in a deterministic manner and selected the first design. Since the true expected present worth of the first design is \$158,685.79, one question might be, what is the probability that the actual cost will be less than or equal to the present worth of \$154,572.58 calculated by the analyst? Assuming present worth is normally distributed with mean \$158,685.79 and variance of \$128,416,795.60,

$$z_{\alpha} = \frac{X-\mu}{\sigma} = \frac{154,572.58 - 158,685.79}{11,332.11} \simeq -.36.$$

Hence, from normal tables (Miller and Freund, 1965:398) $\alpha \simeq .359$. As a result, there is only a 35.9 percent chance that the first design will have an actual cost less than or equal to the cost estimated.

In many applications of this research, multiple cash flows will be seen in the models. As a result, it is possible that no distributional assumptions concerning the random variable of present worth can be made. In such cases it is still possible for certain probability statements to be made via the Chebyshev inequality. Using the Chebyshev inequality for this example, the probability of realizing an actual cost within two standard deviations from the mean is

$$P\ (\,|PW-E(PW)| < 2\,\sigma\,) > 1-\frac{1}{k^2} = 75\%.$$

Another way of stating this is that the probability of being within 2σ($22,664.22) of the expected present worth is greater than 75%.

The strength of Chebyshev's theorem is that it only needs the mean and standard deviation of the distribution. It is also its greatest weakness in that it only provides an upper bound for the probability.

Other tests could be made if we made distributional assumptions; however, the actual distribution is not available to test our assumptions. The suggestion is to then generate the distribution via computer simulation, validate the expected present worth and variance of present worth via the analytical tools given in this research, and then make probability statements after analysis of the distribution of present worth.

The next example is a replacement problem first introduced by Grant (1938:208). The problem is revised to introduce random variables into the problem.

Replacement Economy

In replacement economy, an in-service piece of equipment is considered for either replacement or improvement. The factors which govern the need for this analysis is obsolescence, inadequate capacity, deterioration, improved equipment availability, etc.

The following example is a decision process in extending the life of an item versus immediate replacement.

Example 7.2 A wooden telephone pole has decayed to the point that the pole must be either removed or "stubbed". A stub costs $8.00 with an installation cost which is exponentially distributed with a mean of $20.00. The inspector notes that the upper section of the pole should be good for about 5 years. He estimates that this estimate is normally distributed with a mean of 5 years and a standard deviation of 1 year. A new pole will cost $25.00 with an installation cost which is exponentially distributed with a mean of $40.00. A new pole has a normally distributed life of 20 years with a standard deviation of 4 years. The company has an existing contract to buy all used telephone poles at the rate of $5.00 per pole. The minimum required rate of return is 23 percent.

Figure 7.2 illustrates the decision process. Several model elements are used in the analysis. The initial procurement costs of the stub or new pole is a constant and is simply added to the present worth. To calculate present worth due to installation costs and the salvage value, two models presented in this research are required. Calculating present worth due to installation costs will require the use of the $C_{is}T_{cd}$ model element. The $C_{id}T_{is}$ model element is required to calculate the present worth due to the salvage value.

The expected present worth of the stubbed pole is:

$$E(PW_S) = 8 + 5 + 20(1) - 5(e^{-.23(5)+\frac{1}{2}(-.23)^2(1)^2})$$
$$= \$31.37 \ .$$

and the expected present worth of the new pole is:

$$E(PW_N) = 25 + 40(1) - 5(e^{-.23(20)+\frac{1}{2}(-.23)^2(4)^2}$$
$$= \$64.92 \ .$$

It will apparently pay to stub the old pole. The difference in expected value was really determined by the high cost of installing a new pole. Using the means of this data in a totally deterministic manner would result in the same conclusion with very close figures to those calculated when using the data as random variables. The

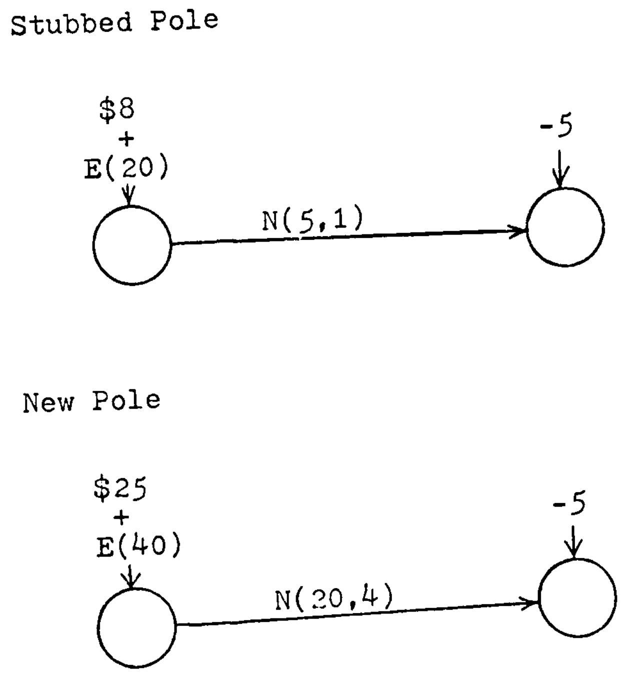


Figure 7.2

Pole Replacement vs. Stubbing

main advantage in using random variables, in this case, is that one can now calculate a variance and standard deviation for the estimated present worth of the stubbed pole:

$$V(PW_S) = (20)^2 + 5^2\left[e^{-2(.23)5+\frac{1}{2}(-2(.23))^2(1)^2}\right]$$

$$- \left[e^{-(.23)5+\frac{1}{2}(-.23)^2(1)^2}\right]^2$$

$$= 400.14$$

$$PW_S = \$20.00$$

Having the standard deviation does give one a great deal more information. If one uses deterministic data, the present worth of the stubbed pole is $31.42. In this case, the advantage in using this research is in being able to obtain variance and standard deviation data.

The next example is a lease or own problem taken from Taylor (1964:210). The problem is revised to introduce random variables.

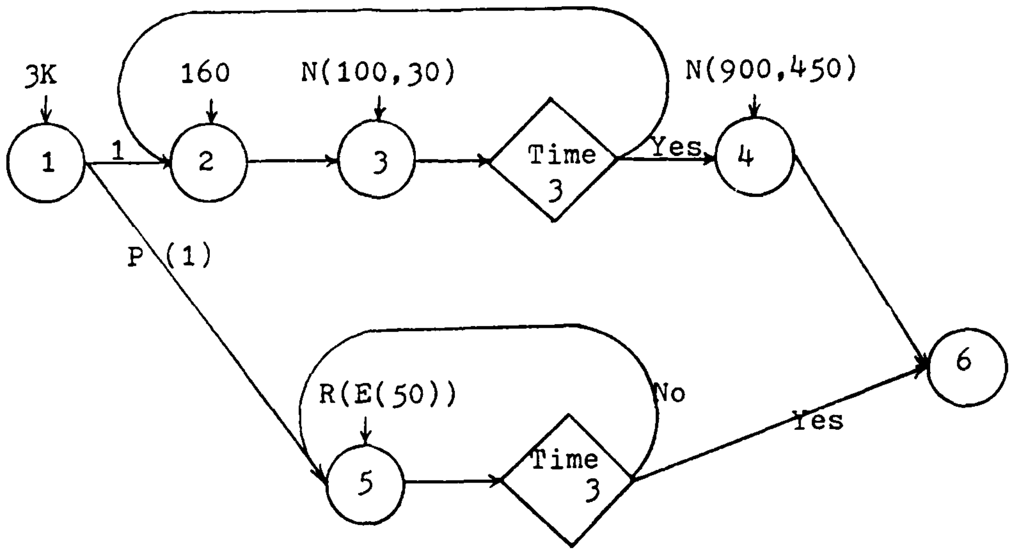
Lease or Buy

To own or lease equipment is a common problem in economic analysis. The obvious advantages of leasing is in avoiding many of the costs of ownership such as obsolescence, repair maintenance, and replacement. The following example is applied to machinery; however, the concepts are applicable to many investments in capital equipment.

Example 7.3 A 2-year-old truck has a net realizable value of $3,000 and is expected to have a salvage value of $900 after its remaining 3-year life. The actual salvage value is normally distributed N(900,450). Its operating disbursements for taxes, insurance, and registration are $160 a year. Annual inspection, maintenance, and repair costs are estimated to be composed of two costs. The first cost is for preventive maintenance and is estimated to be normally distributed with a mean of $100 and variance of $900. The second cost is a ramp function which is expected to increase at a rate of $50 per year. The $50 rate is exponentially distributed with a mean of $50. This second cost is primarily due to breakdowns with the seriousness of each breakdown increasing with time. The time between these breakdowns is poisson distributed with a mean of one year. An equivalent truck can be leased for 20 cents a mile plus $15 a day for every day that the customer keeps it, whether it is driven or not. The annual utilization cost, based upon past records, is expected to be normally distributed with a mean of $1,050 and a standard deviation of $500. Minimum required rate of return is 15%.

Figure 7.3 illustrates the network for owning the truck and the network for leasing a truck. At node 2, in both networks, the variable N is increased by the constant 1

Owned Truck



Leased Truck

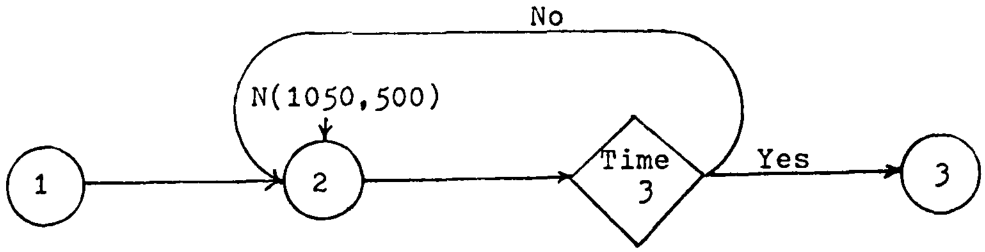


Figure 7.3

Owned vs. Leased Truck

to designate the period of "current" analysis. The input of "R(E(50))" at node 5 designates a ramp function with a cost parameter which is exponentially distributed with a mean of $50. The method of branching is tested after node 3 and 5 in the "owned truck" illustration and after node 2 in the "leased truck" illustration. This test is on the "current" time for the alternative. The actual network arrangement is discussed in more detail under Suggestions for Further Research in the next chapter.

The analysis for expected present worth for the owned truck is:

$$E(PW_0) = 3{,}000 + (100+160)\left[\frac{M_{t1}(-r) - \left[M_{t1}(-r)\right]^4}{1-M_{t1}(-r)}\right]$$

$$+ 50\left[-M'_{t2}(-r)\left[\frac{1+M_{t2}(-r)^3\left[(M_{t2}(-r))3-4\right]}{\left[1-M_{t2}(-r)\right]^2}\right]\right]$$

$$- 900\left[M_{t3}(-r)\right],$$

where $M_{t1}(-r) = e^{-.15}$

$$M_{t2}(-r) = e^{(e^{-.15}-1)}$$

$$M'_{t2}(-r) = -e^{(e^{-.15}-1-.15)}$$

$$M_{t3}(-r) = e^{-.45}\ .$$

The resulting calculation yields a present worth of \$3,195.90. The analysis for the expected present worth for the leased truck is

$$E(PW_L) = 1050 \left[\frac{M_t(-r) - \left[M_t(-r)\right]^4}{1-M_t(-r)} \right],$$

where $M_t(-r) = e^{-.15}$.

The resulting calculation yields a present worth of \$2,351.11. In this case, present worth is dealing with costs and the leased truck is the cheaper alternative.

These examples have shown the usefulness of this research in extending traditional or classical engineering economic problems. These are two primary benefits that are seen by the author in this effort. First, it is rare that one can forecast costs or profits with certainty. This research enables one to use a more realistic approach to using the known data in a problem. The second benefit is to the student in that there should be an increased motivation in dealing with analytical tools which approach real-world problems with fewer assumptions and/or simplifications.

The next area of discussion for the use of this research material is for the validation and verification of simulation models.

Validation and Verification of Simulation Models

One of the drawbacks in attempting to solve real-world problems is the complexity of the available mathematical tools. Another drawback is that appropriate mathematical tools may not be available. An approach which can be used to address both of these problems is simulation. Also, simulation of an alternative can produce results beyond the analytical calculations of mean and variance. As pointed out earilier, using Chebyshev's inequality to make probability statements can at best produce weak probability statements. Making other assumptions such as normality will produce errors if the present worth distribution is abnormally skewed or peaked. In using simulation, the actual distribution of present worth is available to make probability statements.

To have any faith in the output results of a simulation program, one must be able to validate and verify the model. The analytical tools presented in this research should aid the model builder in both of these efforts by allowing validation and verification of expected value and variance data.

The next area where this research could be put to practical use is in the development of a simulation language for engineering economic problems.

Development of a Simulation Language

Simulation is a relatively new technique in solving capital investment problems. Hess and Quigley (1963) were among the first to use Monte Carlo simulation techniques for the construction of output distributions for present worth. Other authors which have used simulation include Hertz (1964), Bussey and Stevens (1971), and Whitehouse (1974).

When dealing with series type cash flows, there are two basic approaches which are used in the literature. One approach is rather brute force in repeating a node for the number of times required for the series. An alternate method would be to "loop" back to the node. Both of these methods are time consuming as far as computer time is concerned. An alternate approach using this research is to use a single node to calculate the present worth of a series of cash flows. As an example, let us assume that both the cash flows and the time between cash flows are random variables. For a specific run of the simulation program, which would generate one value for present worth, two random number generators take a sample from the cash flow and time distributions. The present worth for this specific realization would then be:

$$PW = C \sum_{t=1}^{n} e^{-rt} .$$

The variable n represents the total number of cash flows allowed, and it could be determined by dividing the economic life by the time parameter. Since this would generate a real number in most cases, a simple procedure would be to truncate n to an integer. For example, if the time parameter was .3 years and the economic life was 10 years, n could be 33. Repeated runs through this node would produce the same expected value of present worth and variance of present worth as calculated using formulas from this research. Using this type of a node will allow the analyst to validate segments of the model. Since the actual distribution of present worth would not be available, the actual use of these nodes would be in building a model. Once the model is "correct", these nodes would be replaced with appropriate nodes to collect data.

In this chapter the main objective was to examine applications for this research. Two basic areas were discussed, one area being the direct use of this research to analyze problems and the other area being the indirect use of this research in additional research areas. The next chapter is a summary of this research and includes suggestions for further research.

Chapter 8

SUMMARY, SUGGESTIONS FOR FURTHER RESEARCH, AND CONCLUSION

The objective of this research was to develop analytical tools to solve specific classes of capital investment problems in engineering economics involving mutually exclusive alternatives. A total of 16 basic models were developed for impulsive and series cash flow model elements under the assumption of deterministic, constant interest rates. A review of Table 3.7 indicates a great deal additional research is required. This chapter summarizes the research and makes suggestions for further research.

Summary

The first phase of this research was to develop a detailed classification for capital investment models. A spanning set of models found in the literature were reviewed to determine types of models and required parameters. Next, a taxonomic structure for capital investment problems was developed. This area of the research extended Fleischer and Ward's work (1977) to include a distinction between models involving one-time cash flows (impulsive) and series cash flows. A further extension in the taxonomic structure was made by looking at independent and dependent models. A short form classification was also developed in Chapter 3. A total of sixteen basic models were developed for impulsive and series cash flow model elements under the assumption of deterministic, constant interest rates (see Tables 4.4-4.11). Dependent model elements were extended to allow for a finite number of cash flows (see Table 4.12). In addition, a general model for analysis was developed in Chapter 3 and expanded in Chapter 4. Chapter 5 is an indepth discussion of the contribution of this research. Chapter 6 introduced a network logic to modeling economic problems and Chapter 7 discussed application of the research.

This research has highlighted various areas where additional research is required. The next section of this chapter is a review of these suggestions and the introduction of other research areas not covered in this research.

Suggestions for Further Research

Although a great deal of work has been accomplished by many authors in the areas of simulation, risk analysis, utility theory, etc., a large void still exists for many of the basic models which are required in analysis of economic problems. Further research is required in the development of these models and the use of these models.

Non-constant Interest Rate Models

As pointed out in this research, with the exception of Reisman and Rao (1973), stochastic interest rate models have not been developed. The argument for considering the interest rate as a constant is founded upon the concept of a minimum attractive rate-of-return. Currently, interest rates are changing rapidly. There is reason to believe that management may not be able to either determine a minimum attractive rate-of-return or must use a very high rate to cover risk in the analysis process.

It would seem that an area for additional research would be to develop models for stepwise constant and variable interest rates with stochastic parameters. As models become more and more complex, simulation becomes an important tool; however, analytical tools are still required to validate and verify the simulation output.

Alternate Decision Criteria

The decision as to the best alternative has been assumed to be based upon the alternative with the minimum expected present worth. Other decision criteria need to be researched. Annual cost analysis is commonly found in economic analysis. It would not be too difficult to extend this research to introduce this decision criterion.

Other criteria such as the aspiration level principle and the most probable future principle should also be explored for use in the analysis of data generated by the use of the analytical tools developed in this research.

Introduction of Additional Parameters

The basic parameters used in this research were first cost (P), revenue (R), expense (E), and salvage value (S). Other parameters such as costs due to taxes and profits due to depreciation could also be introduced. One method to introduce these parameters is to look at expenses and costs which appear in a given year and calculate an expected net cash flow and variance of net cash flow for each year. These data calculations could then be used to calculate the expected present worth for the alternatives. The data calculations would have to be handled as n impulsive cash flows. The suggested formulas for calculating expected net cash flow and variance for net cash flow are:

$$E(NCF_1) = (1-\alpha)\left(\sum_{i=1}^{j} R_j - \sum_{i=1}^{k} E_k\right) + \alpha(D)$$

and

$$V(NCF_1) = (1-\alpha)^2 \left(\sum_{i=1}^{j} V(R_j) - \sum_{i=1}^{k} V(E_k)\right) + \alpha^2 V(D)$$

where:

$E(NCF)$ = expanded value of net cash flow in year 1

α = tax rate

D = depreciation

$V(NCF_1)$ = variance of net cash flow in year 1.

Simulation Model

Several areas have been pointed out in this research where simulation could be used as an aid in the analysis of an economic problem. The complexity of many engineering economic analysis problems appear to require the use of simulation.

A suggested computer flow diagram is illustrated in Figure 8.1. Blocks such as "compute a gross revenue" and "compute a gross expense" could contain complex networks. If discrete time units are used in the simulation, the dependence between different cash flows could be introduced into the program. The blocks such as "generate a set of initial costs" and "generate a salvage value" are areas where impulsive models developed in this research could be useful. The block titled "perform analysis" would contain programs to evaluate the data under selected criteria and print out selected information.

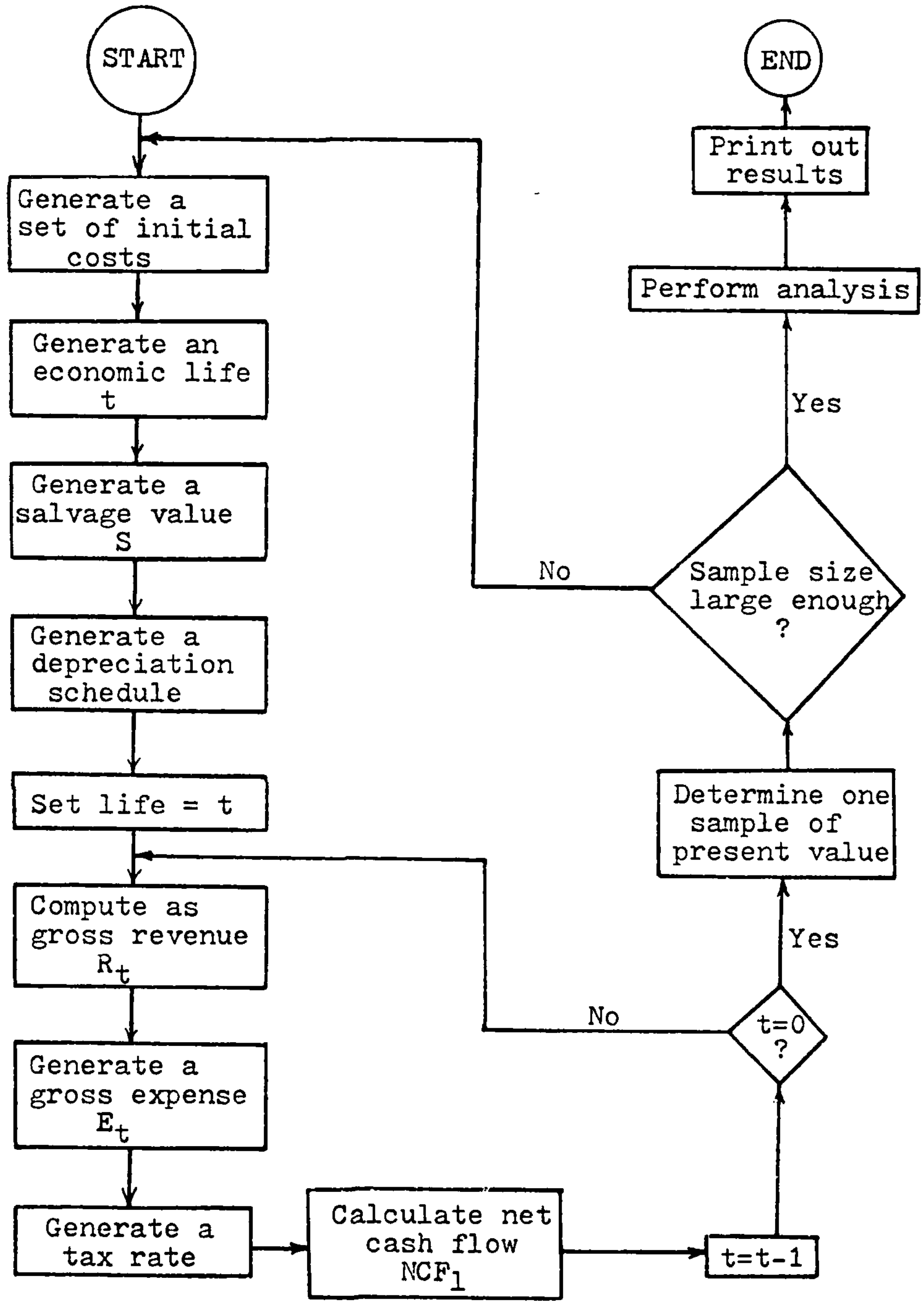


Figure 8.1

Simulation Flow Model

Conclusion

This research has expanded the analytical tools available to accomplish the analysis of economic models involving mutually exclusive alternatives with impulsive and series cash flows under the assumption of deterministic, constant interest rates. The use of these tools allows the analyst to use available data in a more realistic mode and calculate a more accurate and, in some cases, more correct solution.

BIBLIOGRAPHY

BIBLIOGRAPHY

Alchian, A. Economic Replacement Policy. Santa Monica, CA., Rand Corporation, Report R-224, Rm. 2153, 1952.

Alderfer, Clayton P. and Harold Bierman, Jr. "Choices with Risk: Beyond the Mean and Variance." The Journal of Business, 43, No. 3 (1970), 341-53.

American National Standard, Industrial Engineering Terminology, Engineering Economy, ANSI Z94.5-1972. New York: The American Society of Mechanical Engineers, (1973).

Apostol, Tom M. Mathematical Analysis, A Modern Approach to Advanced Calculus. Reading: Addison-Wesley, 1957.

Archer, Earnest R. "A Decision Making Model." Industrial Engineering, 7, No. 4 (1975), 27-29.

Barish, N. N. Economic Analysis for Engineering and Managerial Decision Making: A Book of Readings. New York: McGraw-Hill, 1962.

Bernhard, Richard H. "Discount Methods of Expenditure Evaluation-A Classification of their Assumptions." The Journal of Industrial Engineering, 13, No. 1 (1962), 19-27.

Bierman, H. J. and W. H. Hausman. "The Resolution of Investment Uncertainty Through Time." Management Science, 18, No. 8 (1972), 654-662.

Bierman, Harold, Jr. and Seymour Smidt. The Capital Budgeting Decision. 3rd ed. New York: Macmillan, 1971.

Bobillier, P. A., B. C. Kahan, and A. R. Probst. Simulation with GPSS and GPSS V. Englewood Cliffs: Prentice-Hall, 1976.

Bowman, E. H. and R. Fetter. Analysis of Production Management. Homewood: Richard D. Irwin, 1957.

Buck, James R. "Practical Considerations of New Cash Flow Techniques." Proceedings of the 1975 Spring Annual Conference, AIIE, 81-86.

Bussey, Lynn E. The Economic Analysis of Industrial Projects. Englewood Cliffs: Prentice-Hall, 1978.

________, and G. T. Stevens, Jr. "Net Present Value from Complex Cash Flow Streams by Simulation." AIIE Transactions, 3, No. 1 (1971), 81-89.

Canada, John R. "Decision Flow Networks." Journal of American Institute of Industrial Engineers, C, No. 1 (1974), 30-37.

________, and Harrison M. Wadsworth. "Methods for Qualifying Risk in Economic Analysis of Capital Projects." The Journal of Industrial Engineering, 19, No. 1 (1968), 32-37.

Coprath, David W. "From Statistical Decision Theory to Practice: Some Problems with the Transition." Management Science, 19, No. 8 (1973), 873-883.

Dean, Joel. Capital Budgeting. New York: Columbia University Press, 1951.

DeFaro, Clovis. "A Sufficient Condition for a Unique Nonnegative Internal Rate of Return: Further Comments." Journal of Financial and Quantitative Analysis, 13, No. 3 (1978), 577.

DeGarmo, Paul E. and John R. Canada. Engineering Economy. 5th ed. New York: Macmillan, 1973.

Dreyfus, S. E. "A Generalized Equipment Replacement Study." Journal of the Society of Industrial and Applied Mathematics, 8, No. 3 (1960), 425-435.

Duncan, J. Acheson. Quality Control and Industrial Statistics. Homewood: Richard D. Irwin, 1965.

Fabrycky, W. J. and H. G. Thuesen. Engineering Economy. Englewood Cliffs: Prentice-Hall, 1964.

Fish, John C. L. Engineering Economics. New York: McGraw-Hill, 1923.

Fishborn, Peter C. "Decision Under Uncertainty: An Introductory Exposition." The Journal of Industrial Engineering, 17, No. 7 (1966), 341-353.

Fisher, Irving. The Theory of Interest. New York: Macmillan, 1930; reprinted, New York: Kelly and Milan, 1954.

Fleischer, Gerald A., ed. Risk and Uncertainty: Non-Deterministic Decision Making in Engineering Economy. Norcross: Engineering Economy Division, American Society of Industrial Engineers, Monograph Series No. 2, (1975).

________. "Two Major Issues Associated with the Rate of Return Method for Capital Allocation: The 'Ranking Error' and 'Preliminary Selection'." The Journal of Industrial Engineering, 17, No. 4 (1966), 202-8.

________, and T. L. Ward. "Classification of Compound Interest Models in Economic Analysis." Engineering Economist, (Fall, 1977), 13-20.

Gibbons, Jean Dickinson. Nonparametric Methods for Quantitative Analysis. New York: Holt, Rinehart and Winston, 1976.

Goldman, O. B. Financial Engineering. New York: John Wiley and Sons, 1920.

Grant, Eugene L. Principles of Engineering Economy. 3rd ed. New York: Ronald Press, 1950.

________. Principles of Engineering Economy. revised ed. New York: Ronald Press, 1938.

________, W. Grant Ireson, and Richard S. Leavenworth. Principles of Engineering Economy. 6th ed. New York: John Wiley and Sons, 1976.

Grubbstrom, R. W. "On the Application of the Laplace Transform to Certain Economic Problems." Management Science, 13, No. 7 (1967), 558-567.

Henry, Claude. "Investment Decisions Under Uncertainty: The 'Irreversability Effect'." American Economics Review, 64, No. 6 (1974), 1006-1012.

Henry, Claude. "Investment Decisions Under Uncertainty: The 'Irreversability Effect'." American Economics Review, 64, No. 6 (1974), 1006-1012.

Hertz, David B. "Investment Policies that Pay Off." Harvard Business Review, (January-February, 1968), 96-108.

________. "Risk Analysis in Capital Investment." Harvard Business Review, (January-February, 1964), 95-106.

Hespos, Richard F. and Paul A. Strassman. "Stochastic Decision Trees for the Analysis of Investment Decision." Management Science, 11, No. 10 (1965), 244-259.

Hess, Sidney W. and Harry A. Quigley. "Analysis of Risk in Investments Using Monte Carlo Techniques." Chemical Engineering Progress Symposium Series 42: Statistics and Numerical Methods in Chemical Engineering, New York: American Institute of Chemical Engineers, 1963.

Hill, Thomas W. and James R. Buck. "Zeta Transforms, Present Value, and Economic Analysis." AIIE Transactions, 6, No. 2 (1974), 120-125.

Hillier, Frederick S. The Evaluation of Risky Interrelated Investments. Amsterdam: North-Holland, 1969.

________. "Supplement to 'The Derivation of Probabilistic Information for the Evaluation of Risky Investments'." Management Science, 11, No. 3 (1966), 485-487.

________. "The Derivation of Probabilistic Information for the Evaluation of Risky Investment." Management Science, 9, No. 4 (1963), 443-457.

________, and Gerald J. Lieberman. Operations Research. San Francisco: Holden Day, 1974.

Hirshleifer, J. Price Theory and Applications. Englewood Cliffs: Prentice-Hall, 1976.

Kaplan, S. "Computer Algorithms for Finding Exact Rates of Return." Journal of Business, 40, No. 4 (1967), 389-392.

________. "A Note on a Method for Precisely Determining the Uniqueness of Non-uniqueness of the Internal Rate of Return for a Proposed Investment." Journal of Industrial Engineering, 16, No. 1 (1965), 70-71.

Knight, Donald Orlin. "Application of GERT to Contingent Decisions." PHD Dissertation, Arizona State University, (1969).

Lesser, Arthur. "Engineering Economy in the United States in Retrospect-An Analysis." Engineering Economist, 14, No. 2 (1969), 109-115.

Luce, R. D. and H. Raiffa. Games and Decisions. New York: John Wiley and Sons, 1957.

Mallik, Arup K. Engineering Economy with Computer Applications. Mahomet: Engineering Technology, 1979.

Mantell, Edmund H. "A Central Limit Theorem for Present Values of Discounted Cash Flows." Management Science, 19, No. 3 (1972), 314-318.

Mao, James C. T. "The Internal Rate of Return as a Ranking Criterion." The Engineering Economist, 11, No. 4 (1966), 1-13.

Miller, Irwin and John E. Freund. Probability and Statistics for Engineers. Englewood Cliffs: Prentice-Hall, 1965.

Morris, William T. Decision Analysis. Columbus: Grid, 1977.

________. The Analysis of Management Decisions. Homewood: Richard D. Irwin, 1964.

________. Engineering Economics. Homewood: Richard D. Irwin, 1960.

Naddor, Eliezer. Inventory Systems. New York: John Wiley and Sons, 1966.

Norstrom, Carl L. "A Sufficient Condition for a Unique Nonnegative Internal Rate of Return." Journal of Financial and Quantitative Analysis, 7, No. 6 (1973), 1835-1839.

Oakford R. V., Arturo Salazar, and H. A. DiGiulio. "The Long Term Effectiveness of Expected Net Present Value Maximization." Proceedings of the 1979 Fall Annual Conference, AIIE, 71-84.

Park, C. S. and G. J. Thuesen. "The Effectiveness of Using Uncertainty Resolution to Measure Investment Flexibility in Sequential Capital Rationing." AIIE Transactions, 11, No. 3 (1979), 207-219.

Pazner, Elisha A. and Assaf Razin. "On Expected Value Vs. Expected Future Value." The Journal of Finance, 30, No. 3 (1975), 875-877.

Perrakis, Stylianos and Claude Henin. "The Evaluation of Risky Investments with Random Timing of Cash Returns." Management Science, 21, No. 1 (1974), 79-86.

Pritsker, A. Alan B. Modeling and Analysis Using Q-Gert Networks. New York: John Wiley and Sons, 1979.

Raiffa, Howard. Decision Analysis-Introductory Lectures on Choices Under Uncertainty. Reading: Addison-Wesley, 1970.

________. Decision Analysis. Reading: Addison Wesley, 1968.

Reisman, Arnold. Managerial and Engineering Economics. Boston: Allyn and Bacon, 1971.

________, and Arza K. Rao. Discounted Cash Flow Analysis: Stochastic Extensions. Norcross: Engineering Division, American Institute of Industrial Engineers, Monograph Series No. 1, (1973).

________, and Arza K. Rao. "Stochastic Cash Flow Formulas Under Conditions of Inflation." The Engineering Economist, 18, No. 1 (1972), 49-69.

Selby, Samuel M., ed. Standard Mathematical Tables. Cleveland: CRC Press, 1973.

Settles, Frank Stanley, Jr. "GERT Network Models of Production Economics." PHD Dissertation, Arizona State University, (1969).

Sivazlian, B. D. "On a Discounted Replacement Problem with Arbitrary Repair Time Distribution." Management Science, 19, No. 11 (1973), 1301-1309.

Spiegel, Murray R. Applied Differential Equations. 2nd ed. Englewood Cliffs: Prentice-Hall, 1967.

Taylor, George A. Managerial and Engineering Economy. Princeton: D. Van Nostrand, 1964.

Terborgh, George. Business Investment Management. Washington D. C.: Machinery and Allied Product Institute, 1967.

________. Business Investment Policy. Washington D. C.: Machinery and Allied Product Institute, 1958.

________. Dynamic Equipment Policy. New York: McGraw Hill, 1949.

Ward, Thomas L. "Discovered Stochastic Cash Flow Analysis." PHD Dissertation, University of Southern California, (1975).

Wellington, Arthur M. The Economic Theory of the Location of Railways. New York: John Wiley and Sons, 1908.

Whitehouse, G. W. "Extensions, New Developments, and Applications of GERT: Graphical Evaluation and Review Technique." PHD Dissertation, Arizona State University, (1966).

Whitehouse, Gary E. "Using Decision Flow Networks." Industrial Engineering, 6, No. 7 (1974), 18-25.

________. System Analysis and Design Using Network Techniques. Englewood Cliffs: Prentice-Hall, 1973.

Young, Donovan and Luis E. Contreras. "Expected Present Worths of Cash Flows Under Uncertain Timing." Engineering Economist, 20, No. 4 (1975), 257-268.

APPENDIX I

SELECTED MATHEMATICAL PROOFS AND DERIVATIONS

(1) Proof that $E(e^{-rt}) \geq e^{-r\,E(t)}$

Given: T is a random variable

To prove: $E(e^{-rT}) \geq e^{-r\,E(T)}$

Solution: The Taylor series expansion for e^{-rT} about the point $T = \mu_t$ is:

$$e^{-rT} = e^{-r\mu_t} - r(T-\mu_t)e^{-r\mu_t} + \frac{r^2(T-\mu_t)^2\, e^{-r\mu_t}}{2!} + \ldots$$

Taking the expected value of both sides of the previous equation leads to:

$$E(e^{-rT}) = e^{-r\mu_t} + E(T-\mu_t)re^{-r\mu_t} + \frac{E(T-\mu_t)^2}{2!} + \ldots$$

Recognizing that

$$E(T-\mu_t) = 0$$

and

$$E(T-\mu_t)^2 = \sigma_t^2 ,$$

$$E(e^{-rT}) = e^{-r\mu_t}\left[\, 1 + \frac{r^2\sigma^2{}_t}{2} + \ldots\right]$$

Since all terms in the expansion are greater than or equal to 0, the right side of this equation is greater than $e^{-r\mu_t}$.

Since

$$e^{-r\,E(T)} = e^{-r\mu_t} ,$$

$$E(e^{-rT}) \geq e^{-rE(T)} .$$

(2) Derivation of the variance of present worth for $C_{id}T_{is}$ Independent Model

Given: $PW = C\ e^{-rT}$

T is an independently distributed random variable

C is a constant

$V(PW) = E(PW^2) - \left[E(PW)\right]^2$

To find: V(PW)

Solution:

$$(PW)^2 = C^2(e^{-rT})^2 = C^2\ e^{-2rT}$$

$$E(PW^2) = (C^2)\cdot E(e^{-2rT})$$

$$E(PW) = (C)\cdot E(e^{-rT}$$

$$(E(PW))^2 = C^2\cdot(E(e^{-rT}))^2 \ .$$

Hence:

$$V(PW) = C^2\ E(e^{-2rT}) - C^2(E(e^{-rT}))^2$$

$$= C^2\left[E(e^{-2rt}) - (E(e^{-rt}))^2\right] \ .$$

Since

$$E(e^{-rt}) = M_t(-r) \ ,$$

$$(E(e^{-rt}))^2 = (M_t(-r))^2 \ ,$$

And

$$E(e^{-rT})^2 = \sum_{t=0}^{\infty} (e^{-rt})^2\ P(t)$$

$$= \sum_{t=0}^{\infty} (e^{-2rt})\ P(t)$$

$$= M_t(-2r) \ ,$$

Then

$$V(PW) = C^2\left[M_t(-2r)-(M_t(-r))^2 \right]$$

(3) Derivation for the mean and variance for the present worth for a $C_{sd}T_{ss}$ model element.

E(PW) derivation

Given: $PW = C \sum_{j=1}^{\infty} e^{-r\sum_{i=1}^{j} t_i}$

t_i are independent, identically distributed random variables

C is a constant

To find: E(PW)

Solution:

$$PW = C \sum_{j=1}^{\infty} e^{-r\sum_{i=1}^{j} t_i}$$

$$E(PW) = E(C \sum_{j=1}^{\infty} e^{-r\sum_{i=1}^{j} t_i}) \; .$$

Since C is independent of t_i ,

$$E(PW) = E(C) \cdot E(\sum_{j=1}^{\infty} e^{-r\sum_{i=1}^{j} t_i})$$

Since C is a constant,

$$E(PW) = C \cdot E(\sum_{j=1}^{\infty} e^{-r\sum_{i=1}^{j} t_i}).$$

By expanding,

$$E(PW) = C\left[E(e^{-rt_1}) + E(e^{-r(t_1+t_2)}) + E(e^{-r(t_1+t_2+t_3)}) + \ldots\right]$$

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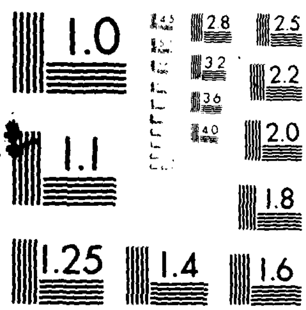

MICROCOPY RESOLUTION TEST CHART

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For all i and k,

$$E(e^{-rt_i}) = E(e^{-rt_k}) = E(e^{-rT}) .$$

Setting $E(e^{-rT}) = X$,

$$E(PW) = C \sum_{n=1}^{\infty} X^n .$$

Since $|e^{-rT}| < 1$ for all $T < 0$, X 1, and this geometric series converges to

$$C\left[\frac{1}{1-X} - 1\right] = \frac{CX}{1-X}$$

Hence,

$$E(PW) = C \cdot \frac{E(e^{-rT})}{1-E(e^{-rT})} .$$

Using

$$E(e^{-rT}) = M_t(-r) ,$$

$$E(PW) = C \cdot \frac{M_t(-r)}{1-M_t(-r)} .$$

To find: V(PW)

By definition of variance,

$$V(PW) = E(PW^2) - \left[E(PW)\right]^2 .$$

By squaring PW,

$$PW^2 = \left(C \sum_{j=1}^{\infty} e^{-r \sum_{i=1}^{j} t_i}\right)^2$$

$$= C^2\left[\sum_{j=1}^{\infty} e^{-r \sum_{i=1}^{j} t_i}\right]\left[\sum_{k=1}^{\infty} e^{-r \sum_{i=1}^{j} t_i}\right]$$

$$= C^2(e^{-rt_1} + e^{-r(t_1+t_2)} + e^{-r(t_1+t_2+t_3)} + \ldots)$$

$$\cdot(e^{-rt_1} + e^{-r(t_1+t_2)} + e^{-r(t_1+t_2+t_3)} + \ldots)$$

$$= C^2(e^{-r2t_1} + e^{-r(2t_1+t_2)} + e^{-(2t_1+t_2+t_3)} + \ldots)$$

$$+ C^2(e^{-r(2t_1+t_2)} + e^{-r(2t_1+2t_2)} + e^{-r(2t_1+2t_2+t_3)} + \ldots)$$

$$+ C^2(e^{-r(2t_1+t_2+t_3)} + e^{-r(2t_1+2t_2+t_3)} + e^{-r(2t_1+2t_2+2t_3)}$$

$$+ e^{-r(2t_1+2t_2+2t_3+t_4)} + \ldots) .$$

Taking the expected value of both sides and letting

$X = E(e^{-2rt})$ and $Y = E(e^{-rt})$ leads to:

$$E(PW^2) = C^2\Big[\left[X + XY + XY^2 + XY^3 + XY^4 + \ldots\right]$$

$$+ \left[XY + X^2 + X^2Y + X^2Y^2 + X^2Y^3 + X^2Y^4 + \ldots\right]$$

$$+ \left[XY^2 + X^2Y + X^3 + X^3Y + X^3Y^2 + X^3Y^3 + X^3Y^4 + \ldots\right]$$

$$+ \left[XY^3 + X^2Y^2 + X^3Y + X^4 + X^4Y + X^4Y^2 + X^4Y^3 + \ldots\right]$$

$$+ \left[\ldots\right]\Big] .$$

$$= \sum_{k=1}^{\infty} X^k \sum_{j=0}^{\infty} Y^j + \sum_{k=1}^{\infty} X^k \sum_{j=1}^{\infty} Y^j$$

$$= C^2 \left[\frac{X}{1-X} \cdot \frac{1}{1-Y} + \frac{X}{1-X} \cdot \frac{Y}{1-Y} \right]$$

$$= C^2 \left[\frac{X+XY}{(1-X)(1-Y)}\right] = C^2 \frac{X(1+Y)}{(1-X)(1-Y)} .$$

Now, substituting $X = E(e^{-2rt})$ and $Y = E(e^{-rt})$,

$$E(PW^2) = C^2 \frac{E(e^{-2rt})\quad(1+E(e^{-rt}))}{(1-E(e^{-2rt}))(1-E(e^{-rt}))} .$$

Noting $E(e^{-2rt}) = M_t(-2r)$ and $E(e^{-rt}) = M_t(-r)$,

$$E(PW^2) = C^2 \frac{M_t(-2r)(1+M_t(-r))}{(1-M_t(-2r))(1-M_t(-r))} .$$

Since (from p. 181)

$$E(PW) = C\cdot\frac{M_t(-r)}{1-M_t(-r)} ,$$

$$(E(PW))^2 = C^2\left[\frac{M_t(-r)}{1-M_t(-r)}\right]^2 .$$

Hence,

$$V(PW) = E(PW^2)-(E(PW))^2$$

$$= C^2 \frac{M_t(-2r)(1+M_t(-r))}{(1-M_t(-2r))(1-M_t(-r))}$$

$$- C^2\left[\frac{M_t(-r)}{1-M_t(-r)}\right]^2 ,$$

which can be reduced to

$$V(PW) = C^2\left[\frac{M_t(-2r)-(M_t(-r))^2}{\left[1-M_t(-2r)\right]\left[1-M_t(-r)\right]^2}\right] .$$

(4) Derivation for the variance for the present worth for a $C_{ss}T_{ss}$ model element.

Given: $PW = c_1e^{-rt_1} + c_2e^{-r(t_1+t_2)} + c_3e^{-r(t_1+t_2+t_3)} + \ldots$

t_i are independent, identically distributed random variables

c_i are independent, identically distributed random variables.

t_i and c_i are independent for all i.

To find: $V(PW)$

Solution:

By definition

$V(PW) = E(PW^2) - \left[E(PW)\right]^2$.

By squaring PW:

$$PW^2 = c_1^2e^{-r2t_1} + c_1c_2e^{-r(2t_1+t_2)} + c_1c_3e^{-r(2t_1+t_2+t_3)} + \ldots$$

$$+ c_1c_2e^{-r(2t_1+t_2)} + c_2^2e^{-r(2t_1+2t_2)}$$

$$+ c_2c_3e^{-r(2t_1+2t_2+t_3)} + \ldots + c_1c_3e^{-r(2t_1+t_2+t_3)}$$

$$+ c_2c_3e^{-r(2t_1+2t_2+t_3)} + c_3^2e^{-r(2t_1+2t_2+2t_3)} + \ldots$$

Taking the expected value of both sides and letting $X = E(e^{-2rt})$ and $Y = E(e^{-rt})$, leads to:

$$E(PW^2) = E(c_1^2)X+E(c_1c_2)XY+E(c_1c_3)XY^2+\ldots$$

$$+ E(c_1c_2)XY+E(c_2^2)X^2+E(c_2c_3)X^2Y+\ldots$$

$$+ E(c_1c_3)XY^2+E(c_2c_3)X^2Y+E(c_3^2)X^3+\ldots$$

Those terms of the form

$$E(e_j^2)x^j$$

can be grouped as:

$$\sum_{j=1}^{\infty} E(e_j^2)x^j .$$

Since e_j are independent, identically distributed random variables for all j,

$$E(e_j^2) = E(e_k^2) = E(e^2) \text{ for all } j \text{ and } k.$$

By definition,

$$E(e^2) = V(e) = [SE(e)]^2 .$$

Hence,

$$\sum_{j=1}^{\infty} E(e_j^2)x^j = (V(e)) = [SE(e)]^2 \sum_{j=1}^{\infty} x^j .$$

As previously shown,

$$\sum_{j=1}^{\infty} x^j = \sum_{j=1}^{\infty} [B_x e^{\text{[illegible]}}]^j$$

$$= \sum_{j=1}^{\infty} (B_x\text{[illegible]})^j = \frac{B_x\text{[illegible]}}{1 - B_x\text{[illegible]}} .$$

Hence,

$$\sum_{j=1}^{\infty} E(e_j^2)x^j = (V(e)) = [SE(e)]^2 \frac{B_x\text{[illegible]}}{1 - B_x\text{[illegible]}} .$$

Since the e_i are all independent, identically distributed random variables, then $E(e_i e_j)$ can be expressed as $E(e_i)E(e_j) = E(e_i)^2$, for i not equal to j. The remaining terms in the expression of $E(\hat{\lambda})^2$ can now be expressed as:

[illegible]

[illegible] terms of moment generating functions, the remaining terms can be expressed as:

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Substituting [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] Derivation for [illegible] for the [illegible] dependent [illegible] model element.

[illegible] for the model is

[illegible]

where

[illegible]

[illegible]

Since [illegible] is a constant then one can divide by [illegible] to obtain

[illegible]

[illegible]

[illegible]

$$+ n_1^2 e^{-\theta(2n_1+2n_2+2n_3)} + 2n_1^2 e^{-\theta(2n_1+2n_2+2n_3+n_4)} + \cdots$$

$$+ n_2^2 e^{-\theta(2n_1+2n_2)} + 2n_2^2 e^{-\theta(2n_1+2n_2+n_3)}$$

$+ 2n_2^2$ [illegible] $+ \cdots$

$+ n_3^2$ [illegible] $+ 2n_3^2$ [illegible] $+ \cdots$

$= n_1^2$ [illegible] $+ \cdots]$

$+ n_1^2$ [illegible] $+ \cdots]$

$+ n_1^2$ [illegible] $+ \cdots]$

$+ \cdots$

$+ n_2^2$ [illegible] $+ \cdots]$

$+ n_2^2$ [illegible] $+ \cdots]$

$+ \cdots$

$+ n_3^2$ [illegible] $+ \cdots]$

$+ n_3^2$ [illegible] $+ \cdots]$

$+ \cdots$

The expected value for the n_i^2 terms would then be

[illegible] $\sum$ [illegible] $\sum$ [illegible]

$+$ [illegible] $\sum$ [illegible] $\sum$ [illegible]

$+$ [illegible] $\sum$ [illegible] $\sum$ [illegible]

$+ \cdots$.

After rearranging terms, the expected value of the $r_i^{\,2}$ terms would be

$$\left[\sum_{j=1}^{\infty} (E(e^{-2rT}))^{j}\right]\cdot\left[E(T^{2}e^{-2rT})\sum_{k=0}^{\infty} (E(e^{-2rT}))^{k}\right]$$

$$\cdot\left[1+2\sum_{i=1}^{\infty} (E(e^{-rT}))^{i}\right],$$

which is equal to

$$\frac{1}{1-M_T(-2r)}\cdot\frac{M_T''(-2r)}{4}\cdot\frac{1}{1-M_T(-2r)}\cdot\left[1+2\,\frac{M_T(-r)}{1-M_T(-r)}\right].$$

The second set of equations for the present work (denoted by [illegible]) is composed of $r_i\, r_j$ $(i \neq j)$ terms.

[illegible]

$+ t_1 e^{-r2t_1} t_3 e^{-r2t_3} e^{-r2t_2} \left[2 + 4e^{-rt_4} + 4e^{-r(t_4+t_5)} + \dots\right]$

$+ t_1 e^{-r2t_1} t_3 e^{-r2t_3} e^{-r2t_2} e^{-r2t_4} \left[2 + 4e^{-rt_5} + 4e^{-r(t_5+t_6)} + \dots\right]$

$+ \dots$

$+ t_1 e^{-r2t_1} t_4 e^{-rt_4} e^{-rt_2} e^{-rt_3} \left[2 + 2e^{-rt_5} + \dots\right]$

$+ t_1 e^{-r2t_1} t_4 e^{-rt_4} e^{-r2t_2} e^{-rt_3} \left[2 + 2e^{-rt_5} + \dots\right]$

$+ t_1 e^{-r2t_1} t_4 e^{-rt_4} e^{-r2t_2} e^{-r2t_3} \left[2 + 2e^{-rt_4} + \dots\right]$

$+ t_1 e^{-r2t_1} t_4 e^{-r2t_4} e^{-r2t_2} e^{-r2t_3} \left[2 + 4e^{-rt_5} + \dots\right]$

$+ t_1 e^{-r2t_1} t_4 e^{-r2t_4} e^{-r2t_2} e^{-r2t_3} e^{-r2t_5} \left[2 + 4e^{-rt_6} + \dots\right]$

$+ \dots$

$+ t_2 e^{-r2t_2} t_3 e^{-rt_3} e^{-r2t_1} \left[2 + 2e^{-rt_4} + \dots\right]$

$+ t_2 e^{-r2t_2} t_3 e^{-r2t_3} e^{-r2t_1} \left[2 + 4e^{-rt_4} + \dots\right]$

$+ t_2 e^{-r2t_2} t_3 e^{-r2t_3} e^{-r2t_1} e^{-r2t_4} \left[2 + 4e^{-rt_5} + \dots\right]$

$+ \dots$

After rearranging these terms, the expected value of the $t_i t_j$ terms is:

$$E(te^{-2rt}) \cdot E(te^{-rt}) \cdot \sum_{k=0}^{\infty} (E(e^{-rt}))^k$$

$$\cdot \sum_{j=0}^{\infty} (E(e^{-2rt}))^j \cdot \sum_{i=0}^{\infty} (E(e^{-rt}))^i$$

$$\cdot 2 \sum_{n=0}^{\infty} E(e^{-rt})^n$$

$$+\left[\sum_{n=2}^{\infty}\binom{n}{2} E(te^{-2rt})\cdot E(te^{-2rt})\cdot(E(e^{-2rt}))^{n-2}\right]$$

$$\cdot\left[2+4\sum_{k=1}^{\infty}(E(e^{-rt}))^{k}\right],$$

which is equal to:

$$\left[\frac{-M_t'(-2r)}{2}\right]\cdot\left[-M_t'(-r)\right]\cdot\left[\frac{1}{1-M_t(-r)}\right]$$

$$\cdot\left[\frac{1}{1-M_t(-2r)}\right]\cdot\left[\frac{1}{1-M_t(-r)}\right]\cdot 2\left[\frac{1}{1-M_t(-r)}\right]$$

$$+(A)\cdot\left[2+4\left[\frac{M_t(-r)}{1-M_t(-r)}\right]\right],$$

where

$$A=\sum_{n=2}^{\infty}\binom{n}{2}(E(te^{-2rt}))(E(te^{-2rt}))\cdot(E(e^{-2rt}))^{n-2}.$$

The expression for A can now be written as:

$$A=(E(te^{-2rt}))^{2}\cdot\sum_{n=2}^{\infty}\binom{n}{2}(E(e^{-2rt}))^{n-2}.$$

$\sum_{n=2}^{\infty}\binom{n}{2}(E(e^{-2rt}))^{n-2}$ can be solved using the following arguments:

let $\rho = E(e^{-2rt})$.

then $f(\rho) = \sum_{n=0}^{\infty}\rho^{n} = \frac{1}{1-\rho}$,

$$f'(\rho)=\sum_{n=1}^{\infty}n\rho^{(n-1)}=\frac{1}{(1-\rho)^{2}}.$$

and

$$f''(\rho) = \sum_{n=2}^{\infty} n(n-1)\rho^{n-2} = \frac{2}{(1-\rho)^3} .$$

Since

$$\sum_{n=2}^{\infty} \binom{n}{2} \rho^{n-2}$$

can be written as

$$\sum_{n=2}^{\infty} n(n-1)\rho^{n-2} ,$$

$$\sum_{n=2}^{\infty} \binom{n}{2} (E(e^{-2rt}))^{n-2} = \frac{2}{(1-E(e^{-2rt}))^3} .$$

A can then be written as:

$$(E(te^{-2rt}))^2 \cdot \frac{2}{(1-E(e^{-2rt}))^3} ,$$

and can be solved using moment generating functions as:

$$A = \left[\frac{-M_t'(-2r)}{2}\right]^2 \cdot \frac{2}{(1-M_t(-2r))^3} .$$

Combining both sets of terms (t_i^2 and $t_i t_j$) simply leads to:

$$E\left(\frac{PW^2}{C^2}\right) = \frac{M_t''(-2r)}{4(1-M_t-(2r))^2} \cdot \left[1 + \frac{2M_t(-r)}{1-M_t(-r)}\right]$$

$$+ \frac{M_t'(-2r) \cdot M_t(-r)}{(1-M_t(-2r))(1-M_t(-r)^3}$$

$$+ \frac{(M_t'(-2r))^2}{(1-M_t(-2r))^3} \cdot \left[1 + \frac{2M_t(-r)}{1-M_t(-r)}\right]$$

Hence

$$E(PW^2) = C^2\left[\left[\frac{M_t''(-2r)}{4(1-M_t(-2r))^2} + \frac{(M_t'(-2r))^2}{(1-M_t(-2r))^3}\right]\left[1 + \frac{2M_t(-r)}{1-M_t(-r)}\right] + \frac{M_t'(-2r)\cdot M_t(-r)}{(1-M_t(-2r))(1-M_t(-r))^3}\right].$$

(6) Derivation of the expected present worth for the $C_{ss}T_{ss}$ dependent ramp model element.

Given: $PW = c_1 t_1 e^{-rt_1} + c_2(t_1+t_2)\, e^{-r(t_1+t_2)} + \ldots$

$$c_n(t_1+t_2+\ldots t_n)\, e^{-r(t_1+t_2+\ldots t_n)} + \ldots .$$

c_i and t_i are independent and identically distributed random variables.

To find: E(PW)

Solution:

Since c_i is independent of t_i,

$$E(PW) = E(c_1)\cdot E(t_1 e^{-rt_1}) + E(c_2)\cdot E(t_1 e^{-r(t_1+t_2)} + t_2 e^{-r(t_1+t_2)})$$
$$+ \ldots + E(c_n)\cdot E(t_1 e^{-r(t_1+t_2+\ldots+t_n)} + t_2 e^{-r(t_1+t_2+\ldots+t_n)}$$
$$+ \ldots t_n e^{-r(t_1+t_2+\ldots+t_n)}) + \ldots$$

Since $E(c_i) = E(c_j) = \mu_c$ for all i and j,

$$E(PW) = \mu_c \left[E(t_1 e^{-rt_1}) + E(t_1 e^{-r(t_1+t_2)}) + E(t_2 e^{-r(t_1+t_2)}) + \ldots\right.$$
$$+ E(t_1 e^{-r(t_1+t_2+\ldots+t_n)}) + E(t_2 e^{-r(t_1+t_2+\ldots+t_n)}) + \ldots$$
$$\left. + E(t_n e^{-r(t_1+t_2+\ldots+t_n)}) + \ldots\right] .$$

Since the t_i's are independently distributed random variables,

$$E(PW) = \mu_c \cdot E(t_1 e^{-rt_1}) \left[1 + E(e^{-rt_2}) + E(e^{-r(t_2+t_3)}) + \ldots\right]$$

$$+ \mu_c \cdot E(t_2 e^{-rt_2}) \cdot E(e^{-rt_1}) \left[1 + E(e^{-rt_3}) + E(e^{-r(t_3+t_4)}) + \ldots \right]$$

$$+ \ldots + \mu_c E(t_n e^{-rt_n}) \cdot E(e^{-rt_1}) \cdot E(e^{-rt_2}) \ldots E(e^{-r(t_{n-1})})$$

$$\left[1 + E(e^{-r(t_{n+1})}) + E(e^{-r(t_{n+1}+t_{n+2})}) + \ldots\right]$$

$$+ \ldots$$

However, $E(t_i e^{-rt_i}) = E(t_j e^{-rt_j}) = - M_t'(-r)$

and $E(e^{-rt_i}) = E(e^{-rt_j}) = M_t(-r)$ for all i and j.

Hence:

$$E(PW) = \mu_c \left[-M_t'(-r)\right] \cdot \sum_{n=0}^{\infty} \left[M_t(-r)\right]^n \cdot \sum_{k=0}^{\infty} \left[M_t(-r)\right]^k$$

$$= \mu_c \, (-M_t'(-r)) \cdot \frac{1}{1-M_t(-r)} \cdot \frac{1}{1-M_t(-r)}$$

$$= \frac{-\mu_c M_t'(-r)}{\left[1-M_t(-r)\right]^2} \; .$$

(7) Derivation of $C_{ss}T_{ss}$ Dependent ramp, finite n cash flow model element

$$PW = c_1t_1e^{-rt_1} + c_2(t_1+t_2)e^{-r(t_1+t_2)}$$
$$+ c_3(t_1+t_2+t_3)e^{-r(t_1+t_2+t_3)} + \ldots$$
$$+ c_n(t_1+t_2+t_3+\ldots+t_n)e^{-r(t_1+t_2+t_3+\ldots t_n)}$$
$$= c_1t_1e^{-rt_1} + c_2t_1e^{-r(t_1+t_2)} + c_3t_1e^{-r(t_1+t_2+t_3)}$$
$$+\ldots+ c_nt_1e^{-r(t_1+t_2+t_3+\ldots+t_n)} + c_2t_2e^{-r(t_1+t_2)}$$
$$+ c_3t_2e^{-r(t_1+t_2+t_3)} +\ldots+ c_nt_2e^{-r(t_1+t_2+t_3+\ldots+t_n)}$$
$$+ c_3t_3e^{-r(t_1+t_2+t_3)} +\ldots+ c_nt_3e^{-r(t_1+t_2+t_3+\ldots+t_n)}$$
$$+\ldots+ c_nt_ne^{-r(t_1+t_2+t_3+\ldots+t_n)}$$

c_i and t_i are independent and identically distributed random variables.

Solving in terms of expected values:

$$E(PW) = E(C)\cdot E(te^{-rt}) + 2E(C)\cdot E(te^{-rt})\cdot E(e^{-rt})$$
$$+ 3\cdot E(C)\cdot E(te^{-rt})\cdot \left[E(e^{-rt})\right]^2$$
$$+\ldots+ n\,E(C)\cdot E(te^{-rt})\cdot \left[E(e^{-rt})\right]^{n-1}$$
$$= E(C)\cdot E(te^{-rt})\cdot \sum_{k=1}^{n} k\left[E(e^{-rt})\right]^{k-1}$$

Let

$$X = E(e^{-rt}) .$$

Then

$$\sum_{k=1}^{n} k(E(e^{-rt}))^{k-1} = X^0 + 2X^1 + 3X^2 + \ldots + nX^{n-1}$$

$$= \frac{d}{dx} \sum_{k=1}^{n} X^k .$$

Now,

$$\sum_{k=1}^{n} X^k = \frac{X - X^{n+1}}{1 - X}$$

$$\frac{d}{dx} S_n = \frac{(1-X)(1-(n+1)X^n) + (X-X^{n+1})}{(1-X)^2}$$

$$= \frac{1-X - (n+1)X^n + X(n+1)X^n + X - X^{n+1}}{(1-X)^2}$$

$$= \frac{1 + X^n(-n-1+nX+X-X)}{(1-X)^2}$$

$$= \frac{1 + X^n(nX-n-1)}{(1-X)^2}$$

$$\sum_{k=1}^{n} k\left[Ee^{-rt}\right]^{k-1} = \left[\frac{1 + \left[E(e^{-rt})\right]^n \left[nE(e^{-rt})-n-1\right]}{(1-E(e^{-rt}))^2}\right]$$

Since: $E(te^{-rt}) = -M_t'(-r)$,

$E(e^{-rt}) = M_t(-r)$,

and $E(C) = \mu_c$

$$E(PW) = \mu_c(-M_t'(-r))\left[\frac{1 + \left[M_t(-r)\right]^n \left[nM_t(-r)-n-1\right]}{\left[1-M_t(-r)\right]^2}\right]$$

(3) Derivation of $C_{ss}T_{ss}$ Dependent decay, finite n cash flow model element

$$PW = c_1 e^{-(a+r)t_1} + c_2 e^{-(a+r)(t_1+t_2)}$$

$$+ \ldots + c_n e^{-(a+r)(t_1+t_2+\ldots+t_n)}$$

Taking expected values, while noting that the c_i's and t_i's are independent, identically distributed random variables, leads to

$$E(PW) = \mu_c \sum_{k=1}^{n} \left[E(e^{-(a+r)t_i})\right]^k$$

Setting $E(e^{-(a+r)t_i}) = X$,

$$\sum_{k=1}^{n} E(e^{-(a+r)t_i})^k = \sum_{k=1}^{n} X^k = X \sum_{j=0}^{n-1} X^j = X\left[\frac{1-X^n}{1-X}\right]$$

Hence:

$$E(PW) = \mu_c \left[\frac{E(e^{-(a+r)t_i}) - \left[E(e^{-(a+r)t_i})\right]^{n+1}}{1 - E(e^{-(a+r)t_i})}\right].$$

and in terms of moment generating functions:

$$E(PW) = \mu_c \left[\frac{M_t(-(a+r)) - \left[M_t(-(a+r))\right]^{n+1}}{1 - M_t(-(a+r))}\right].$$

(?) Derivation of [illegible] Dependent growth, [illegible] case flow model element

$$PW = [illegible]$$

$$+ \ldots + [illegible]$$

The growth function is composed of the step function and the decay functions. The combination of these functions yields

$$E(PW) = [illegible]$$

and in terms of moment generating functions

$$E(PW) = [illegible]$$

BIOGRAPHICAL SKETCH

[illegible] Howard [illegible] was born in El Paso, Texas, on April 15, 194[illegible]. He received his elementary and secondary education in the El Paso Public Schools. In [illegible] he entered Baylor University in Waco, Texas and left without completing his degree in [illegible]. He enlisted in the United States Air Force in 1969 and was selected under the Airman Education and Commissioning Program in 1970 to attend Arizona State University in Tempe, Arizona. He completed his Bachelor of Science degree at Arizona State University in 1972 and received his commission as a Second Lieutenant in the United States Air Force the same year. In September of 1972 he again entered Arizona State University, this time under the Air Force Institute of Technology to complete a Master of Science degree in Industrial Engineering. Following the completion of his Master's degree in January of 1974, he was assigned to Air Force bases in California, Guam, and Colorado. As an Assistant Professor of Mathematics at the United States Air Force Academy in Colorado Springs, he was selected to return to Arizona State University in [illegible] to pursue a degree of Doctor of Philosophy. He is presently a Captain in the United States Air Force and is a member of Tau Beta Pi (Engineering Honor Society), Alpha Pi Mu (Industrial Engineering Honor Society), Air Force Association, and American Institute of Industrial Engineers-Student Chapter. He is married and the father of one son and one daughter.

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