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LARGE TIME STEP SHOCK-CAPTURING TECHNIQUES FOR SCALAR CONSERVAT--ETC(U)
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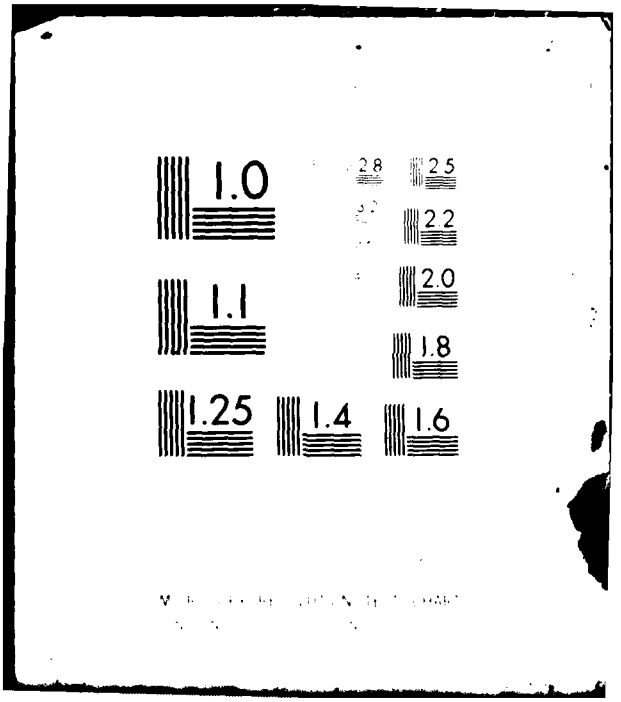
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Large time-step shock-capturing techniques for scalar conservation laws

Randall J. LeVeque*

Abstract. For a scalar conservation law $u_t = f(u)_x$ with f'' of constant sign, the first order upwind difference scheme is a special case of Godunov's method. The method is equivalent to solving a sequence of Riemann problems at each step and averaging the resulting solution over each cell in order to obtain the numerical solution at the next time level. The difference scheme is stable (and the solutions to the associated sequence of Riemann problems do not interact) provided the Courant number ν is less than 1. By allowing and explicitly handling such interactions, it is possible to obtain a generalized method which is stable for ν much larger than 1. In many cases the resulting solution is considerably more accurate than solutions obtained by other numerical methods. In particular, shocks can be correctly computed with virtually no smearing. The generalized method is rather unorthodox and still has some problems associated with it. Nonetheless, preliminary results are quite encouraging.

*Department of Computer Science, Stanford University, Stanford, CA 94305.

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1. Introduction.

A scalar conservation law is a partial differential equation of the form

$$u_t(x,t) = f(u(x,t))_x \tag{1.1}$$

where $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. When u is a density and f a flux, (1.1) states that the integral of u over some interval (x_1, x_2) changes only due to the flux through the endpoints, since

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2} u(x,t)dx &= \int_{x_1}^{x_2} \frac{\partial}{\partial x}(f(u(x,t)))dx \\ &= f(u(x_2,t)) - f(u(x_1,t)).\end{aligned}$$

The theory of conservation laws is described, for example, by Lax[7] and Whitham[10].

The most basic problem for a conservation law is the *Riemann problem*, which is (1.1) together with piecewise constant initial conditions with a single discontinuity,

$$u(x,0) = \begin{cases}\alpha & x < x_0 \\ \beta & x \geq x_0.\end{cases} \tag{1.2}$$

The solution of a Riemann problem can often be found analytically. Various numerical schemes for solving (1.1) with arbitrary initial data are based on solving a sequence of Riemann problems exactly. As usual with finite difference schemes, we choose a spatial stepsize h and a time-step k and set $x_j = jh$, $t_n = nk$. We then approximate $u(x_j, t_n)$ by u_j^n. But now we view the discrete solution $\{u_j^n\}$ not as an approximation to some smooth function but rather as a representation of a step function

$$u(x,t_n) = u_j^n \qquad \text{for} \quad x_{j-1/2} < x < x_{j+1/2} \tag{1.3}$$

where $x_{j+1/2} = x_j + h/2$. Each discontinuity defines a single Riemann problem. If the time-step k is sufficiently small, the solutions to these Riemann problems do not interact in time k and the solution to (1.1) with initial conditions (1.3) can be computed exactly. This is the case if the *Courant number*,

$$\nu = \frac{k}{h}\max_x f'(u(x,t_n)),$$

is less than 1. In order to continue this process we must project the exact solutions of the associated Riemann problems onto the computational grid. This mapping can be defined in various ways. Setting u_j^{n+1} to the average value of $u(x,t_{n+1})$ over the cell $(x_{j-1/2}, x_{j+1/2})$, i.e.

$$u_j^{n+1} = h\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})dx$$

gives *Godunov's method*[5]. Setting u_j^{n+1} to the value of $u(x,t_{n+1})$ at some randomly chosen point in $(x_{j-1/2}, x_{j+1/2})$ gives Chorin's version of *Glimm's scheme*, also known as the *random choice method*[1][3]. This method has the advantage that shocks and contact discontinuities in the solution remain sharp. However, their positions are no longer correct in general, although they are in an average sense. Both of these methods can also be used more generally for systems of conservation laws.

The aim of the present paper is to show how, for scalar conservation laws, the restriction $\nu < 1$ can frequently be dropped by explicitly handling the interactions of the solutions to the various Riemann problems. This has the obvious advantage of requiring less work to advance the

solution to a given time. It also turns out to be more accurate in many situations. In particular, for large values of ν, sharp shocks can be maintained even when using Godunov-type averaging. The problems associated with random choice methods can thus be avoided. The resulting scheme is conservative and shock locations are correct, at least for problems involving only shocks. The procedure described below has also given very good results on numerous problems involving shock-rarefaction interactions, but there are still some difficulties to be resolved in this area. These will be discussed in section 5.

In some ways the method is best thought of as a generalization of the first order upwind difference scheme for a constant coefficient linear problem. Recently Roe[9] has advocated viewing difference schemes for such equations in an "increment form" in order to generalize them to new methods for systems of conservation laws. We will begin in the same framework, and generalize in a different direction.

2. The constant coefficient linear problem.

We begin by considering the constant coefficient linear advection equation

$$u_t = cu_x \qquad c > 0. \tag{2.1}$$

We define

$$\Delta_j^n = u_{j+1}^n - u_j^n,$$
$$\nu = ck/h, \qquad \text{the Courant number.}$$

Any explicit linear two-level difference scheme can be written in increment form as

$$u_j^{n+1} - u_j^n = -\nu \sum_i \gamma_i \Delta_{j+i}^n \tag{2.2}$$

for some choice of the coefficients γ_i. The scheme is at least first order accurate if

$$\sum_i \gamma_i = 1.$$

Further conditions on the γ_i give higher order methods. See Roe[9] for a more general discussion. We now think of implementing (2.2) in the following way: for each j we compute $\nu\Delta_j^n$, split it up into fractions proportional to γ_i, and add the ith fraction to u_{j-i}^n. Thus rather than interpreting (2.2) as a formula for computing u_j^{n+1}, we can take it as a prescription for distributing the incremental data $\nu\Delta_j^n$ over nearby meshpoints.

The choice of coefficients γ_i can be motivated in the following way. For simplicity consider a single Riemann problem

$$u_j^n = \begin{cases} \alpha & j \le J \\ \beta & j > J. \end{cases} \tag{2.3}$$

This is interpreted as a discretization of

$$u(x, t_n) = \begin{cases} \alpha & x < x_{J+1/2} \\ \beta & x \ge x_{J+1/2}. \end{cases}$$

At time t_{n+1} the solution of this Riemann problem is

$$u(x, t_{n+1}) = \begin{cases} \alpha & x < x_{J+1/2} - \nu h \\ \beta & x \ge x_{J+1/2} - \nu h. \end{cases}$$

We would like our approximation u^{n+1} to be some discrete representation of this jump. If $\nu = 1$, the jump is at $x_{J-1/2}$ and is represented by

$$u_j^{n+1} = \begin{cases} \alpha & j \le J-1 \\ \beta & j > J-1. \end{cases} \qquad (\nu = 1) \tag{2.4}$$

On the other hand, if $\nu < 1$, the true position of the shock can be represented by smearing the jump over more meshpoints. One possiblility is

$$u_j^{n+1} = \begin{cases} \alpha & j < J \\ \alpha + \nu(\beta - \alpha) & j = J \\ \beta & j > J. \end{cases} \qquad (\nu \le 1) \tag{2.5}$$

This agrees with (2.4) for $\nu = 1$ and has been chosen so that the discrete conservation law holds,

$$\sum_j h u_j^{n+1} = \sum_j h u_j^n + kc(\beta - \alpha).$$

Note that each u_j is the average value of $u(x,t)$ over the interval $[x_{j-1/2}, x_{j+1/2}]$.

For arbitrary $\{u_j^n\}$ we apply (2.5) to each discontinuity in the step function (1.3). This yields the standard first order upwind difference scheme

$$u_j^{n+1} = u_j^n + \nu \Delta_j^n \tag{2.6}$$

obtained from (2.2) by taking $\gamma_0 = 1$ and all other $\gamma_i = 0$.

Other conservative representations of a jump at $x_{J+1/2} - \nu h$ are also possible. These lead to other special cases of (2.2). For example, replacing (2.5) by

$$u_j^{n+1} = \begin{cases} \alpha & j \le J-1 \\ \alpha + \frac{1}{2}\nu(1+\nu)(\beta - \alpha) & j = J \\ \beta + \frac{1}{2}\nu(1-\nu)(\beta - \alpha) & j = J+1 \\ \beta & j \ge J+2 \end{cases} \tag{2.7}$$

leads to the general scheme

$$u_j^{n+1} = u_j^n + \frac{1}{2}\nu(1+\nu)\Delta_j^n + \frac{1}{2}\nu(1-\nu)\Delta_{j-1}^n$$

which is the Lax-Wendroff scheme for (2.1). For smooth solutions this is often preferable to (2.6), being second order accurate. For a single time-step on the Riemann problem, however, (2.7) is clearly inferior to (2.5) since the discontinuity has been smeared over more mesh points than necessary and since the approximation is no longer monotonic.

We will call (2.5) the *optimal representation* of a jump at $x_{J+1/2} - \nu h$, since it is monotonic and involves as few mesh points as possible.

Implementing (2.6) can now be described as follows: *interpret* the data $\{u_j^n\}$ as being a sequence of Riemann problems, *solve* the Riemann problems exactly, and then *represent* the resulting discontinuities on the finite mesh via (2.5). This viewpoint will prove fruitful when dealing with more general conservation laws.

3. Upwind differencing with $\nu > 1$.

Adopting the viewpoint proposed in the previous section allows us to extend the scheme (2.6) for use with large time steps. Attempting to apply (2.6) directly with $\nu > 1$ is unstable. Instead we again interpret the discrete data as a representation of a step function and solve a sequence of Riemann problems. Again consider the single discontinuity

$$u_j^n = \begin{cases} \alpha & j \le J \\ \beta & j > J. \end{cases}$$

The discontinuity $\Delta_J^n = \beta - \alpha$ at $x_{J+1/2}$ propagates under (2.1) to $x_{J+1/2} - \nu h$ at time t_{n+1}. Let $\mu = \lfloor \nu \rfloor$, the integer part of ν. Then $x_{J+1/2} - \nu h = x_{J-\mu+1/2} - (\nu - \mu)h$. The solution to the Riemann problem is

$$u(x, t_{n+1}) = \begin{cases} \alpha & x < x_{J-\mu+1/2} - (\nu - \mu)h \\ \beta & x \ge x_{J-\mu+1/2} - (\nu - \mu)h. \end{cases} \tag{3.1}$$

which we must now represent on the discrete mesh. By again averaging the solution over each cell we obtain the representation

$$u_j^{n+1} = \begin{cases} \alpha & j < J - \mu \\ \alpha + (\nu - \mu)(\beta - \alpha) & j = J - \mu \\ \beta & j > J - \mu. \end{cases} \tag{3.2}$$

This is also the natural generalization of (2.6) if we read (2.6) not as "add ν times Δ_j^n to the next mesh point over" but rather as "add Δ_j^n to the next ν mesh points over". Applying (3.2) at each discontinuity gives the following scheme for general $\{u_j^n\}$:

$$u_j^{n+1} = u_j^n + \Delta_j^n + \Delta_{j+1}^n + \cdots + \Delta_{j+\mu}^n + (\nu - \mu)\Delta_{j+\mu+1}^n.$$

Most of these terms cancel, leaving

$$u_j^{n+1} = (1 - (\nu - \mu))u_{j+\mu+1}^n + (\nu - \mu)u_{j+\mu+2}^n.$$

This is simply the method of characteristics with linear interpolation.

4. General scalar conservation laws.

Now consider a scalar conservation law

$$u_t = f(u)_x. \tag{4.1}$$

We will always assume that $f''(u)$ has constant sign for all u of interest. Suppose, to begin with, that $f'' > 0$ and that $u(x, 0)$ is nondecreasing.

Again let $\Delta_j^n = u_{j+1}^n - u_j^n$ and set

$$\begin{aligned} c_j^n &= (f(u_{j+1}^n) - f(u_j^n))/\Delta_j^n \qquad \text{for} \quad \Delta_j^n \neq 0, \\ \nu_j^n &= c_j^n k/h, \\ \mu_j^n &= \lfloor \nu_j^n \rfloor. \end{aligned}$$

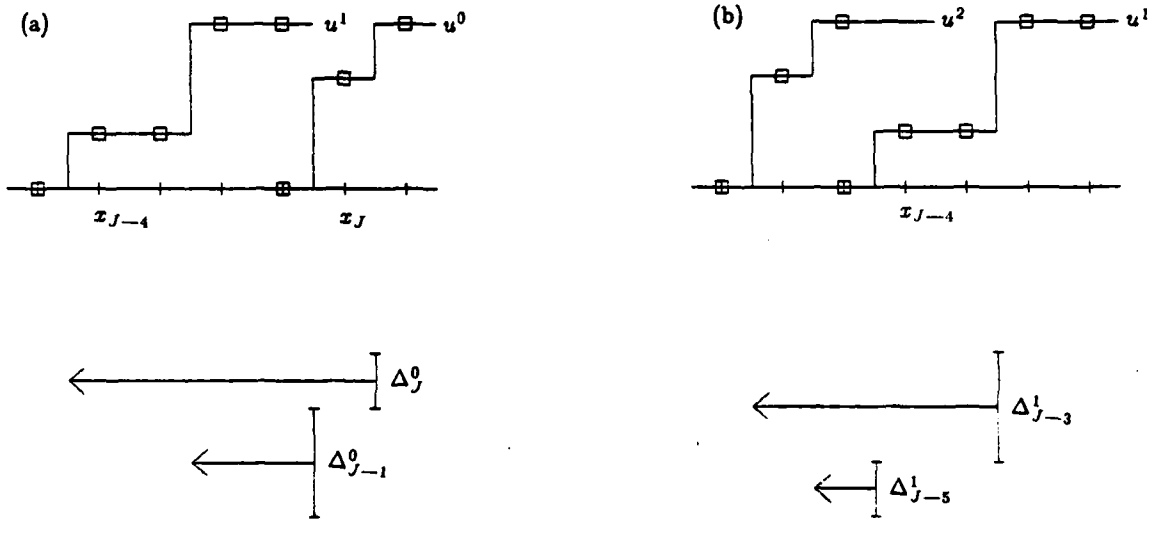

Figure 4.1. *An example of shock propagation without explicit handling of interactions. The squares represent the grid function, the solid line its interpretation. The arrows show the propagation of the* Δ_j.

Each c_j^n is the propagation speed of the discontinuity Δ_j^n according to the Rankine-Hugoniot relation. We do not need to define c_j^n if $\Delta_j^n = 0$; there is no need to propagate jumps of height zero. In general the discontinuity Δ_j^n is now a shock which travels at speed c_j^n and should propagate ν_j^n grid points in one time step. For concreteness assume that it is propagating to the left. As a first attempt at generalizing the procedure of section 3 we try adding Δ_j^n to $u_j^n, u_{j-1}^n, \ldots, u_{j-\mu_j^n+1}^n$ and $(\nu_j^n - \mu_j^n)\Delta_j^n$ to $u_{j-\mu_j^n}^n$. This gives a conservative scheme, since

$$\sum_j u_j^{n+1} = \sum_j u_j^n + \sum_j \nu_j^n \Delta_j^n$$

so

$$\begin{aligned}\sum_j h u_j^{n+1} - \sum_j h u_j^n &= k \sum_j (f(u_j^{n+1}) - f(u_j^n)) \\ &= k(f(+\infty) - f(-\infty)).\end{aligned}$$

Again this scheme can be interpreted as the method of characteristics with linear interpolation. The problem is that when some $\nu_j^n > 1$ we may be allowing characteristics to cross, i.e. some shocks may propagate right past their neighbors, rather than coalescing into a single shock as should happen. An example will illustrate this.

Example 4.1. Consider $f(u) = \frac{1}{2}u^2$ in (4.1), i.e. $u_t = uu_x$. Take the initial conditions

$$u_j^0 = \begin{cases} 0 & j < J \\ 2 & j = J \\ 3 & j > J \end{cases} \tag{4.2}$$

and $k = 2h$. Then

$$\begin{aligned} \Delta_{J-1}^0 = 2 \qquad c_{J-1}^0 = 1 \qquad \nu_{J-1}^0 = 2 \\ \Delta_J^0 = 1 \qquad c_J^0 = 5/2 \qquad \nu_J^0 = 5 \end{aligned}$$

We thus add Δ_{J-1}^0 to the two mesh points u_{J-1}^0 and u_{J-2}^0 and Δ_J^0 to the five mesh points $u_J^0, \ldots, u_{J-4}^0$. We find that

$$u_j^1 = \begin{cases} 0 & j < J-4 \\ 1 & J-4 \le j \le J-3 \\ 3 & j \ge J-2 \end{cases} \tag{4.3}$$

This is illustrated in Figure 4.1a. Rather than coalescing into a single shock, Δ_J^0 has "passed right by" Δ_{J-1}^0. The resulting u^1 is not the optimal representation of the true solution at time t_1, although it is a conservative representation. (The true solution is a single shock at $x_{J-3} - h/6$.)

Remarkably, the shock does not become further smeared out at subsequent times. In fact, starting from (4.3) we compute that

$$u_j^2 = \begin{cases} 0 & j < J-7 \\ 2 & j = J-7 \\ 3 & j > J-7 \end{cases}$$

as shown in Figure 4.2b. In this case the shock has sharpened and is, in fact, the optimal representation of the true solution: a shock of height 3 at $x_{J-6} - h/6$. It has precisely the same shape as u^0 and at future times $u^3, u^4, u^5, \ldots$ will alternate between the shape of u^1 and that of u^0, always being a conservative and monotonic representation of the true solution (although not always the optimal representation). In this example the shock is never badly smeared and the solution is quite acceptable for all t_n. If we had taken k/h much larger, however, the resulting solution would have been badly smeared, at least for some t_n.

This problem can be greatly alleviated by explicitly handling the coalescing of shocks. This is not as difficult as it sounds due to the following result.

PROPOSITION 4.1. *Consider the conservation law $u_t = (f(u))_x$ with $f''(u) > 0$ for $\alpha < u < \gamma$ and initial conditions*

$$u(x,0) = \begin{cases} \alpha & x < \xi_1 \\ \beta & \xi_1 \le x \le \xi_2 \\ \gamma & x > \xi_2 \end{cases}$$

for some $\xi_1 < \xi_2$, $\alpha < \beta < \gamma$. Let t_c be the time at which the two shocks coalesce. Consider the same problem $v_t = (f(v))_x$ with initial conditions

$$v(x,0) = \begin{cases} \alpha & x < \xi_0 \\ \gamma & x \ge \xi_0 \end{cases}$$

where $\xi_0 = ((\beta - \alpha)\xi_1 + (\gamma - \beta)\xi_2)/(\gamma - \alpha)$. Then, for $t \ge t_c$,

$$u(x,t) = v(x,t) \qquad \forall x.$$

The same result holds if $\alpha > \beta > \gamma$ and $f''(u) < 0$ for $\gamma < u < \alpha$.

This result is an immediate consequence of conservation. The implication is that whenever two adjacent shocks Δ^n_{J-1} and Δ^n_J are going to coalesce at some time between t_n and t_{n+1} (i.e. when $kc^n_J > kc^n_{J-1} + h$) we can simply merge them into a single shock $\hat{\Delta} = \Delta^n_{J-1} + \Delta^n_J$ at time t_n and propagate this merged shock. At time t_n, $\hat{\Delta}$ is located at $\hat{x} = (\Delta^n_{J-1} x_{J-1/2} + \Delta^n_J x_{J+1/2})/\hat{\Delta}$ and it propagates at speed $\hat{c} = (f(u^n_{J+1}) - f(u^n_{J-1}))/(u^n_{J+1} - u^n_{J-1})$. We then optimally represent the resulting shock solution at time t_{n+1} on the discrete grid. Applying this procedure to example 4.1 gives

$$u^n_j = \begin{cases} 0 & j < J - 3n \\ 2 & j = J - 3n \\ 3 & j > J - 3n \end{cases} \qquad \forall n$$

which is always the optimal representation of the true solution.

For general data $\{u^n_j\}$ it may be necessary to merge several shocks at a time. Since the propagation speeds of the shocks change as they coalesce, it is not always easy to write out the appropriate equivalent problem *a priori*. The following procedure can be used, assuming Δ^n_j is nonzero for a finite number N of points x_j. Let $\mathcal{T} = \{T_i\}$ be the set of ordered triples

$$T_i = (\hat{x}_i, u_i^-, u_i^+)$$

where each $\hat{x}_i$ is the location of a shock, a discontinuous jump in u from the value u_i^- on the left to u_i^+ on the right. The T_i are ordered from left to right so that $\hat{x}_i < \hat{x}_{i+1}$. For each triple we define

$$\hat{c}_i = (f(u_i^+) - f(u_i^-))/(u_i^+ - u_i^-)$$
$$\hat{\Delta}_i = u_i^+ - u_i^-.$$

Initially the T_i are chosen as the triples $(x_{j+1/2}, u^n_j, u^n_{j+1})$ corresponding to nonzero Δ^n_j. We wish to replace $\mathcal{T}$ by a new set of shocks S which do not interact in time k and which yield the same exact solution at time t_{n+1} as the original set. This can be accomplished by Algorithm 4.1, which works from left to right and employs a stack of triples S_i for $i = 1, 2, \ldots, I$. *The rightmost shock* on the stack is represented by S_I, which is on top of the stack. The stack has the property that at the beginning of each iteration the shocks represented by the S_i would not interact in time k if there were no disturbance to the right of $\hat{x}_I$ (i.e. if $u \equiv u_I^+$ for $x > \hat{x}_I$). Each new shock is put on top of the stack and then is merged as necessary with shocks already on the stack.

ALGORITHM 4.1.

$I := 1$
$S_1 := T_1$
for $j := 2, 3, \ldots, N$ **do**
  $I := I + 1$
  $S_I := T_j$
  while $0 < (\hat{x}_I - \hat{x}_{I-1})/(\hat{c}_I - \hat{c}_{I-1}) < k$ **do**
    remove S_I and S_{I-1} from the stack and replace them by a single triple with
      $\hat{x} = (\hat{x}_{I-1}\hat{\Delta}_{I-1} + \hat{x}_I\hat{\Delta}_I)/(\hat{\Delta}_{I-1} + \hat{\Delta}_I)$
      $u^- = u^-_{I-1}$
      $u^+ = u^+_I$
    $I := I - 1$

It looks as if this algorithm may require $O(N^2)$ steps in the worst case, because of the nested inner loop. It is always linear, however, because the innermost step merges two shocks. Since we begin with only N shocks this will be executed at most N times.

PROPOSITION 4.2. *For any stepsize k, Algorithm 4.1 computes the exact solution to (4.1) for piecewise constant initial data with a finite number of shocks.*

Proof. Two shocks which appear to interact in time k when considered in isolation must also interact in time k in the presence of other shocks. Since the merging procedure is associative, the order in which we merge them is irrelevant. ∎

In general, errors will be introduced interpreting and representing the solution. For "smooth" portions of the solution the method can again be viewed as the method of characteristics with linear interpolation. The error in a single time step is thus $O(h^2)$, giving $O(h^2/k)$ global error. A true discontinuity in the solution is propagated to exactly the correct location, providing the solution is properly interpreted.

5. Rarefaction waves.

Now consider a jump for which $\Delta_j^n f''(u) < 0$. Such a discontinuity should spread out into a rarefaction wave. Unfortunately, applying the procedure of section 4 directly will cause Δ_j^n to propagate as a sharp discontinuity.

In dealing with shocks we sometimes found it necessary to merge several discontinuities into a single shock. Rarefaction waves can be approximated by taking the opposite approach. The discontinuity Δ_j^n is broken up into several (say m) discontinuities each of height Δ_j^n/m and all located at $x_{j+1/2}$. These discontinuities will then travel at different speeds, spreading out to approximate the true rarefaction. We should choose m large enough that the resulting solution looks smooth. Since the true rarefaction spreads over $(f'(u^n_{j+1}) - f'(u^n_j))k/h$ meshpoints, it is reasonable to choose m as an approximation to this quantity, e.g. as the integer part of some finite difference approximation. In general rarefaction and shock waves may interact. It will again be necessary to handle this interaction explicitly in order to obtain good results for large values of k/h. These interactions can also be handled by Algorithm 4.1, at least approximately, if we define m triples T_i, $i = 1, 2, \ldots, m$ corresponding to each partitioned discontinuity with

$$\hat{x}_i = x_{j+1/2}$$
$$u_i^- = u_j^n + \frac{i-1}{m}\Delta_j^n$$
$$u_i^+ = u_j^n + \frac{i}{m}\Delta_j^n.$$

See Figure 5.1 for an example of this.

Example 5.1. In order to better understand this shock–rarefaction interaction, consider the problem $u_t = (\frac{1}{2}u^2)_x$ with initial data

$$u_j^0 = \begin{cases} 4 & j = J \\ 0 & j \neq J. \end{cases} \tag{5.1}$$

This is a representation of

$$u(x,0) = \begin{cases} 4 & x_{J-1/2} < x < x_{J+1/2} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Take $k = h$. Then the true solution at time k is

$$u(x,k) = \begin{cases} (x_{J+1/2} - x)/h & x_{J+1/2} - \sqrt{8}h < x < x_{J+1/2} \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

The rarefaction has overtaken the shock, slowing it down and decreasing its strength. In the spirit of Proposition 4.1 we can attempt to find a new set of initial conditions which give the same solution at time k without any interaction. The desired initial conditions are

$$v(x,0) = \begin{cases} \sqrt{8} & x_{J+1/2} - \sqrt{2}h < x < x_{J+1/2} \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

This pulse is wider and shorter than the original pulse (see Figure 5.1). In the solution $v(x,t)$ there is no interaction before time k at which point the rarefaction wave just catches up to the shock. For $t \geq k$, $v(x,t) = u(x,t)$ for all x. Computing the exact equivalent set of noninteracting discontinuities (such as (5.4)) for a general scalar conservation law with step function initial conditions would be difficult but could be done. Taking this approach and then solving the resulting Riemann problems exactly would yield the exact solution to such a problem, just as Algorithm 4.1 gives the exact solution for a pure shock problem.

Instead of following this course, we will concentrate on the more easily obtainable approximate solution given by Algorithm 4.1 together with the previously described rarefaction partitioning procedure. Taking $m = 4$ in the rarefaction gives us 5 triples T_i in the set $\mathcal{T}$. These are

$$\begin{aligned} T_1 &= (x_{J-1/2},\ 0,\ 4), \\ T_2 &= (x_{J+1/2},\ 4,\ 3), \\ T_3 &= (x_{J+1/2},\ 3,\ 2), \\ T_4 &= (x_{J+1/2},\ 2,\ 1), \\ T_5 &= (x_{J+1/2},\ 1,\ 0). \end{aligned}$$

In applying Algorithm 4.1, we see that T_1 and T_2 interact. This is the only interaction and the resulting set S consists of 4 triples. The first of these, S_1, is derived from T_1 and T_2 and has

$$\begin{aligned} \hat{x}_1 &= \tfrac{1}{3}(4x_{J-1/2} - x_{J+1/2}) \\ &= x_J - \tfrac{5}{6}h \\ u_1^- &= 0 \\ u_1^+ &= 3. \end{aligned}$$

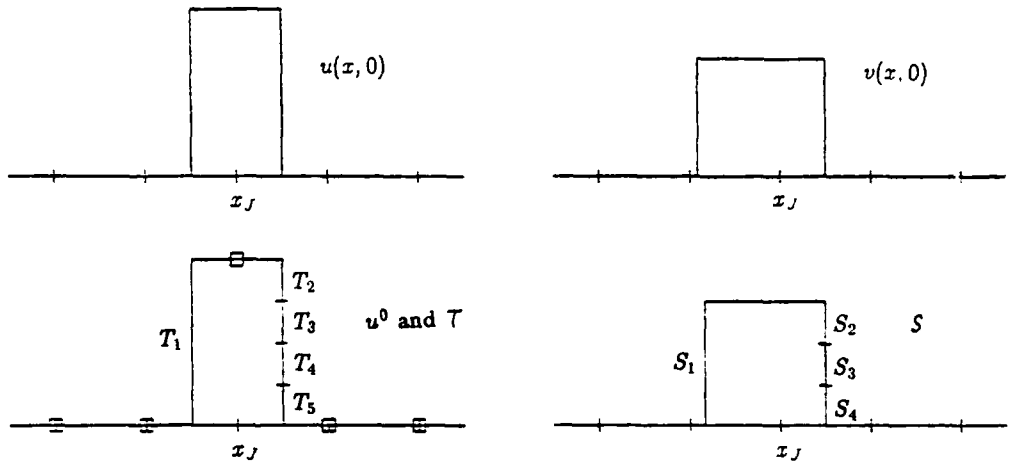

Figure 5.1. *$u(x,0)$ are the original initial conditions for Example 5.1. These can be replaced by the noninteracting initial conditions $v(x,0)$ which produce the same solution at time k. $\mathcal{T}$ is the original set of discontinuities. Algorithm 4.1 replaces these by the noninteracting set S.*

Figure 5.2. *(a) shows the set S at time k. (b) shows the representation of S (as squares) and the true solution at time k.*

Figure 5.1 shows the initial data both before and after merging. The original pulse has been replaced by a wider and shorter pulse which approximates (5.4). We advance the solution to time k by moving each S_i at the appropriate speed. The final positions are

$$\begin{aligned} S_1: &\quad (x_J - \tfrac{5}{6}h) - \tfrac{3}{2}h &&= x_{J-2} - \tfrac{1}{3}h \\ S_2: &\quad x_{J+1/2} - \tfrac{5}{2}h &&= x_{J-2} \\ S_3: &\quad x_{J+1/2} - \tfrac{3}{2}h &&= x_{J-1} \\ S_4: &\quad x_{J+1/2} - \tfrac{1}{2}h &&= x_J \end{aligned}$$

This is shown in Figure 5.2a. These discontinuities are then represented on the mesh in the usual way, giving the solution

$$u_j^1 = \begin{cases} 2 & j = J-2 \\ 3/2 & j = J-1 \\ 1/2 & j = J \\ 0 & \text{otherwise} \end{cases}$$

which is shown together with the exact solution in Figure 5.2b.

In practice it is necessary to modify this procedure slightly in order to maintain smoothness in the rarefaction waves over several time steps. Rather than putting each jump in the partitioned rarefaction at $x_{j+1/2}$, we spread them out between x_j and x_{j+1} by setting

$$\hat{x}_i = x_j + \frac{ih}{m+1} \qquad i = 1, 2, \ldots, m.$$

This procedure, together with Algorithm 4.1, has given quite satisfactory results for a variety of problems. It must be stressed, however, that Algorithm 4.1 may not handle interactions properly when several shocks and rarefactions are present. Consider, for example a problem involving two shocks followed by a rarefaction. It may be that the two shocks, considered in isolation, would coalesce in time k, whereas in reality the the rarefaction catches up to the second shock and slows it down sufficiently that no further interaction occurs. Algorithm 4.1, which works from left to right, would produce an incorrect result which may be quite bad for large Courant numbers.

Even more disturbing is the fact that monotonicity of the $\hat{x}_i$ may be lost when merging a shock with a rarefaction. If $\hat{\Delta}_I$ and $\hat{\Delta}_{I-1}$ have opposite signs then the new $\hat{x}$ defined in Algorithm 4.1 does not lie between $\hat{x}_I$ and $\hat{x}_{I-1}$ and hence may lie to the left of $\hat{x}_{I-2}$ or to the right of the next $x_{j+1/2}$. Currently this is handled by aborting execution whenever $\hat{x}_I < \hat{x}_{I-1}$. This has never happened for moderate values of the Courant number on "reasonable" initial data.

6. Numerical results for 1D scalar conservation laws

Before proceeding to briefly discuss several space dimensions and systems of conservation laws, we pause to present some computational examples.

Figure 6.1 shows several sets of initial data. The remaining figures show the results of applying the method to problems with those initial conditions. In Figures 6.2 through 6.4, the true solutions are known and are shown as solid lines. The numerical solutions are represented by the squares.

The other examples begin with more complicated initial data and use periodic boundary conditions (achieved by applying Algorithm 4.1 to a larger interval with the mesh function suitably extended). The numerical solutions are shown together with an "exact" solution computed on a much finer mesh. These examples are included to show that the procedure can indeed handle problems involving many interactions.

The method does have its limitations, of course. For problems beginning with the initial conditions shown in Figure 6.1e, taking values of k/h larger than 10 sometimes led to nonmonotonicity of the $\hat{x}_i$ as discussed in section 5.

7. Several space dimensions and systems of conservation laws.

Scalar conservation laws in several space dimensions can be handled by means of a spacial splitting or fractional step method. This is a standard way of extending one dimensional methods to higher space dimensions. Crandall and Majda[2] discuss the use of fractional step methods for computing weak solutions of conservation laws.

In two space dimensions, the general conservation law has the form

$$u_t(x,y,t) = \big(f(u(x,y,t))\big)_x + \big(g(u(x,y,t))\big)_y. \tag{7.1}$$

Let $S_x(k)$ denote the approximate solution operator for the problem $u_t = (f(u))_x$ on a time-step of length k as defined in the preceeding sections. Similarly, let $S_y(k)$ denote the approximate solution operator for $u_t = (g(u))_y$. Then u^{n+1} is computed by solving a sequence of one-dimensional problems, alternating between the x- and y-directions. Using a second order Strang-type splitting as described in [8], we set

$$u^{n+1} = S_x(k/2)S_y(k)S_x(k/2)u^n. \tag{7.2}$$

We now have to contend with errors both in the 1D difference scheme and in the splitting (7.2).

Example 7.1. Consider the problem

$$u_t = (\frac{1}{2}u^2)_x + (\frac{1}{2}u^2)_y \qquad \text{for} \quad 0 \le x \le 1, \ 0 \le y \le 1$$

with initial data as shown in Figure 7.1. This test problem has been used by Gropp[4]. The true solution at time 1 is shown in Figure 7.2.

In this case each 1D problem involves a single shock, and can be solved "exactly" by using sufficiently large time steps. The resulting shocks in the full 2D problem are still sharp, but are all parallel to the x- or y-axes. The result is that shocks which have some other orientation, such as the "diagonal" shocks in Figure 7.2, are represented by zig-zagging shocks. The number of zigs and zags is directly proportional to the number of 1D problems solved to reach the given time. In this case the best results are obtained by using moderately small values of ν. But we still want $\nu > 1$, or else the 1D procedure reduces to upwind differencing and the shocks are badly smeared. Figure 7.3 shows some examples for various values of ν.

At the present time it is not clear how to extend this method to handle arbitrary systems of conservation laws. In general, a result like Proposition 4.1 will no longer hold, so that interactions cannot be handled in the same simple manner. For problems in which the eigenvectors of the Jacobian matrix $\partial f/\partial u$ are nearly constant (as in some weak-shock problems), it may be possible to handle interactions sufficiently well to generalize the methods of Roe[9]. Research is continuing on this approach.

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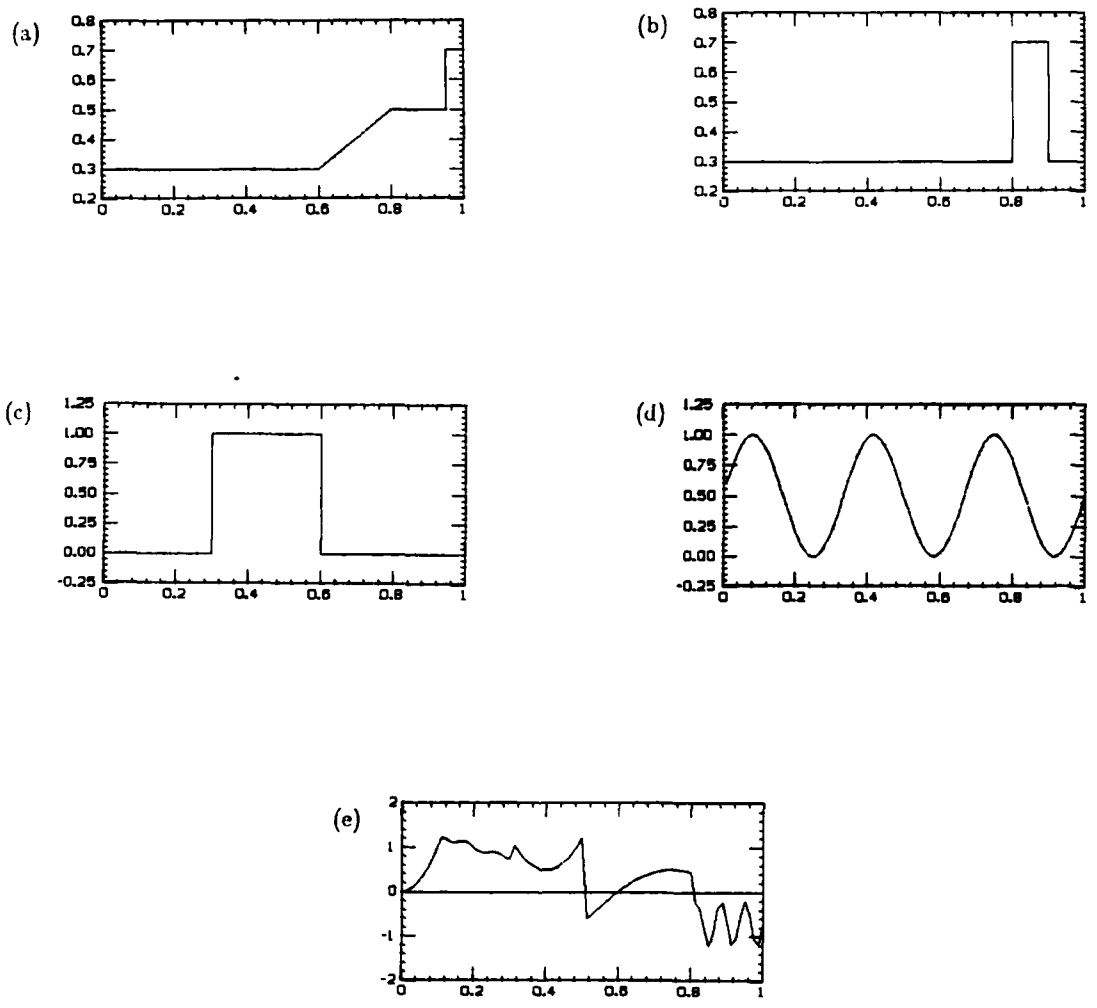

Figure 6.1. *Initial conditions for the test problems.*

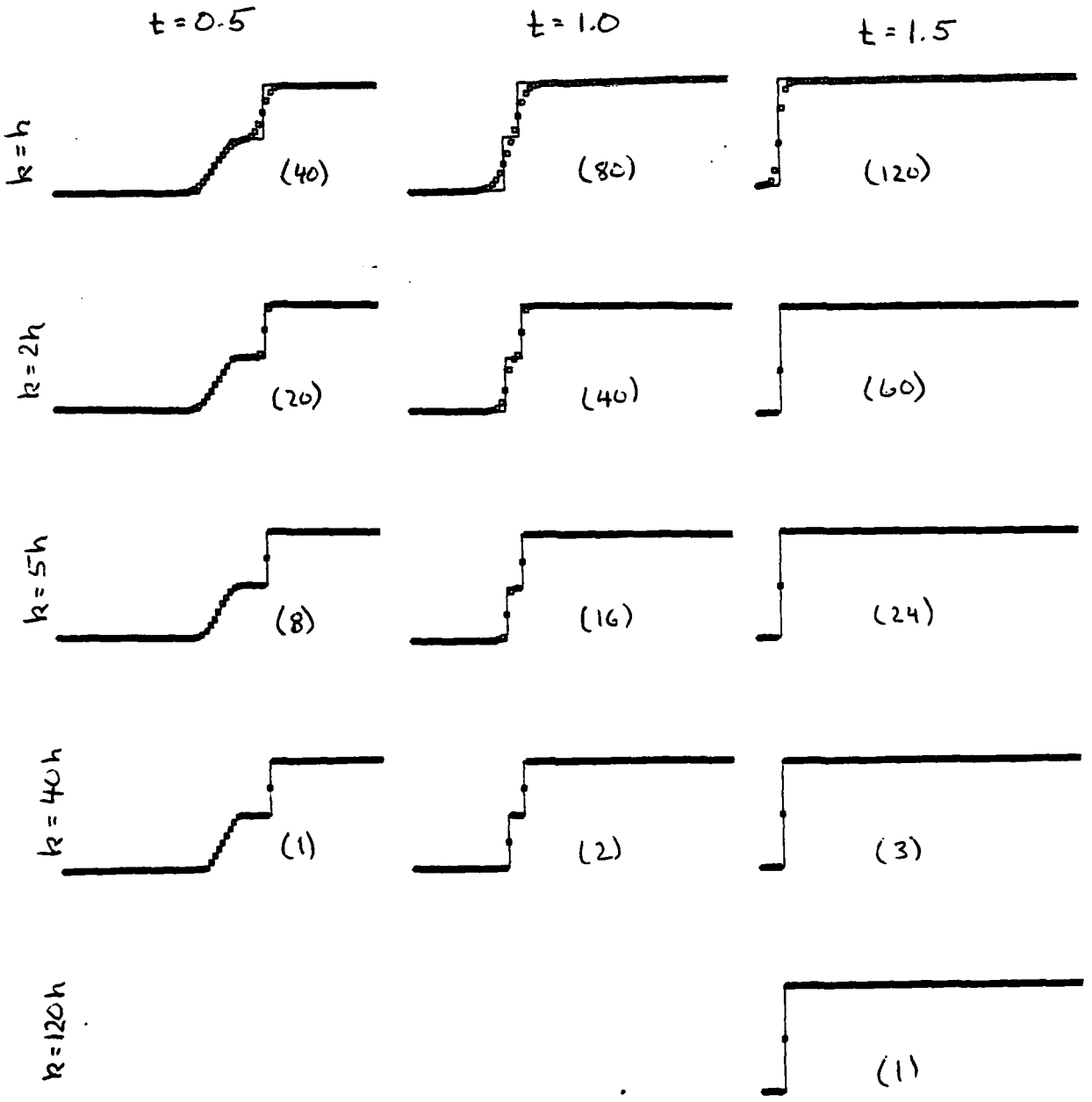

Figure 6.2. *True and computed solutions for* $u_t = (\frac{1}{2}u^2)_x$ *and initial conditions from Figure 6.1a. In all cases* $h = 1/80$. *The numbers in parentheses indicate the number of time-steps needed to compute each solution.*

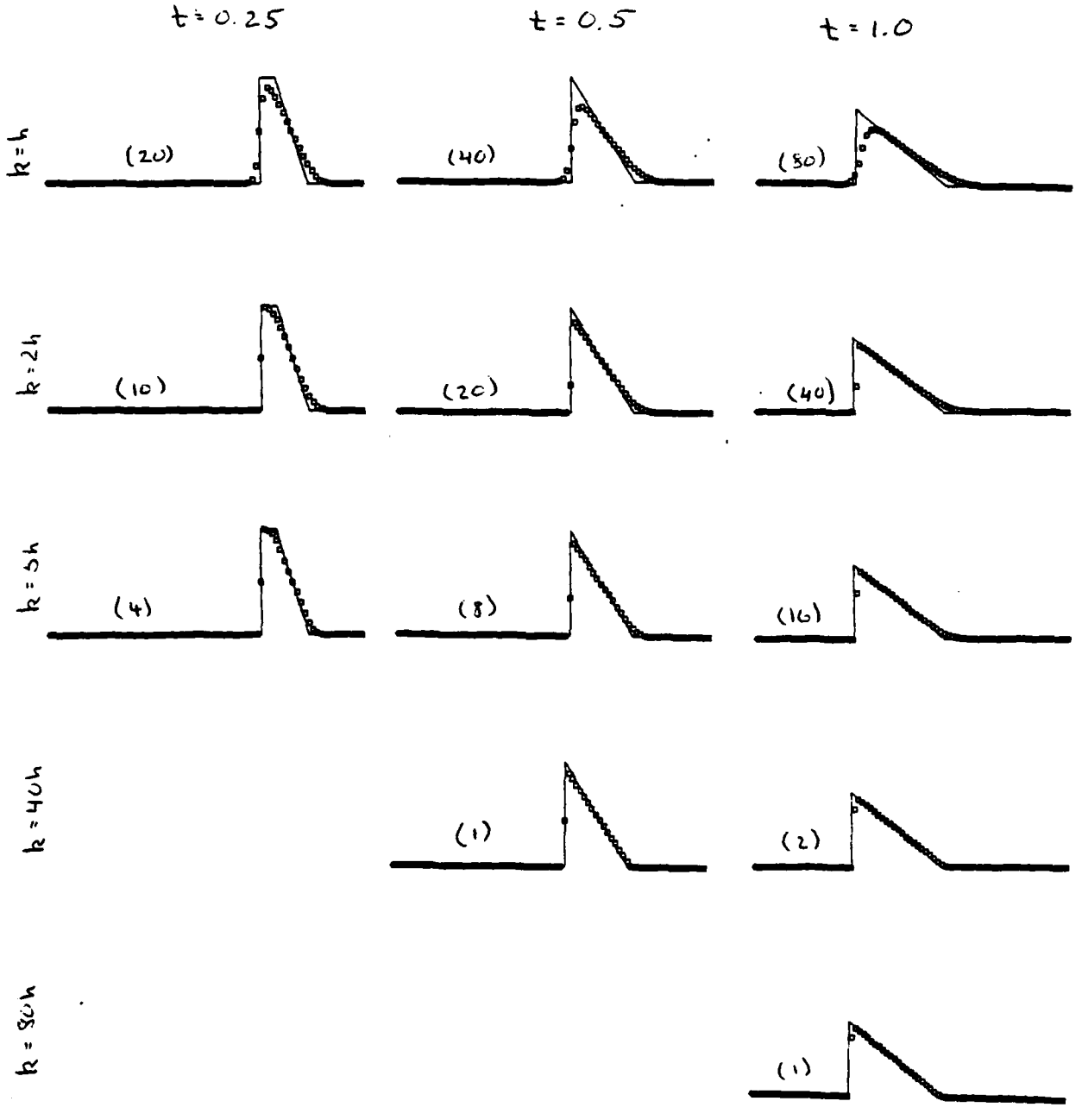

Figure 6.3. *True and computed solutions for* $u_t = (\frac{1}{2}u^2)_x$ *and initial conditions from Figure 6.1b. In all cases* $h = 1/80$. *The numbers in parentheses indicate the number of time-steps needed to compute each solution.*

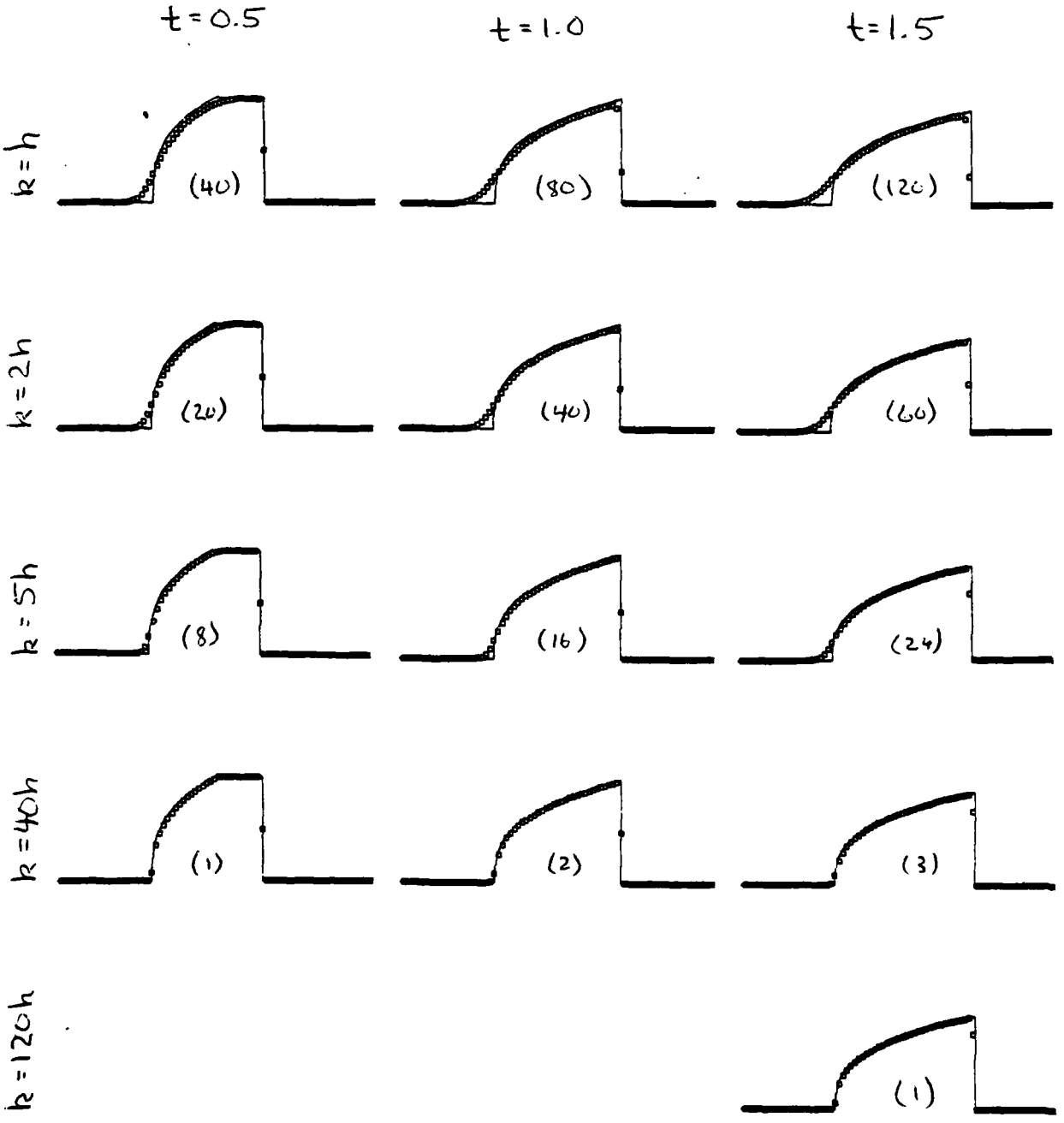

Figure 6.4. *True and computed solutions for* $u_t = (-0.1u^3)_x$ *and initial conditions from Figure 6.1c. In all cases* $h = 1/80$. *The numbers in parentheses indicate the number of time-steps needed to compute each solution.*

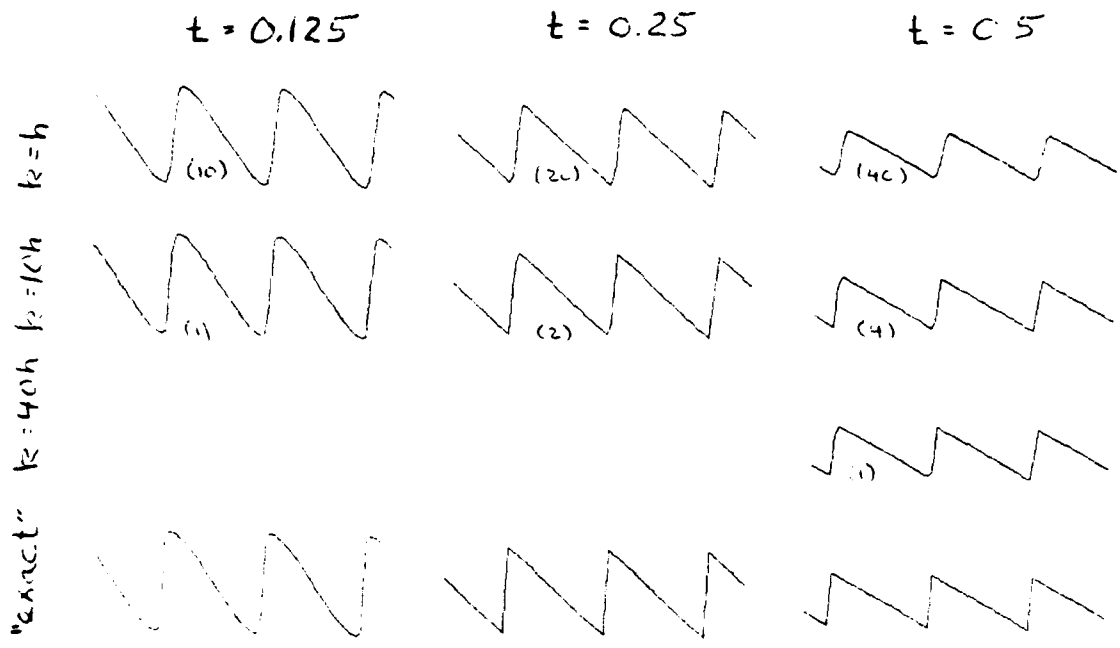

Figure 6.5 *Computed solutions for* $u_t = (\frac{1}{2}u^2)_x$ *and initial conditions from Figure 6.1d with* $h = 1/80$. *The "exact" solution shown at the bottom was computed using* $h = 1/240$. *The numbers in parentheses indicate the number of time-steps needed to compute each solution.*

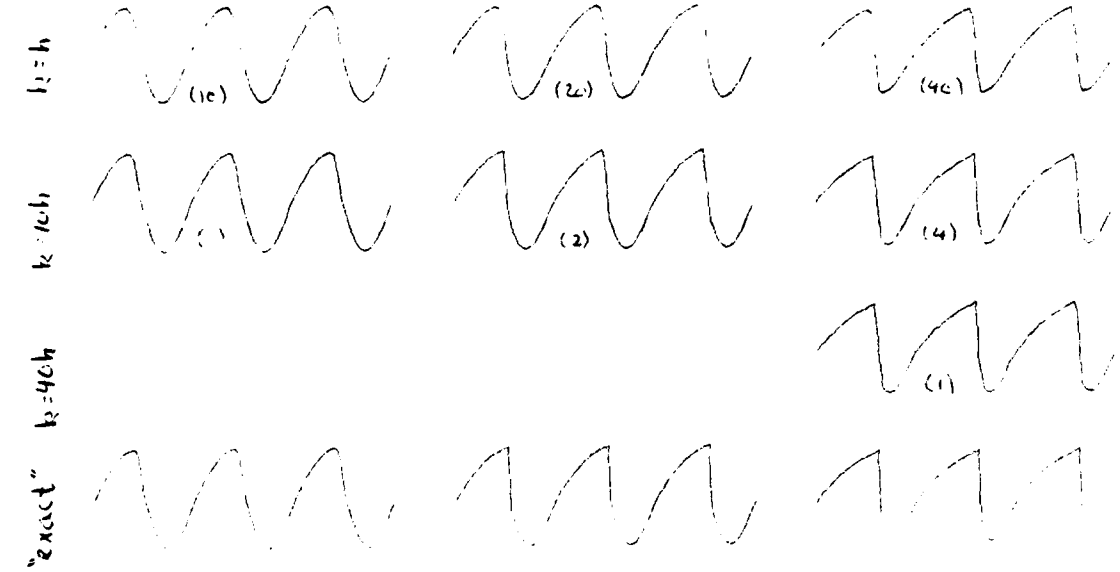

Figure 6.6. *Computed solutions for* $u_t = (-0.1u^4)_x$ *and initial conditions from Figure 6.1d with* $h = 1/80$. *The "exact" solution shown at the bottom was computed using* $h = 1/240$. *The numbers in parentheses indicate the number of time-steps needed to compute each solution.*

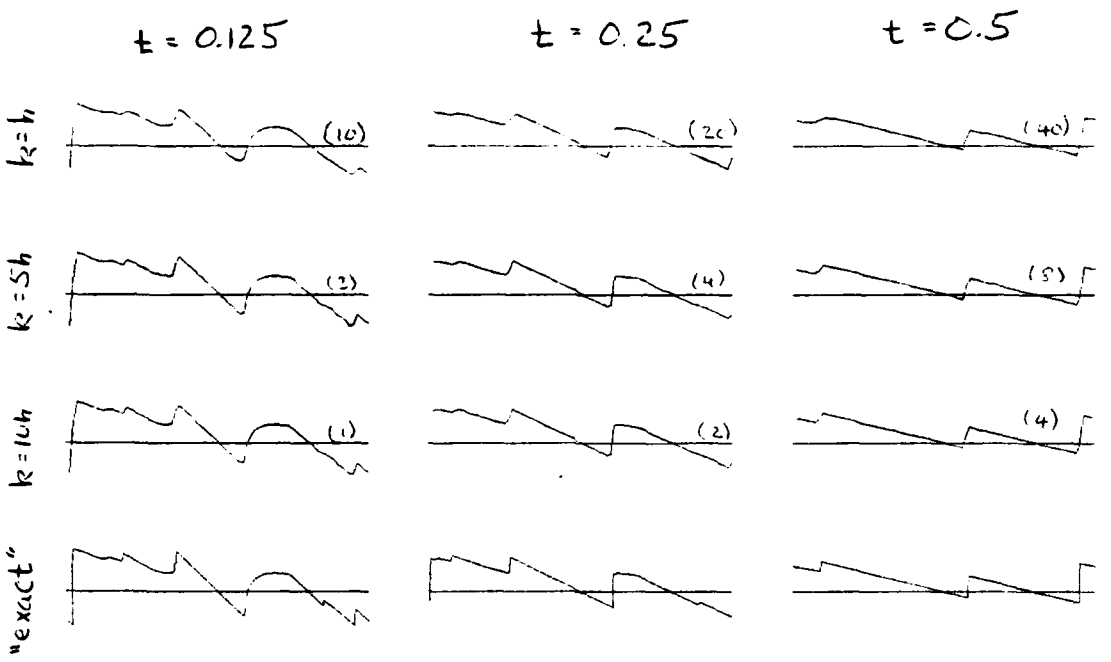

Figure 6.7 *Computed solutions for* $u_t = (\frac{1}{2}u^2)_x$ *and initial conditions from Figure 6.1e with* $h = 1/80$. *The "exact" solution shown at the bottom was computed using* $h = 1/480$. *The numbers in parentheses indicate the number of time-steps needed to compute each solution.*

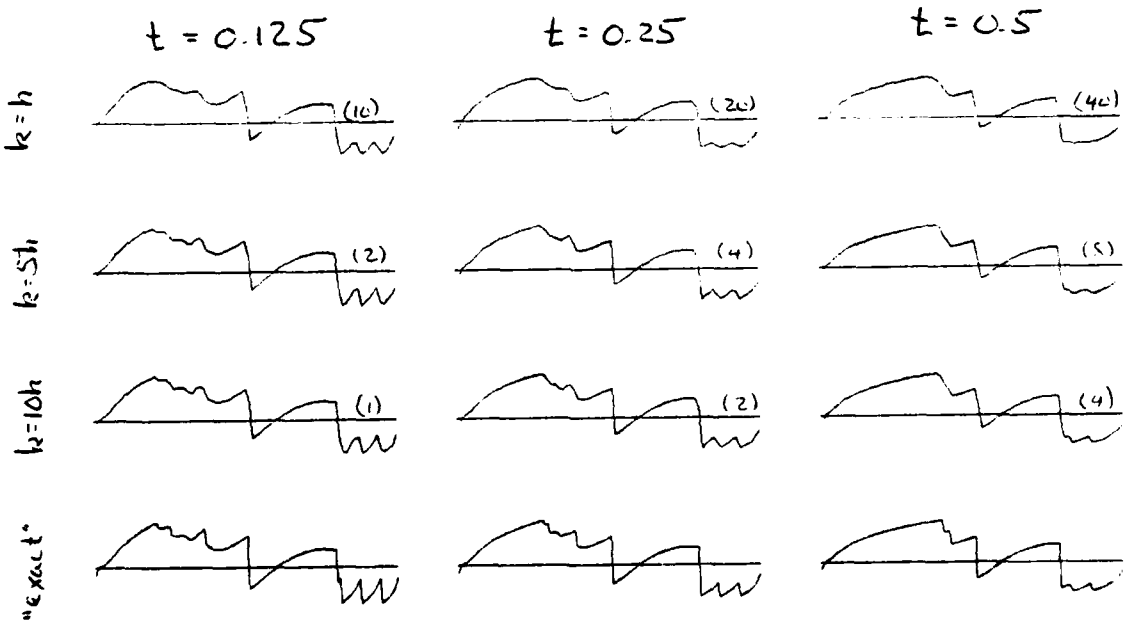

Figure 6.8. *Computed solutions for* $u_t = (-0.1u^4)_x$ *and initial conditions from Figure 6.1e with* $h = 1/80$. *The "exact" solution shown at the bottom was computed using* $h = 1/480$. *The numbers in parentheses indicate the number of time-steps needed to compute each solution.*

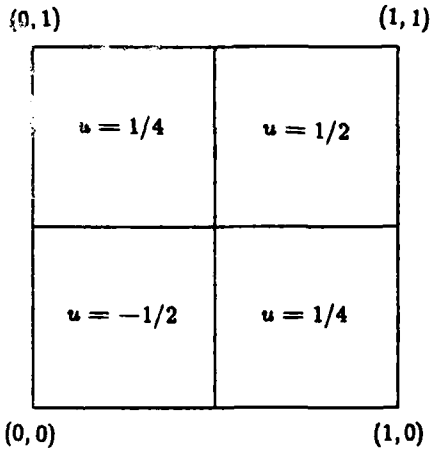

Figure 7.1. *Initial conditions for Example 7.1*

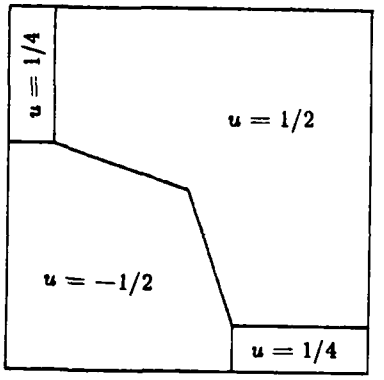

Figure 7.2. *Solution at time* $t = 1$.

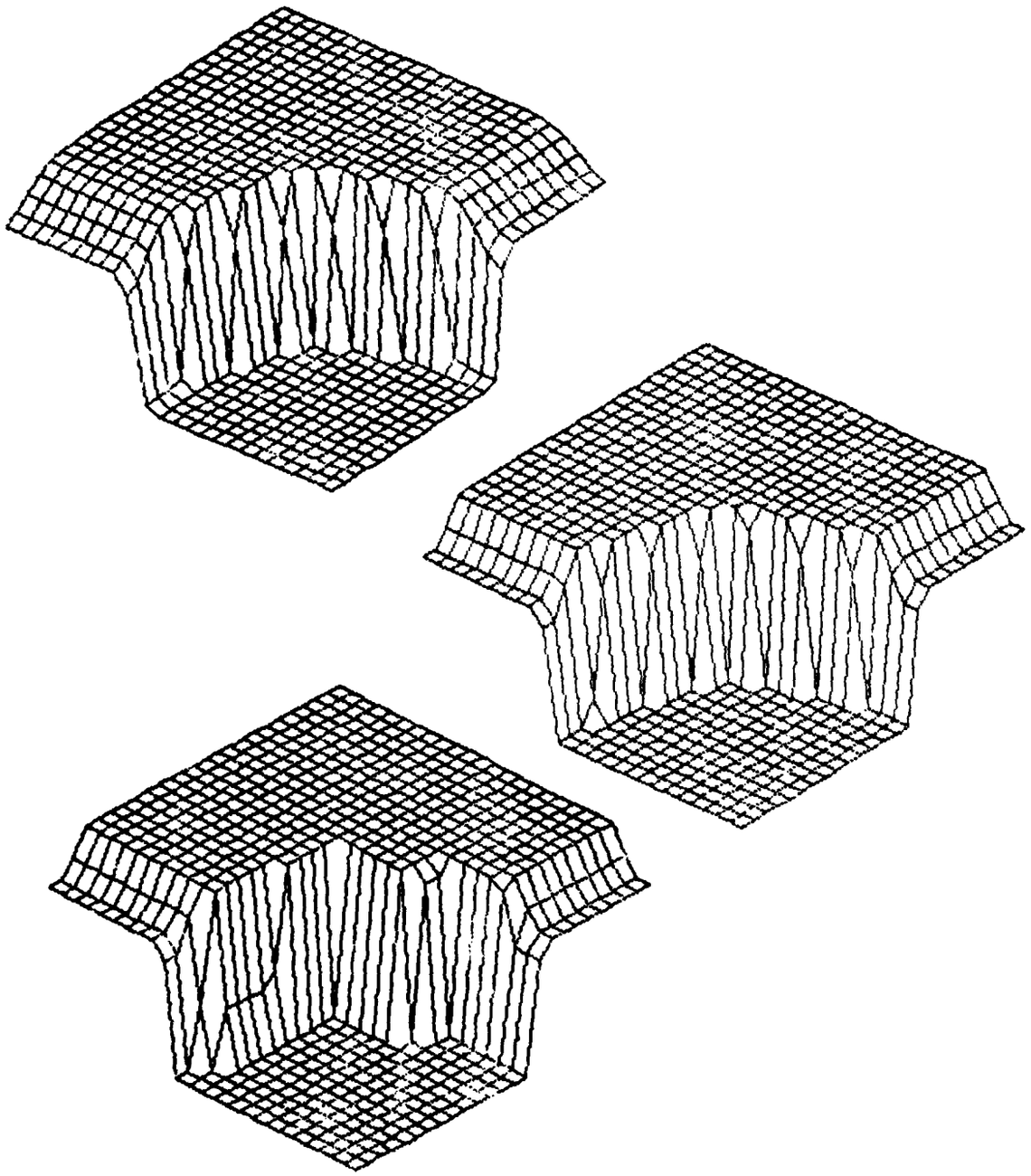

Figure 7.3 *Computed solutions to Example 7.1 at* $t = 1$. *In all cases* $h = 1/25$. *The three cases shown are* $k = h$, $k = 5h$, *and* $k = 25h$.

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