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  TO THE WE   L PROCESS MODEL

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A system testing program may consist of two kinds of test phases: those in which system configuration is continually modified to reduce the frequency of failures and those in which system configuration does not change. The system failure rate may be modeled as the intensity function of a Weibull process in the former case and a homogeneous Poisson process in the latter case. This paper considers a nonhomogeneous Poisson process consisting of a time truncated Weibull process followed by a time truncated homogeneous Poisson process and illustrates how such an extended model can be used to plan and evaluate a reliability demonstra-

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Abstract: (continued)

tion test. The resulting methodology allows a unified treatment of failure data from two consecutive test phases and can reduce consumer and producer risks while at the same time requiring less test time.

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AN EXTENSION TO THE WEIBULL PROCESS MODEL

## 1. INTRODUCTION

Recent papers by Bain and Engelhardt (1980)[1] and Crow (1977)[2] have investigated inferential procedures for the time truncated Weibull process, a nonhomogeneous Poisson process with intensity $R(t/T)^{\beta-1}$. An intensity function of this form is often used to model the failure rate of a repairable system that is being tested for reliability growth (see Figure 1, test phase A). In such applications, a test phase in which the failure rate is changing is sometimes followed by a test phase in which system configuration is fixed and the failure rate is therefore constant (Figure 1, test phase B). This latter test phase is modeled as a homogeneous Poisson process with intensity R and is usually intended to demonstrate the reliability of the final system configuration with adequate confidence. In this paper, we show how to conduct a unified analysis of the failure data from the two consecutive test phases so that the data from the Weibull process can be combined with the data from the homogeneous Poisson process to evaluate the final system failure rate R.

Our primary objective in studying this compound model is to reduce the length of test phase B while controlling the overall risks to consumer and producer. Optimal procedures for accomplishing this objective are introduced in Section 3 and illustrated in Sections 4 and 5. Also provided are point estimators for the model parameters (Section 2) and a test for goodness-of-fit (Section 6).

## 2. POINT ESTIMATORS

The Weibull process begins at time 0 and terminates at time T; the homogeneous Poisson process with intensity R then commences and continues until time T + S. The intensity function of the resulting nonhomogeneous Poisson process is shown in Figure 1 and may be written

$$r(t) = \begin{cases} R(t/T)^{\beta-1} & \text{if } 0 \leq t \leq T, \\ R & \text{if } T \leq t \leq T + S, \end{cases} \tag{1}$$

where $R > 0$ and $\beta > 0$. Let K be the total number of failures up to time T + S, N the number of failures that occur before time T, and $T_1, \ldots, T_N$ the failure times for the time truncated Weibull process ($0 < T_1 < \ldots < T_N < T$). Then the Poisson process with intensity function r(t) has a sample function density given by

$$f_{K,N,T_1,\ldots,T_N}(k,n,t_1,\ldots,t_n)$$

---

1 Bain, L. J. and M. Engelhardt, "Inferences on the Parameters and Current System Reliability for a Time Truncated Weibull Process," Technometrics, Vol. 22, pp. 421-426, August 1980.

2 Crow, L. H., Confidence Interval Procedures for Reliability Growth Analysis, Technical Report No. 197, US Army Materiel Systems Analysis Activity, Aberdeen Proving Ground, MD, June 1977.

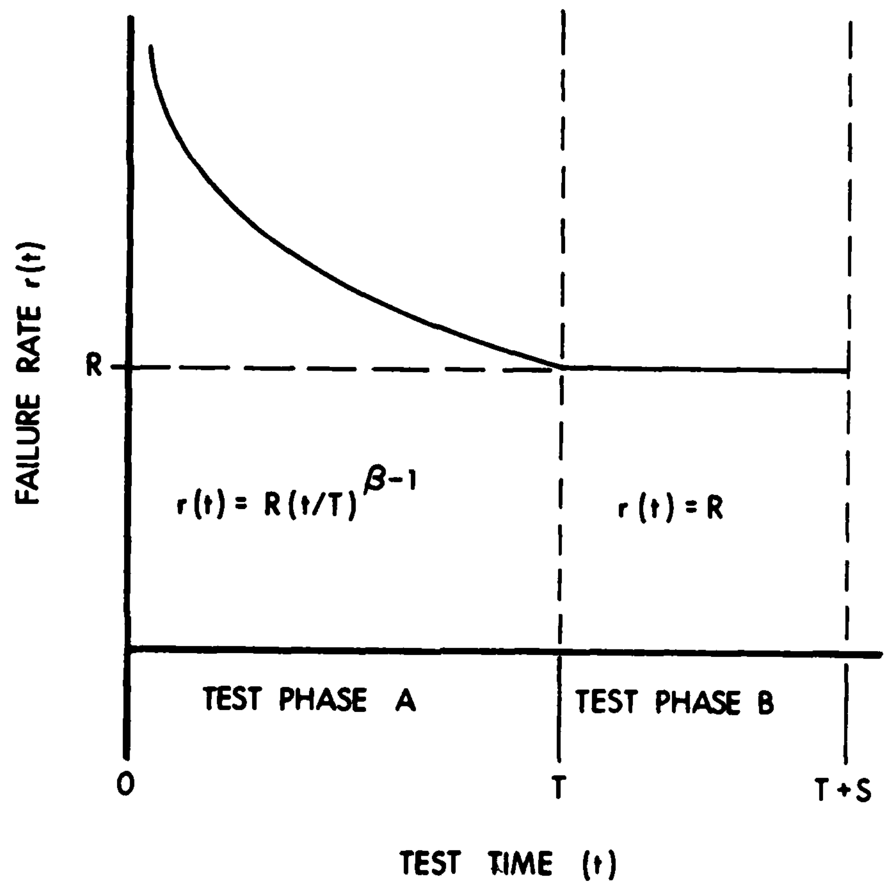


Figure 1. Intensity Function for the Case $0<\beta<1$.

$$= \begin{cases} \exp(-RT/\beta - RS)\ (RS)^k/k! & \text{if } N=0, \quad (2.1) \\ R^k \left[\prod_{i=1}^{n} (T/t_i)\right]^{1-\beta} \dfrac{S^{k-n}\exp(-RT/\beta - RS)}{(k-n)!} & \text{if } N=n>0, \quad (2.2) \end{cases}$$

where $n=1,2,\ldots$ and $0 < t_1 < \ldots < t_n < T$, $k=0,1,2,\ldots$.

This two-parameter model is used in applications where $P(N=0)$ is quite small, and therefore the likelihood expression in equation (2.2) can be maximized to obtain the following estimators for $\beta$ and $R$:

$$\hat{\beta} = [-T + (T^2 + 4\ KST/Y)^{1/2}]/2S, \quad (3)$$

$$\hat{R} = K/(S + T/\hat{\beta}), \quad (4)$$

where $Y = \sum_{i=1}^{N} \ln(T/T_i)$. These point estimators may be compared with those for a time truncated Weibull process alone in Bain and Engelhardt (1980, Equations (4) and (5)).

## 3. INFERENCES ON FINAL FAILURE RATE R

In order to deal with this two parameter model, we henceforth condition on the event $N > 0$ and obtain from Equation (2.2) the conditional density

$$f_c\ (k,n,t_1,\ldots,t_n)$$

$$= R^k \left[\prod_{i=1}^{n} (T/t_i)\right]^{1-\beta} \exp(-RT/\beta - RS)/(1-\exp(-RT/\beta))(k-n)!\ S^{n-k}, \quad (5)$$

where $n=1,2,\ldots$ and $0 < t_1 < \ldots < t_n < T$. As in Bain and Engelhardt (1980, equation (7)), it is apparent that $Y$ is sufficient for $\beta$ and that the conditional distribution of $K$ given $Y = y$ can be used to construct uniformly most powerful unbiased (UMPU) tests for $R$ (Reference 3, pp. 134-140).

To obtain this distribution, consider first the conditional distribution of $N$ given $Y = y$ which appears in Bain and Engelhardt (1980, equation (8)):

$$P(N=n|Y=y) = G(RTy)\ (RTy)^n/n!(n-1)!, \quad (6)$$

where $n=1,2,\ldots$ and $G(x) = \left[\sum_{j=1}^{\infty} x^j/j!(j-1)!\right]^{-1}$.

---

[3] Lehmann, E. L., Testing Statistical Hypotheses, John Wiley and Sons, New York, NY, 1959.

Test phases A and B constitute two independent experiments, and therefore

$$P(K = k,\ N = n \mid Y = y)$$

$$= P(N = n \mid Y = y)\ (RS)^{k-n} \exp(-RS)/(k-n)!, \qquad (7)$$

where $k = 1,2,\ldots$ and $n=1,\ldots,k$. Summing over n, we obtain the desired probability distribution

$$P(K = k \mid Y = y)$$

$$= G(RTy)\ (RS)^k \exp(-RS) \sum_{n=1}^{k} (Ty/S)^n/n!(n-1)!(k-n)! \qquad (8)$$

and the cumulative distribution function

$$F(k;\ R \mid y) = \sum_{j=1}^{k} P(K = j \mid Y = y), \qquad (9)$$

where $k = 1,2,\ldots$ . It is noted that the conditional probability distributions specified in Equations (6) and (8) can be readily computed at minimal cost.

The distribution function F can be used to construct a variety of inferential procedures for the parameter R. For example, confidence bounds on R can be constructed by techniques similar to the procedures in Reference 4. In this paper, we concentrate on properties of the conditional operating characteristic curves and show how they may be used to appraise the need for test phase B.

Let $k_0$ be the maximum total number of failures acceptable during the two test phases, and suppose we observe $Y = y$ during test phase A. Then the conditional operating characteristic curve for this test may be depicted as the probability of acceptance, $F(k_0;R|y)$, as a function of R. The conditioning variable Y has an important effect on the properties of this test. Since y and T occur everywhere in equations (6) and (8) as a product, it is clear that the test time for test phase A will appear longer or shorter, in some sense, accordingly as Y is larger or smaller. A significant consequence of this phenomenon is that producer and consumer risks will generally be smaller if a relatively larger value of Y is observed.

In the following two sections we consider a testing situation in which test phase A is first completed and afterwards the length of test phase B is to be determined. The observed value of Y can then be used to decide how much, if any, additional test time S is desirable in order to control the statistical risks of the overall test. A suitable procedure

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[4] Miller, G., Confidence Intervals for the Reliability of a Future System Configuration, Technical Report No. 343, US Army Materiel Systems Analysis Activity, Aberdeen Proving Ground, MD, September 1981.

for conducting this decision-making process is illustrated by a detailed example in Section 4. To simplify the presentation, we consider a case in which the proposed test phase B is of a given length S and the only decision to be made is whether to conduct test phase B or not. The more complicated case in which test phase B is of variable length can be handled in a similar manner by examining an appropriate selection of constant length test phases.

## 4. DECISION PROCEDURE

A system is scheduled to undergo a reliability growth test phase of T = 2000 hours and could be slated, if necessary, for a follow-on fixed configuration test phase of S = 750 hours. The minimum acceptable value (MAV) for reliability is a mean time between failures (MTBF) of 51 hours, which corresponds to a failure rate of $51^{-1}$ per hour. The consumer's risk is the probability of accepting the system when it barely attains the MAV. The producer is required by contract to achieve 92 hours MTBF; so the producer's risk is the probability that the system will be rejected even though this contractual requirement is met.

Our objective in this example is to eliminate the 750 hour test phase B whenever doing so will not result in excessive risks to either consumer or producer. These risks have been computed through the use of equation (6) for a single reliability growth test phase of length T = 2000, and the results are shown in Table 1 over a representative range of values for the conditional variable Y. (As a guide in choosing a range of likely outcomes for Y, note that the conditional distribution of $2\beta Y$ given N = n is a chi-square with 2n degrees of freedom). The acceptable numbers of failures in this illustration were selected so as to balance the risks, though other choices could have been made.

TABLE 1 - PERCENTAGE RISKS FOR TEST PHASE A ALONE.

| y | Acceptable Number of Failures | Consumer's Risk | Producer's Risk | Sum of Risks |
|---|---|---|---|---|
| 10.0 | 17 | 21.0 | 17.6 | 38.6 |
| 20.0 | 24 | 15.7 | 14.6 | 30.3 |
| 30.0 | 29 | 10.9 | 14.9 | 25.8 |
| 40.0 | 34 | 11.2 | 10.9 | 22.1 |
| 50.0 | 38 | 9.7 | 9.9 | 19.6 |
| 60.0 | 42 | 10.0 | 7.7 | 17.7 |
| 70.0 | 45 | 7.8 | 8.1 | 15.9 |
| 80.0 | 48 | 6.8 | 7.8 | 14.6 |

It is apparent in Table 1 that the risks tend to decrease as y increases. This tendency is a manifestation of the phenomenon previously mentioned, that the test will appear longer for larger values of y. Our technique in this example is to select a critical value $y_0$ such that if $Y > y_0$, then test phase A alone will be sufficient to evaluate the system reliability.

Before selecting $y_0$, we may wish to compare the risks involved when both test phases A and B are conducted. These risks are computed through the use of equations (8) and (9) for T = 2000 and S = 750, and the results are shown in Table 2. As expected, the additional testing yields lower risks, and a decreasing trend in risks is apparent as y increases.

TABLE 2 - PERCENTAGE RISKS FOR THE COMBINATION OF TEST PHASES A AND B.

| y | Acceptable Number of Failures | Consumer's Risk | Producer's Risk | Sum of Risks |
|---|---|---|---|---|
| 10.0 | 28 | 10.0 | 9.1 | 19.1 |
| 20.0 | 35 | 7.8 | 7.7 | 15.5 |
| 30.0 | 41 | 8.1 | 5.3 | 13.4 |
| 40.0 | 45 | 5.8 | 6.0 | 11.8 |
| 50.0 | 49 | 5.1 | 5.4 | 10.5 |
| 60.0 | 53 | 5.2 | 4.3 | 9.5 |
| 70.0 | 56 | 4.1 | 4.5 | 8.6 |
| 80.0 | 59 | 3.6 | 4.4 | 8.0 |

The problem at hand is to choose $y_0$, the smallest value of y at which the risks shown in Table 1 are worth taking in order to save the time and expense of conducting test phase B. This choice of $y_0$ is the only difficult step in applying the methodology discussed herein and may require a decision by top management. The program manager will usually have budgetary constraints to consider, and the contract may require that risks not exceed certain levels. For our purposes, we shall choose $y_0$ = 45.0 and accordingly say that additional testing is desirable if $Y < 45.0$, but that otherwise ($Y \geq 45.0$) test phase B need not be conducted.

Once $y_0$ is chosen, our conditional procedure for scheduling test phase B is well-defined with risks as stated in Table 3. These risks are the same as in Table 2 for y less than 45.0, since test phase B would then be scheduled. For y greater than 45.0 the risks are taken from Table 1, since in that instance test phase B would not be conducted. Note that all of the analysis up to this point has been accomplished without knowledge of any test results, and in particular, the choice of $y_0$ does not depend on the results from test phase A.

TABLE 3 - PERCENTAGE RISKS FOR PROCEDURE WITH $y_0$ = 45.0.

| y | Acceptable Number of Failures | Consumer's Risk | Producer's Risk | Sum of Risks |
|---|---|---|---|---|
| 10.0 | 28 | 10.0 | 9.1 | 19.1 |
| 20.0 | 35 | 7.8 | 7.7 | 15.5 |
| 30.0 | 41 | 8.1 | 5.3 | 13.4 |
| 40.0 | 45 | 5.8 | 6.0 | 11.8 |
| 50.0 | 38 | 9.7 | 9.9 | 19.6 |
| 60.0 | 42 | 10.0 | 7.7 | 17.7 |
| 70.0 | 45 | 7.8 | 8.1 | 15.9 |
| 80.0 | 48 | 6.8 | 7.8 | 14.6 |

Table 3 shows risks that are reasonable over a wide range of values for y. These risks may be compared with those from a conventional analysis of a 750 hour, fixed configuration, reliability demonstration test: 20.5 percent consumer's risk and 12.3 percent producer's risk (when 11 failures are acceptable). By utilizing information from the previous reliability growth testing, our procedure reduces risks and, in many cases, eliminates the need for a follow-on test phase.

## 5. ANALYSIS OF DATA

The 2000 hour reliability growth test phase in Section 4 was eventually conducted, and the following 35 failure times were recorded:

| | | | | |
|---|---|---|---|---|
| 13.4 | 329.5 | 811.4 | 1213.6 | 1585.1 |
| 49.9 | 417.0 | 815.2 | 1330.3 | 1605.9 |
| 72.8 | 609.8 | 824.2 | 1336.4 | 1672.9 |
| 86.9 | 612.9 | 830.7 | 1429.5 | 1715.9 |
| 177.4 | 669.7 | 895.2 | 1477.2 | 1756.2 |
| 191.3 | 704.1 | 944.6 | 1512.6 | 1825.3 |
| 243.9 | 749.3 | 1156.7 | 1548.2 | 1875.3 |

The observed value of Y was therefore $y = \sum_{i=1}^{35} \ln(2000/t_i) = 40.1$, which is less than the critical value $y_0 = 45.0$. Consequently, the follow-on fixed configuration test of 750 hours was conducted, and 10 more failures were observed.

Using the observed values y = 40.1 and k = 45, we can obtain from Equation (9)

$$F(k;\ 51^{-1} \mid y) = .057$$

and infer with 94.3 percent confidence that the producer did achieve the MAV of 51 hours MTBF. The point estimates obtained from equations (3) and (4) are $\beta = .851$ and $R^{-1} = 68.9$ hours MTBF.

## 6. GOODNESS-OF-FIT TEST

Let M denote the number of failures that occur during the fixed configuration test phase B. As derived in Section 8, the conditional probability distribution of M given K = k and Y = y is

$$P(M = m \mid K = k,\ Y = y) = H(k,y)(S/yT)^m/m!(k-m)!k-m-1)!, \qquad (10)$$

where $y > 0$; $k = 1,2,\ldots$; $m = 0,1,\ldots,k-1$; and

$H(k,y) = [\sum_{j=0}^{k-1} (S/yT)^j/j!(k-j)!(k-j-1)!]^{-1}$. This probability distribution is shown in Table 4 for the data set considered in the previous section (k=45, y=40.1).

TABLE 4 - CONDITIONAL PROBABILITY DISTRIBUTION OF M.

| m | Prob (m) | $\sum_{j=0}^{m}$ Prob (j) |
|---|---|---|
| 0 | 0.000 | 0.000 |
| 1 | 0.000 | 0.000 |
| 2 | 0.000 | 0.000 |
| 3 | 0.001 | 0.001 |
| 4 | 0.004 | 0.005 |
| 5 | 0.012 | 0.017 |
| 6 | 0.028 | 0.045 |
| 7 | 0.056 | 0.101 |
| 8 | 0.093 | 0.194 |
| 9 | 0.128 | 0.322 |
| 10 | 0.151 | 0.473 |
| 11 | 0.153 | 0.626 |
| 12 | 0.133 | 0.759 |
| 13 | 0.101 | 0.860 |
| 14 | 0.067 | 0.927 |
| 15 | 0.039 | 0.966 |
| 16 | 0.020 | 0.986 |
| 17 | 0.009 | 0.995 |
| 18 | 0.003 | 0.998 |
| 19 | 0.001 | 0.999 |
| 20 | 0.001 | 1.000 |

For this particular example, Table 4 indicates that the number of failures observed during test phase B will be in the range from 6 to 16, inclusive, with probability 0.969. Therefore to obtain a goodness-of-fit test with a 3.1 percent significance level, we reject the model for $M < 6$ or $M > 16$. This kind of test can be used to detect an unexpected change in reliability during test phase B.

## 7. CONCLUSIONS AND SUMMARY

The efficiency of a reliability assessment is greatly increased by the use of an extended model which allows a unified treatment of the data from two separate test phases. The amount of system testing may be reduced, and in some cases an entire test phase may be eliminated. At the same time, both consumer and producer risks can be held below levels often encountered in conventional reliability demonstration tests.

A workable model for realizing these beneficial results is presented in Sections 1 and 2, and in Section 3 the appropriate conditional distribution is obtained for performing UMPU hypothesis tests on the final system reliability. The method is illustrated by a realistic example in Sections 4 and 5, and a test for goodness-of-fit is introduced and applied in Section 6.

# 8. APPENDIX

Conditional on $N = n > 0$, the random variables $\ln(T/T_i)$, $i = 1, \ldots, n$, are distributed as the (reversed) order statistics from $n$ independent exponential distributions with mean $\beta^{-1}$, and therefore the statistic $Y = \sum_{i=1}^{n} \ln(T/T_i)$ is distributed as a gamma variable with probability density function

$$f_{Y|N}(y|n) = \beta^n y^{n-1} \exp(-\beta y)/(n-1)! \qquad (11)$$

for $y > 0$. From equation (11) it is straightforward to obtain the joint probability density function of Y, K, and M as

$$R^k \exp(-\beta y - RT/\beta - RS)\, y^{k-m-1}\, T^{k-m}\, S^m/m!(k-m)!(k-m-1)!, \qquad (12)$$

where $y > 0$, $k = 1,2,\ldots,$ and $m = 0,1,\ldots, k-1$. Equation (10) can now be derived from Expression (12). Notice in (12) that K and Y are sufficient statistics for R and $\beta$, so that these parameters do not appear in the conditional probability distribution (10).

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