	AD-A109 533 UNCLASSIFIED	SOUTHERN UTILIZING DEC 81 J TR-154	METHODI COMPLE D HART	ST UNIV X-VALUE , H L G	DALLA S S-ARA RAY	S TEX ( AYS IN	DEPT OF THE MO	STATIS DELING NO	OF ARM	F/G A PROCE 5-C-043 NL	12/1 ETC(1	1)
	Lor AgiA ∿a≻as											
			END DATE FILMED 02-182 DTIC									



UTILIZING COMPLEX-VALUES S-ARRAYS IN THE MODELING OF ARMA PROCESSES

Ъy

Jeffrey D. Hart and Henry L. Gray

Technical Report No. 154 Department of Statistics ONR Contract

December, 1981

Research sponsored by the Office of Naval Research Contract N00014-75-C-0439

Reproduction in whole or in part is permitted for any purpose of the United States Government

1



82 01 06027

t

The document has been approved for public release and sale; its distribution is unlimited

DEPARTMENT OF STATISTICS Southern Methodist University Dallas, Texas 75275

FILE COPY

and make and rable to its fodes weiß und/er 59:01.81

Utilizing Complex-Valued S-Arrays in

the Modeling of ARMA Processes

Jeffrey D. Hart Department of Mathematics University of Arkansas Fayetteville, Arkansas 72701 Henry L. Gray Department of Statistics Southern Methodist University Dallas, Texas 75275

Key Words: ARMA processes; Box-Jenkins method; S-array method.

## SUMMARY

The problem of estimating the order, (p,q), of an ARMA (p,q) process is considered. An extension of the Gray, Kelley, and McIntire (1978) method of estimating (p,q) is proposed and shown to be particularly useful for processes whose spectra have a certain form. Simulated data is used to illustrate the usefulness of the extension.

### 1. Introduction and Definitions

Gray, Kelley, and McIntire (1978) have illustrated how a certain transform,  $S_n(\cdot)$ , of the autocorrelation function may be used to estimate p and q from a realization of an ARMA(p,q) process. Their technique hinges on the following two facts.

If  $\rho_k$  is the autocorrelation function of an ARMA(p,q) process, then (1)

k>q.

$$\rho_{k} - \phi_{1}\rho_{k-1} - \dots - \phi_{p}\rho_{k-p} = 0$$
 for

Under quite general conditions on the real

sequence {f<sub>k</sub>},

$$S_n(f_m) = C$$
 for  $m \ge m_0$  iff  $\{f_m\} \in L(n,\Delta)$  for  $m > m_0$ .

These two facts imply that p and q can always be determined if the autocorrelation sequence of the process is known. This important "consistency" property is not possessed by the popular Box-Jenkins (1976) method in the case of the mixed process (i.e. p > 0 and q > 0).

The Gray, et al method of estimating p and q involves the examination of an array, known as the S-array, which contains values of  $S_n(\hat{\rho}_k)$ , where usually

$$\hat{\rho}_{k} = \frac{\sum_{t=1}^{N-k} (\bar{x}_{t} - \bar{\bar{x}}) (\bar{x}_{t+k} - \bar{\bar{x}})}{\sum_{t=1}^{N} (\bar{x}_{t} - \bar{\bar{x}})^{2}}$$

A constancy pattern in the S-array consistent with (2) leads to estimates of p and q. Numerous simulation studies and use of the S-array method on real data have indicated that this constancy pattern is sometimes more apparent in an array based on  $S_n[(-1)^k \hat{\rho}_k]$ . Since  $\{\rho_k\} \in L(p, \Delta)$  for k > q if and only if  $\{(-1)^k \rho_k\} \in L(p, \Delta)$  for k > q, the same theoretical justification exists for using  $S_n[(-1)^k \hat{\rho}_k]$  to estimate p and q as for  $S_n(\hat{\rho}_k)$ . The differing statistical properties of these two transforms will be discussed later.

 $S_n(\rho_k)$  and  $S_n[(-1)^k \rho_k]$  are members of the following class of transforms of  $\rho_k$ :

 $\{S_n(e^{2\pi i\omega k}\rho_k): 0 \leq \omega \leq \frac{1}{2}\}.$ 

The properties of these transforms when  $\rho_k$  is the autocorrelation function of an ARMA(p,q) process, and the estimation of p and q by means of  $S_n(e^{2\pi i \omega k} \hat{\rho}_k)$ , are the subject of the remainder of this paper. 2

(2)

Before proceeding to the next section, the following definitions and notation are given.

A stochastic process  $\{X_t\}$ ,  $t = 0, \pm 1, \pm 2, ...,$  is said to be autoregressive of order p and moving average of order q, or ARMA(p,q), if

$$X_{t} = \sum_{k=1}^{p} \phi_{k} X_{t-k} + Z_{t} - \sum_{k=1}^{q} \theta_{k} Z_{t-k} \text{ for all } t,$$

where the  $\phi_k$  and  $\theta_k$  are constants and  $\{Z_t\}$  is a white noise process with finite variance. If the operator B is defined by  $BX_t = X_{t-1}$ , then the above may be written as

$$\phi(B)X_{t} = \theta(B)Z_{t}, \text{ where}$$
  
$$\phi(B) = 1 - \phi_{1}B - \phi_{2}B^{2} - \dots - \phi_{p}B^{p}$$

and

$$\theta(\mathbf{B}) = 1 - \theta_1 \mathbf{B} - \theta_2 \mathbf{B}^2 - \dots - \theta_q \mathbf{B}^q .$$

It is well known that  $\{X_t\}$  is stationary if and only if all of the roots of  $\phi(x) = 0$  lie outside the unit circle.

Let m be an integer and f be a complex-valued function of a real variable. Further, let  $f_m = f(m)$ ,

$$H_{n}(f_{m}) = \begin{cases} f_{m-n+1} & f_{m-n+2} & \cdots & f_{m} \\ f_{m-n+2} & f_{m-n+3} & \cdots & f_{m+1} \\ \vdots & \vdots & \vdots & \vdots \\ f_{m} & f_{m+1} & \cdots & f_{m+n-1} \end{cases}$$

$$H_{n+1}(1; f_{m}) = \begin{cases} 1 & 1 & \dots & 1 \\ f_{m-n+1} & f_{m-n+2} & \dots & f_{m+1} \\ \vdots & \vdots & & \vdots \\ f_{m} & f_{m+1} & \dots & f_{m+n-1} \end{cases}$$

J

where n is a positive integer. Now define

$$S_n(f_m) = \frac{H_{n+1}(1; f_m)}{H_n(f_m)}$$

and

$$R_{n}(f_{m}) = \begin{cases} \frac{H_{n}(f_{m})}{H_{n}(1;f_{m})} , & n = 2,3,... \\ f_{m} & , & n = 1 \end{cases}$$

The S-array for the function f is the following array of complex numbers:

m/n	1	2	<u></u>	<u>k</u>	•••
	:	:		•	
-j	S <sub>1</sub> (f <sub>-j</sub> )	S <sub>2</sub> (f_j)	• • •	S <sub>k</sub> (f-j)	•••
•	:	•		• •	
-2	s <sub>1</sub> (f <sub>-2</sub> )	s <sub>2</sub> (f <sub>-2</sub> )	•••	$S_k(f_{-2})$	• • •
-1	\$1 <sup>(f</sup> -1)	S <sub>2</sub> (f <sub>-1</sub> )	• • •	$S_{k}(f_{-1})$	• • •
0	\$1(f0)	s <sub>2</sub> (f <sub>0</sub> )	•••	$s_{k}^{(f_{0})}$	•••
1	S <sub>1</sub> (f <sub>1</sub> )	\$ <sub>2</sub> (f <sub>1</sub> )	• • •	$s_k(f_1)$	•••
2	\$1(f <sup>2</sup> )	$S_2(f_2)$	•••	$s_{k}(f_{2})$	• • •
:		•		• •	
Ė.	S <sub>1</sub> (f <sub>j</sub> )	s <sub>2</sub> (fj)	• • •	S <sub>k</sub> (fj)	•••
•		:		:	

The following recursion relations, due to Pye and Atchison (1973), are quite helpful in calculating S-arrays. If  $S_0(f_m)$  is defined to be 1 for all m, then

$$\mathbb{R}_{n+1}(\mathbf{f}_{m}) = \mathbb{R}_{n}(\mathbf{f}_{m+1}) \left[ \frac{\mathbf{S}_{n}(\mathbf{f}_{m+1})}{\mathbf{S}_{n}(\mathbf{f}_{m})} - 1 \right]$$

and

$$S_{n}(f_{m}) = S_{n-1}(f_{m+1}) \left[ \frac{R_{n}(f_{m+1})}{R_{n}(f_{m})} - 1 \right] ,$$

where  $n = 1, 2, \ldots$ , and m is any integer.

A complex-valued sequence  $\{f_m\}$  will be said to be an element of  $L(n,\Delta)$  for  $m > m_0$  if there exists a smallest integer n > 0 and a set of  $c_i$ 's such that

$$f_k + c_1 f_{k-1} + \dots + c_n f_{k-n} = 0, m > m_0$$

2. Properties of 
$$S_n(e^{2\pi i \omega k} \rho_k)$$

Theorem 1 is now stated in order to given an explicit form for each of the transforms in  $\{S_n(e^{2\pi i\omega k}\rho_k): 0 \le \omega \le \frac{1}{2}\}$ . <u>Theorem 1</u>. Let  $\{a_k: k=0, \pm 1, \ldots\}$  be any sequence of real or complex numbers. Then for any positive integer  $n_0$  and any integer  $k_0$ , we have

$$S_{n_{0}}(e^{2\pi i\omega k_{0}}a_{k_{0}}) = e^{2\pi i\omega n_{0}} \begin{vmatrix} 1 & e^{-2\pi i\omega} & \dots & e^{-2\pi i\omega n_{0}} \\ a_{k_{0}}-n_{0}+1 & a_{k_{0}}-n_{0}+2 & \dots & a_{k_{0}}+1 \\ a_{k_{0}}-n_{0}+2 & a_{k_{0}}-n_{0}+3 & \dots & a_{k_{0}}+2 \\ \vdots & \vdots & & \vdots \\ a_{k_{0}} & a_{k_{0}}+1 & \dots & a_{k_{0}}+n_{0} \end{vmatrix}$$

Proof: By definition

$$S_{n_{0}}(e^{2\pi i\omega k_{0}}a_{k_{0}}) = \frac{\frac{H_{n_{0}+1}(1;e^{2\pi i\omega k_{0}}a_{k_{0}})}{H_{n_{0}}(e^{2\pi i\omega k_{0}}a_{k_{0}})}$$

$$H_{n_{0}+1}[1;e^{2\pi i\omega k_{0}}a_{k_{0}}] = 1 \qquad 1 \qquad \dots \qquad 1$$

$$e^{2\pi i\omega (k_{0}-n_{0}+1)}a_{k_{0}-n_{0}+1} e^{2\pi i\omega (k_{0}-n_{0}+2)}a_{k_{0}-n_{0}+2} \qquad \dots \qquad e^{2\pi i\omega (k_{0}+1)}a_{k_{0}+1}$$

$$e^{2\pi i\omega (k_{0}-n_{0}+2)}a_{k_{0}-n_{0}+2} e^{2\pi i\omega (k_{0}-n_{0}+3)}a_{k_{0}-n_{0}+3} \qquad \dots \qquad e^{2\pi i\omega (k_{0}+2)}a_{k_{0}+2}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$e^{2\pi i\omega k_{0}}a_{k_{0}} e^{2\pi i\omega (k_{0}+1)}a_{k_{0}+1} \qquad \dots \qquad e^{2\pi i\omega (k_{0}+n_{0})}a_{k_{0}+n_{0}}$$

The above follows from simple row and column operations. A

similar set of operations yields

$$H_{n_0}(e^{2\pi i\omega k_0}a_{k_0}) = e^{2\pi i\omega n_0} H_{n_0}(a_{k_0}),$$

and the result follows.

From Theorem 1 we see that  $S_n(e^{2\pi i\omega k}a_k)$  depends on k only through the sequence  $\{a_m\}$ . This fact will be helpful later when we consider  $\{a_m\} = \{\hat{\rho}_m\}$ .

As noted previously, the autocorrelation function of an ARMA(p,q) process satisfies the following relationship:

$$\rho_k - \phi_1 \rho_{k-1} - \dots - \phi_p \rho_{k-p} = 0, \ k > q.$$

This relationship, however, is equivalent to

$$2\pi i \omega k [\rho_k^{-\phi_1} \rho_{k-1}^{-} \dots^{-\phi_k} \rho_{k-p}] = 0, k > q$$

or

$$(e^{2\pi i\omega k}\rho_{k}) - \phi_{1}e^{2\pi i\omega}(e^{2\pi i\omega (k-1)}\rho_{k-1}) - \dots - \phi_{p}e^{2\pi i\omega p}(e^{2\pi i\omega (k-p)}\rho_{k-p}) = 0,$$
  
k > q.

In other words, for k > q,  $f_k(\omega) = e^{2\pi i \omega k} \rho_k$ , is the solution of a pth order, homogeneous difference equation with complex constant coefficients. If (2) holds for complex sequences, then  $f_k(\omega) \in L(p, \Delta)$  for k > q allows the complex-valued transforms  $S_n(e^{2\pi i \omega k} \hat{\rho}_k)$ ,  $0 < \omega < \frac{1}{2}$ , to be candidates for use in the estimation of p and q. The validity of (2) for complex sequences is established in the following theorem.

<u>Theorem 2</u> Let  $\{f_k\}$  be a sequence of complex numbers. Suppose  $S_{n_0}(f_m)$  and  $R_{n_0}(f_m)$  are defined and  $|S_{n_0}(f_m)| > 0$  for  $m \ge m_0$ .

Then

$$f_{m} \in L(n_{0}, \Delta) \text{ for } m \ge m_{0} \text{ iff}$$

$$S_{n_{0}}(f_{m}) = C \text{ for } m \ge m_{0}.$$
Further  $C = (-1)^{n_{0}}(1-a_{1}-a_{2}-\dots-a_{n_{0}}) \text{ where}$ 

$$f_{m} - a_{1}f_{m-1} - \dots - a_{n_{0}}f_{m-n_{0}} = 0, m \ge m_{0}.$$

Proof: The proof of this theorem by Gray, Houston, and Morgan (1978) for real  $\{f_k\}$  depends only upon properties of determinants and systems of linear equations. These properties hold so long as the elements of  $\{f_k\}$  belong to a field. Since the complex number system satisfies the field properties, the result follows.

The basis for the remainder of this work is established in Corollary 1 and Theorem 3.

<u>Corollary 1.</u> Suppose the time series  $\{X_t: t = 0, \pm 1, ...\}$  is a stationary ARMA(p,q) process with autocorrelation  $\rho_m$ . Suppose  $S_{n_0}(f_m(\omega))$  and  $R_{n_0}(f_m(\omega))$  are defined, p > 0 and  $|S_{n_0}(f_m(\omega))| > 0$ . Then for some integer  $m_0$ 

$$S_{n_0}(f_m(\omega)) = C_{1,\omega}, \quad m \ge m_0 \text{ and}$$
$$S_{n_0}(f_{m_0-1}(\omega)) \neq C_{1,\omega}$$

if and only if  $n_0 = p$  and  $m_0 = q$ . In addition  $C_{1,\omega} = (-1)^p (1 - \phi_1 e^{2\pi i \omega} - \phi_2 e^{4\pi i \omega} - \dots - \phi_p e^{2\pi i \omega p}).$ 

**Proof:** Since  $\{f_m(\omega)\} \in L(p, \Delta)$  and satisfies the equation

$$f_{m}(\omega) - (\phi_1 e^{2\pi i \omega}) f_{m-1}(\omega) - \dots - (\phi_p e^{2\pi i \omega p}) f_{m-p}(\omega) = 0, m > q,$$

the results follow immediately from Theorem 2.

Theorem 3 below is the analog of Theorem 3 in Gray, Kelley, and McIntire (1978). Its validity follows from the fact that, since  $\rho_k = \rho_{-k}$ , we have

$$p_{-k} = \phi_1 \rho_{-k+1} = \cdots = \phi_p \rho_{-k+p} = 0, -k < -q.$$

Theorem 3 Under the conditions of Theorem 2

$$S_{n_0}(f_{m}(\omega)) = C_{2,\omega}, \quad m \leq m_1 \text{ and}$$
$$S_{n_0}(f_{m_1}+1(\omega)) \neq C_{2,\omega}$$

iff  $n_0 = p$  and  $m_1 = -q-1$ . In addition,

$$C_{2,\omega} = \frac{-\overline{C}_{1,\omega}e^{2\pi i\omega p}}{\phi_p}$$

Proof: The proof is completely analogous to the proof of Theorem 3 in Gray, Kelley, and McIntire (1978). Only trivial modifications are required to adjust for the fact that  $S_n(f_m(\omega))$ is complex-valued.

Corollary 1 and Theorem 3 show that there is no inherent reason for considering only the real valued transforms  $S_n(\hat{\rho}_k)$  and  $S_n[(-1)^k \hat{\rho}_k]$  in the estimation of p and q. The question remains, however, as to which transform from the class  $\{S_n(e^{2\pi i\omega k} \hat{\rho}_k):$  $0 \le \omega \le \frac{1}{2}\}$  is (in some sense) best for estimating p and q. The best transform must obviously depend on the nature of the process being observed, and thus some method is required to identify this transform for a given process. One immediate clue towards the solution of this problem comes from Corollary 1, which shows that

$$S_p(f_m(\omega)) = (-1)^p (1 - \phi_1 e^{2\pi i \omega} - \dots - \phi_p e^{2\pi i \omega p}), m \ge q.$$
 (3)

Since the spectrum,  $s_x(\cdot)$ , of an ARMA (p,q) process is such that

$$\mathbf{s}_{\mathbf{x}}(\boldsymbol{\omega}) \propto \frac{\left|1-\theta_{1}e^{2\pi i\boldsymbol{\omega}}-\ldots-\theta_{q}e^{2\pi i\boldsymbol{\omega}q}\right|^{2}}{\left|1-\phi_{1}e^{2\pi i\boldsymbol{\omega}}-\ldots-\phi_{p}e^{2\pi i\boldsymbol{\omega}p}\right|^{2}} \quad (0 \leq \omega \leq \frac{1}{2}),$$

it follows that for  $m \ge q$ ,  $S_p(f_m(\omega))$  is closely related to  $s_x(\omega)$ . In fact, if q = 0, we have

$$s_{\mathbf{x}}(\omega) \propto \frac{1}{|S_{\mathbf{p}}(f_{\mathbf{m}}(\omega))|^2}, \quad \mathbf{m} \geq q.$$

This relationship will be exploited in the next section when the problem of choosing an optimal transform  $S_n(e^{2\pi i\omega}0^k\hat{\rho}_k)$  is formulated.

## 3 Formulation of the Optimal Frequency Problem

Estimating p and q by the S-array method involves examining an array for the presence of a certain constancy pattern. The data at hand supports the estimates  $\hat{p}$  and  $\hat{q}$  if

$$S_{p}^{*}(e^{2\pi i\omega k}\hat{\rho}_{k}) \approx C_{1}, \quad k \geq \hat{q} \text{ and}$$
$$S_{p}^{*}(e^{2\pi i\omega k}\hat{\rho}_{k}) \approx C_{2}, \quad k \leq -\hat{q} - 1.$$

For a given process, the transform to be used to estimate p and q should be the one which, on the average, makes the correct constancy pattern the most apparent. The ability of  $S_n(e^{2\pi i\omega k\hat{\rho}_k})$  to evidence this constancy depends on two things:

(i) the variability of  $S_p(e^{2\pi i\omega k\hat{\rho}}_k)$  for  $k \ge q$  and  $k \le -q-1$ , and

(ii) the magnitude of the two constants being estimated by  $S_p(e^{2\pi i\omega k\hat{\rho}_k})$  for  $k \ge q$  and  $k \le -q - 1$ .

These considerations lead to the following definition.

<u>Definition 1</u> Suppose  $\{X_t\}$  is a stationary ARMA (p,q) process. Then the frequency  $\omega_0$  will be referred to as optimal for the estimation of p and q iff

$$\frac{\operatorname{var}[S_{p}(e^{2\pi i\omega_{0}q_{p}})]}{|1-\phi_{1}e^{2\pi i\omega_{0}}-\dots-\phi_{p}e^{2\pi i\omega_{0}p}|^{2}} \leq \frac{\operatorname{var}[S_{p}(e^{2\pi i\omega_{p}})]}{|1-\phi_{1}e^{2\pi i\omega_{0}}-\dots-\phi_{p}e^{2\pi i\omega_{p}}|^{2}}$$

for all  $\omega \in [0, \frac{1}{2}]$ .

In light of (3), our definition of the optimal frequency is consistent with (i) and (ii) above, although we consider the variance of only positive lag S-array values. Note that the quantity considered in the definition is essentially the squared coefficient of variation of the random variable  $S_p(e^{2\pi i \omega} 0^q \hat{\rho}_q)$ .

In order to obtain an expression for  $var[S_p(e^{2\pi i \omega q_p^2})]$ , recall that, by Theorem 1

 $S_{p}(e^{2\pi i \omega q} \hat{\rho}_{q}) = e^{2\pi i \omega p}$   $\begin{vmatrix}
1 & e^{-2\pi i \omega} & \cdots & e^{-2\pi i \omega p} \\
\hat{\rho}_{q-p+1} & \hat{\rho}_{q-p+2} & \cdots & \hat{\rho}_{q+1} \\
\hat{\rho}_{q-p+2} & \hat{\rho}_{q-p+3} & \cdots & \hat{\rho}_{q+2} \\
\vdots & \vdots & \vdots \\
\hat{\rho}_{q} & \hat{\rho}_{q+1} & \cdots & \hat{\rho}_{q+p} \\
H_{p}(\hat{\rho}_{q})
\end{vmatrix}$ 

By performing appropriate row and column operations and expanding the numerator by cofactors of the first row, we have

$$S_{p}(e^{2\pi i \omega q} \hat{\rho}_{q}) = (-1)^{p}(1 - \hat{\phi}_{1}^{(p,q)} e^{2\pi i \omega} - \dots - \hat{\phi}_{p}^{(p,q)} e^{2\pi i \omega p})$$

where  $(\hat{\phi}_{1}^{(p,q)}, \hat{\phi}_{2}^{(p,q)}, \dots, \hat{\phi}_{p}^{(p,q)})$  is the solution of

$$\begin{bmatrix} \hat{\rho}_{q} & \hat{\rho}_{q-1} & \cdots & \hat{\rho}_{q-p+1} \\ \hat{\rho}_{q+1} & \hat{\rho}_{q} & \cdots & \hat{\rho}_{q-p+2} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\rho} & \hat{\rho} & \cdots & \hat{\rho}_{q} \end{bmatrix} \mathbf{y} = \begin{bmatrix} \hat{\rho}_{q+1} \\ \hat{\rho}_{q+2} \\ \vdots \\ \hat{\rho}_{q+p-1} \\ q+p-2 \end{bmatrix}$$

By the definition of variance of a complex valued random variable (i.e., var(X + iY) = var(X) + var(Y))we thus have

$$\operatorname{var}[S_{p}(e^{2\pi i \omega q} \hat{\rho}_{q})] = \sum_{k=1}^{p} \operatorname{var}(\hat{\phi}_{k}^{(p,q)})[\cos^{2} 2\pi \omega k + \sin^{2} 2\pi \omega k]$$

+ 2 ΣΣ {(cos2πωjcos2πωk + sin2πωjsin2πωk) j<k

× 
$$\operatorname{cov}(\hat{\phi}_{j}^{(p,q)}, \hat{\phi}_{k}^{(p,q)})$$
  
=  $\sum_{k=1}^{p} \operatorname{var}(\hat{\phi}_{k}^{(p,q)})$   
+ 2  $\sum \operatorname{cos}[2\pi\omega(k-j)]\operatorname{cov}(\hat{\phi}_{j}^{(p,q)}, \hat{\phi}_{k}^{(p,q)})$   
j

 $\psi_{0}^{p-1} = \psi_{0}^{p-1} + 2 \sum_{k=1}^{p-k} \psi_{k} \cos 2\pi \omega k , \qquad (4)$ where  $\psi_{k} = \sum_{j=1}^{p-k} \cos(\hat{\phi}_{j}^{(p,q)}, \hat{\phi}_{j+k}^{(p,q)}), (k = 0, 1, ..., p-1).$ 

Therefore, the frequency which is optimal for the estimation of p and q is the value of  $\omega$  which minimizes

$$C(\omega) = \frac{\psi_0 + 2\Sigma \psi_k \cos 2\pi \omega k}{|1 - \phi_1 e^{2\pi i \omega} - \dots - \phi_p e^{2\pi i \omega p}|^2} .$$
 (5)

Interestingly, the above expression shows that our criterion for identifying  $\omega_0$  is essentially equivalent to the problem of finding the frequency at which the spectrum of an ARMA (p,p-1) process has its minimum.

The above considerations illustrate that for many processes there exists  $\omega_0 \neq 0$ ,  $\frac{1}{2}$  for which  $S_n(e^{2\pi i\omega_0 k} \hat{\rho}_k)$  is optimal for the estimation of p and q. We have not, however, addressed the problem of identifying  $\omega_0$  given a record of finite length from an ARMA process. A less than optimal, but useful, solution to this problem will be discussed in the next section.

# 4. Estimating the Optimal Frequency $\omega_0$

The dependence of (5) upon p makes the estimation of  $\omega_0$ difficult. Suppose for the moment, however, that for the process under consideration q = 0. Then the quantity C( $\omega$ ) is proportional to

$$C^{\star}(\omega) = s_{x}(\omega) \left( \psi_{0} + 2 \sum_{k=1}^{p-1} \psi_{k} \cos 2\pi \omega k \right).$$

Experience has shown that for most processes the minimum of  $C^{\star}(\omega)$ occurs at about the same frequency as does the minimum of  $s_{x}(\omega)$ . This is because  $\psi_{0} + 2 \sum \psi_{k} \cos 2\pi \omega k$  is almost flat in relation to k=1 $s_{x}(\omega)$ . Stated another way, if  $\{S_{p}(e^{2\pi i \omega k} \hat{\rho}_{k})\}$  is viewed as a stochastic process with

 $S_{p}(e^{2\pi i\omega k}\hat{\rho}_{k}) = S_{p}(e^{2\pi i\omega k}\rho_{k}) + e_{p,k}(\omega),$ 

t

then the errors  $e_{p,k}(\omega)$  are essentially homoscedastic.

In order to partially substantiate the previous claim, an approximation to  $ln(NC(\omega))$  has been calculated for two different autoregressive processes and plotted (see Figures 1 and 2) for comparison with

$$\ln \left\{ \frac{1}{|1-\phi_1|^{2\pi i\omega}-\ldots-\phi_p|^{2\pi i\omega p}|^2} \right\}$$

The approximation of  $ln(NC(\omega))$  uses the Box and Jenkins (1976) approximation for the variance-covariance matrix of

$$\hat{\phi}' = (\hat{\phi}_1^{(p,0)}, \hat{\phi}_2^{(p,0)}, \dots, \hat{\phi}_p^{(p,0)}).$$

Assuming that the noise process  $\{a_t\}$  is composed of independent and identically distributed  $N(0,\sigma^2)$  random variables, it can be shown that

 $\operatorname{var}(\hat{\phi}) \simeq \frac{1}{N}(1 - \rho' \phi) P^{-1},$ 

where

$$\underline{\rho}'=(\rho_1,\rho_2,\ldots,\rho_p),\ \underline{\phi}'=(\phi_1,\phi_2,\ldots,\phi_p),$$

$$P = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-3} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} \cdots & 1 \end{bmatrix},$$

and N is the sample size.

The processes associated with Figures 1 and 2 are, respectively,

14

f

$$(1-.60B)(1-1.0607B+.5625B^2)(1+.4944B+.64B^2)X_t = Z_t$$
 (6)

and

$$(1-.95B)(1+.95B)(1-1.3435B+.9025B^{2})X'_{t} = Z'_{t}.$$
(7)  
As was conjectured,  $ln(NC(\omega))$  and  $-2ln(1-\phi_{1}e^{2\pi i\omega}-...-\phi_{p}e^{2\pi i\omega p}|)$   
are minimized at the same value of  $\omega$  for process (6) and at  
approximately the same value for process (7).

The above considerations suggest that, for autoregressive processes, a reasonable estimate of  $\omega_0$  would be the value  $\hat{\omega}_0$  for which a window spectral estimate  $\hat{s}_x(\omega)$  is minimum. It should be pointed out, though, that since  $\omega_0$  is of interest only because of its utility in estimating p and q, it is actually not important to have a precise estimate of this parameter. If the array associated with  $\hat{\omega}_0$  indicates obvious estimates of p and q, then  $\hat{\omega}_0$  has performed its intended function.

The purpose of our discussion to this point has been twofold:

- (i) to illustrate theoretically the possible value of complex-valued S-arrays in the estimation
  - of p and q
- (ii) to address the problem of estimating a frequency  $\omega_0$  for which  $S_n(e^{2\pi i\omega_0 k_{\rho_k}^2})$  is optimal for the estimation of p and q.

Still open for research is a complete solution to the problem in (ii). For the present, however, it is suggested to initially examine the S-array associated with  $\hat{\omega}_0$ , unless  $\hat{\omega}_0$  occurs at a sharp dip in the estimate of the spectrum. Such a sharp dip





is evidence of a near noninvertible moving average factor, and thus is not indicative of a small value of

$$\frac{1}{\left|1-\phi_{1}e^{2\pi i\omega}-\dots-\phi_{p}e^{2\pi i\omega p}\right|^{2}}$$

If  $\omega_0$  occurs at a sharp dip, then the S-array associated with the next smallest local minimum should be examined initially. Additional S-arrays may be examined if estimates for p and q are not apparent in the first array. Such an examination of several different arrays may seem prohibitive in terms of computing time, but this is actually not the case due to the recursive algorithm defined in Section 1.

### 5. Examples

The results of Section 4 imply that S-arrays based on frequencies satisfying  $0 < \omega < \frac{1}{2}$  will be the most valuable for processes whose spectra have relatively large power at  $\omega = 0$ and  $\omega = \frac{1}{2}$ . Therefore, since the purpose of this section is to illustrate the practical importance of complex-valued S-arrays, the two examples which follow will involve such processes.

In each of the two examples to be considered, five independent realizations of an autoregressive process were generated. The process in Example 1 (whose log-spectrum appears in Figure 3) is

$$(1 - .95B)(1 + .95B)X_{\mu} = Z_{\mu}$$
, (8)

and the process in Example 2 is process (7). Realizations of length  $N_1 = 75$  and  $N_2 = 50$ , respectively, were generated from



(8) and (7). In each case the values of the noise process  $\{Z_t\}$  were obtained by generating random samples from the N(0,1) distribution using IMSL subroutine GGNPM.

For each of the ten realizations an estimate of the spectrum was calculated. The estimates in each case utilized a Parzen window based in Examples 1 and 2, respectively, on 11 and 8 values of the estimated autocorrelation function. The following empirical measures of constancy in the S-array were then computed for each realization in Example j:

$$z_{j}(\omega) = \frac{\frac{1}{6}\sum_{k=0}^{5} |s_{p_{j}}(\hat{\rho}_{k}e^{2\pi i\omega k}) - \overline{s}_{p_{j}}(\omega)|^{2}}{|\overline{s}_{p_{j}}(\omega)|^{2}},$$

where  $p_1 = 2$ ,  $p_2 = 4$ ,  $\overline{S}_{p_j}(\omega) = \frac{1}{6} \sum_{k=0}^{5} S_{p_j}(\hat{\rho}_k e^{2\pi i \omega k})$ , and  $\omega = 0$ ,  $\frac{1}{2}$ , and  $\hat{\omega}_0$ , the frequency at which the estimated spectrum is minimized. Note that  $c_j(\omega)$  is a sample analog of

the quantity  $C(\omega)$ .

Table 1 contains the average value of  $c_j(0)$ ,  $c_j(\frac{1}{2})$ , and  $c_j(\hat{w}_0)$  over the five realizations of Example j (j = 1,2). In addition, Tables 2 and 3 show the S-arrays associated with  $\omega = 0$ ,  $\frac{1}{2}$ , and  $\hat{\omega}_0$  for typical realizations in the two examples.

In each example, the average value of  $c_j(\hat{\omega}_0)$  is seen to be smaller than the average of either  $c_j(0)$  or  $c_j(\frac{1}{2})$ , and further,  $c_j(\hat{\omega}_0)$  was the smallest of the three values for all five realizations in both examples. The numerical evidence in Table 1 favoring S-arrays based on  $\hat{\omega}_0$  is presented visually

## TABLE 1

AVERAGE VALUES OF  $c_1(\omega)$  and  $c_2(\omega)$ IN EXAMPLES 1 AND 2

t	c,(0)	$c_1(\tilde{\omega}_0)$	c,(.5)
1	.0878126	.0005025	.0361297
2	3.1317649	.1238482	.2255245

in Tables 2 and 3. Although constancy is apparent in all three arrays in both examples, it is most apparent (for Example j) in the array based on  $\hat{w}_0$  due to the magnitude of the quantities being estimated in column  $p_i$ .

## 6. Summary

A refinement of the S-array method of modeling ARMA processes has been introduced in this chapter. In so doing, the theory of complex-valued S-arrays was developed, and the problem of identifying a frequency whose associated S-array is optimal for estimating the order of an ARMA process was formulated. Additionally, two examples involving simulated data were considered in which S-arrays based on estimated optimal frequencies gave clearer determinations of the order of the underlying processes than did real-valued S-arrays.

The true usefulness of the method discussed in this chapter cannot be ascertained until it has been utilized on real data. Constancy patterns in the S-arrays of simulated data tend to appear quite good at all frequencies, and thus the potential worth of complex-valued S-arrays may have been understated in the examples of the previous section. The possibility exists that for certain data, constancy which is virtually hidden in the two real-valued S-arrays is readily apparent in some complex-valued array.

An aspect of the S-array method which has not been discussed here is the information it contains about the

## TABLE 2

# S-ARRAYS FOR A TYPICAL REALIZATION IN EXAMPLE 1

ω = 0

m/n	1	L		2		3		4
<i>c</i>	507	0.000	1/5	0.000	601	0.000	202	0.000
-0	507	0.000	145	0.000	.491	0.000	-, 393	0.000
-5	1.286	0.000	125	0.000	.046	0.000	.015	0.000
-4	510	0.000	112	0.000	042	0.000	094	0.000
-3	1.261	0,000	100	0.000	.148	0.000	106	0.000
-2	524	0.000	105	0.000	.533	0.000	1.318	0.000
-1	1.346	0.000	084	0.000	658	0.000	-1.608	0.000
0	574	0.000	.074	0.000	~.083	0.000	.079	0.000
1	1.100	0.000	.093	0.000	111	0.000	.222	0.000
2	558	0.000	.089	0.000	312	0.000	078	0.000
3	1.039	0.000	.102	0.000	030	0.000	053	0.000
4	563	0.000	.111	0.000	.077	0.000	.044	0.000
5	1.030	0.000	.131	0.000	176	0.000	.199	0.000

ω = .50

<u>n/n</u>	1			2		3	<u> </u>	4	
-6	-1.493	.000	069	.000	044	.000	198	.000	
-5	-3.286	.000	130	.000	.168	000	162	.000	
-4	-1.490	.000	098	.000	.260	000	.236	000	
-3	-3.261	.000	135	.000	.093	000	.162	000	
-2	-1.476	.000	151	.000	431	.000	737	.000	
-1	-3.346	.000	210	.000	1.271	000	-3.103	.000	
0	-1.426	.000	.183	.000	160	000	.152	.000	
1	-3.100	.000	.134	.000	090	000	124	000	
2	-1.442	.000	.121	.000	.197	.000	.118	.000	
3	-3.039	.000	.089	.000	186	000	.133	.000	
4	-1.437	.000	.116	.000	280	000	465	000	
5	-3.030	.000	.062	.000	016	000	.101	.000	

in the second second

23

<u>m/n</u>		1		2	<u>.</u>	3		4
-6	-1.145	.471	1.927	.586	.906	-4.841	543	-1.883
-5	-1.672	2.185	1.932	.636	-2.540	.265	3.085	.023
-4	-1.144	.469	1.916	.615	-3.373	1.439	1.912	5.973
-3	-1.665	2.161	1.919	.645	-1.426	-1.353	1.803	4.932
-2	-1.140	.455	1.926	.656	3.149	-7.722	18.318	-4.37
-1	-1.690	2.242	1,930	.705	-10.334	10.906	~17.091	-33.340
0	-1.125	.407	1.737	.438	-1.873	258	1.828	.177
1	-1.617	2.007	1.739	.479	-1.517	845	.450	3.139
2	-1.130	.423	1.745	.488	.563	-4.110	3.831	102
3	-1.600	1.949	1.746	.514	-2.530	.706	3.538	238
4	-1.129	.418	1.734	.496	-3.593	2.311	3.315	8.20
5	-1.597	1.941	1.737	.541	-1.339	-1.152	.990	09

ω = .298

÷

فكالصفائك والشعبان بديناهم وبال

ŧ

TABLE 3

S-ARRAYS FOR A TYPICAL REALIZATION IN EXAMPLE 2

					0 = 0					
u/m			2				4		5	
ę	8.316	0.000	.897	0.000	-1.065	0.000	228	0.000	.259	0.000
<u>5</u>	.094	0.000	.633	0.000	-2.458	0.000	243	0.000	691	0.000
4-	- 563	0.000	.703	0.000	-1.232	0.000	298	0.000	.837	0.000
ĥ	-2.791	0.000	769.	0.000	-2.245	0.000	200	0.000	.248	0.000
-7	1.307	0.000	.281	0.000	806	0.000	214	0.00	.525	0.000
7	.296	0.000	.583	0.000	-1.156	0.000	197	0.000	-4.731	0.000
0	-,228	0.000	.376	0.000	557	0.000	.146	0.000	150	0.000
~1	- ,566	0.000	.276	0.000	- , 999	0.000	.156	0.000	250	0.000
7	-1.558	0.000	.651	0.000	- , 744	0.000	.148	0.000	-1.107	0.000
m	1.288	0.000	.433	0.000	-1,808	0.000	.212	0.000	278	0.000
4	086	0.000	<i><b><i><b>۲</b><i></i></i></b></i>	0.000	-1.088	0.000	.179	0.000	125	0.000
ŝ	893	0.000	.808	0.000	-1.992	0.000	.169	0.000	-2.798	0.000
					ω = .50					
m/m	T		2				4		ŝ	
9-	-10.316	.000	3.323	- ,000	-1.396	000	247	000.	.210	000
ŝ	-2.094	.000	2.997	000	3.769	000	227	000.	1.235	- 000
-4	-1.437	000.	4.540	000	-2.035	000.	311	000.	193	.000
٣	167.	000	3.942	000	4.510	000	472	000.	.423	000
- 2	-3.307	.000	3.756	000	-1.548	000	501	000.	232	000.
-1	-2.296	000.	4.522	000	3.134	- 0000	534	.000	12.043	000
0	-1.772	.000	2.913	000	-1.510	000	. 395	.000	382	-,000
-	-1.434	000.	3.688	000	1.920	000	.365	000.	110	000
7	442	000	2.739	000	-1.494	000	.349	· 000	1.890	.000
~	-3.288	.000	2.800	000	2.986	000	.221	.000	064	- 000
4	-1.914	000.	3.677	- ,000	-1.668	.000	.168	.000	224	000
ŝ	-1.107	.000	2.994	- 000	2.612	-,000	.183	.000	2.275	.000

25

TABLE 3 (con't)

ŝ
34
1
I
3

				3						
		1		2		3		4		5
9-	-6.115	7.788	1.225	-2.033	-1.819	.705	4.448	-2.191	-4.412	1.880
ŝ	-1.601	.915	1 325	-1.736	-3.415	-2.622	4.454	-2.218	-6.951	22.342
4	-1.240	.365	1 409	-3.092	-2.014	1.369	4.544	-2.249	-3.232	-8.707
<b>.</b>	016	-1.497	1 253	-2.577	-4.624	-2.777	4.547	-2.056	-4.450	1.256
-2	-2.266	1.928	1 550	-2.387	-2.371	1.008	4.578	-2.071	-3.373	-5.569
7	-1.712	1.083	1.465	-3.071	-3.798	-2.240	4.568	-2.016	-22.550	116.096
0	-1.424	.645	1,441	-1.651	-1.987	.749	3.395	-1.440	-3.414	1.555
٦	-1.238	.362	1.547	-2.327	-2.681	-1.737	3.386	-1.400	-2.959	944
7	- 694	467	1.297	-1.510	-1.667	.643	3.410	-1.410	.814	-20.647
e	-2.256	1.913	1.405	-1.555	-2.567	-2.488	3.378	-1.261	-3.066	369
4	-1.502	.764	1.309	-2.338	-1.688	.883	3.441	-1.275	-3.645	2.171
Ś	-1.059	060.	1.242	-1.741	-3.126	-1.885	3.442	-1.292	8.636	-51.149

2ó

ł

possible nonstationarity of the observed time series (see Gray, Kelley, and McIntire (1978)). An analog of this important feature of the S-array method is not possessed by automatic order selection techniques such as the AIC criterion of Akaike (1969). The behavior of complex-valued S-arrays under an assumption of nonstationarity is an area for future research. Some unforseen application of these arrays to the nonstationary problem may well exist.

#### REFERENCES

- Akaike, H. (1969). "Fitting autoregressive models for prediction," Annals of the Institute of Statistical Mathematics, 21, 243-247.
- Box, G.E.P. and Jenkins, G.M. (1976). <u>Time Series Analysis: Fore-</u> casting and <u>Control</u> revised edition, Holden-Day Inc., San Francisco.
- Gray, H. L., Houston, A.G., and Morgan, F.W. (1978). "On G-spectral estimation," in <u>Proceedings of the 1976 Tulsa Symposium on</u> <u>Applied Time Series</u>, New York: Academic Press.
- Gray, H. L., Kelley, G.D., and McIntire, D.D. (1978). "A new approach to ARMA modeling," <u>Communications in Statistics</u>, <u>B7</u>, 1-78.
- Pye, W.C. and Atchison, T.A. (1973). "An algorithm for the computation of the higher order G-transformation," <u>SIAM</u> Journal on Numerical Analysis, 10, 1-7.

t

REPORT DOCUMENTATION BACE	READ INSTRUCTIONS
REPURT DUCUMENTATION PAGE	BEFORE COMPLETING FORM
$\frac{1}{2} \operatorname{GOV} A \operatorname{CCESSION} HO$	A RECIPIENT SCATALOG NUMBER
134 /////3.33	
ITLE (and Subility)	3. THE OF REPORT & PERIOD COVERED
Utilizing Complex-Values S-Arrays in	TECHNICAL REPORT
the modeling of AMA Process	6 PERFORMING ORG. REPORT NUMBER
Jeffrey D. Hart and Henry L. Gray	N00014-75-C-0439
ERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Southern Methodist University Dallas, Texas 75275	
CONTROLLING OFFICE NAME AND ADDRESS	December 1981
Office of Naval Research	13. NUMBER OF PAGES
Arlington, VA 2221/	29
MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)	IB. SECURITY CLASS, (of this report)
	15. DECLASSIFICATION DOWN GRADING
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State	ease and sale; its ole or in part is s Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observect entered in Block 20, 11 different in	ease and sale; its ole or in part is s Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed on Block 20, 11 different in	ease and sale; its ole or in part is s Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observect onlored in Block 20, 11 different in SUPPLEMENTARY NOTES	ease and sale; its ole or in part is os Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed on Block 20, 11 different in SUPPLEMENTARY NOTES	ease and sale; its ole or in part is ss Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observect entered in Block 20, if different in SUPPLEMENTARY NOTES	ease and sale; its ole or in part is ss Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed on Block 20, 11 different in SUPPLEMENTARY NOTES SUPPLEMENTARY NOTES	ease and sale; its ole or in part is s Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed entered in Block 20, 11 different in SUPPLEMENTARY NOTES SUPPLEMENTARY NOTES ARMA Processes; Box-Jenkins method; S-array me	ease and sale; its ole or in part is s Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the abstract entered in Block 20, 11 different in SUPPLEMENTARY NOTES REY WORDS (Continue on reverse eide 11 necessory and identify by block number ARMA Processes; Box-Jenkins method; S-array me	ease and sale; its ole or in part is is Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed on Block 30, 11 different in SUPPLEMENTARY NOTES SUPPLEMENTARY NOTES ARMA Processes; Box-Jenkins method; S-array me	ease and sale; its ole or in part is s Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed on Block 20, 11 different in SUPPLEMENTARY NOTES KEY WORDS (Continue on reverse side 11 necessary and identify by block number ARMA Processes; Box-Jenkins method; S-array me	ease and sale; its ole or in part is ss Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed entered in Block 20, if different in SUPPLEMENTARY NOTES KEY WORDS (Continue on reverse side if necessary and identify by block number ARMA Processes; Box-Jenkins method; S-array me METRACT (Continue on reverse side if necessary and identify by block number The pro-lem of estimating order. (p.g) of an A	ease and sale; its ole or in part is ss Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the abstract entered in Block 20, 11 different in SUPPLEMENTARY NOTES SUPPLEMENTARY NOTES ARMA Processes; Box-Jenkins method; S-array me TRACT (Centinue on reverse elde II necessary and identify by block number, The pro-lem of estimating order, (p,q) of an A considered. An extension of the Gray, Kelley,	ease and sale; its ole or in part is is Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed on Block 20, 11 different in SUPPLEMENTARY NOTES KEY WORDS (Continue on reverse side 11 necessary and identify by block number ARMA Processes; Box-Jenkins method; S-array me ARMA Processes; Box-Jenkins method; S-array me Considered. An extension of the Gray, Kelley, method of estimating (p,q) is proposed and sho	ease and sale; its ole or in part is s Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the abstract misred in Black 20, 11 dillerent in SUPPLEMENTARY NOTES REY WORDS (Continue on reverse elde 11 necessary and identify by black number ARMA Processes; Box-Jenkins method; S-array me STRACT (Continue on reverse elde 11 necessary and identify by black number The pro-lem of estimating order, (p,q) of an A considered. An extension of the Gray, Kelley, method of estimating (p,q) is proposed and sho useful for processes whose spectra have a cert	ease and sale; its ole or in part is is Government.
DISTRIBUTION STATEMENT (of this Report) This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the abstract entered in Block 20, 11 dillorent in SUPPLEMENTARY NOTES SUPPLEMENTARY NOTES ARMA Processes; Box-Jenkins method; S-array me The pro-lem of estimating order, (p,q) of an A considered. An extension of the Gray, Kelley, method of estimating (p,q) is proposed and sho useful for processes whose spectra have a cert data is used to illustrate the usefulness of t	ease and sale; its ole or in part is is Government.
This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the obstract entered in Block 20, 11 dillerent in SUPPLEMENTARY NOTES SUPPLEMENTARY NOTES ARMA Processes; Box-Jenkins method; S-array me ARMA Processes; Box-Jenkins method; S-array me Considered. An extension of the Gray, Kelley, method of estimating (p,q) is proposed and sho useful for processes whose spectra have a cert data is used to illustrate the usefulness of t	ease and sale; its ole or in part is is Government.
TRACT (Continue on reverse side II necessary and identify by block number ARMA Processes; Box-Jenkins method; S-array me The pro-lem of estimating order, (p,q) of an A considered. An extension of the Gray, Kelley, method of estimating (p,q) is proposed and sho useful for processes whose spectra have a cert data is used to illustrate the usefulness of t	ease and sale; its ole or in part is s Government.
This document has been approved for public rel distribution is unlimited. Reproduction in wh permitted for any purposes of the United State DISTRIBUTION STATEMENT (of the observed in Block 20, 11 different in DISTRIBUTION STATEMENT (of the observed in Block 20, 11 different in DUPPLEMENTARY NOTES EY WORDS (Continue on reverse olde 11 necessary and identify by block number ARMA Processes; Box-Jenkins method; S-array me STRACT (Continue on reverse olde 11 necessary and identify by block number Considered. An extension of the Gray, Kelley, method of estimating (p,q) is proposed and sho useful for processes whose spectra have a cert data is used to illustrate the usefulness of t	ease and sale; its ole or in part is is Government. The Report thod. RMA (p,q) process is and McInitre (1978) won to be particularly ain form. Simulated the extension.

والمحاد المنبي الراجع

and the second second

