LEVEL II

AD A109369

DTIC FILE COPY

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | |
|---|---|
| 1. REPORT NUMBER N00014-76-C-0034 / 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) An Explicit Method of Solution of the Geomagnetic Induction Problem | 5. TYPE OF REPORT & PERIOD COVERED Technical Report, Nov.,'81 |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) Victor Barcilon | 8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0034 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS The University of Chicago | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 041-476 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 North Quincy Street Arlington, VA 22217 | 12. REPORT DATE Nov. 81 |
| | 13. NUMBER OF PAGES 22 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (see block 11) | 15. SECURITY CLASS. (of this report) Unclassified |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

DTIC ELECTE JAN 7 1982 S D

D

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Earth Conductivity
Inverse Problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The problem of inferring the Earth conductivity as a function of depth from surface measurements of the magnetic field is considered. Two versions of a method analogous to that of Bailey (1970) are presented. The main feature of the method consists in the fact that the conductivity is given explicitly in terms to two auxiliary sequences of functions which are found by integrating a set of first order nonlinear differential equations.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

81 12 14 011

An Explicit Method of Solution

of the Geomagnetic Induction Problem

by

Victor Barcilon
Department of the Geophysical Sciences
The University of Chicago
Chicago, IL 60637

ABSTRACT

The problem of inferring the Earth conductivity as a function of depth from surface measurements of the magnetic field is considered. Two versions of a method analogous to that of Bailey (1970) are presented. The main feature of the method consists in the fact that the conductivity is given explicitly in terms to two auxiliary sequences of functions which are found by integrating a set of first order nonlinear differential equations.

## 1. Introduction

The inverse geomagnetic problem has come to refer to the problem of retrieving the Earth conductivity from temporal measurements of the magnetic field at the Earth surface. For the case in which the conductivity can be assumed to be solely a function of depth, Bailey (1970) and Weidelt (1972) have provided explicit solutions of the governing mathematical problem. Their results are remarkable in view of the fact that this inverse problem is non-linear.

The mathematical formulation of the inverse geomagnetic problem leads to an inverse Sturm-Liouville problem: consequently, any advance on one front can be carried over to the other. As a matter of fact, the paper of Weidelt (1972) is an ingenious adaptation of a method proposed by Gelfand & Levitan (1956) for solving the canonical Sturm-Liouville problem. The present paper falls in the same category, and is concerned with the adaptation of a method for solving the Sturm-Liouville which I have proposed (Barcilon 1982).

The method of solution which I shall present, has some features in common with that of Bailey (1972). In particular, it provides an explicit, closed form expression for the conductivity. Also, its implementation requires the integration of a set of first order nonlinear equations. Finally, it requires some a-priori assumptions on the smoothness of the conductivity.

## 2. The Geomagnetic Induction Problem

We shall review the induction problem very succinctly and refer the readers to, for example, Le Mouël's (1976) article for an extensive discussion of the topic.

In a nutshell the situation is as follows: time varying magnetic fields of external origin, induce electric fields, and hence electric currents, inside the Earth. These electric currents, which are also time varying, in turn generate magnetic fields of internal origin. Surface measurements of magnetic fields can be processed in such a way as to distinguish between the externally and the internally generated parts. The ratios of these fields contain information about the conductivity of regions through which the currents have flowed.

The following simplified version of Maxwell's equations describes the situation:

$$\underset{\sim}{B}_t + \nabla \times \underset{\sim}{E} = 0 \ , \qquad (1)$$

$$\frac{1}{\mu_o} \nabla \times \underset{\sim}{B} = \sigma \underset{\sim}{E} \ , \qquad (2)$$

$$\nabla \cdot \underset{\sim}{B} = 0 \ . \qquad (3)$$

In view of the characteristic time scale of the phenomenon, we have made the quasi-static approximation in (2) and neglected the displacement current. Also in (2), the electric current is written via Ohm's law, $\sigma$ being the conductivity which is solely a function of radius r. The magnetic permeability has also been assumed to be a constant $\mu_o$, the free space permeability.

Because of the solenoidal nature of the magnetic field $\underset{\sim}{B}$, it is well known that (see e.g. Stern 1976)

$$\underset{\sim}{B} = - \nabla \times \nabla \times p \, \underset{\sim}{r} - \nabla \times q \, \underset{\sim}{r} \qquad (4)$$

where p $(r,\theta,\phi,t)$ and $q(r,\theta,\phi,t)$ are the poloidal and toroidal scalar potentials. Only the poloidal potential enters into the induction problem. Substituting the above expression for $\underset{\sim}{B}$ in terms of p only in (1) and (2) and eliminating $\underset{\sim}{E}$, we get:

$$\nabla^2 p = \mu_o \sigma \, p_t \, , \quad \text{for} \quad r < a \, , \tag{5}$$

a being the Earth radius, whereas in the non-conducting exterior:

$$\nabla^2 p = 0 \, , \quad \text{for} \quad r > a \, . \tag{6}$$

At the Earth surface, the three components of $\underset{\sim}{B}$ must be continuous because of our assumption about the permeability. As a result

$$[B_r] = [B_\theta] = [B_\phi] = 0 \, , \tag{7}$$

where [ ] stands for "jump across r=a", and the subscript $r,\theta,\phi$ refer to the spherical coordinates. Since $\sigma$ depends solely on r, we write the poloidal scalar as a series of spherical harmonics $\{Y_\ell^m(\theta,\phi)\}$ and as a Fourier transform integral over frequency, namely

$$p = \begin{cases} \dfrac{a}{2\pi} \displaystyle\int_{-\infty}^{+\infty} e^{-i\omega t} \, d\omega \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m(r,\omega) \, Y_\ell^m(\theta,\phi) \, , \quad r < a \, , \\[2em] \dfrac{a}{2\pi} \displaystyle\int_{-\infty}^{+\infty} e^{-i\omega t} \, d\omega \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \frac{E_\ell^m(\omega)}{\ell+1} \left(\frac{r}{a}\right)^{\ell} - \frac{I_\ell^m(\omega)}{\ell} \left(\frac{a}{r}\right)^{\ell+1} \right] Y_\ell^m(\theta,\phi) , \quad r>a \, . \end{cases} \tag{8}$$

In the above formula, $\omega$ is the frequency, $E_\ell^m(\omega)$ and $I_\ell^m(\omega)$ are for the time being coefficients entering in the most general solution of Laplace's equation (6). In order for p to satisfy (5) we must require that

$$\frac{d}{dr} r^2 \frac{df_\ell^m}{dr} - \ell(\ell+1) f_\ell^m + i\mu_0 \omega r^2 \sigma f_\ell^m = 0 \ , \quad \text{for } r < a \ . \tag{9}$$

By means of this representation for p, we can get expressions for the magnetic field outside the Earth, viz.

$$\begin{aligned} \underset{\sim}{B} = - \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} & [\ell E_\ell^m(\omega) \left(\frac{r}{a}\right)^{\ell-1} - (\ell+1) \ I_\ell^m(\omega) \left(\frac{a}{r}\right)^{\ell+2}] \ Y_\ell^m \ \hat{\underset{\sim}{r}} \\ & + [E_\ell^m(\omega) \left(\frac{r}{a}\right)^{\ell-1} + I_\ell^m(\omega) \left(\frac{a}{r}\right)^{\ell+2}] \ [\frac{\partial Y_\ell^m}{\partial\theta} \ \hat{\underset{\sim}{\theta}} + \frac{\partial Y_\ell^m}{\sin\theta \partial\phi} \ \hat{\underset{\sim}{\phi}} \ ] \ , \end{aligned} \tag{10}$$

and inside the Earth

$$\begin{aligned} \underset{\sim}{B} = - \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega \sum_{\ell,m} & \{\ell(\ell+1) \ \frac{a}{r} \ f_\ell^m \ (r,\omega) \ Y_\ell^m \ \hat{\underset{\sim}{r}} \\ & + \frac{a}{r} \frac{d}{dr} r f_\ell^m(r,\omega) \ [\frac{\partial Y_\ell^m}{\partial\theta} \ \hat{\underset{\sim}{\theta}} + \frac{1}{\sin\theta} \ \frac{\partial Y_\ell^m}{\partial\phi} \ \hat{\underset{\sim}{\phi}}]\} \end{aligned} \tag{11}$$

where $\hat{\underset{\sim}{r}}$, $\hat{\underset{\sim}{\theta}}$ and $\hat{\underset{\sim}{\phi}}$ are unit vectors in the r, θ and φ directions.

The continuity of $\underset{\sim}{B}$ across the Earth surface implies that

$$\begin{aligned} (\ell+1) \ f_\ell^m \ (a,\omega) &= \ell E_\ell^m(\omega) - (\ell+1) \ I_\ell^m(\omega) \ , \\ (\frac{d}{dr} \ rf)(a,\omega) &= E_\ell^m(\omega) + I_\ell^m(\omega) \ . \end{aligned} \tag{12}$$

By measuring the magnetic field at r=a, we can deduce $E_\ell^m(\omega)$ and $I_\ell^m(\omega)$: we shall look upon these quantities as the data for the inverse induction problem.

3. Properties of the Poloidal Field.

The solution of (9) has some properties which will play a crucial role in the reconstruction of $\sigma(r)$. These properties are associated with the dependence of $f_\ell^m(r,\omega)$ on the frequency $\omega$. In the process of deriving these properties we shall keep the angular numbers $\ell$ and m fixed. In fact, from now on we shall focus our attention on a single pair $(\ell,m)$ and drop all superscripts or superscripts referring to it.

We shall assume that the conductivity $\sigma$ is a differentiable function of r. One might be able to relax this assumption at the cost of complicating our presentation. For the sake of simplicity, we shall adopt this smoothness requirement. We should note at this stage that this assumption is weaker than the analyticity assumed by Bailey. Incidentally, in view of this assumption it follows that $\sigma(r)$ is bounded, i.e.

$$\sigma(r) \leq M \text{ , for } 0 \leq r \leq a \text{ .} \tag{13}$$

It also follows that the differential equation (9) has a regular singular point at r=0 and hence it has one solution which is analytic in that neighborhood (Coddington & Levinson, 1955, ). It is this analytic solution which we shall denote by $f(r,\omega)$. More specifically, $f(r,\omega)$ is that solution of (9) such that

$$f(r,\omega) \sim Ar^{\ell} \text{ , } \quad r \to 0 \tag{14}$$

where A is a constant.

We shall need a better representation of f in the neighborhood of the origin. It is obtained by replacing $\sigma(r)$ in (9) by its value at r=0, viz.,

$$f(r,\omega) \sim 1.3....(2\ell+1)A\, j_\ell\, [e^{i\pi/4}\, \mu_o^{\frac{1}{2}}\, \omega^{\frac{1}{2}}\, \sigma^{\frac{1}{2}}(0)r] \ , \quad r \to 0 \tag{15}$$

where $j_\ell(\ )$ is the spherical Bessel function of order $\ell$ [Abramowitz & Stegun 1972, p. 437].

We next turn to the dependence of f on $\omega$ and consider first the asymptotic behavior for large $\omega$'s. We resort to the WKBJ method (Olver 1974) and look for $f(r,\omega)$ in the following form:

$$f(r,\omega) \sim \exp[\pm ie^{i\pi/4}\, \mu_o^{\frac{1}{2}}\, \omega^{\frac{1}{2}} \int_o^r \sigma^{\frac{1}{2}}\, dr]\ \{a_o^{\pm}(r) + \frac{a_1^{\pm}(r)}{\omega^{\frac{1}{2}}} + \ldots\} \quad . \tag{16}$$

Substituting (16) in (9) and sorting terms according to powers of $\omega^{\frac{1}{2}}$ we deduce that†

---

†Use of the differentiability of $\sigma$ is made at this stage.

---

$$a_o^{\pm}(r) = \frac{\text{const}}{r\sigma^{\frac{1}{4}}(r)} \quad . \tag{17}$$

The asymptotic representation (16) is suitable for <u>both</u> solutions of (19). Therefore, we must select the constants entering in (17) in such a way as to pick out that solution which we are calling $f(r,\omega)$. To do so we must match the asymptotic representation for large $\omega$ with the representation (15) for small r.

We proceed as follows: first we note that (16) is only valid for r in the interval $(\mu_o^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}(o)\omega^{-\frac{1}{2}},a)$. This suggests that we replace r in (13) by $(\mu_o\sigma(o)\omega)^{-\frac{1}{2}}\alpha$ and consider $\alpha>>1$. Making use of standard results for Bessel functions (Abramowitz & Stegun 1972, p. 364) we deduce that

$$f([\mu_o\sigma(o)\omega]^{-\frac{1}{2}}\alpha,\omega) \sim 1.3\ldots(2\ell+1)A\, e^{-i\pi/4}\, \alpha^{-1} \cos[e^{i\pi/4}\alpha-(\ell+1)\tfrac{\pi}{2}]\ . \qquad (18)$$

This implies that

$$a_o^{\pm}(r) = 1.3\ldots(2\ell+1)A\, e^{-i\pi/4}\, \mu_o^{-\frac{1}{2}}\omega^{-\frac{1}{2}}\sigma^{-\frac{1}{4}}(o)\ \frac{e^{\pm i(\ell+1)\frac{\pi}{2}}}{2}\ . \qquad (19)$$

Indeed, the asymptotic representation

$$f(r,\omega) \sim \frac{1.3\ldots(2\ell+1)A\, e^{-i\pi/4}}{\mu_o^{\frac{1}{2}}\sigma^{\frac{1}{4}}(o)\, r\,\sigma^{\frac{1}{4}}(r)\,\omega^{\frac{1}{2}}}\ \cos[e^{\frac{i\pi}{4}}\ \mu_o^{\frac{1}{2}}\omega^{\frac{1}{2}}\int^{r}\sigma^{\frac{1}{2}}(\rho)d\rho\ -\ (\ell+1)\tfrac{\pi}{2}]\ , \qquad (20)$$

agrees with (18), and hence (15), for small r's. We shall also need the asymptotic form of $\frac{d}{dr}$ (rf) which is obtained by differentiation:

$$\frac{d}{dr}(rf) \sim -1.3\ldots(2\ell+1)A\ \sigma^{-\frac{1}{4}}(o)\ \sigma^{\frac{1}{4}}(r)\ \sin[e^{i\pi/4}\mu_o^{\frac{1}{2}}\omega^{\frac{1}{2}}\int_o^{r}\sigma^{\frac{1}{2}}dr\ -\ (\ell+1)\tfrac{\pi}{2}]\ . \qquad (21)$$

We next turn out attention to another aspect of the same question we have been investigating, namely the dependence of $f(r,\omega)$ on the frequency. We state that f is an entire function of $\omega$, i.e. an analytic function of $\omega$ over the entire complex $\omega$-plane. The proof follows a classical procedure (see e.g. Titchmarsh 1962, p. 6). We write f as a Taylor series in $\omega$:

$$f(r,\omega) = \sum_o^{\infty} f_n(r)\ \omega^n \qquad (22)$$

where

$$f_o'' + \frac{2}{r} f_o' - \frac{\ell(\ell+1)}{r^2} f_o = 0 \qquad (23a)$$

and

$$f_n'' + \frac{2}{r} f_n' - \frac{\ell(\ell+1)}{r^2} f_n = -i\mu_o \sigma f_{n-1}; \quad (n\geq 1) \tag{23b}$$

also

$$\lim_{r\to o} r^{-\ell} f_n(r) = A\, \delta_{on} . \tag{24}$$

Clearly

$$f_o = Ar^{\ell}$$

and

$$f_n = \frac{i\mu_o}{2\ell+1} \int_o^r \left( \frac{\rho^{\ell+2}}{r^{\ell+1}} - \frac{r^{\ell}}{\rho^{\ell-1}} \right) \sigma(\rho) f_{n-1}(\rho)\, d\rho . \tag{25}$$

It is a simple matter to show that

$$|f_n(r)| \leq \frac{\mu_o^n M^n r^{\ell+2n} A}{2.4\ldots(2n) \cdot (2\ell+3)(2\ell+5)\ldots(2\ell+2n+1)}$$

and hence that the Taylor series converges everywhere in the complex $\omega$-plane. Actually, we can go further and state that $f(r,\omega)$ is an entire function of $\omega$ of order ½[†].

---

[†]See (Boas 1954, p. 8) for a definition of "order".

This can be seen from our previous discussion of the asymptotic behavior of $f(r,\omega)$. The same can be said about $\frac{d}{dr}(rf)$, i.e. it is also an entire function of order ½.

The above results have an important consequence. They enable us to write $f(r,\omega)$ and $\frac{d}{dr}(rf)$ as infinite products (Boas 1954, p. 22) of very specific form, namely

$$f(r,\omega) = f(r,o) \prod_{n=1}^{\infty} \left(1 + \frac{\omega}{i\lambda_n(r)}\right) , \tag{26}$$

$$\psi(r,\omega) \equiv \frac{d}{dr}(rf) = \psi(r,o) \prod_{n=1}^{\infty} \left(1 + \frac{\omega}{i\mu_n(r)}\right) . \tag{27}$$

For a fixed value of r, say $r=\rho$, $\{-i\lambda_n(\rho)\}_1^{\infty}$ and $\{-i\mu_n(\rho)\}_1^{\infty}$ are the zeros of f and $\psi$ viewed as functions of $\omega$. In other words, $\{\lambda_n(\rho)\}_1^{\infty}$ and $\{\mu_n(\rho)\}_1^{\infty}$ are the eigenvalues of the following two eigenvalue problems:

$$\begin{aligned} &u_n'' + \frac{2}{r}u_n' - \frac{\ell(\ell+1)}{r^2}u_n + \mu_o\lambda_n\sigma u_n = 0 , \quad r\varepsilon(o,\rho) \\ &u \quad \text{finite at } r = 0 , \\ &u = 0 \quad \text{at } r = \rho , \end{aligned} \tag{28a}$$

and

$$\begin{aligned} &v_n'' + \frac{2}{r}v_n' - \frac{\ell(\ell+1)}{r^2}v_n + \mu_o\nu_n\sigma v_n = 0 , \quad r\varepsilon(o,\rho) \\ &v \quad \text{finite at } r = 0 , \\ &(rv)' = 0 \quad \text{at } r = \rho . \end{aligned} \tag{28b}$$

The reason for making this connection with eigenvalue problems lies in the fact that (28a) and (28b) enable us to show that

$$\lambda_n(\rho) > 0 \ , \quad \text{for} \quad n=1,2,\ldots \tag{29}$$

and

$$\nu_n(\rho) > 0 \ , \quad \text{for } n=1,2,\ldots$$

Finally, we can easily find the functions $f(r,o)$ and $\psi(r,o)$ appearing in (26) and (27). Thus, regardless of the conductivity profile $\sigma(r)$, we can say that the radial poloidal field has the following form:

$$f(r,\omega) = Ar^{\ell} \prod_{n=1}^{\infty} \left(1 + \frac{\omega}{i\lambda_n(r)}\right) \tag{26'}$$

where $\{\lambda_n(r)\}_1^{\infty}$ is an increasing sequence of positive numbers. Similarly

$$\psi(r,\omega) = (\ell+1)\ A\ r^{\ell} \prod_{n=1}^{\infty} \left(1 + \frac{\omega}{i\nu_n(r)}\right) \ . \tag{27'}$$

4. An Explicit Formula for the Conductivity

There are several ways of obtaining formulas expressing $\sigma(r)$ in terms of $\{\lambda_n(r)\}_1^{\infty}$ and $\{\nu_n(r)\}_1^{\infty}$. We shall present two formulas.

Let us consider the following contour integral:

$$J(r) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f^2(r,\omega)}{\psi^2(r,\omega)} \, d\omega \tag{30}$$

where $\Gamma$ is a circle in the $\omega$-plane of infinite radius. Using the calculus of residues, we can deduce that

$$J(r) = \sum_{n=1}^{\infty} \frac{2f(r,-i\nu_n)\dot{f}(r,-i\nu_n)}{\dot{\psi}^2(r,-i\nu_n)} \tag{31}$$

where a dot stands for differentiation with respect to $\omega$. On the other hand, since the path of integration is an infinitely large circle, we can replace f and $\psi$ by their asymptotic representations as given in (20) and (21), i.e.

$$J(r) = \frac{1}{2\pi i} \oint_{|\omega| \to \infty} \frac{e^{-i\pi/2}}{\mu_o r^2 \sigma(r) \omega} \cdot \cot \left[ e^{i\pi/4} \mu_o^{1/2} \omega^{1/2} \int_0^r \sigma^{1/2} d\rho - (\ell+1) \frac{\pi}{4} \right] d\omega$$

Clearly

$$J(r) = \frac{-i}{\mu_o r^2 \sigma(r)} . \tag{32}$$

Equating these two evaluations of J(r) we conclude that

$$\frac{1}{\sigma(r)} = -2\mu_o r^2 \sum_{n=1}^{\infty} \nu_n^2(r) \frac{\prod_k \left(1 - \frac{\nu_n(r)}{\lambda_k(r)}\right) \sum_j \frac{1}{\lambda_j(r)} \prod_{k \neq j} \left(1 - \frac{\nu_n(r)}{\lambda_k(r)}\right)}{\prod_{k \neq n} \left(1 - \frac{\nu_n(r)}{\nu_k(r)}\right)^2} . \tag{33}$$

A second formula for $\sigma(r)$ can be derived in terms of $\{\lambda_n(r)\}_1^\infty$ and a third auxiliary spectrum $\{\kappa_n(r)\}_1^\infty$. To that effect, it is convenient to introduce two different combinations of f and $\psi$:

$$F(r,\omega) = r^{-\ell} f(r,\omega) \tag{34}$$

and

$$\Theta(r,\omega) = r^{\ell+1}\{\psi(r,\omega) - (\ell+1)f(r,\omega)\} \tag{35}$$

It follows from these definitions that F and $\Theta$ are also entire functions of $\omega$ of order ½. Their product representations which are

$$F(r,\omega) = A\prod_1^\infty (1 + \frac{\omega}{i\lambda_n(r)}) \tag{36}$$

and

$$\Theta(r,\omega) = -i\mu_o A\ S(r)\ \omega\prod_1^\infty (1 + \frac{\omega}{ik_n(r)}) , \tag{37}$$

where

$$S(r) = \int_o^r \rho^{2(\ell+1)}\sigma(\rho)d\rho , \tag{38}$$

are obtained partly from the definitions (34) and (35), partly from considering the equations

$$F' = r^{-2(\ell+1)}\Theta ,$$
$$\Theta' = -i\mu_o \omega r^{2(\ell+1)}\sigma F , \tag{39}$$

and partly from the eigenvalue problem:

$$w_n'' + \frac{2}{r} w_n' - \frac{\ell(\ell+1)}{r^2} w + \mu_o \kappa_n(\rho)\sigma w_n = 0 ,$$

$$w \text{ finite at } r = 0 ,$$

$$rw' - \ell w = 0 \text{ at } r = \rho.$$

The next step consists in the discretization of (39). This is accomplished by replacing $\sigma(r)$ in (39) by an expression involving sums of delta functions:

$$\sigma(r) = \sum_{j=1}^{N} s_j \ \delta(r-r_j) \ . \tag{40}$$

In other words, we consider model Earths made up of N conducting spherical shells located at $r_1$, $r_2$, ... (Parker 1980). It is convenient to define $r_o$=0 and $r_{N=1}$=a. Denoting by $F_j(\omega)$ and $\Theta_j(\omega)$ the values of $F(r,\omega)$ and $\Theta(r,\omega)$ at $r=r_j+0$ and substituting (40) in (39) we deduce that

$$F_{j+1} = F_j + \frac{1}{2\ell+1} \left[ \frac{1}{r_j^{2\ell+1}} - \frac{1}{r_{j+1}^{2\ell+1}} \right] \Theta_j , \tag{41a}$$

$$\Theta_{j+1} = \Theta_j - i\mu_o(\omega) \ r_{j+1}^{2(\ell+1)} \ s_{j+1} \ F_{j+1} . \tag{41b}$$

Our goal once again is to obtain a formula for the conductivity at the

Earth surface. We shall accomplish our goal by considering the limit of the expression for $s_N/a-r_N$ as N tends to infinity, i.e. as we approach the continuum limit.

In analogy with the continuous case, we set

$$F_o = A \ ,$$

$$\Theta_o = 0 \ .$$

Then, it follows from (41) that $F_{N=1}$ and $\Theta_N$ are polynomials in $\omega$ with the following structures:

$$F_{N+1}(\omega) = A \prod_{k=1}^{N} \left(1 + \frac{\omega}{i\lambda_k^{(N)}(a)}\right) \ , \tag{42a}$$

$$\Theta_N(\omega) = -i\mu_o A\left\{\sum_{j=1}^{N} r_j^{2(\ell+1)} s_j\right\}\omega \prod_{k=1}^{N-1} \left(1 + \frac{\omega}{i\kappa_k^{(N)}(a)}\right) \ . \tag{42b}$$

(We have used notations which are reminiscent of those for the continuous case; e.g. $\{-i\lambda_\kappa^{(N)}(a)\}_1^N$ are the zeros of $F_{N+1}$ and as $N\to\infty$, these zeros will coincide with those of $F(\omega,a)$). Substituting (42a), (42b) in (41a) and performing a long division in $\omega$, we deduce that

$$\frac{1}{r_N^{2\ell+1}} = \frac{1}{a^{2\ell+1}} + \frac{2\ell+1}{\mu_o \sum_1^N r_j^{2(\ell+1)} s_j} \cdot \frac{\prod_1^{N-1} \kappa_k^{(N)}(a)}{\prod_1^{N} \lambda_k^{(N)}(a)} \ . \tag{43}$$

In the process, we can also find $F_N(\omega)$ which has a form similar to that of $F_{N+1}(\omega)$, namely

$$F_N(\omega) = A \prod_{k=1}^{N-1} \left(1 + \frac{\omega}{i\lambda_k^{(N-1)}(r_N)}\right) . \tag{44}$$

Substituting (42b) and (44) in (41b) and performing another division in $\omega$, we get

$$s_N = \frac{\sum_1^N r_j^{2(\ell+1)} s_j}{r_N^{2(\ +1)}} \prod_1^{N-1} \frac{\lambda_k^{(N-1)}(r_N)}{\kappa_k^{(N)}(a)} . \tag{45}$$

The desired formula for the conductivity is therefore

$$\sigma(a) = \lim_{N\to\infty} \frac{s_N}{a-r_N}$$

$$= \lim_{N\to\infty} \frac{(\sum_1^N r_j^{2(\ell+1)} s_j)}{(a-r_N) r_N^{2(\ell+1)}} \cdot \prod_1^{N-1} \frac{\lambda_k^{(N-1)}(r_N)}{\kappa_k^{(N)}(a)}$$

But, as $N\to\infty$ we can show that $r_N \to a$. Therefore we can approximate (43) as follows

$$r_N \doteq a\left[1 - \frac{a^{2\ell+1}}{\mu_o \sum_1^N r_j^{2(\ell+1)} s_j} \frac{\prod^{N-1} \kappa_k^{(N)}(a)}{\prod_1^N \lambda_k^{(N)}(a)}\right]$$

and consequently

$$\sigma(a) = \lim_{N\to\infty} \frac{\mu_o \left(\sum_1^N r_j^{2(\ell+1)} s_j\right)^2}{(r_N a)^{2\ell+2}} \lambda_1^{(N)}(a) \prod_1^{N-1} \frac{\lambda_{k+1}^{(N)}(a)\lambda_k^{(N-1)}(r_N)}{\kappa_k^{(N)^2}(a)}$$

i.e.

$$\sigma(a) = \mu_o \frac{S^2(a)}{a^{4(\ell+1)}} \lambda_1(a) \prod_1^{\infty} \frac{\lambda_{k+1}(a)\lambda_k(a)}{\kappa_k^2(a)} . \tag{46}$$

A similar formula can be written for an arbitrary radius:

$$\sigma(r) = \mu_o \frac{S^2(r)}{r^{4(\ell+1)}} \lambda_1(r) \prod_1^{\infty} \frac{\lambda_{k+1}(r)\lambda_k(r)}{\kappa_k^2(r)} . \tag{47}$$

5. Solution of the Inverse Geomagnetic Problem.

So far we have derived results which hold for general conductivity profiles. We would like to exploit these results to retrieve $\sigma(r)$.

Since the auxiliary spectra $\{\lambda_n\}$ and $\{\nu_n\}$ on the one hand or $\{\lambda_n\}$ and $\{\kappa_n\}$ on the other, enter so prominently in the formulas for $\sigma$, we express the data (12) in terms of these spectra:

$$\ell E(\omega)-(\ell+1)I(\omega) = A\ell(\ell+1)\ a^{\ell} \prod_{n=1}^{\infty} \left(1 + \frac{\omega}{i\lambda_n(a)}\right)$$

$$E(\omega)+I(\omega) = A(\ell+1)\ a^{\ell} \prod_{n=1}^{\infty} \left(1 + \frac{\omega}{i\nu_n(a)}\right) \tag{48}$$

$$-(\ell^2+\ell-1)E(\omega) + (\ell^2+2\ell+2)I(\omega) = -i\mu_o a^{-(\ell+1)} AS(a)\omega \prod \left(1 + \frac{\omega}{ik_n(a)}\right)$$

or better still

$$|\ell E(\omega)-(\ell+1)I(\omega)|^2 = A^2\ell^2(\ell+1)^2a^{2\ell}\prod_1^\infty\left(1-\frac{\omega^2}{\lambda_n^2(a)}\right),$$

$$|E(\omega)+I(\omega)|^2 = A^2(\ell+1)^2a^{2\ell}\prod_1^\infty\left(1-\frac{\omega^2}{\nu_n^2(a)}\right), \tag{49}$$

$$|-(\ell^2+\ell-1)E(\omega)+(\ell^2+2\ell+2)I(\omega)|^2 = \mu_o a^{-2(\ell+1)}A^2S^2(a)\omega^2\prod_1^\infty\left(1-\frac{\omega^2}{\kappa_n^2(a)}\right).$$

In practice, approximations to the first few values of $\lambda_n(a)$, $\nu_n(a)$ and $\kappa_n(\omega)$ could be obtained by approximating the left hand sides of (49) by polynomials in $\omega^2$ with real, positive zeros. Note also that we can extract the value S(a) of the moment of $\sigma$.

In order to generate $\{\lambda_n(r)\}$, $\{\nu_n(r)\}$ and $\{\kappa_n(r)\}$ for arbitrary values of r, we note that

$$f(r,-i\lambda_n(r)) = 0 ,$$

$$\psi(r,-i\nu_n(r)) = 0 ,$$

$$\Theta(r,-i\kappa_n(r)) = 0 ,$$

Differentiating with respect to r and making use of the product representations (26'), (27') and (36) we infer that

$$\frac{d\lambda_n}{dr} = -(\ell+1)\frac{\lambda_n(r)}{r}\frac{\prod_{k=1}^\infty(1-\lambda_n(r)/\nu_k(r))}{\prod_{k\neq n}^\infty(1-\lambda_n(r)/\lambda_k(r))}, \tag{50}$$

$$\frac{d\nu_n}{dr} = \frac{\nu_n(r)}{\ell+1}\left\{\frac{\ell(\ell+1)}{r} - \mu_o\nu_n(r)r\sigma(r)\right\}\frac{\prod_{k=1}^{\infty}(1-\nu_n(r)/\lambda_k(r))}{\prod_{k\neq n}^{\infty}(1-\nu_n(r)/\nu_k(r))}, \tag{51}$$

and

$$\frac{d\kappa_n}{dr} = r^{2(\ell+1)}\frac{\sigma(r)}{S(r)}\kappa_n(r)\frac{\prod_{k=1}^{\infty}(1-\kappa_n(r)/\lambda_k(r))}{\prod_{k\neq n}^{\infty}(1-\kappa_n(r)/\kappa_k(r))}. \tag{52}$$

Incidentally, by starting from the identity

$$F(r,-i\lambda_n(r)) = 0 ,$$

we could deduce similarly that

$$\frac{d\lambda_n}{dr} = -\frac{\mu_o\lambda_n^2(r)}{r^{2(\ell+1)}}S(r)\frac{\prod_{k=1}^{\infty}(1-\lambda_n(r)/\kappa_k(r))}{\prod_{k\neq n}^{\infty}(1-\lambda_n(r)/\lambda_k(r))}. \tag{53}$$

Two routes are available to solve the inverse induction problem. The first consists in integrating the first order differential equations (50) and (51) for $\{\lambda_n(r)\}_1^\infty$ and $\{\nu_n(r)\}_1^\infty$ subject to the initial conditions $\{\lambda_n(a)\}_1^\infty$ and $\{\nu_n(a)\}_1^\infty$ deduced from the data. The conductivity, which incidentally enters in (51), is given by (33).

The second approach consists in integrating (52) and (53) and of relying upon (47) for an explicit calculation of $\sigma(r)$. The fact that S(r) enters in (52) and (53) is handled by writing

$$\frac{dS}{dr} = r^{2(\ell+1)}\sigma(r) \tag{54}$$

and recalling that S(a) can be inferred from the data.


**Acknowledgment**

This research was supported by the Office of Naval Research under Contract N00014-76-C-0034. Reproduction in whole or in part is permitted for any purpose of the United States Government.

## References

Abramowitz, M. & Stegun, I.A., 1972. *Handbook of Mathematical Functions*, Dover, N.Y.

Bailey, R.C., 1970. Inversion of the geomagnetic induction problem, *Proc. R. Soc.*, 135A, 185-194.

Barcilon, V., 1982. Explicit solution of the inverse problem for a vibrating string. *J. Math. Anal. Appl.* (to appear).

Boas, R.P., 1954. *Entire Functions*, Acad. Press, N.Y.

Coddington, E.A. & Levinson, N., 1955. *Theory of Ordinary Differential Equations*, McGraw-Hill, N.Y.

Gelfand, I.M. & Levitan, B.M., 1956. On the Determination of a Differential Equation from its Spectral Function, *Amer. Math. Soc. Trans.* 1, 253-304.

Le-Mouël, J.-L., 1976. L'induction dans le globe, Chap. 29, pp. 129-159 in *Traite' de Geophysique Interne*, eds. Coulumb, J. & Jobert, G., vol. 2, Masson Paris.

Olver, F.W.J., 1974. *Introduction to Asymptotics and Special Functions*, Academic Press, N.Y.

Parker, R.L., 1980. The Inverse Problem of Electromagnetic Induction, *JGR*, 85, 4421-4428.

Stern, D.P., 1976. Representation of Magnetic Fields in Space, *Rev. Geophys. Space Phys.*, 14, 199-214.

Titchmarch, E.C., 1969. *Eigenfunction Expansions, Vol. 1*, Oxford.

Weidelt, P., 1972. The Inverse Problem of Geomagnetic Induction, *Z. Geophys.*, 38, 257-289.